Cyclically fully commutative elements in affine Coxeter groups

Mathias Pétréolle

ICJ

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Plan

1. Introduction
2. Cyclically fully commutative elements and heaps
3. Characterization and enumeration in finite and affine types

Coxeter groups

Coxeter group W given by Coxeter matrix $(m_{s,t})_{s,t\in S}$

$$\text{Relations} \begin{cases} s^2 = 1 \\ \underbrace{sts\cdots}_{m_{s,t}} = \underbrace{tst\cdots}_{m_{s,t}} \end{cases}$$

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Theorem (Matsumoto, 1964)

Given two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other.

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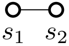

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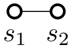

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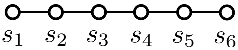

A_6

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- Biagioli-Jouhet-Nadeau (2013): characterizations in terms of heaps, computation of $W^{FC}(q) := \sum_{w \in W^{FC}} q^{\ell(w)}$

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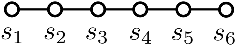
Examples in 

A_6

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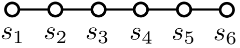
Examples in 

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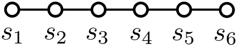
Examples in 

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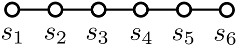
Examples in 

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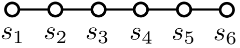
Examples in 

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Motivation for introducing CFC elements: looking for a cyclic version of Matsumoto's theorem.

Heaps

Proposition (Stembridge, 1995)

A reduced word represents a FC element if and only if no element of its commutation class contains a factor $\underbrace{sts\cdots}_{m_{s,t}}$, for a $m_{s,t} \geq 3$

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$\Rightarrow$ We encode the whole commutation class of a FC elements by its heap.

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The heap of a word **w** is a poset $(H, \prec)$ labelled by generators s_i of W. If two words are commutation equivalent, their heaps are isomorphic.

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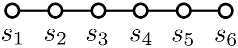

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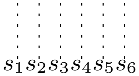
$H =$ 

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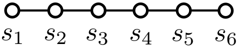

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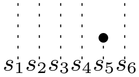

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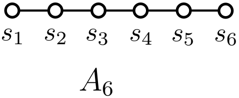
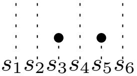


$H =$ 

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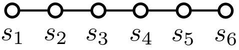

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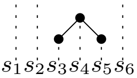
$H =$ 

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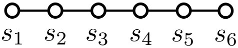

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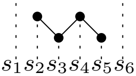
$H =$ 

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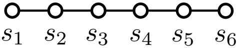

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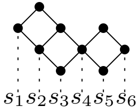
$H =$ 

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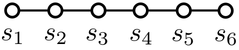

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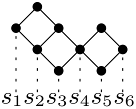
$H =$ 

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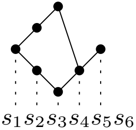

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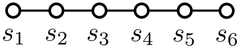

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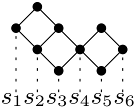
$H =$ 

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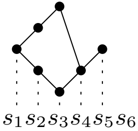

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We write $x \prec_c y$ if x and y are connected by an edge in H (chain covering relation)

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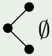
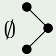
- No convex chain $i_1 \prec \cdots \prec i_{m_{s,t}}$ in H such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $m_{s,t} \geq 3$  and 

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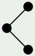
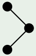
- No convex chain $i_1 \prec \cdots \prec i_{m_{s,t}}$ in H such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $m_{s,t} \geq 3$  $\emptyset$ and $\emptyset$ 

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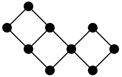
$H =$ 

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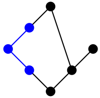

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Cylindric transformation

Let H be a heap. The cylindric transformation H^c is defined by the same points, labellings and chain covering relations $\prec_c$ as H, and some new relations:

- for each generator s, consider the minimal point a and the maximal point b in the chain H_s (for the partial order $\prec$). If a is minimal and b is maximal in the poset H, we add a new relation $b \prec_c a$.
- for each pair of generators (s,t) such that $m_{s,t} \geq 3$, consider the minimal point a and the maximal point b in the chain $H_{\{s,t\}}$ (for the partial order $\prec$). If one has label s and the other has label t, we add a new relation $b \prec_c a$.

