Greatest Common Divisors of Specialized Schur Functions

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Plan:

- Greatest common divisor of $s_\lambda(1^k)$ with $\lambda \vdash n$.
- Existence of generalized parking spaces.
- Greatest common divisor of $s_\lambda(1, q, \cdots, q^{k-1})$ with $\lambda \vdash n$.

Greatest Common Divisors of $s_\lambda(1^k)$

Schur Functions

A partition of a positive integer n is a weakly decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots), \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

of non-negative integers with $\sum_i \lambda_i = n$. Then we write $\lambda \vdash n$. The length $l(\lambda)$ of a partition λ is defined by

$$l(\lambda) = \#\{i : \lambda_i > 0\}.$$

Let k be a positive integer and let λ be a partition with length $\leq k$. The Schur function $s_\lambda(x_1, \cdots, x_k)$ corresponding to λ is defined by

$$s_\lambda(x_1, \cdots, x_k) = \frac{\det\left(x_i^{\lambda_j+k-j}\right)_{1\leq i,j\leq k}}{\det\left(x_i^{k-j}\right)_{1\leq i,j\leq k}}.$$

The Schur functions are symmetric polynomials in $x_1, \cdots, x_k$ with non-negative integer coefficients.

Specialized Schur Functions

We are interested in the greatest common divisors of the special values

$$s_\lambda(1^k) = s_\lambda(\underbrace{1, \cdots, 1}_{k}), \quad \text{and} \quad s_\lambda(1, q, q^2, \cdots, q^{k-1}).$$

The special values $s_\lambda(1^k)$ can be interpreted as follows:

$$\begin{aligned} s_\lambda(1^k) &= \text{the number of semistandard tableaux of shape } \lambda \\ &\quad\ \text{with entries in } \{1, 2, \cdots, k\} \\ &= \text{the dimension of the irreducible representation} \\ &\quad\ \text{of } \mathbf{GL}_k \text{ with highest weight } \lambda \\ &= \prod_{x \in D(\lambda)} \frac{k + c(x)}{h(x)}, \end{aligned}$$

where $D(\lambda)$ is the Young diagram of λ, and $c(x)$ and $h(x)$ denote the content and the hook length of x respectively.

Theorem 1 Let k and n be positive integers. Then we have

$$\gcd_{\mathbb{Z}}\left\{s_\lambda(1^k) : \lambda \vdash n\right\} = \frac{k}{\gcd(n,k)}.$$

Example If $n = 4$, then we have

k	$s_{(4)}(1^k)$	$s_{(3,1)}(1^k)$	$s_{(2^2)}(1^k)$	$s_{(2,1^2)}(1^k)$	$s_{(1^4)}(1^k)$	GCD
1	1	0	0	0	0	1
2	5	3	1	0	0	1
3	15	15	6	3	0	3
4	35	45	20	15	1	1
5	70	105	50	45	5	5
6	126	210	105	105	15	3
7	210	378	196	210	35	7
8	330	630	336	378	70	2
9	495	990	540	630	126	9
10	715	1485	825	990	210	5

Theorem 1 Let k and n be positive integers. Then we have

$$\gcd_{\mathbb{Z}}\left\{s_\lambda(1^k) : \lambda \vdash n\right\} = \frac{k}{\gcd(n,k)}.$$

Proof follows from the following two claims.

Claim 1 For any partition λ of n, the integer $s_\lambda(1^k)$ is divisible by $k/\gcd(n,k)$.

Claim 2 The integer $k/\gcd(n,k)$ is an element of the ideal of $\mathbb{Z}$ generated by $s_\lambda(1^k)$'s ($\lambda \vdash n$).

Proof of Theorem 1 (1/4)

Claim 1 For any partition λ of n, the integer $s_\lambda(1^k)$ is divisible by $k/\gcd(n,k)$.

