On m-Cover Posets and Their Applications

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Outline

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The m-Cover Poset

- let $\mathcal{P} = (P, \leq)$ be a poset
- bounded poset: a poset with a least and a greatest element, denoted by $\hat{0}$ and $\hat{1}$
- for $m > 0$, consider m-tuples

$$\mathbf{p} = (\underbrace{\hat{0}, \hat{0}, \ldots, \hat{0}}_{l_1}, \underbrace{p, p, \ldots, p}_{l_2}, \underbrace{q, q, \ldots, q}_{l_3})$$

for $p, q \in P$ with $\hat{0} \neq p \lessdot q$

where $\lessdot$ is the covering relation of $\mathcal{P}$

The m-Cover Poset

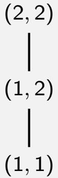
- write $\mathbf{p} = \left(\hat{0}^{l_1}, p^{l_2}, q^{l_3}\right)$ instead
- define $P^{\langle m\rangle} = \left\{\left(\hat{0}^{l_1}, p^{l_2}, q^{l_3}\right) \mid 0_P \neq p \lessdot q, l_1 + l_2 + l_3 = m\right\}$
- m-cover poset of $\mathcal{P}$: the poset $\mathcal{P}^{\langle m\rangle} = \left(P^{\langle m\rangle}, \leq\right)$

  where $\leq$ means componentwise order

Example

$\mathcal{P}$

$\mathcal{P}^{\langle 2\rangle}$



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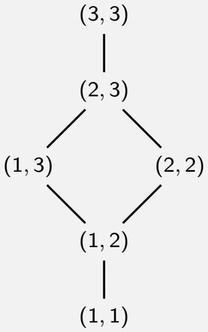

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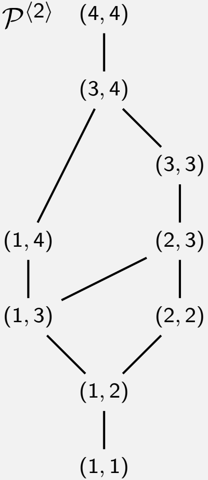

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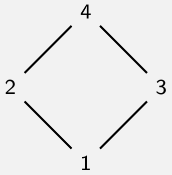
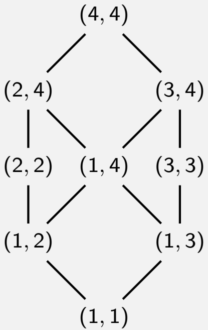
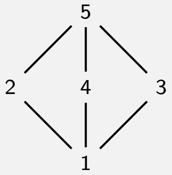


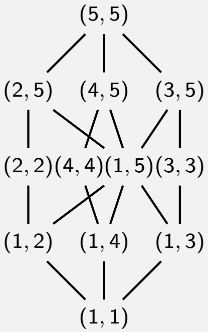


Example

$\mathcal{P}$

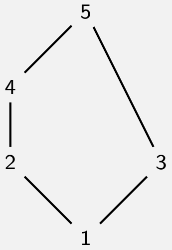


$\mathcal{P}^{\langle 2\rangle}$

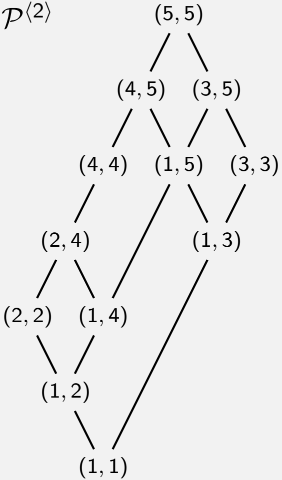


Example

$\mathcal{P}$



$\mathcal{P}^{\langle 2\rangle}$



A Characterization

Theorem (Kallipoliti & , 2013)

Let $\mathcal{P}$ be a bounded poset. Then, $\mathcal{P}^{\langle m\rangle}$ is a lattice for all $m > 0$ if and only if $\mathcal{P}$ is a lattice and the Hasse diagram of $\mathcal{P}$ with $\hat{0}$ removed is a tree rooted at $\hat{1}$.