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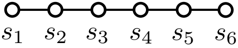

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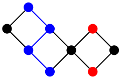
$H =$ 

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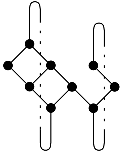
$H^c =$ 

Cylindric convex chain

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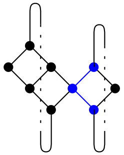
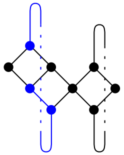


not cylindric convex chain



cylindric convex chain

Characterization of CFC elements

Theorem (P., 2014)

Cylindric transformed heaps H^c of CFC elements are characterized by:

- No chain covering relation $i \prec_c j$ in H^c such that $s_i = s_j$ and $i \neq j$
- No cylindric convex chain $i_1 \prec_c \cdots \prec i_{m_{s,t}}$ in H^c such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $m_{s,t} \geq 3$

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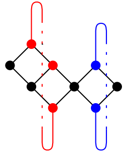
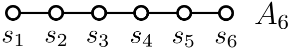


$\mathbf{w} = s_5 s_3 s_4 s_2 s_1 s_3 s_2 s_6 s_5$



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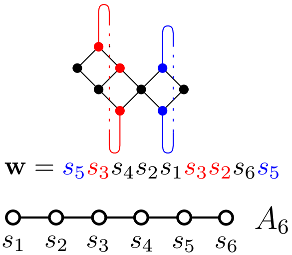
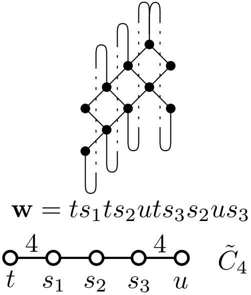
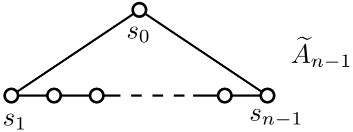




Type $\tilde{A}$



Theorem (P., 2014)

$w \in \tilde{A}_{n-1}$ is CFC if and only if one (equivalently, any) of its reduced expressions $\mathbf{w}$ verifies one of these conditions:

(a) each generator occurs at most once in $\mathbf{w}$,

(b) $\mathbf{w}$ is an alternating word and $|\mathbf{w}_{s_0}| = |\mathbf{w}_{s_1}| = \cdots = |\mathbf{w}_{s_{n-1}}| \geq 2$.

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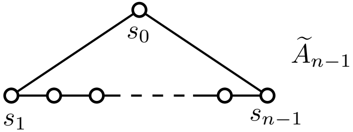

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$$\tilde{A}^{CFC}_{n-1}(q) := \sum_{w \in \tilde{A}_{n-1}} q^{\ell(w)} = P_{n-1}(q) + \frac{2^n - 2}{1 - q^n}\, q^{2n},$$

where $P_{n-1}(q)$ is a computable polynomial.

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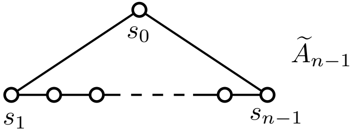

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The coefficients of $\tilde{A}^{CFC}_{n-1}(q)$ are ultimately periodic of exact period n, and the periodicity starts at length n.

Other types

Theorem (P., 2014)

For W of any affine types, we have an explicit characterization and the enumeration of CFC elements. In all these types, the coefficients of $W^{CFC}(q) := \sum_{w \in W^{CFC}} q^{\ell(w)}$ are ultimately periodic.

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The CFC elements in type A_{n-1} are those having reduced expressions in which each generator occurs at most once.
Moreover, for $n \geq 3$,

$$A_{n-1}^{CFC}(q) = (2q+1)A_{n-2}^{CFC}(q) - qA_{n-3}^{CFC}(q).$$

where $A_0^{CFC}(q) = 1$, $A_1^{CFC}(q) = 1 + q$.

Logarithmic elements

We say that an element w is logarithmic if and only if the equality $\ell(w^k) = k\ell(w)$ holds for all positive integer k.

Theorem (Marquis, 2013 - P., 2014)

For W=$\tilde{A}$, $\tilde{B}$, $\tilde{C}$, or $\tilde{D}$, if w is a CFC element, w is logarithmic if and only if a (equivalently, any) reduced expression $\mathbf{w}$ of w has full support (i.e all generators occur in $\mathbf{w}$).

Thank you for your attention