Proof of Claim 1

Let $d = \gcd(n,k)$. It follows from the Frobenius formula that

$$\sum_{\lambda \vdash n} \frac{s_\lambda(1^k)}{k/d} \chi^\lambda(\sigma) = \frac{1}{k/d} \cdot k^{l(\mathrm{type}(\sigma))} \quad (\sigma \in \mathfrak{S}_n),$$

where χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to λ, and $\mathrm{type}(\sigma)$ is the cycle type of σ. Hence it is enough to show that there exists a representation of $\mathfrak{S}_n$ whose character θ is given by

$$\theta(\sigma) = \frac{1}{k/d} \cdot k^{l(\mathrm{type}(\sigma))} \quad (\sigma \in \mathfrak{S}_n).$$

Proof of Theorem 1 (2/4)

Consider the permutation representation of $\mathfrak{S}_n$ on $X = (\mathbb{Z}/k\mathbb{Z})^n$, and put

$$X_p = \{(x_i) \in (\mathbb{Z}/k\mathbb{Z})^n : x_1 + \cdots + x_n - pd \in \{0, 1, \cdots, d-1\}\}$$

for $p = 0, 1, \cdots, k/d-1$, where we identify $\mathbb{Z}/k\mathbb{Z}$ with $\{0, 1, \cdots, k-1\}$. If we denote by ψ and ψ_p the permutation character of X and X_p, then we have

$$\psi(\sigma) = k^{l(\mathrm{type}(\sigma))}, \quad \text{and} \quad \psi = \psi_0 + \psi_1 + \cdots + \psi_{k/d-1}.$$

Since $\gcd(k/d, n/d) = 1$, we can find an equivariant bijection between X_0 and X_p, so we have

$$\psi_0 = \psi_1 = \cdots = \psi_{k/d-1}.$$

Hence we conclude that θ is the permutation character ψ_0 of X_0, and that $\frac{s_\lambda(1^k)}{k/d}$ is an integer.

Proof of Theorem 1 (3/4)

Claim 2 The integer $k/\gcd(n,k)$ is an element of the ideal of $\mathbb{Z}$ generated by $s_\lambda(1^k)$'s ($\lambda \vdash n$).

Proof of Claim 2

We have the following relation among ideals of $\mathbb{Z}$:

$$\left\langle s_\lambda(1^k) : \lambda \vdash n \right\rangle = \left\langle m_\lambda(1^k) : \lambda \vdash n \right\rangle \supset \left\langle m_{(f^{n/f})}(1^k) : f \mid d \right\rangle,$$

where $m_\lambda(x_1, \cdots, x_k)$ is the monomial symmetric polynomial corresponding to λ, and $d = \gcd(k,n)$. Since we have

$$m_{(f^{n/f})}(1^k) = \binom{k}{f},$$

it is enough to show that

$$\frac{k}{d} \in \left\langle \binom{k}{f} : f \mid d \right\rangle.$$

Proof of Theorem 1 (4/4)

Lemma If e divides k, then

$$\frac{k}{e} \in \left\langle \binom{k}{f} : f \mid e \right\rangle.$$

This lemma can be shown by using the induction on e and the relation

$$\binom{p^a l}{p^a} - \frac{p^a l}{p^a} \equiv 0 \bmod pl,$$

where p is a prime.

Generalized Parking Spaces

Parking Functions

A parking function of length n is a sequence $(a_1, a_2, \cdots, a_n)$ of positive integers satisfying

- $a_i \in \{1, 2, \cdots, n\}$, and
- $\#\{i : a_i \leq k\} \geq k$ for $k = 1, 2, \cdots, n$.

Imagine that there are n cars $C_1, C_2, \cdots, C_n$ and n parking spaces $1, 2, \cdots, n$ in a one-way street. Car C_i prefers the parking space a_i and approaches its preferred parking space.

- If it is free, then C_i parks there.
- If it is occupied, then C_i parks in the next available space if possible.

Then the sequence $(a_1, \cdots, a_n)$ is a parking function if and only if all cars can park.

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We put

$$\mathrm{PF}_n = \text{the set of parking functions of length } n.$$

Example

$$\mathrm{PF}_2 = \{\, 11,\ 12,\ 21 \,\},$$

$$\mathrm{PF}_3 = \left\{ \begin{matrix} 111, & 112, & 121, & 211, & 113, & 131, & 311, & 122 \\ 212, & 221, & 123, & 132, & 213, & 231, & 312, & 321 \end{matrix} \right\}$$

The symmetric group $\mathfrak{S}_n$ acts on the set PF_n by permuting entries:

$$\sigma \cdot (a_1, \cdots, a_n) = (a_{\sigma(1)}, \cdots, a_{\sigma(n)}) \quad (\sigma \in \mathfrak{S}_n).$$

It is known that the corresponding permutation character is given by

$$\varphi(\sigma) = (n+1)^{l(\mathrm{type}(\sigma))-1} \quad (\sigma \in \mathfrak{S}_n).$$

More generally, given a positive integer k, we consider the class function on $\mathfrak{S}_n$ defined by

$$\varphi_k(\sigma) = k^{l(\mathrm{type}(\sigma))-1} \quad (\sigma \in \mathfrak{S}_n).$$

Question When is φ_k the character of some representation of $\mathfrak{S}_n$?

