Isomorphisms of Hecke modules and parabolic Kazhdan-Lusztig polynomials

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Main results

Theorem

Let $I, J \subseteq S$, $u \in {}^{I}W^{J}$ *and* $v, w \in (W_I)^{I \cap u^{-1}Ju}$; *then*

$$R^{J,x}_{uv,uw} = R^{I \cap u^{-1}Ju,x}_{v,w},$$

$$P^{J,x}_{uv,uw} = P^{I \cap u^{-1}Ju,x}_{v,w}.$$

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Let $J \subseteq S$; then there exists a Coxeter system $(W', S \cup \{ s' \})$, where $s' \notin S$, such that

$$R^{J,x}_{v,w}[W] = R^{S,x}_{s'v,s'w}[W']$$

and

$$P^{J,x}_{v,w}[W] = P^{S,x}_{s'v,s'w}[W'],$$

for all $v, w \in W^J$.

Let S be a finite set and $m : S \times S \to \{ 1, 2, ..., \infty \}$ be a map such that

$$m(s,t) = 1, \text{ if } s = t,$$

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We define the Coxeter group W relative to the Coxeter matrix m by the presentation

$$(st)^{m(s,t)} = e,$$

for every $s, t \in S$, where e is the identity of the group and S is its generator set.

Coxeter groups

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We define a partial order on W, and we call it the Bruhat order of W, in the following way: given $u, v \in W$ and $v = s_1 s_2 ... s_q$ a reduced expression for v,

$$u \leqslant v \Leftrightarrow \text{there exists a reduced expression } u = s_{i_1} ... s_{i_k},\ 1 \leqslant i_1 < ... < i_k \leqslant q.$$

Coxeter groups - Descent sets and quotients

A right descent of $w \in W$ is an element $s \in S$ such that $\ell(ws) < \ell(w)$.

$$D_R(w) := \{ s \in S \mid \ell(ws) < \ell(w) \}.$$

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For any $J \subseteq S$ we define

$$W^J := \{ w \in W \mid \ell(sw) > \ell(w) \ \forall \ s \in J \},$$
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For $J \subseteq S$, each element $w \in W$ has a unique expression

$$w = w_J w^J,$$

where $w^J \in W^J$ and $w_J \in W_J$, and the subgroup $W_J \subseteq W$ is the group with $J \subseteq S$ as generator set. In particular $W_S = W$ and $W_\varnothing = \{ e \}$.

On W^J we consider the induced order. With $[u, v]^J$ we denote an interval in W^J, i.e., if $v, w \in W^J$,

$$[v, w]^J := \left\{ z \in W^J \;\middle|\; v \leqslant z \leqslant w \right\}.$$

Coxeter groups - Quotients and projections

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Define the canonical projection $P^J : W \to W^J$ by

$$P^J(w) = w^J,$$

and analogously $Q^J : W \to {}^J W$.

Coxeter groups - Projections

P^J and Q^J are morphisms of posets, i.e.

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Moreover, for any $I, J \subseteq S$,

$$P^J \circ Q^I = Q^I \circ P^J.$$

Hecke algebras

Let $A := \mathbb{Z}[q^{-1/2}, q^{1/2}]$ be the ring of Laurent polynomials in the indeterminate $q^{1/2}$. Recall that the Hecke algebra $\mathcal{H}(W)$ is the free A-module generated by the set $\{ T_w \mid w \in W \}$ with product defined by

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } s \notin D_R(w), \\ qT_{ws} + (q-1)T_w, & \text{otherwise,} \end{cases}$$

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For $s \in S$ we can easily see that

$$T_s^{-1} = (q^{-1} - 1)T_e + q^{-1}T_s$$

and then use this to invert all the elements T_w, where $w \in W$.

Hecke algebras - Involution

On $\mathcal{H}(W)$ there is an involution ι such that

$$\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}, \quad \iota(T_w) = T_{w^{-1}}^{-1},$$

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$$\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}, \quad \iota(T_w) = T_{w^{-1}}^{-1},$$

for all $w \in W$. Furthermore this map is a ring automorphism, i.e.

$$\iota(T_v T_w) = \iota(T_v)\iota(T_w) \;\; \forall \; v, w \in W.$$

The expansion of $\iota(T_w)$ in terms of the basis $\{ T_w \mid w \in W \}$ and the introduction of a ι-invariant basis $\{ C_w \}_{w \in W}$ of $\mathcal{H}(W)$ have lead to the definition of two families of polynomials $\{ R_{y,w} \}_{y,w \in W} \subseteq \mathbb{Z}[q]$ and $\{ P_{y,w} \}_{y,w \in W} \subseteq \mathbb{Z}[q]$ such that

$$\iota(T_w) = q^{-\ell(w)} \sum_{y \leqslant w} (-1)^{\ell(y,w)} R_{y,w}(q) T_y,$$

$$C_w = q^{\frac{\ell(w)}{2}} \sum_{y \leqslant w} (-1)^{\ell(y,w)} q^{-\ell(y)} P_{y,w}(q^{-1}) T_y,$$

for all $w \in W$.

