A Bialgebra on Hypertree and Partition Posets

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SLC 72

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Incidence Hopf Algebra of a Family of Bounded Posets

Bounded poset = a poset with a least and a greatest element.

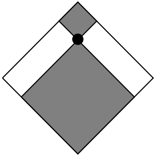
We consider posets up to isomorphisms of posets.
Considered a family $\mathcal{P}$ of bounded posets which is

- Interval closed,
- Stable under direct product.

We endow the $\mathbb{Q}$-vector space $V_{\mathcal{P}}$ generated by $\mathcal{P}$ with

- a coproduct defined for all $P \in V_{\mathcal{P}}$ by:

$$\Delta[P] = \sum_{x \in P} [0_P, x] \otimes [x, 1_P],$$



- the direct product of posets.

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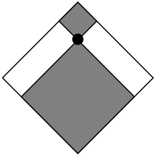


- the direct product of posets.

Theorem (W.R. Schmitt, 1994)

$(V_{\mathcal{P}}, \Delta, \times)$ *is a Hopf Algebra, called Incidence Hopf algebra.*

Moebius number

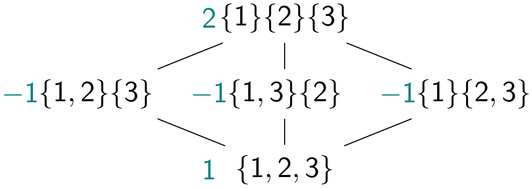
Definition

For any poset P the Moebius function is defined by :

$$\mu(x,x) = 1, \qquad \forall x \in P$$

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z), \qquad \forall x < y \in P.$$

If P is bounded, the Moebius number of P is $\mu(P) := \mu(\hat{0}, \hat{1})$



Link between Moebius numbers and Incidence Hopf algebra

Idea :

The coproduct on the Incidence Hopf algebra enables us to compute Moebius numbers of posets in this algebra !

Hypergraphs and hypertrees

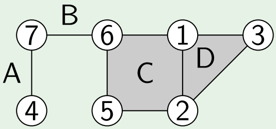
Definition (Berge, 1989)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- *V is a finite set (vertices)*
- *E is a collection of subsets of cardinality at least two of elements of V (edges).*

The valency of a vertex v in H is the number of edges containing v.

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

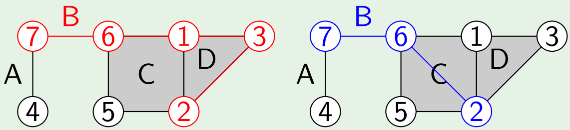
Definition

Let $H = (V, E)$ be a hypergraph.
A walk from d to f in H is an alternating sequence of vertices and edges beginning by d and ending by f:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.

Examples of walks



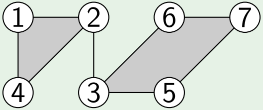
Hypertrees

Definition

A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- *there exists a walk from v to w in H with distinct edges e_i, (H is connected),*
- *and this walk is unique, (H has no cycles).*

Example of a hypertree



The hypertree poset

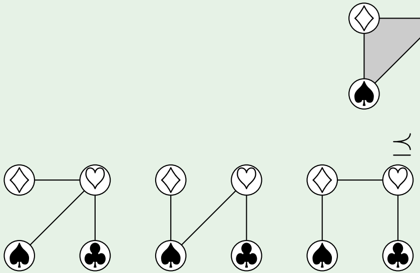
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

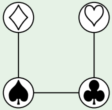
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices



but not



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- Triangle-like poset
- $\mathrm{HT_n}$ = hypertree poset on n vertices.
- Möbius number : $(n-1)^{n-2}$ [McCammond and Meier 2004]

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Goal :

Construction of an analogue of Incidence Hopf algebra which enables us to compute again Moebius numbers of posets.

From the Incidence Hopf Algebra to a simpler Bialgebra

- Add a maximum element to triangle posets
- Close by interval and product

$\Rightarrow$ Incidence Hopf algebra $\mathcal{H}$

Construction of a smaller bialgebra in which computation will be easier.

THE Bialgebra

Lemma (McCammond, Meier, 2004)

Let τ be a hypertree on n vertices.

(a) *The interval $[\hat{0}, \tau]$ is a direct product of partition posets (bounded posets),*

(b) *The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets.*

Family of direct products of hypertree posets and partition posets is interval closed and closed by direct product $\rightsquigarrow$ associated algebra $\mathcal{B}$

We endow this algebra with the following coproduct :

$$\Delta(d) = \sum_{x \in d} [\hat{0}_d, x] \otimes [x, \hat{1}_d] \quad \text{and} \quad \Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_{\widehat{t}}),$$

for a bounded poset $d \in \mathcal{B}$ and a triangle poset $t \in \mathcal{B}$,
where $\widehat{t}$ is the bounded poset obtained from t by adding a greatest element.