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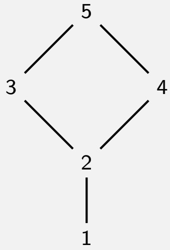
- these posets are (in principle) so-called chord posets

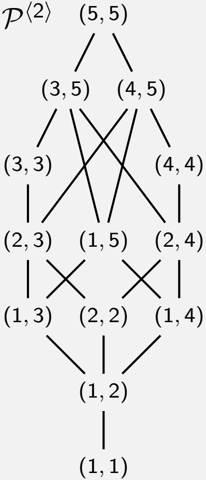
  see Kim, Mészáros, Panova, Wilson: "Dyck Tilings, Increasing Trees, Descents and Inversions" (JCTA 2014)
- they have a natural connection to Dyck paths

Example

$\mathcal{P}$



$\mathcal{P}^{\langle 2\rangle}$



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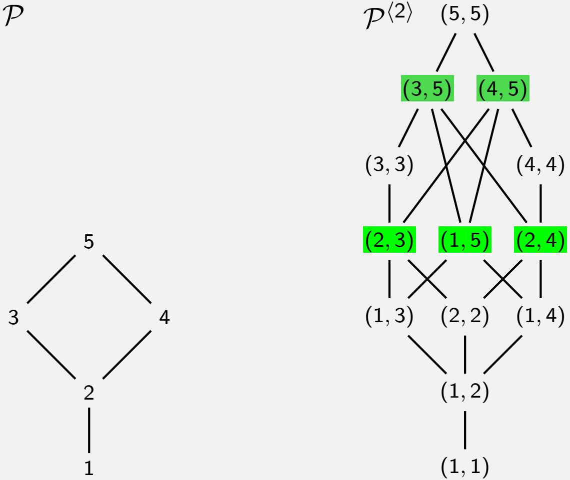

$\mathcal{P}$
5
3
4
2
1
$\mathcal{P}^{\langle 2\rangle}$
(5, 5)
(3, 5)
(4, 5)
(3, 3)
(4, 4)
(2, 3)
(1, 5)
(2, 4)
(1, 3)
(2, 2)
(1, 4)
(1, 2)
(1, 1)

Irreducible Elements

- join-irreducible element of $\mathcal{P}$: a non-minimal element $p \in P$ with a unique lower cover $p_\star$
- meet-irreducible element of $\mathcal{P}$: a non-maximal element $p \in P$ with a unique upper cover $p^\star$
- $\mathcal{J}(\mathcal{P})$: set of all join-irreducible elements of $\mathcal{P}$
- $\mathcal{M}(\mathcal{P})$: set of all meet-irreducible elements of $\mathcal{P}$

  this is of course abuse of notation!

Irreducible Elements

Proposition (Kallipoliti & Mühle, 2013)

Let $\mathcal{P}$ be a bounded poset with $\hat{0} \notin \mathcal{M}(\mathcal{P})$ and $\hat{1} \notin \mathcal{J}(\mathcal{P})$, and let $m > 0$. Then,

$$\mathcal{J}\left(\mathcal{P}^{\langle m\rangle}\right) = \left\{\left(\hat{0}^s, p^{m-s}\right) \mid p \in \mathcal{J}(\mathcal{P}) \text{ and } 0 \leq s < m\right\}, \quad \text{and}$$

$$\mathcal{M}\left(\mathcal{P}^{\langle m\rangle}\right) = \left\{\left(p^s, (p^\star)^{m-s}\right) \mid p \in \mathcal{M}(\mathcal{P}) \text{ and } 1 \leq s \leq m\right\}.$$

Irreducible Elements

Corollary (Kallipoliti & , 2013)

Let $\mathcal{P}$ be a bounded poset with $\hat{0} \notin \mathcal{M}(\mathcal{P})$ and $\hat{1} \notin \mathcal{J}(\mathcal{P})$, and let $m > 0$. Then,

$$\left|\mathcal{J}\left(\mathcal{P}^{\langle m\rangle}\right)\right| = m \cdot \left|\mathcal{J}(\mathcal{P})\right| \quad \text{and} \quad \left|\mathcal{M}\left(\mathcal{P}^{\langle m\rangle}\right)\right| = m \cdot \left|\mathcal{M}(\mathcal{P})\right|.$$

Cardinality

Proposition (Kallipoliti & , 2013)

