q-Jacobi-Stirling numbers and q-differential equations for q-classical polynomials

Ana F. Loureiro (University of Kent) and Jiang Zeng (U. Lyon-I)

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Outline

- q-classical polynomials
- q-differential equations
- q-Jacobi-Stirling numbers
- q-Stirling numbers, signed partitions

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q-differential equations for q-classical polynomials and q-Jacobi-Stirling numbers, arXiv :1309.4968

A Monic Orthogonal Polynomial Sequence (MOPS) $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle u_0, P_n P_k \rangle = N_n \delta_{n,k} \ , \ \text{with} \ \ N_n \neq 0.$$

where u_0 is the first element of the corresponding dual sequence.

- In this case u_0 is said to be **regular**.

- $\{P_n\}_{n\geqslant 0}$ the second order recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)$$

with $P_0 = 1$ and $P_{-1} = 0$ and

$$\beta_n = \frac{\langle u_0, xP_n^2 \rangle}{\langle u_0, P_n^2 \rangle} \qquad \text{and} \qquad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2 \rangle}{\langle u_0, P_n^2 \rangle} \neq 0, \ n \in \mathbb{N}$$

in this case...

the Hankel determinant

$$\Delta_n(u_0) = \det\left[(u_0)_{i+j}\right]_{0\leqslant i,j\leqslant n} \neq 0\ ,\ \ n \geqslant 0, \quad \text{with } (u_0)_k = < u_0, x^k >,$$

and

$$P_n(x) = \frac{1}{\Delta_{n-1}(u_0)} \begin{vmatrix} 1 & (u_0)_1 & \dots & (u_0)_{n-1} & (u_0)_n \\ (u_0)_1 & (u_0)_2 & \dots & (u_0)_n & (u_0)_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (u_0)_{n-1} & (u_0)_n & \dots & (u_0)_{2n-2} & (u_0)_{2n-1} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix}\ ,\ \ n \geqslant 0,$$

Classical polynomials

Theorem. For any MOPS $\{P_n\}_{n\geqslant 0}$ the following statements are equivalent.

(a) $\{P_n\}_{n\geqslant 0}$ is classical , *i.e.*, $\left\{P_n^{[1]}(x) := \frac{1}{n+1} DP_{n+1}(x)\right\}_{n\geqslant 0}$ is a MOPS.

(Hahn, 1937)

(b) There exists a pair of polynomials (Φ, Ψ), with Φ monic, $\deg\Phi \leqslant 2$, $\deg\Psi = 1$, and such that

$$D(\Phi u_0) + \Psi u_0 = 0$$

(c) There exists a pair of polynomials (Φ, Ψ) such that $\{P_n\}_{n\geq 0}$ satisfies

$$\mathcal{L}[P_n](x) = \chi_n P_n(x) \ , \ n \geq 1, \quad \text{with} \quad \mathcal{L} := \Phi(x)D^2 - \Psi(x)D,$$

with $\chi_n \neq 0$ given by

$$\chi_n = \begin{cases} -n\Psi'(0) & \text{if} \quad \deg\Phi = 0,1 \\ n\left(n-1-\Psi'(0)\right) & \text{if} \quad \deg\Phi = 2 \end{cases} \ , \ n \geq 0. \qquad (1)$$

(Bochner, 1929)

(d) There is a monic polynomial Φ with $\deg\Phi \leq 2$ and a sequence of nonzero numbers $\{\vartheta_n\}_{n\geq 0}$ such that

$$P_n u_0 = \vartheta_n D^n\left(\left(\Phi(x)\right)^n u_0\right) \ , \ n \geqslant 0. \qquad (2)$$

q-Classical Polynomials

Definition. A MOPS $\{P_n\}_{n\geqslant 0}$ is q-classical iff

$$\{P_n^{[1]}(x) := \frac{1}{[n+1]_q} D_q P_{n+1}(x)\}_{n\geqslant 0}$$

is also a MOPS. (Hahn, 1949)

Here, $q \neq 0$ and $|q| \neq 1$,

$$\big(D_q f\big)(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad \text{if} \quad x \neq 0,$$

$$\big(D_q f\big)(0) := f'(0)$$

and

$$[a]_q := \frac{q^a - 1}{q - 1}$$

q-Classical polynomials

Theorem. For any MOPS $\{P_n\}_{n\geqslant 0}$ the following statements are equivalent.

