Graph Properties of Graph Associahedra

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joint work with **Vincent Pilaud** (CNRS, LIX Polytechnique)

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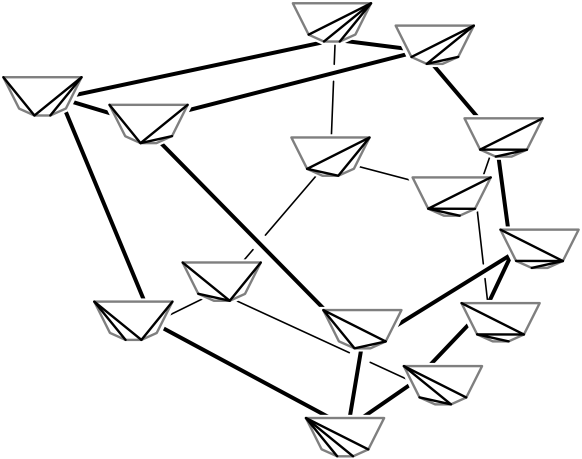
Definition

An ***associahedron*** is a polytope whose graph is the flip graph of triangulations of a convex polygon.

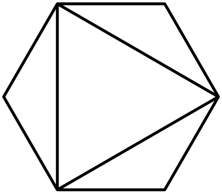


Faces $\leftrightarrow$ dissections of the polygon

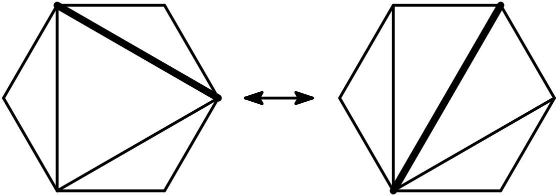
Focus on graphs

Flip graph on the triangulations of the polygon:

Vertices: *triangulations*



Edges: *flips*



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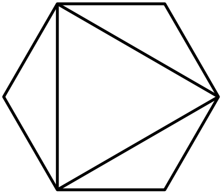
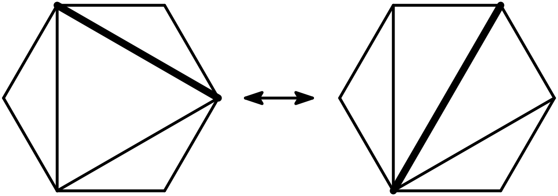
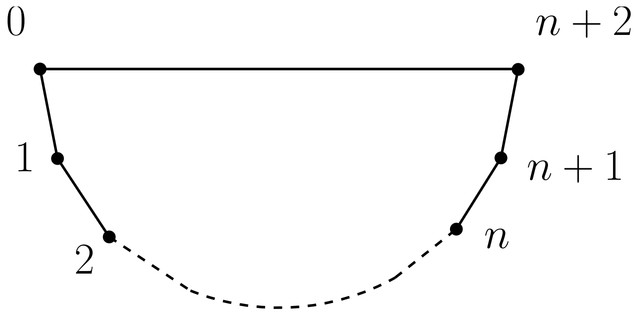




n diagonals $\Rightarrow$ the flip graph is n-regular.

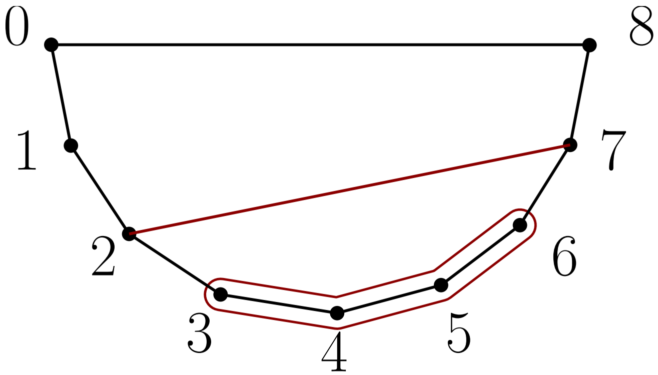
Useful configuration (Loday's)

$$G_{n+3} =$$



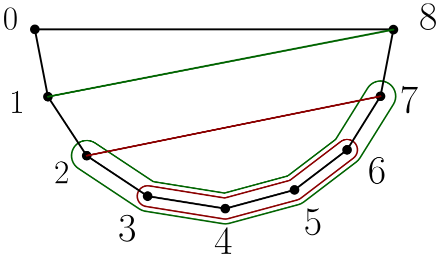
Graph point of view

$\{\text{diagonals of } G_{n+3}\} \longleftrightarrow \{\text{strict subpaths of the path } [n+1]\}$