Let $\mathcal{P}$ be a bounded poset with n elements, c cover relations and a atoms. Then for $m > 0$, we have

$$\left|P^{\langle m\rangle}\right| = (c-a)\binom{m}{2} + m(n-1) + 1.$$

atoms are elements covering $\hat{0}$

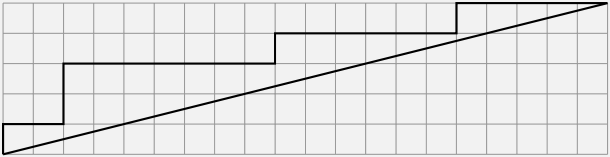
Outline

m-Dyck Paths

- m-Dyck path: a lattice path from $(0,0)$ to (mn, n) consisting only of up-steps $(0,1)$ and right-steps $(1,0)$ and staying weakly above $x = my$
- $\mathcal{D}_n^{(m)}$: set of all m-Dyck paths of parameter n
- we have $\left|\mathcal{D}_n^{(m)}\right| = \mathrm{Cat}^{(m)}(n) = \frac{1}{n}\binom{mn+n}{n-1}$
  these are the Fuss-Catalan numbers
- step sequence: $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ with $u_1 \leq u_2 \leq \cdots \leq u_n$ and $u_i \leq m(i-1)$ for $1 \leq i \leq n$

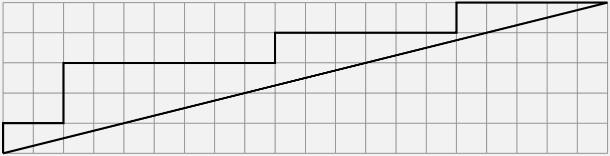
Example

$\mathfrak{p} \in \mathcal{D}_5^{(4)}$



Example

$$\mathfrak{p} \in \mathcal{D}_5^{(4)}$$



$$\mathbf{u}_{\mathfrak{p}} = (0, 2, 2, 9, 15)$$

Rotation Order on $\mathcal{D}_n^{(m)}$

- rotation order: exchange a right-step of $\mathfrak{p} \in \mathcal{D}_n^{(m)}$, which is followed by an up-step, with the subpath of $\mathfrak{p}$ starting with this up-step
- m-Tamari lattice: the lattice $\mathcal{T}_n^{(m)} = \left(\mathcal{D}_n^{(m)}, \leq_R\right)$
  where $\leq_R$ denotes the rotation order
- we omit superscripts, when $m = 1$

Example

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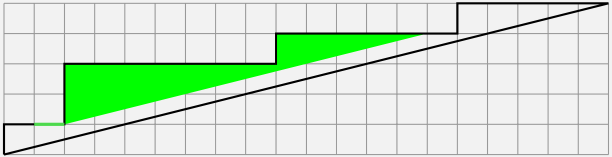

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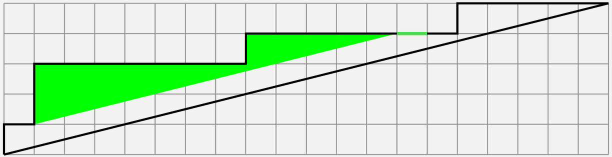

Example

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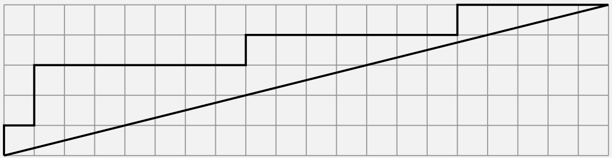
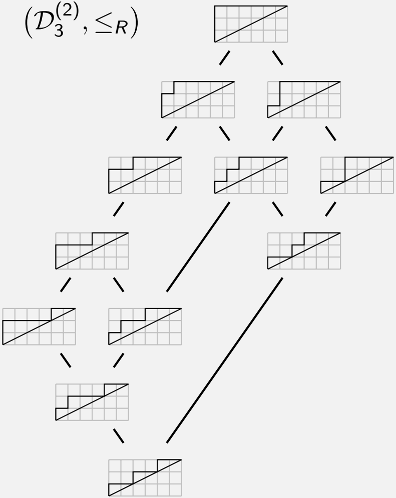


Example

$\left(\mathcal{D}_3^{(2)}, \leq_R\right)$



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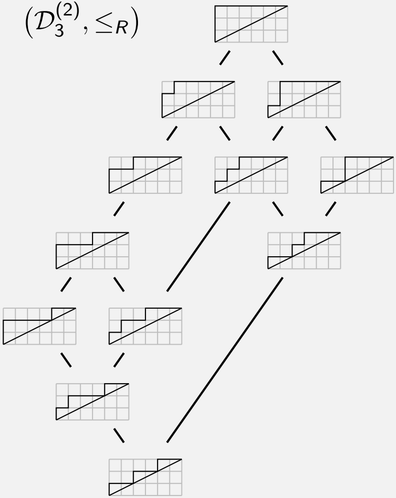

Behold: this is the 2-cover poset of the pentagon lattice!

