Polynomiality of the structure coefficients of double-class algebras

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72nd Séminaire Lotharingien de Combinatoire
Université Claude Bernard Lyon 1, March 2014

Plan.

I. **Introduction: structure coefficients of an algebra**

II. **Partitions**

III. **Two polynomiality results**

   1. $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$
   2. Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$

IV. **Structure coefficients of the double-class algebra**

V. **Conclusions and further applications**

I. Introduction: structure coefficients of an algebra

- **Problem:** Let $\mathcal{A}$ be an algebra over a field F with basis $b_1, b_2, \cdots, b_n$. For two basis elements, say b_i and b_j, write:

$$b_i b_j = \sum_k c_{i,j}^k b_k,$$

where $c_{i,j}^k \in F$. The elements $c_{i,j}^k$ are called the structure coefficients of $\mathcal{A}$ and there is no explicit formula for them, even in the particular algebras we will consider.

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- **Our work:**
  1- A framework in which one can obtain the form of the structure coefficients of the double-class algebra.[1]
  2- A polynomiality property of these coefficients in some specific cases.

[1] These coefficients "contain" structure coefficients of centres of groups algebras.

II. Partitions

- A partition λ is a list of integers $(\lambda_1, \lambda_2, \ldots)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 1$. The λ_i are called the parts of λ. The size of a partition λ (noted $|\lambda|$) is the sum of all its parts.
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  The proper partitions will be used to index bases of the algebras considered in this talk.

III. Two polynomiality results

1. Center of the symmetric group algebra $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$

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- Every element of $\mathcal{S}_n$ can be written (in a unique way) as a product of disjoint cycles.
- For a permutation $\omega \in \mathcal{S}_n$, we define the cycle-type of ω, $ct(\omega)$, to be the partition of n with parts equal to the lengths of the cycles that appear in its decomposition.

  Example: $\omega = 2\quad 6\quad 5\quad 4\quad 3\quad 1 = \begin{pmatrix}1 & 2 & 6\end{pmatrix}\begin{pmatrix}5 & 3\end{pmatrix}\begin{pmatrix}4\end{pmatrix}.$

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  $ct(\omega) = (3, 2, 1)$.
- The center of the symmetric group algebra, $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$, is:

$$\mathcal{Z}(\mathbb{C}[\mathcal{S}_n]) = \{x \in \mathbb{C}[\mathcal{S}_n] \mid x \cdot y = y \cdot x \;\forall y \in \mathbb{C}[\mathcal{S}_n]\}.$$

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- The family $(S_\lambda(n))_{|\lambda| \leqslant n}$ indexed by proper partitions of size at most n forms a basis for $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$, where,

$$S_\lambda(n) = \sum_{\substack{\omega \in \mathcal{S}_n, \\ ct(\omega) = \lambda \cup (1^{n-|\lambda|})}} \omega$$

- For λ and δ two proper partitions with size at most n,

$$S_\lambda(n) \cdot S_\delta(n) = \sum_{\substack{\rho \text{ proper partition} \\ |\rho| \leqslant n}} c_{\lambda,\delta}^{\rho}(n) S_\rho(n),$$

  the numbers $c_{\lambda,\delta}^{\rho}(n)$ are the structure coefficients of the center of the symmetric group algebra $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$.

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  the numbers $c_{\lambda,\delta}^{\rho}(n)$ are the structure coefficients of the center of the symmetric group algebra $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$.

- Motivation (Cori [1975]:) The structure coefficients of the center of the symmetric group algebra count the number of embedded graphs into orientable surfaces with some conditions.