It is not hard to show that, if k is relatively prime to n, then φ_k is the permutation character on

$$\{x \in (\mathbb{Z}/k\mathbb{Z})^n : x_1 + \cdots + x_n = 0\}.$$

By using Theorem 1, we can prove

Corollary

$$\varphi_k \text{ is the character of a representation of } \mathfrak{S}_n \iff k \text{ is relatively prime to } n.$$

Proof It follows from the Frobenius formula that

$$\varphi_k = \sum_{\lambda \vdash n} \frac{s_\lambda(1^k)}{k} \chi^\lambda.$$

Hence we have

$$\begin{aligned} &\varphi_k \text{ is the character of a representation of } \mathfrak{S}_n \\ &\iff \frac{s_\lambda(1^k)}{k} \in \mathbb{Z} \text{ for all } \lambda \vdash n \\ &\iff k \text{ is relatively prime to } n, \end{aligned}$$

since $\gcd\{s_\lambda(1^k) : \lambda \vdash n\} = k / \gcd(n, k)$.

Generalization to Coxeter groups

Let (W, S) be a finite Coxeter system and V its geometric representation.

Example (Type A_{n-1})

$$\begin{aligned} W &= \mathfrak{S}_n, \\ S &= \{s_i = (i, i+1) : 1 \le i \le n-1\}, \\ V &= \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}. \end{aligned}$$

In this setting, we have

$$l(\mathrm{type}(\sigma)) - 1 = \dim V^\sigma \quad (\sigma \in \mathfrak{S}_n),$$

where $V^\sigma = \{v \in V : \sigma v = v\}$.

Generalization to Coxeter groups

Let (W, S) be a finite Coxeter system and V its geometric representation. Let k be a positive integer and consider the class function φ_k^W on W given by

$$\varphi_k^W(w) = k^{\dim V^w} \quad (w \in W),$$

where V^w is the fixed-point subspace of w.

Question When is φ_k^W is the character of a representation of W?

A W-module U is called a generalized parking space if its character is given by φ_k^W for some positive integer k. For example, the vector space $\mathbb{C}\,\mathrm{PF}_n$ with basis PF_n is a parking space for $\mathfrak{S}_n$.

Theorem 2 Let W be an irreducible Coxeter group. Then φ_k^W is a character of some representation of W if and only if the following condition is satisfied:

type	condition on k
A_{n-1}	k is relatively prime to n
B_n, D_n	k is odd
E_6, E_7, F_4	k is not divisible by 2 and 3
E_8	k is not divisible by 2, 3, and 5
H_3	$k \equiv 1, 5, 9 \bmod 10$
H_4	$k \equiv 1, 11, 19, 29 \bmod 30$
$I_2(m)$ (m is even)	$k = 1$ or "$k \geq m - 1$ and $k^2 \equiv 1 \bmod 2m$"
$I_2(m)$ (m is odd)	$k = 1$ or "$k \geq m - 1$ and $k^2 \equiv 1 \bmod m$"

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$I_2(m)$ (m is odd)	$k = 1$ or "$k \geq m - 1$ and $k^2 \equiv 1 \bmod m$"

Remark E. Sommers proved that, if W is a Weyl group and k satisfies the above condition, then the permutation representation on Q/kQ has the character φ_k^W, where Q is the root lattice.

If W is not of type $I_2(m)$ with $m = 5$ or $m \geq 7$, then the condition in Theorem 2 can be stated in terms of "generalized q-Catalan number" $C_k^W(q)$:

$$C_k^W(q) = \prod_{i=1}^{r} \frac{[k + e_i]_q}{[1 + e_i]_q},$$

where $e_1, \cdots, e_r$ are the exponents of W and $[m]_q = (1 - q^m)/(1 - q)$.

Example (Type A_{n-1}) If $W = \mathfrak{S}_n$, then the exponents are $1, 2, \cdots, n-1$, and

$$C_k^{\mathfrak{S}_n}(q) = \frac{1}{[n]_q} \begin{bmatrix} k + n - 1 \\ k \end{bmatrix}_q,$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_q$ is the q-binomial coefficient. If $k = n+1$ (Coxeter number), then $C_{n+1}^{\mathfrak{S}_n}(q)$ is a q-analogue of the Catalan number C_n.