Example

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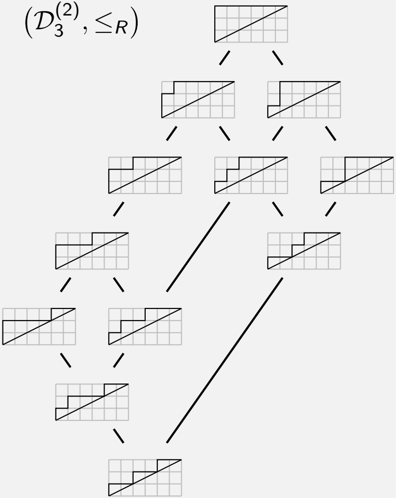


Behold: this is the 2-cover poset of the pentagon lattice!

The pentagon lattice is isomorphic to $\mathcal{T}_3$.

The Posets $\mathcal{T}_n^{\langle m \rangle}$

- the Hasse diagram of $\mathcal{T}_n$ with $\mathfrak{o}$ removed is a tree if and only if $n \leq 3$

Observation

The poset $\mathcal{T}_n^{\langle m \rangle}$ is a lattice for all $m > 0$ if and only if $n \leq 3$.

The Posets $\mathcal{T}_n^{\langle m\rangle}$

- $\mathcal{T}_n$ has $\operatorname{Cat}(n)$ elements, $n-1$ atoms, and $\frac{n-1}{2}\operatorname{Cat}(n)$ cover relations

Observation

We have

$$\left|\mathcal{D}_n^{\langle m\rangle}\right| = \frac{n-1}{2}\Big(Cat(n)-2\Big)\binom{m}{2} + m\cdot Cat(n) - m + 1.$$

The Posets $\mathcal{T}_n^{\langle m\rangle}$

- for $n > 3$ and $m > 1$: $\mathcal{T}_n^{\langle m\rangle}$ is not a lattice and $\left|\mathcal{D}_n^{\langle m\rangle}\right| < \mathrm{Cat}^{(m)}(n)$
- idea: consider a lattice completion of $\mathcal{T}_n^{\langle m\rangle}$
- Dedekind-MacNeille completion: the smallest lattice containing a given poset, denoted by $\mathbf{DM}$

Theorem (Kallipoliti & , 2013)

For $m, n > 0$, we have $\mathcal{T}_n^{(m)} \cong \mathbf{DM}\left(\mathcal{T}_n^{\langle m\rangle}\right)$.

Sketch of Proof

- how do you prove such a statement?

Sketch of Proof

- how do you prove such a statement?
- recall the following result ...

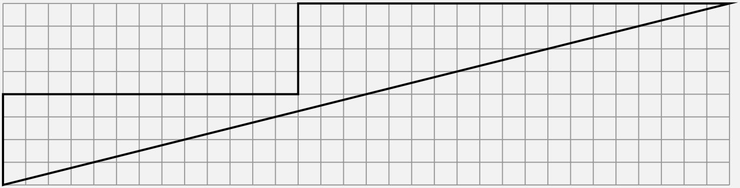
Theorem (Banaschewski, 1956)

If $\mathcal{P}$ is a finite lattice, then $\mathcal{P} \cong \mathbf{DM}\big(\mathcal{J}(\mathcal{P}) \cup \mathcal{M}(\mathcal{P})\big)$.

