Partitioning maps and Hopf algebras

The 72th Séminaire Lotharingien de Combinatoire
Lyon

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Université Paris-Sud

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Introduction

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Generalized Parking Functions and (a_n)-parkization

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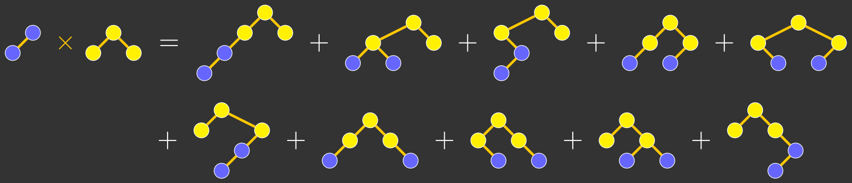

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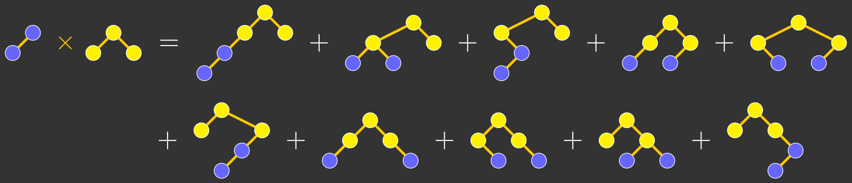

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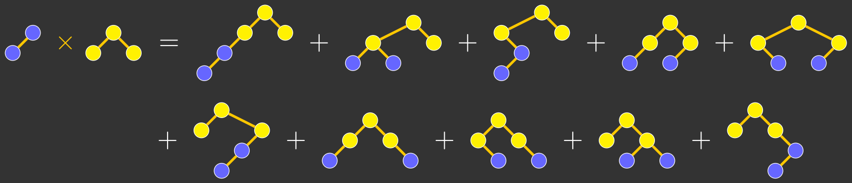

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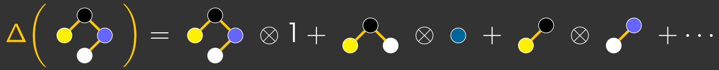

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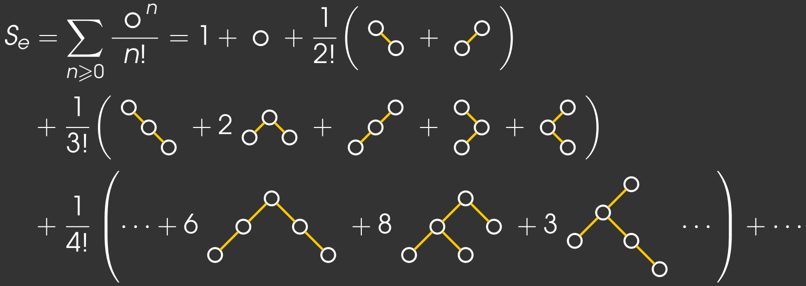

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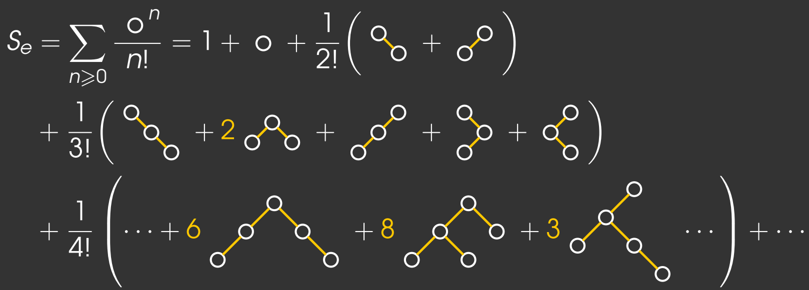

More information :

What is the coefficient of the tree T ?
$\rightarrow$ *The number of permutations giving T by binary search tree insertion*

Outline

Polynomial realization + "alphabet doubling trick" :

- Define lots of Hopf algebras : ***FQSym***, ***WQSym***, ***PQSym***, ...

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- *Partitioning maps*
- *Type of alphabet*

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Definition (Condition 1)

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- *surjective* : if $\mathfrak{A}$ infinite.

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$S_\varphi(\mathfrak{A}) := \left\langle m_\sigma \mid m_\sigma := \sum_{\varphi(w)=\sigma} w \right\rangle$ has bases indexed by $\mathcal{C}$.

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$\rightarrow (m_\pi)_{\pi\in\mathbb{S}}$: basis indexed by *set partition*.