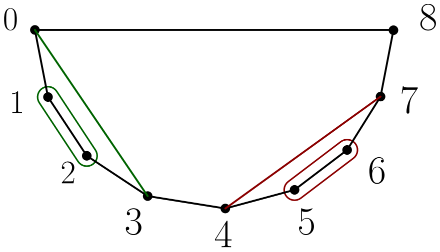


Non-crossing diagonals

Two ways to be non-crossing in Loday's configuration:



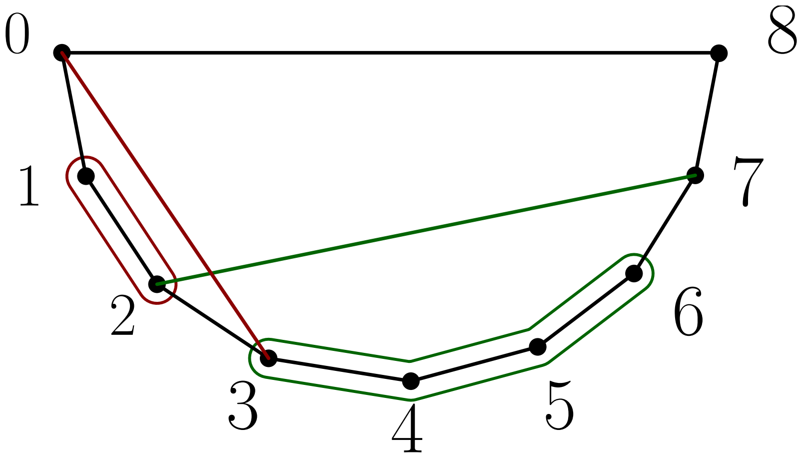
nested subpaths



non-adjacent subpaths

Pay attention to the second case:

The right condition is indeed *non-adjacent*, disjoint is not enough!



Now do it on graphs

$G = (V, E)$ a (connected) graph.

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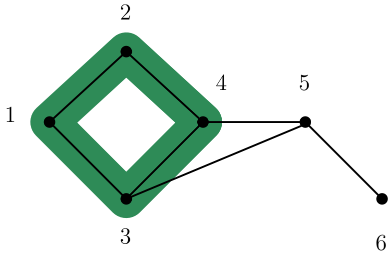
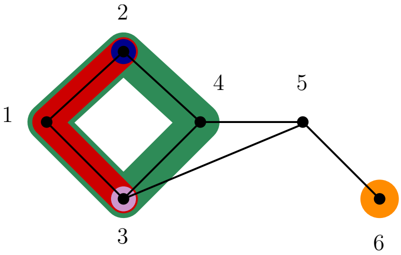
Definition

- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;
- t and t' are ***compatible*** if they are nested or non-adjacent;
- A ***tubing*** of G is a set of pairwise compatible tubes of G.



A tube

(generalizes a diagonal)



A maximal tubing

(generalizes a triangulation)

Graph associahedra

The simplicial complex of tubings is spherical

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The simplicial complex of tubings is spherical $\Rightarrow$ **flip graph** !

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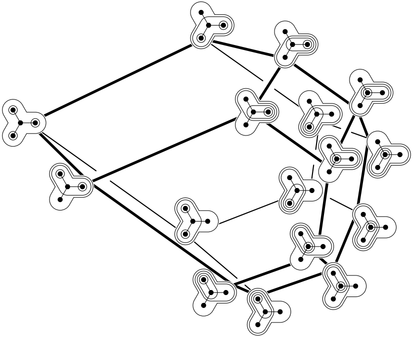
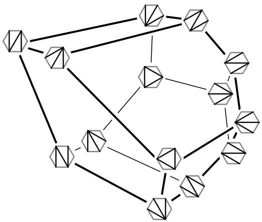
Theorem (Carr-Devadoss '06)

There exists a polytope called ***graph associahedron*** *of* G*, denoted* $\mathbf{Asso}_G$*, whose graph is this flip graph.*

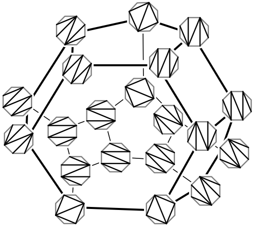


Faces $\leftrightarrow$ tubings of G.

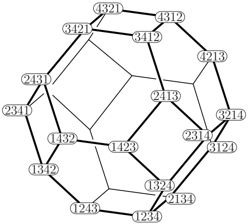
Classical polytopes...