- ... and investigate the irreducibles

Irreducibles of $\mathcal{T}_n^{(m)}$

- a meet-irreducible element of $\mathcal{T}_8^{(4)}$:



Irreducibles of $\mathcal{T}_n^{(m)}$

Proposition (Kallipoliti & , 2013)

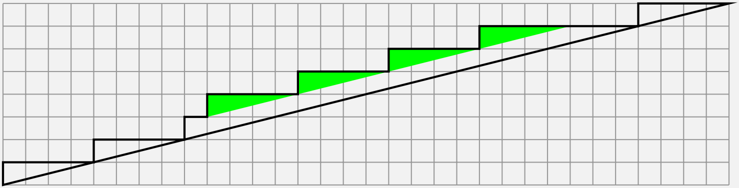
Let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$. *Then,* $\mathfrak{p} \in \mathcal{M}\big(\mathcal{T}_n^{(m)}\big)$ *if and only if its step sequence* $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ *satisfies*

$$u_j = \begin{cases} 0, & \text{for } j \leq i, \\ a, & \text{for } j > i, \end{cases}$$

where $1 \leq a \leq mi$ *and* $1 \leq i < n$.

Irreducibles of $\mathcal{T}_n^{(m)}$

- a join-irreducible element of $\mathcal{T}_8^{(4)}$:



Irreducibles of $\mathcal{T}_n^{(m)}$

Proposition (Kallipoliti & , 2013)

Let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$. *Then,* $\mathfrak{p} \in \mathcal{J}\left(\mathcal{T}_n^{(m)}\right)$ *if and only if its step sequence* $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ *satisfies*

$$u_j = \begin{cases} m(j-1), & \text{for } j \notin \{i, i+1, \ldots, k\}, \\ m(j-1) - s, & \text{for } j \in \{i, i+1, \ldots, k\}, \end{cases}$$

for exactly one $i \in \{1, 2, \ldots, n\}$, *where* $k > i$ *and* $1 \leq s \leq m$.

Irreducibles of $\mathcal{T}_n^{(m)}$

Corollary (Kallipoliti & , 2013)

$\left|\mathcal{M}\left(\mathcal{T}_n^{(m)}\right)\right| = m\binom{n}{2}$ *for every* $m, n > 0$.

Corollary (Kallipoliti & , 2013)

$\left|\mathcal{J}\left(\mathcal{T}_n^{(m)}\right)\right| = m\binom{n}{2}$ *for every* $m, n > 0$.

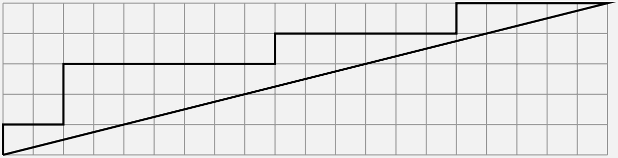
Irreducibles of $\mathcal{T}_n^{\langle m\rangle}$

- the previous results also imply what the irreducibles of $\mathcal{T}_n$ look like
- we have characterized the irreducibles of $\mathcal{P}^{\langle m\rangle}$ for arbitrary bounded posets earlier
- put these things together!

Irreducibles of $\mathcal{T}_n^{\langle m\rangle}$

- the previous results also imply what the irreducibles of $\mathcal{T}_n$ look like
- we have characterized the irreducibles of $\mathcal{P}^{\langle m\rangle}$ for arbitrary bounded posets earlier
- put these things together!
- but how?
  - elements of $\mathcal{T}_n^{(m)}$: m-Dyck paths
  - elements of $\mathcal{T}_n^{\langle m\rangle}$: m-tuples of Dyck paths

The Strip-Decomposition



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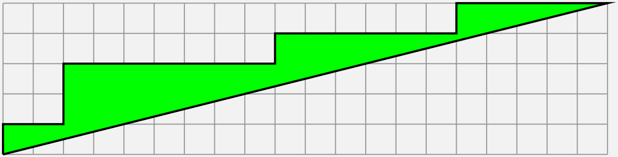

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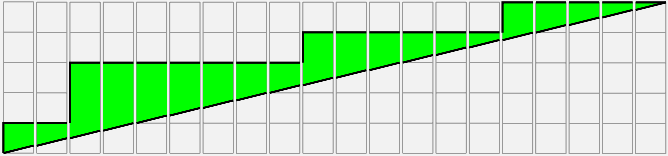

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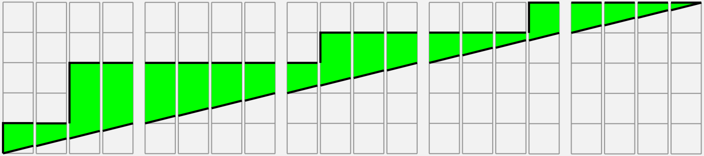