Parabolic Kazhdan-Lusztig and R-polynomials

Theorem (V. Deodhar)

Let (W, S) a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1, q\}$, there is a unique family of polynomials $\left\{ R^{J,x}_{v,w} \right\}_{v,w \in W^J} \subseteq \mathbb{Z}[q]$ such that, for all $v, w \in W^J$:

1. $R^{J,x}_{v,w} = 0$ *if* $v \not\leq w$;
2. $R^{J,x}_{w,w} = 1$;
3. *if* $v < w$ *and* $s \in D_R(w)$ *then*

$$R^{J,x}_{v,w} = \begin{cases} R^{J,x}_{vs,ws}, & \text{if } s \in D_R(v), \\ qR^{J,x}_{vs,ws} + (q-1)R^{J,x}_{v,ws}, & \text{if } s \notin D_R(v) \text{ and } vs \in W^J, \\ (q-1-x)R^{J,x}_{v,ws}, & \text{if } s \notin D_R(v) \text{ and } vs \notin W^J. \end{cases}$$

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2. $P^{J,x}_{w,w} = 1$;
3. $\deg(P^{J,x}_{v,w}) \leqslant \frac{\ell(v,w)-1}{2}$, *if* $v < w$;
4. $$q^{\ell(v,w)} P^{J,x}_{v,w}(q^{-1}) = \sum_{z \in [v,w]^J} R^{J,x}_{v,z}(q) P^{J,x}_{z,w}(q),$$
   if $v \leqslant w$.

The Hecke modules $M^{J,I}_{u,x}$

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$$W^{J,I}_{u} := W^{J} \cap uW_{I}.$$

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$$M_u^{J,I} := \text{span}_A \left\{ m_v^{J,I} \;\middle|\; v \in W_u^{J,I} \right\}.$$

The Hecke module $M_{u,x}^{J,I}$

For all $w \in W_I$, there is an A-module morphism $\phi_u^{J,x} : \mathcal{H}(W_I) \to M_u^{J,I}$ defined by

$$\phi_u^{J,x}(T_w) = x^{\ell(P_J(uw))} m_{P^J(uw)}^{J,I},$$

where x is any element of the ring A.

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where x is any element of the ring A.

Definition (P. Sentinelli)

If $I, J \subseteq S$, $u \in {}^I W^J$ and $x \in \{-1, q\}$, we call $M_{u,x}^{J,I}$ the right $\mathcal{H}(W_I)$-module with A-basis indexed by the set $W_u^{J,I}$ and with $\mathcal{H}(W_I)$-action defined by

$$m_{uv}^{J,I} T_w := \phi_u^{J,x}(T_v T_w)$$

where $s_1 \cdots s_k$ is a reduced expression of $w \in W_I$ and $v \in (W_I)^{I \cap u^{-1} J u}$.

The Hecke module $M_{u,x}^{J,I}$

The isomorphism $W_u^{J,I} \simeq (W_I)^{I \cap u^{-1}Ju}$ induces an isomorphism of A-modules $\psi : M_{u,x}^{J,I} \to M_{e,x}^{I \cap u^{-1}Ju, I}$ defined by

$$\psi(m_{uv}^{J,I}) = m_v^{I \cap u^{-1}Ju, I},$$

for all $v \in (W_I)^{I \cap u^{-1}Ju}$.

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Theorem (P. Sentinelli)

The isomorphism of A-modules $\psi : M_{u,x}^{J,I} \to M_{e,x}^{I\cap u^{-1}Ju,I}$ is an isomorphism of right $\mathcal{H}(W_I)$-modules, for $x \in \{-1, q\}$. Moreover there is an isomorphism of $\mathcal{H}(W_I)$-modules

$$M_{e,x}^{J,S} \simeq \bigoplus_{v\in {}^I W^J} M_{e,x}^{I\cap v^{-1}Jv,I},$$

for $x \in \{-1, q\}$.