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for a bounded poset $d \in \mathcal{B}$ and a triangle poset $t \in \mathcal{B}$,
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$\mathcal{B}$ is a bialgebra.

Comparison between coproducts

- Same coproducts on bounded posets.
- In $\mathcal{H}$

$$\Delta(\widehat{t}) = \sum_{x \in \widehat{t}} [\hat{0}, x] \otimes [x, \hat{1}]$$

- In $\mathcal{B}$

$$\Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_{\widehat{t}})$$

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Why working in $\mathcal{B}$?

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- Same coproducts on bounded posets.
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$$\Delta(t) = \sum_{x \in t} [\hat{0}_t, x] \otimes [x, \hat{1}_{\widehat{t}})$$

Why working in $\mathcal{B}$?

Because $[x, \hat{1}_{\widehat{t}})$ can be written as a product of hypertree posets whereas $[x, \hat{1}]$ cannot !

Computation of the Coproduct in this Bialgebra

Lemma (McCammond, Meier, 2004)

Let τ be a hypertree on n vertices.

(a) *The interval $[\hat{0}, \tau]$ is a direct product of partition posets, with one factor p_j for each vertex in τ with valency j.*

(b) *The half-open interval $[\tau, \hat{1})$ is a direct product of hypertree posets, with one factor $\mathrm{HT_j} = h_j$ for each edge in τ with size j.*

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$$\Delta(h_n) = \sum_{(\alpha,\pi)\in\mathcal{P}_n} c^n_{\alpha,\pi} p_\alpha \otimes h_\pi,$$

where for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\pi = (\pi_2, \pi_3, \ldots, \pi_l)$, $p_\alpha = 1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ and $h_\pi = h_2^{\pi_2} h_3^{\pi_3} \ldots h_l^{\pi_l}$.

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$c^n_{\alpha,\pi} =$ number of hypertrees in h_n with :

- α_i vertices of valency i, $\forall i \geq 1$
- π_j edges of size j, $\forall j \geq 2$

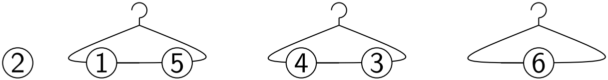
First criterion

Criterion for the vanishing of $c^n_{\alpha,\pi}$

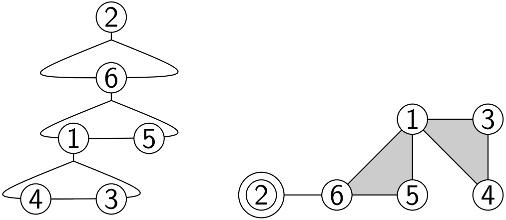
$$c^n_{\alpha,\pi} \neq 0 \Longleftrightarrow \sum_{i=1}^{k} \alpha_i = n, \quad \sum_{j=2}^{l} (j-1)\pi_j = n-1 \text{ and } \sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{l} \pi_j - 1.$$

Counting hypertrees

A π-hooked partition P, for $\pi = (1, 2)$:



Associated hypertree



Prüfer code

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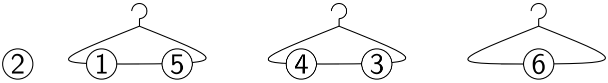
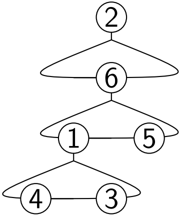
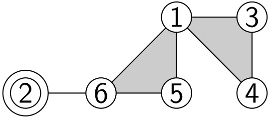


Code :





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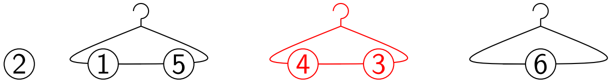

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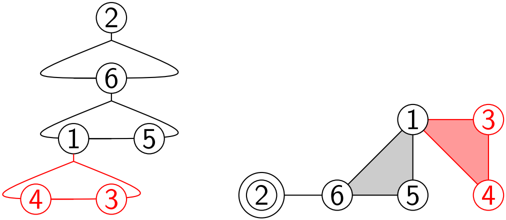

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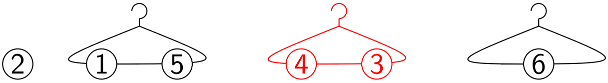
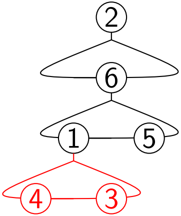
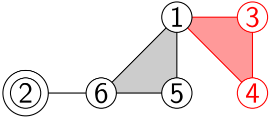