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Partitioning maps (Condition 2/3)

Definition (Condition 2)

$$\text{If} \quad \varphi(uv) = \varphi(u'v') \quad \text{then} \quad \begin{cases} \varphi(u) = \varphi(u'), \\ \varphi(v) = \varphi(v'). \end{cases}$$

(for $|u| = |u'|$ et $|v| = |v'|$)

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Partitioning maps (Condition 2/3) - 2

Lemma

$S_\varphi(\mathfrak{A})$ is a sub-algebra of $\mathbb{K}\langle\mathfrak{A}\rangle$.

$$m_\sigma m_\mu = \sum_{\substack{\exists uv \in \varphi^{-1}(\tau) \\ \varphi(u)=\sigma;\ \ \varphi(v)=\mu}} m_\tau \quad \text{for} \quad \tau \in \mathcal{C}.$$

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$\rightarrow \simeq$ ***WSym*** as algebra.

Type of alphabet **Alph**

For example :

$$\text{red integers } (\mathbb{N}, \leqslant) \quad \text{and} \quad \text{blue integers } (\mathbb{N}, \leqslant);$$

such that $\forall x, y \in \mathbb{N}$,

$$\zeta(x) \leqslant \zeta(y) \quad \text{whenever} \quad x \leqslant y;$$

and $\mathfrak{A} := \mathbb{N} \oplus \mathbb{N}$ endowed with the *order structure* $\leqslant$:

$$x \leqslant y \iff \begin{cases} x \leqslant y & \text{with } x, y \in \mathbb{N}, \\ x \leqslant y & \text{with } x, y \in \mathbb{N}, \text{ or} \\ x \in \mathbb{N} & \text{and} \quad y \in \mathbb{N}. \end{cases}$$

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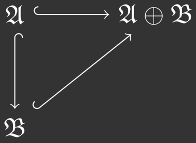
Ingredients :

Formally : *category* of *sets* with *injective maps* and $\oplus$.



Partitioning maps (3/3)

Let $\mathfrak{A} := \mathfrak{B} \oplus \mathfrak{C}$ be an alphabet.

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For any $b, b' \in \mathfrak{B}^m$ and any $c, c' \in \mathfrak{C}^n$,

$$\textit{if} \begin{cases} \varphi(b) = \varphi(b') \\ \varphi(c) = \varphi(c') \end{cases} \quad \textit{then}\ \varphi((bc)\sigma) = \varphi((b'c')\sigma)$$

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Exercise : show that *Occ* satisfies Condition 3.
Hint : define $\pi \cdot \sigma$ such that $\varphi(w\sigma) = \varphi(w)\sigma$.

Partitioning maps (Condition 3/3) - 2

Theorem

$S_\varphi(\mathfrak{A})$ is endowed with a *Hopf algebra structure*.

Alphabet doubling trick : for $\mathfrak{A} := \mathfrak{B} \oplus \mathfrak{C}$,

$$\Delta(m_\sigma) \simeq \delta_{\mathfrak{B} \rightleftarrows \mathfrak{C}} \left(\sum_{\varphi(w)=\sigma} w \right) = \sum_{\varphi(w)=\sigma} w_{|\mathfrak{B}} \otimes w_{|\mathfrak{C}} \,.$$

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$\rightarrow \simeq$ ***WSym*** as Hopf algebra.

Some examples

φ-map	structure on $\mathfrak{A}$	Hopf algebra
Occ	none	**WSym**
Std	totally ordered	**FQSym**
BST	totally ordered	**PBT**
Pack	totally ordered	**WQSym**
Park	totally ordered + successor	**PQSym**
...	...	...

(a_n)-Parking Functions

Let $(a_n)_{n\in\mathbb{N}}$ be an *increasing positive sequence* with $a_1 = 1$.