The Hecke module $M^{J,I}_{u,x}$ - An involution

We define a map $\iota^x : M^{J,I}_{u,x} \to M^{J,I}_{u,x}$ by

$$\iota^x(m^{J,I}_{uv}) := \phi^{J,x}_u(\iota(T_v))$$

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$$\iota^x \circ \phi_u^{J,x} = \phi_u^{J,x} \circ \iota.$$

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Moreover ι^x is an involution and

$$\iota^x(m_{uv}^{J,I} T_w) = \iota^x(m_{uv}^{J,I})\iota(T_w),$$

for all $w \in W_I$, $v \in (W_I)^{I \cap u^{-1}Ju}$.

Polynomials

Definition

If $I, J \subseteq S$ and $u \in {}^I W^J$ we define, for each $v, w \in (W_I)^{I \cap u^{-1} J u}$, elements $R^{u,J,I,x}_{uv,uw} \in A$ by

$$\iota^x(m^{J,I}_{uv}) = q^{-\ell(v)} \sum_{w \in (W_I)^{I \cap u^{-1} J u}} (-1)^{\ell(w,v)} R^{u,J,I,x}_{uw,uv} m^{J,I}_{uw}.$$

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Theorem (P. Sentinelli)

Let $I, J \subseteq S$*,* $u \in {}^I W^J$ *and* $v, w \in (W_I)^{I \cap u^{-1} Ju}$*; then*

$$R^{u,J,I,x}_{uv,uw} = R^{J,x}_{uv,uw} = R^{I \cap u^{-1} Ju, x}_{v,w}.$$

Corollary

Let $I, J \subseteq S$ *and* $v, w \in W^J$ *be such that* $Q^I(v) = Q^I(w) = u$*. Then*

$$P^{J,x}_{v,w} = P^{I \cap u^{-1} J u, x}_{Q_I(v), Q_I(w)}$$

Polynomials

Theorem (P. Sentinelli)

Let $J \subseteq S$; then there exists a Coxeter system $(W', S \cup \{ s' \})$, where $s' \notin S$, such that

$$R^{J,x}_{v,w}[W] = R^{S,x}_{s'v,s'w}[W']$$

and

$$P^{J,x}_{v,w}[W] = P^{S,x}_{s'v,s'w}[W'],$$

for all $v, w \in W^J$.

Dimostrazione.

Let m be the Coxeter matrix of (W, S) and let $(W', S \cup \{ s' \})$ be the Coxeter system defined by the following Coxeter matrix m':

$$m'(s,t) = \begin{cases} m(s,t), & \text{if } s, t \in S, \\ 2, & \text{if } s = s' \text{ and } t \in J, \\ 3, & \text{if } s = s' \text{ and } t \in S \setminus J. \end{cases}$$

Now take $u = s'$. Then $u \in {}^{S}W'^{S}$ and $S \cap u^{-1}Su = S \cap s'Ss' = J$. □

Corollaries

Definition

Let (W', S) *be an irreducible and nearly finite Coxeter system. We let*

$$S_f := \{\, s \in S \mid (W, S \setminus \{\, s \,\}) \text{ is irreducible and finite} \,\}\,.$$

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Definition

Let (W', S) be an affine Coxeter system with Coxeter matrix m. Let $s \in S_f$ and $J = \{ t \in S \mid m(s,t) = 2 \}$. The quotient W^J of the finite Coxeter system $(W, S \setminus \{ s \})$ is then called quasi-minuscule.

Corollaries

Corollary

Let W^J *be a quasi-minuscule quotient of a finite Coxeter system* (W, S). *Then*

$$P^{J,x}_{v,w}[W] = P^{S,x}_{s_0 v, s_0 w}[W'],$$

for all $v, w \in W^J$, *where* $(W', S \cup \{ s_0 \})$ *is the affine Coxeter system of* W.

Corollaries

Corollary (from results of P. Mongelli)

Let (W, S) be an affine Coxeter system and $s \in S_f$. Then the polynomial $P^{S\setminus\{s\},q}_{sv,sw}$ is either zero or a monic power of q, for all $v, w \in W_{S\setminus\{s\}}$.

Corollaries

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Take the Coxeter system A_n *with generators* $S = \{ s_1, ..., s_n \}$ *and* B_{n+1} *with generators* $S' = S \cup \{ s_0 \}$*. Then, for all* $j \in \{ 2, ..., n \}$*,*

$$P^{S \setminus \{ s_1, s_j \}, x}_{v,w}[A_n] = P^{S' \setminus \{ s_{j-1} \}, x}_{uv,uw}[B_{n+1}],$$

for all $v, w \in A_n^{S \setminus \{ s_1, s_j \}}$*, where* $u = s_{j-1}s_{j-2}...s_1s_0$*.*

THE END