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- Theorem (Farahat and Higman [1958]): Let λ, δ and ρ be three proper partitions, the function:

$$n\longmapsto c_{\lambda,\delta}^{\rho}(n)$$

  defined for $n\geqslant |\lambda|, |\delta|, |\rho|$ is a polynomial in n.
  Example: One can compute explicitly:
  $S_{(2)}(n)\cdot S_{(2)}(n)=\frac{n(n-1)}{2}S_{\emptyset}(n)+3S_{(3)}(n)+2S_{(2^2)}(n).$

III. Two polynomiality results
2. Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$

- The Hyperoctahedral group $\mathcal{B}_n$ is the subgoup of $\mathcal{S}_{2n}$ consisting of all permutations of $\mathcal{S}_{2n}$ which takes every pair of the form $\{2k-1, 2k\}$ of $[2n]$ to another pair with the same form.
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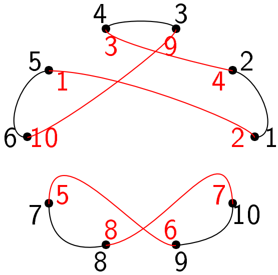
  Example: $\beta = 43\,87\,56\,21 \in \mathcal{B}_4$
- To each permutation ω of $2n$ we associate a graph $\Gamma(\omega)$.
  Example: Take $\omega = 2\,4\,9\,3\,1\,10\,5\,8\,6\,7 \in \mathcal{S}_{10}$.



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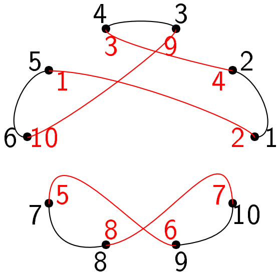

- The coset-type of a permutation x of $\mathcal{S}_{2n}$ is a partition of n with parts equal to half of lengths of the cycles of $\Gamma(x)$.
  Example: coset-type$(\omega) = (3, 2)$.

- Proposition: Let $x \in \mathcal{S}_{2n}$, we have:

$$\begin{aligned} \mathcal{B}_n x \mathcal{B}_n &:= \{bxb' \mid b, b' \in \mathcal{B}_n\} \\ &= \{y \in \mathcal{S}_{2n} \mid coset-type(y) = coset-type(x)\}. \end{aligned}$$

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- The Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ denoted by $\mathbb{C}[\mathcal{B}_n \backslash \mathcal{S}_{2n} / \mathcal{B}_n]$ is the algebra over $\mathbb{C}$ with basis the elements $(S'_\lambda(n))_{|\lambda| \leqslant n}$ indexed by proper partitions with size at most n, where

$$S'_\lambda(n) = \sum_{\substack{\omega \in \mathcal{S}_{2n} \\ coset-type(\omega) = \lambda \cup (1^{n-|\lambda|})}} \omega.$$

- For λ and δ two proper partitions with size at most n,

$$S'_\lambda(n) \cdot S'_\delta(n) = \sum_{\substack{\rho \text{ proper partition} \\ |\rho| \leqslant n}} {c'}^{\rho}_{\lambda,\delta}(n) S'_\rho(n),$$

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  the numbers $c'^{\rho}_{\lambda,\delta}(n)$ are the structure coefficients of the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$.

- Theorem (Dołęga and Féray [2012], T. [2013]): Let λ, δ and ρ be three proper partitions, we have:

$$c'^{\rho}_{\lambda\delta}(n) = \begin{cases} 2^n n! f^{\rho}_{\lambda\delta}(n) & \text{if} \quad n \geqslant |\rho|, \\ 0 & \text{if} \quad n < |\rho|, \end{cases}$$

  where $f^{\rho}_{\lambda\delta}(n)$ is a polynomial in n.
  Example: For every $n \geqslant 4$, we have:

$$S'_{(2)}(n) \cdot S'_{(2)}(n) = 2^n n! \Big(n(n-1) S'_{\emptyset}(n) + 1 S'_{(2)}(n) + 3 S'_{(3)}(n) + 2 S'_{(2^2)}(n) \Big).$$

IV. Structure coefficients of the double-class algebra

- Let $(G_n, K_n)_n$ be a sequence where G_n is a group and K_n is a sub-group of G_n for each n.

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- A double-class of K_n in G_n is a set $\overline{g}^n := K_n g K_n$, for a $g \in G_n$,

$$K_n g K_n = \{kgk' \,;\, k, k' \in K_n\}.$$

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- Let $\Re_n = \{\overline{x_1}^n, \cdots, \overline{x_{l(n)}}^n\}$ be the set of representative elements of the set of double-classes $K_n \backslash G_n / K_n$.