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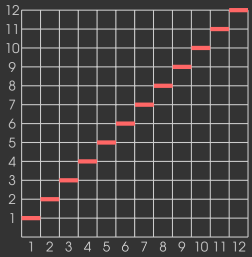

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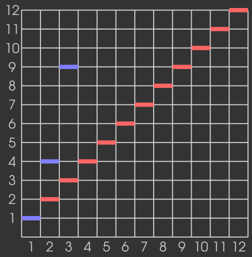

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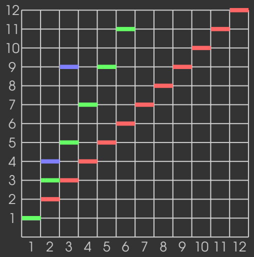

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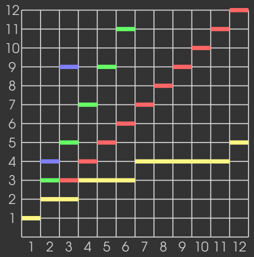

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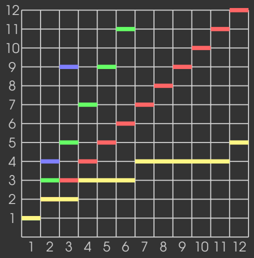

Definition (PITMAN-STANLEY, 99)

Let f be a word on $\mathbb{N}_{>0}$ of size k.
Let $f^{\uparrow}$ be the *non-decreasing arrangment* of f.

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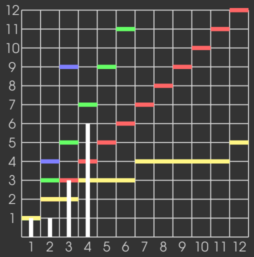

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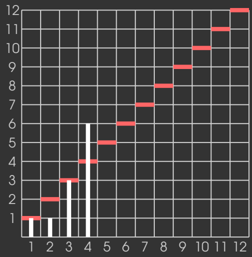

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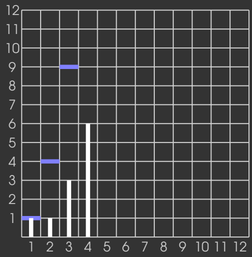

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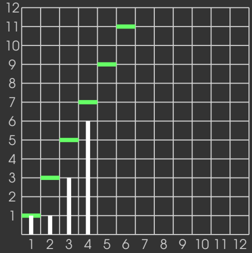

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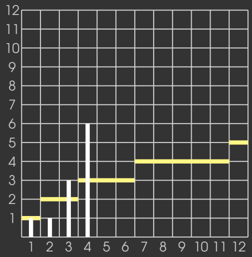

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while $\exists k$ *such that* $f^{\uparrow}_k > a_k$ **do**
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Let $f := 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$ be a word.
Let (c_n) be a sequel defined by $\begin{cases} c_1 = 1\,, \\ c_n = c_{n-1} + 2\,. \end{cases}$

$$park(f, (c_n)) \quad =$$

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Let $f := 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$ be a word.

Let (c_n) be a sequel defined by $\begin{cases} c_1 = 1\,, \\ c_n = c_{n-1} + 2\,. \end{cases}$

$$park(f, (c_n)) = 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$$

i	1	2	3	4	5	6	7	8	9	10	11
$f^{\uparrow}_i$	1	4	7	7	8	8	9	11	22	23	72
c_i	1	3	5	7	9	11	13	15	17	19	21

(a_n)-Parkization

Algorithm park :

Input: $f \in \mathbb{N}^*_{>0}$ a word,
(a_n) an increasing sequence with $a_1 = 1$.
Result: $f \in \mathbf{PF}_{(a_n)}$

1 **while** $\exists k$ *such that* $f^{\uparrow}_k > a_k$ **do**
2 $k \leftarrow \min(k \mid f^{\uparrow}_k > a_k)$;
3 decrement f all letters $\ell > a_k$ of f;
4 **return** f

Let $f := 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$ be a word.
Let (c_n) be a sequel defined by $\begin{cases} c_1 = 1\,, \\ c_n = c_{n-1} + 2\,. \end{cases}$