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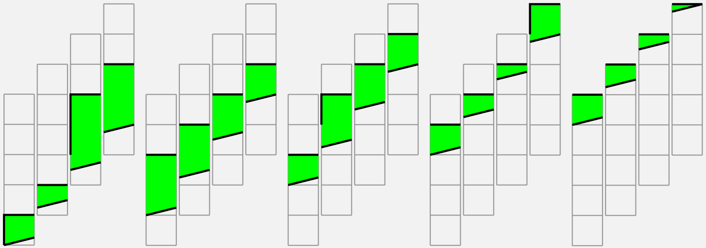

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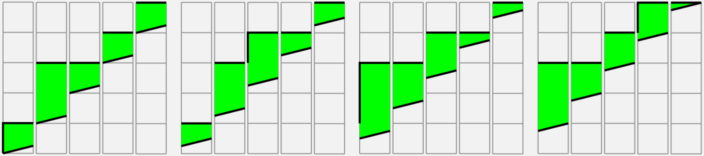

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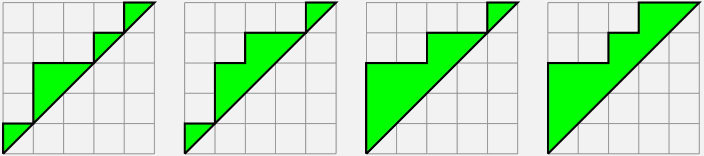

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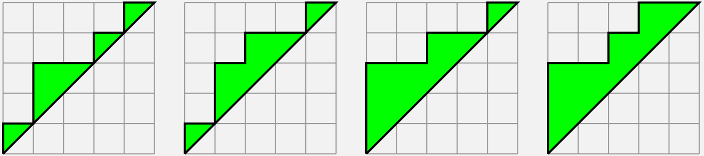

We obtain an injective map $\delta : \mathcal{D}_n^{(m)} \to \left(\mathcal{D}_n\right)^m$!

Irreducibles of $\mathcal{T}_n^{\langle m\rangle}$

Corollary (Kallipoliti & , 2013)

For $m,n>0$, we have $\left|\mathcal{J}\left(\mathcal{T}_n^{\langle m\rangle}\right)\right| = m\binom{n}{2} = \left|\mathcal{M}\left(\mathcal{T}_n^{\langle m\rangle}\right)\right|$.

Proposition (Kallipoliti & , 2013)

If $\mathfrak{p}\in\mathcal{J}\left(\mathcal{T}_n^{(m)}\right)$, *then* $\delta(\mathfrak{p})\in\mathcal{J}\left(\mathcal{T}_n^{\langle m\rangle}\right)$. *If* $\mathfrak{p}\in\mathcal{M}\left(\mathcal{T}_n^{(m)}\right)$, *then* $\delta(\mathfrak{p})\in\mathcal{M}\left(\mathcal{T}_n^{\langle m\rangle}\right)$.

Irreducibles of $\mathcal{T}_n^{\langle m \rangle}$

Proposition (Kallipoliti & , 2013)

The map δ is an poset isomorphism between $\left(\mathcal{J}\bigl(\mathcal{T}_n^{(m)}\bigr), \leq_R\right)$ and $\left(\mathcal{J}\bigl(\mathcal{T}_n^{\langle m \rangle}\bigr), \leq_R\right)$, respectively between $\left(\mathcal{M}\bigl(\mathcal{T}_n^{(m)}\bigr), \leq_R\right)$ and $\left(\mathcal{M}\bigl(\mathcal{T}_n^{\langle m \rangle}\bigr), \leq_R\right)$.

Proposition (Kallipoliti & , 2013)

Every element in $\mathcal{D}_n^{\langle m \rangle}$ can be expressed as a join of elements in $\mathcal{J}\bigl(\mathcal{T}_n^{\langle m \rangle}\bigr)$.

Proving the Connection

Theorem (Kallipoliti & , 2013)

For $m, n > 0$, we have $\mathcal{T}_n^{(m)} \cong \boldsymbol{DM}\bigl(\mathcal{T}_n^{\langle m\rangle}\bigr)$.