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- Let $\Re_n = \{\overline{x_1}^n, \cdots, \overline{x_{l(n)}}^n\}$ be the set of representative elements of the set of double-classes $K_n \backslash G_n / K_n$.
- Let $\overline{\mathbf{x_i}}^n$ be the sum of the elements in $\overline{x_i}^n$. The double-class algebra of K_n in G_n, denoted $\mathbb{C}[K_n \backslash G_n / K_n]$, is the algebra with basis the elements $\overline{\mathbf{x_i}}^n$.
  Example: The Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$, $\mathbb{C}[\mathcal{B}_n \backslash \mathcal{S}_{2n} / \mathcal{B}_n]$, is a double-class algebra.

- The product $\overline{\mathbf{x_i}}^n \cdot \overline{\mathbf{x_j}}^n$ can be written as follows:

$$\overline{\mathbf{x_i}}^n \cdot \overline{\mathbf{x_j}}^n = \sum_{1 \leqslant r \leqslant l(n)} c_{i,j}^r(n) \overline{\mathbf{x_r}}^n.$$

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- There is no explicit formula for these coefficients.
- Goals:
  1. The form of these structure coefficients under some conditions.
  2. Applications to the two specific cases: $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$ and the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$.

Define

$$k(X) := \min_{\substack{k \\ X \cap G_k \neq \emptyset}} k$$

Under some conditions, we have:
Theorem (T.): For $k_1 = k(\bar{x_i}^n)$, $k_2 = k(\bar{x_j}^n)$ and $k_3 = k(\bar{x_r}^n)$ there exists rational numbers $a^r_{i,j}(k)$ all independent of n such that:

$$c^r_{ij}(n) = \frac{|\bar{x_i}^n||\bar{x_j}^n||K_{n-k_1}||K_{n-k_2}|}{|K_n||\bar{x_r}^n|} \sum_{k_3 \leqslant k \leqslant \min(k_1+k_2,n)} \frac{a^r_{i,j}(k)}{|K_{n-k}|}$$

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Remark: We have a similar theorem for the structure coefficients of the centres of groups algebras.

Application to the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$: Let λ be a proper partition of size at most n. The size of its associated double class $S'_\lambda(n)$ is:

$$|S'_\lambda(n)| = \frac{(2^n n!)^2}{z_{2\lambda} 2^{n-|\lambda|}(n-|\lambda|)!},$$

where, $z_\lambda = \prod_{i\geqslant 1} i^{m_i(\lambda)} m_i(\lambda)!$.
Let δ and ρ be two proper partitions with size at most n, we have:

$$c'^{\rho}_{\lambda,\delta}(n) = 2^n n! \frac{z_{2\rho}}{z_{2\lambda} z_{2\delta}} \sum_{|\rho|\leqslant k\leqslant |\lambda|+|\delta|} a^{\rho}_{\lambda\delta}(k) 2^{k-|\rho|} \frac{(n-|\rho|)!}{(n-k)!}.$$

Polynomial!

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Polynomial!

Application to $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$: Let λ be a proper partition of size at most n. The size of its associated conjugacy class $S_\lambda(n)$ is:

$$|S_\lambda(n)| = \frac{n!}{z_\lambda \cdot (n-|\lambda|)!}.$$

Let δ and ρ be two proper partitions with size at most n, we have:

$$c^{\rho}_{\lambda,\delta}(n) = \frac{z_\rho}{z_\lambda z_\delta} \sum_{|\rho|\leqslant k\leqslant |\lambda|+|\delta|} a^{\rho}_{\lambda\delta}(k) \frac{(n-|\rho|)!}{(n-k)!}.$$

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V. Conclusions and further applications

Conclusions:
Under technical conditions,

1. Form of the structure coefficients of double-class algebras.
2. Form of the structure coefficients of centers of groups algebras.
3. We re-obtain the polynomiality property for $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$ and the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$.

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Work in progress:

1. $\mathcal{Z}(\mathbb{C}[GL_n(\mathbb{F}_q)])$, where $GL_n(\mathbb{F}_q)$ is the group of invertible $n \times n$ matrices.
2. Superclasses of unitriangular groups...