$$park(f, (c_n)) \quad = 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$$

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$f^{\uparrow}_i$	1	4	7	7	8	8	9	11	22	23	72
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```
while ∃k such that f_k^↑ > a_k do
    k ← min(k | f_k^↑ > a_k);
    decrement f all letters ℓ > a_k of f;
return f
```

Let $f := 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$ be a word.

Let (c_n) be a sequel defined by $\begin{cases} c_1 = 1\,, \\ c_n = c_{n-1} + 2\,. \end{cases}$

$$park(f, (c_n)) \quad = 3 \cdot 6 \cdot 8 \cdot 7 \cdot 22 \cdot 1 \cdot 71 \cdot 21 \cdot 6 \cdot 7 \cdot 10$$

i	1	2	3	4	5	6	7	8	9	10	11
$f_i^{\uparrow}$	1	3	6	6	7	7	8	10	21	22	71
c_i	1	3	5	7	9	11	13	15	17	19	21

(a_n)-Parkization

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    $k \leftarrow \min(k \mid f_k^{\uparrow} > a_k)$;
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Let $f := 4 \cdot 7 \cdot 9 \cdot 8 \cdot 23 \cdot 1 \cdot 72 \cdot 22 \cdot 7 \cdot 8 \cdot 11$ be a word.
Let (c_n) be a sequel defined by $\begin{cases} c_1 = 1\,, \\ c_n = c_{n-1} + 2\,. \end{cases}$

$$park(f, (c_n)) = 3 \cdot 5 \cdot 7 \cdot 6 \cdot 21 \cdot 1 \cdot 70 \cdot 20 \cdot 5 \cdot 6 \cdot 9$$

i	1	2	3	4	5	6	7	8	9	10	11
$f_i^{\uparrow}$	1	3	5	5	6	6	7	9	20	21	70
c_i	1	3	5	7	9	11	13	15	17	19	21

(a_n)-Parkization

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Let (c_n) be a sequel defined by $\begin{cases} c_1 = 1\,, \\ c_n = c_{n-1} + 2\,. \end{cases}$

$$park(f, (c_n)) \quad = 3 \cdot 5 \cdot 7 \cdot 6 \cdot 18 \cdot 1 \cdot 67 \cdot 17 \cdot 5 \cdot 6 \cdot 9$$

i	1	2	3	4	5	6	7	8	9	10	11
$f^{\uparrow}_i$	1	3	5	5	6	6	7	9	17	18	67
c_i	1	3	5	7	9	11	13	15	17	19	21

(a_n)-Parkization

Algorithm park :

Input: $f \in \mathbb{N}^*_{>0}$ a word,
(a_n) an increasing sequence with $a_1 = 1$.

Result: $f \in \mathbf{PF}_{(a_n)}$

while $\exists k$ *such that* $f^{\uparrow}_k > a_k$ **do**
    $k \leftarrow \min(k \mid f^{\uparrow}_k > a_k)$;
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i	1	2	3	4	5	6	7	8	9	10	11
$f^{\uparrow}_i$	1	3	5	5	6	6	7	9	17	18	21
c_i	1	3	5	7	9	11	13	15	17	19	21

(a_n)-Parkization (2)

Theorem

The *(a_n)-parkization* is a *partitioning map*

if and only if

$$(a_n) := \begin{cases} a_1 = 1\,, \\ a_k = a_{k-1} + b \end{cases} \quad \text{for } k > 1 \text{ with } b \in \mathbb{N}.$$

Special case

For $a_k = k$, one obtains ***PQSym***.

Concluding pitch

Formalism :

3 conditions on words,

Concluding pitch

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3 conditions on words,
\+ *1* operation on alphabets,

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combinatorial Hopf algebra ! ! !

(*versus* 6 complicated mathematical axioms.)

Concluding pitch

Formalism :

3 conditions on words,
+ 1 operation on alphabets,

---

combinatorial Hopf algebra ! ! !

(*versus* 6 complicated mathematical axioms.)

Application :

- New family of Hopf algebras indexed by $\mathbf{PF}_{(a_n)}$.

Perspectives

Plactic like monoids :

- $\mathtt{b} \cdot \mathtt{w} \cdot \mathtt{a} \equiv \mathtt{a} \cdot \mathtt{w} \cdot \mathtt{b}$ if $\mathtt{a}, \mathtt{b}$ have the same breakpoint :

$$S(t) = 1 + t + 3t^2 + 14t^3 + 87t^4 + 674t^5 + 6260t^6 + \dots$$

Perspectives

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Formalize some others specializations ? :

- Difference, product of alphabets, . . .

Perspectives

Plactic like monoids :

- $\texttt{b} \cdot \texttt{w} \cdot \texttt{a} \equiv \texttt{a} \cdot \texttt{w} \cdot \texttt{b}$ if $\texttt{a}, \texttt{b}$ have the same breakpoint :

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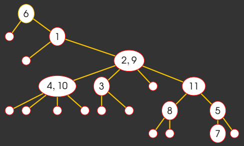
Formalize some others specializations ? :

- Difference, product of alphabets, ...

Links with Bérénice Oger work ? :

- $3 \cdot 5 \cdot 7 \cdot 6 \cdot 18 \cdot 1 \cdot 21 \cdot 17 \cdot 5 \cdot 6 \cdot 9 \in \mathbf{PF}^{11}_{(a_n)} \leftrightarrow$  $\in$ boxed trees.