Proof

$$\begin{aligned}\mathcal{T}_n^{(m)} &\cong \boldsymbol{DM}\Bigl(\mathcal{J}\bigl(\mathcal{T}_n^{(m)}\bigr)\cup\mathcal{M}\bigl(\mathcal{T}_n^{(m)}\bigr)\Bigr)\\ &\cong \boldsymbol{DM}\Bigl(\mathcal{J}\bigl(\mathcal{T}_n^{\langle m\rangle}\bigr)\cup\mathcal{M}\bigl(\mathcal{T}_n^{\langle m\rangle}\bigr)\Bigr)\\ &\cong \boldsymbol{DM}\Bigl(\mathcal{T}_n^{\langle m\rangle}\Bigr).\end{aligned}$$

Bouncing Dyck Paths

- let $\wedge_R$ and $\vee_R$ denote meet and join in $\mathcal{T}_n$
- define a map

$$\beta_{i,j} : (\mathcal{D}_n)^m \to (\mathcal{D}_n)^m,$$
$$(\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m) \mapsto (\mathfrak{q}_1, \ldots, \mathfrak{q}_i \wedge_R \mathfrak{q}_j, \ldots, \mathfrak{q}_i \vee_R \mathfrak{q}_j, \ldots, \mathfrak{q}_m)$$

- bouncing map: $\beta = \beta_{m-1,m} \circ \cdots \circ \beta_{2,3} \circ \beta_{1,m} \circ \cdots \circ \beta_{1,3} \circ \beta_{1,2}$,
- define $\zeta : \mathcal{D}_n^{(m)} \to (\mathcal{D}_n)^m, \quad \mathfrak{p} \mapsto \beta \circ \delta(\mathfrak{p})$

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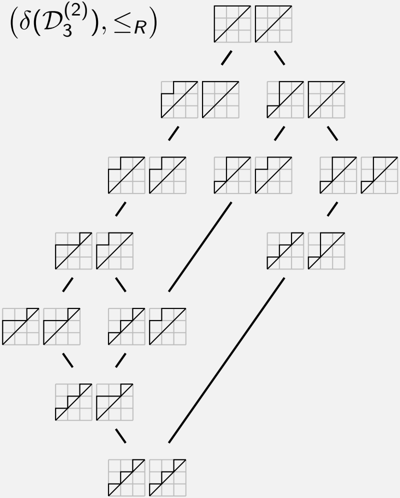
- define $\zeta : \mathcal{D}_n^{(m)} \to (\mathcal{D}_n)^m, \quad \mathfrak{p} \mapsto \beta \circ \delta(\mathfrak{p})$

Conjecture

The posets $\left(\mathcal{D}_n^{(m)}, \leq_R\right)$ and $\left(\zeta\left(\mathcal{D}_n^{(m)}\right), \leq_R\right)$ are isomorphic.

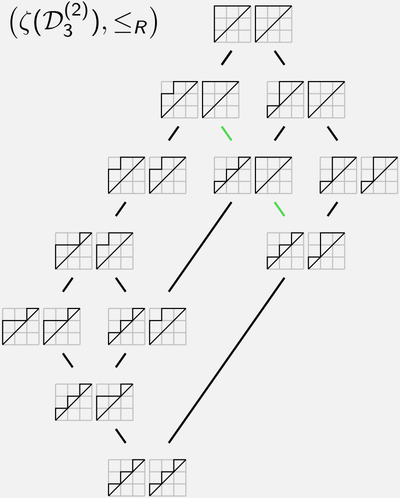
Example

$\left(\delta(\mathcal{D}_3^{(2)}), \leq_R\right)$



Example

$\left(\zeta(\mathcal{D}_3^{(2)}), \leq_R\right)$



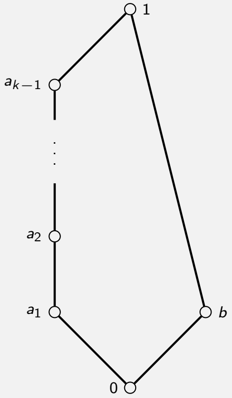
Outline

A Generalization

- $\mathcal{T}_n$ is associated with the Coxeter group A_{n-1}
- Reading's Cambrian lattices provide a generalization of $\mathcal{T}_n$ to the other Coxeter groups
- what about the m-cover posets of other Cambrian lattices?

Cambrian Lattices Associated with the Dihedral Groups

- $\mathfrak{D}_k$: the dihedral group of order $2k$
- $\mathcal{C}_k$: the following poset:



Properties of $\mathcal{C}_k^{\langle m\rangle}$

Proposition (Kallipoliti & , 2013)

For $k > 1$ and $m > 0$, the poset $\mathcal{C}_k^{\langle m\rangle}$ is a lattice with $\binom{m+1}{2}k + m + 1$ elements.