If W is not of type $I_2(m)$ with $m = 5$ or $m \geq 7$, then the condition in Theorem 2 can be stated in terms of "generalized q-Catalan number" $C_k^W(q)$:

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where $e_1, \cdots, e_r$ are the exponents of W and $[m]_q = (1 - q^m)/(1 - q)$.

Corollary Suppose that W is not of type $I_2(m)$ with $m = 5$ or $m \geq 7$. Then

φ_k^W is a character of some representation of W

$\iff C_k^W(q)$ is a polynomial in q.

Greatest Common Divisors of $s_\lambda(1, q, \cdots, q^{k-1})$

Theorem 1 Let k and n be positive integers. Then we have

$$\gcd_{\mathbb{Z}}\left\{s_\lambda(1^k) : \lambda \vdash n\right\} = \frac{k}{\gcd(n,k)}.$$

Theorem 3 Let k and n be positive integers. Then we have

$$\gcd_{\mathbb{Q}[q]}\left\{s_\lambda(1,q,q^2,\cdots,q^{k-1}) : \lambda \vdash n\right\} = \frac{[k]_q}{[\gcd(n,k)]_q},$$

where $[r]_q = (1-q^r)/(1-q)$.

Remark Theorem 3 does not imply Theorem 1 by letting $q=1$. For example,

$$\lim_{q\to 1} \gcd\{(q^2+1)(q+1)^2,(q+1)^3\} = \lim_{q\to 1}(q+1)^2 = 4,$$
$$\gcd\{\lim_{q\to 1}(q^2+1)(q+1)^2, \lim_{q\to 1}(q+1)^3\} = \gcd(8,8) = 8.$$

Theorem 3 Let k and n be positive integers. Then we have

$$\gcd_{\mathbb{Q}[q]} \left\{ s_\lambda(1, q, q^2, \cdots, q^{k-1}) : \lambda \vdash n \right\} = \frac{[k]_q}{[\gcd(n,k)]_q},$$

where $[r]_q = (1 - q^r)/(1 - q)$.

Proof follows from

1.
$$\{z \in \mathbb{C} : z \text{ is a common root of } h_\lambda(1, q, \ldots, q^{k-1}) \ (\lambda \vdash n)\} = \bigsqcup_{d|k,\ d\nmid n} \{z \in \mathbb{C} : z \text{ is a primitive } d\text{-th root of } 1\}.$$

2. If z is a common root of $h_\lambda(1, q, \ldots, q^{k-1})$ $(\lambda \vdash n)$, then z is a simple root of $h_\mu(1, q, \ldots, q^{k-1})$ for some $\mu \vdash n$.

Conjectures

Theorem 3 implies that

$$\frac{s_\lambda(1,q,\cdots,q^{k-1})}{[k]_q/[d]_q} = \frac{s_\lambda(1,q,\cdots,q^{k-1})}{1+q^d+\cdots+q^{k-d}} \in \mathbb{Z}[q],$$

where $\lambda \vdash n$ and $d = \gcd(k,n)$.

Conjecture 1 If λ is a partition of n and $d = \gcd(k,n)$, then

$$\frac{s_\lambda(1,q,\cdots,q^{k-1})}{1+q^d+\cdots+q^{k-d}} \in \mathbb{N}[q],$$

i.e., it is a polynomial with non-negative integer coefficients.

This conjecture is true if

- n is a multiple of k (i.e., $d = k$) (well-known), or
- k/d is relatively prime to n.

Conjecture 1 is now proved.

A finite sequence $(a_0, a_1, \cdots, a_m)$ is called unimodal if there is an index p satisfying

$$a_0 \le a_1 \le \cdots \le a_{p-1} \le a_p \ge a_{p+1} \ge \cdots \ge a_{m-1} \ge a_m.$$

Conjecture 2 Let λ be a partition of n and $d = \gcd(k, n)$. If we write

$$\frac{s_\lambda(1, q, \cdots, q^{k-1})}{1 + q^d + \cdots + q^{k-d}} = \sum_{i \ge 0} a_i q^i,$$

then the sequences

$$(a_0, a_2, a_4, \cdots), \quad \text{and} \quad (a_1, a_3, a_5, \cdots)$$

are both unimodal.

This conjecture is true if

- n is a multiple of k (i.e., $d = k$) (well-known), or
- k is relatively prime to n.