The associahedron

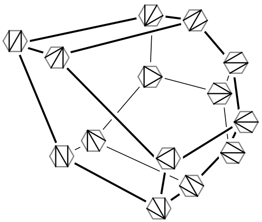


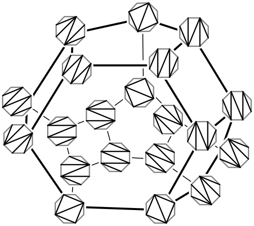
The cyclohedron

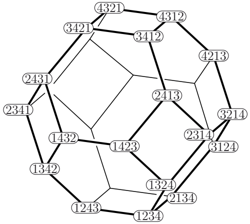


The permutahedron

…can be seen as graph associahedra







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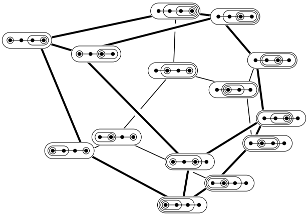

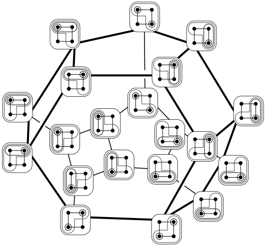

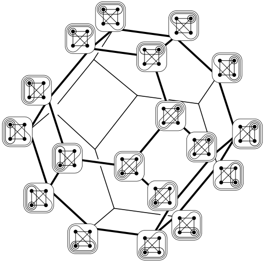

Hamiltonicity of flip graphs

Theorem (Trotter '62, Johnson '63, Steinhaus '64)

The n-dimensional permutahedron is hamiltonian for $n \geq 2$.

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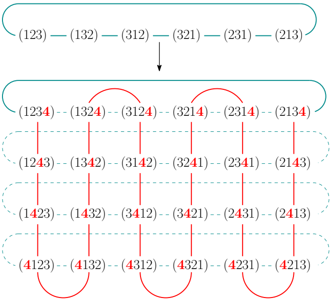

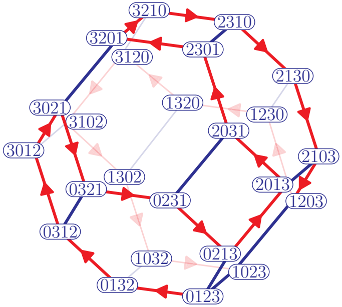

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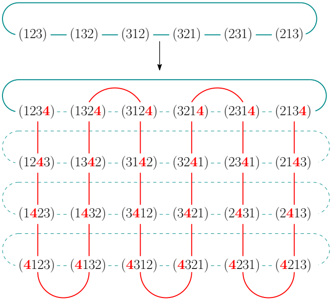

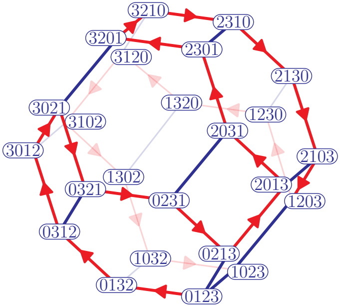

Theorem (Lucas 87, Hurtado-Noy '99)

The n-dimensional associahedron is hamiltonian for $n \geq 2$.

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Any graph associahedron is hamiltonian.

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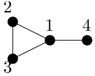

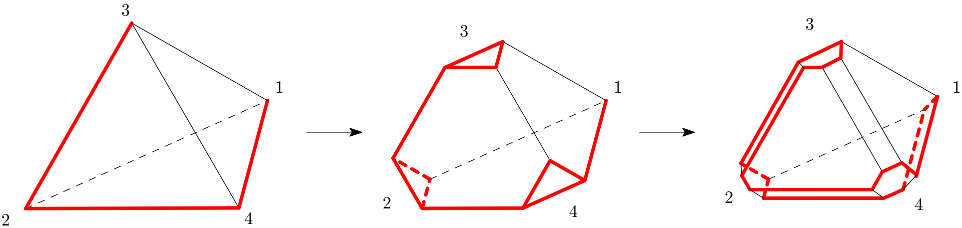

Diameter of flip graphs

Lemma

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Theorem (Sleator-Trajan-Thurston '88, Pournin '12)

The diameter of the n-dimensional associahedron is $2n - 4$ *for* $n \geq 10$.

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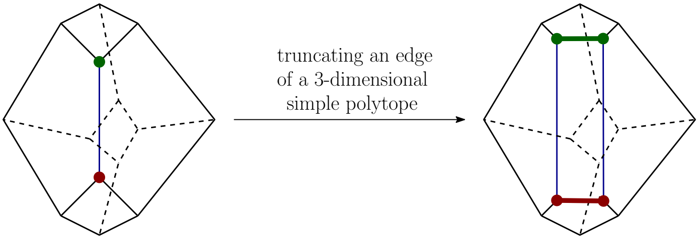
$\rightarrow$ Truncating $\iff$ replacing vertices by complete graphs.



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- Pournin's result for the classical associahedron.

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THANK YOU FOR

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