Code : **1**





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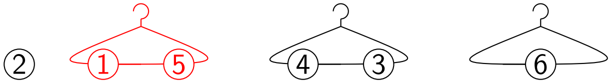
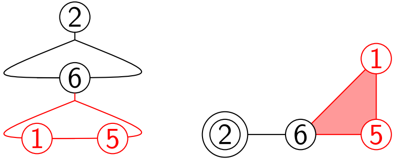


Code : 1,



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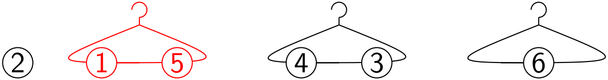
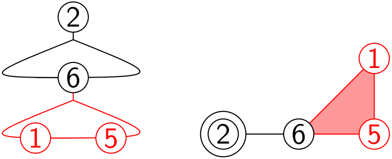
A π-hooked partition P, for $\pi = (1, 2)$:



Code : 1, **6**



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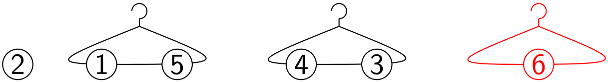
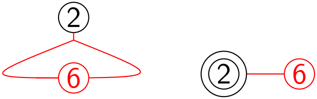


Code : 1, 6,



Prüfer code

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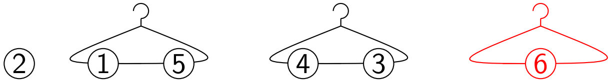
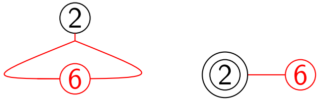


Code : 1, 6, **2**



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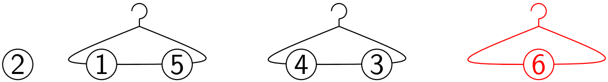
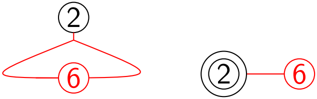


Code : 1, 6, **(2)**



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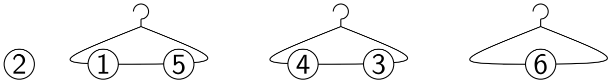
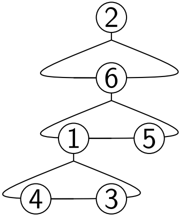
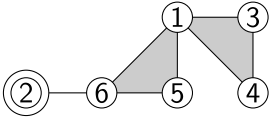


Code : 1, 6





Return of the Prüfer code

constructions of a rooted hypertree of valency set α from P_π

$$\Longleftrightarrow$$

words on $[\![1, n]\!]$, of length $k = \sum_{j\geq 2} \pi_j - 1$, with $\sum_{i\geq 2} \alpha_i$ different letters, where α_i letters appear $i-1$ times, $\forall i \geq 2$

$$\rightsquigarrow \frac{k! \times n!}{\prod_{i\geq 2}(i-1)!^{\alpha_i}\alpha_i!}.$$

Theorem (B.O.)

$$\Delta(h_n) = \frac{1}{n} \times \sum_{(\alpha,\pi)\in\mathcal{P}(n)} \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{k! \times n!}{\prod_{i\geq 1}(i-1)!^{\alpha_i}\alpha_i!} \prod_{i=2}^{k} p_i^{\alpha_i} \otimes \prod_{j=2}^{l} h_j^{\pi_j}.$$

Application : Computation of Moebius numbers of Hypertree Posets

Theorem (McCammond and Meier 2004)

The Moebius number of the augmented hypertree poset on n vertices is given by:

$$\mu(\widehat{HT_n}) = (-1)^{n-1}(n-1)^{n-2}.$$

The following equality holds:

$$(n-1)^{n-2} = \sum_{(\alpha,\pi)\in\mathcal{P}(n)} \frac{(-1)^{i\alpha_i-1}}{n} \times \frac{n!}{\prod_{j\geq 2}(j-1)!^{\pi_j}\pi_j!} \times \frac{k! \times n!}{\prod_{i\geq 1}\alpha_i!},$$

where $\mathcal{P}(n) = (\alpha = (\alpha_1, \ldots, \alpha_k), \pi = (\pi_2, \ldots, \pi_l))$ satisfying:

$$\sum_{i=1}^{k} \alpha_i = n, \quad \sum_{j=2}^{l} (j-1)\pi_j = n-1, \quad \text{and} \quad \sum_{i=1}^{k} i\alpha_i = n + \sum_{j=2}^{l} \pi_j - 1.$$

Thank you very much !