- $\binom{m+1}{2}k + m + 1 = \mathrm{Cat}^{(m)}(\mathfrak{D}_k)$

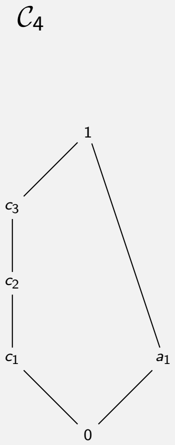
Properties of $\mathcal{C}_k^{\langle m\rangle}$

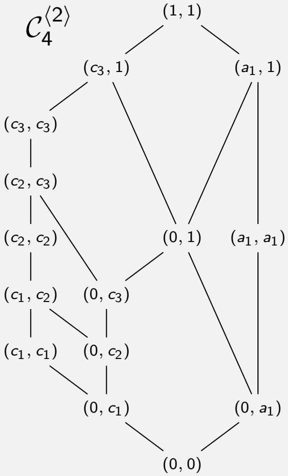
Proposition (Kallipoliti & , 2013)

For $k > 1$ and $m > 0$, the poset $\mathcal{C}_k^{\langle m\rangle}$ is in fact trim, and its Möbius function takes values only in $\{-1, 0, 1\}$.

- this generalizes some structural and topological properties of $\mathcal{C}_k$

Example


$\mathcal{C}_4$
1
c_3
c_2
c_1
a_1
0


$\mathcal{C}_4^{\langle 2\rangle}$
$(1,1)$
$(c_3,1)$
$(a_1,1)$
(c_3,c_3)
(c_2,c_3)
(c_2,c_2)
$(0,1)$
(a_1,a_1)
(c_1,c_2)
$(0,c_3)$
(c_1,c_1)
$(0,c_2)$
$(0,c_1)$
$(0,a_1)$
$(0,0)$

Other Coxeter Groups

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  - the 2-cover poset of the B_3-Tamari lattice has 66 elements ...
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  - ... but $\mathrm{Cat}^{(2)}(B_3) = 84$

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  - the 2-cover poset of the B_3-Tamari lattice has 66 elements ...
  - ... and its Dedekind-MacNeille completion has 88 elements ...
  - ... but $\text{Cat}^{(2)}(B_3) = 84$
- it even fails for the other Cambrian lattices of A_{n-1}

Thank you!

Fuss-Catalan Numbers for Coxeter Groups

- let W be a Coxeter group of rank n, and let $d_1, d_2, \ldots, d_n$ be the degrees of W
- define $\mathrm{Cat}^{(m)}(W) = \prod_{i=1}^{n} \frac{md_n + d_i}{d_i}$

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- define $\mathrm{Cat}^{(m)}(W) = \prod_{i=1}^{n} \frac{md_n + d_i}{d_i}$
- if $W = \mathfrak{S}_n$, then $d_i = i + 1$ for $1 \leq i < n$
- we have $\mathrm{Cat}^{(m)}(\mathfrak{S}_n) = \frac{1}{n}\binom{mn+n}{n-1} = \mathrm{Cat}^{(n)}(m)$

Fuss-Catalan Numbers for Coxeter Groups

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- define $\mathrm{Cat}^{(m)}(W) = \prod_{i=1}^{n} \frac{md_n+d_i}{d_i}$
- if $W = \mathfrak{D}_k$, then $d_1 = 2$ and $d_2 = k$
- we have $\mathrm{Cat}^{(m)}(\mathfrak{D}_k) = \binom{m+1}{2}k + m + 1$

Trim Lattices

- extremal lattice: a lattice $\mathcal{P} = (P, \leq)$ satisfying
  $\bigl|\mathcal{J}(\mathcal{P})\bigr| = \ell(\mathcal{P}) = \bigl|\mathcal{M}(\mathcal{P})\bigr|$
  where $\ell(\mathcal{P})$ is the maximal length of a maximal chain in $\mathcal{P}$
- left-modular element: $x \in P$ satisfying
  $(y \vee x) \wedge z = y \vee (x \wedge z)$ for all $y < z$
- left-modular lattice: a lattice with a maximal chain consisting of left-modular elements
- trim lattice: a left-modular, extremal lattice