(a) $\{P_n\}_{n\geqslant 0}$ is q-classical
(b) There exists a pair of polynomials (Φ, Ψ), with Φ monic, $\deg \Phi \leqslant 2$, $\deg \Psi = 1$, and such that

$$D_q(\Phi u_0) + \Psi u_0 = 0$$

(c) There exists a pair of polynomials (Φ, Ψ) such that $\{P_n\}_{n\geq 0}$ satisfies

$$\mathcal{L}_q[P_n](x) = \chi_n P_n(x) \ , \ \ n \geq 1, \quad \text{with} \quad \mathcal{L}_q := \Phi(x) D_q \circ D_{q^{-1}} - \Psi(x) D_{q^{-1}},$$

with $\chi_n \neq 0$ given by

$$\chi_n = \begin{cases} -[n]_{q^{-1}} \Psi'(0) & \text{if} \quad \deg \Phi = 0, 1 \\ [n]_{q^{-1}} \left([n-1]_q - \Psi'(0)\right) & \text{if} \quad \deg \Phi = 2 \end{cases} \ , \ n \geq 0. \qquad (3)$$

(d) There is a monic polynomial Φ with $\deg \Phi \leq 2$ and a sequence of nonzero numbers $\{\vartheta_n\}_{n\geq 0}$ such that

$$P_n u_0 = \vartheta_n D_q^n \left(\left(\prod_{\sigma=0}^{n-1} q^{-\sigma \deg \Phi} \Phi(q^\sigma x) \right) u_0 \right) \ , \ \ n \geqslant 0. \qquad (4)$$

Corollary. For each integer $k > 0$, if $\{P_n\}_{n\geq 0}$ is q-classical, then so is $\{P_n^{[k]}(x) := \frac{1}{[n+1;q]_k} D_q^k P_{n+k}(x)\}_{n\geq 0}$ and we have

$$\Phi_k(x) D_q \circ D_{q^{-1}} \left(P_n^{[k]}(x) \right) - \Psi_k(x) D_{q^{-1}} \left(P_n^{[k]}(x) \right) = \chi_n^{[k]} P_n^{[k]}(x) \ , \ n \geq 0, \quad (5)$$

where

$$\Phi_k(x) = q^{-k \deg \Phi} \Phi(q^k x), \quad \Psi_k(x) = q^{-k \deg \Phi} \left(\Psi(x) - [k]_q \ (D_{q^k} \Phi)(x) \right)$$

and

$$\chi_n^{[k]} = \begin{cases} -[n]_{q^{-1}} q^{-(\deg \Phi) k} \ \Psi'(0) & \text{if} \quad \deg \Phi = 0, 1, \\ [n]_{q^{-1}} q^{-2k-1} \left([n+2k]_q - (1 + q\Psi'(0)) \right) & \text{if} \quad \deg \Phi = 2 \end{cases} , \ n \geq 0.$$

The corresponding q-classical form $u_0^{[k]}$ fulfills

$$D_q \left(\Phi_k u_0^{[k]} \right) + \Psi_k u_0^{[k]} = 0$$

and $\quad u_0^{[k]} = \zeta_k \left(\prod_{\sigma=0}^{k-1} \Phi_\sigma(x) \right) u_0, \quad$ where $\zeta_k \neq 0$ is such that $\left(u_0^{[k]} \right)_0 = 1$, with the convention $\zeta_0 = 1$.

(Kheriji & Maroni, 2002), (M. Ismail, 2009), (Koekoek *et al.*)

Corollary. For each integer $k > 0$, if $\{P_n\}_{n\geq 0}$ is q-classical, then so is $\{P_n^{[k]}(x) := \frac{1}{[n+1;q]_k} D_q^k P_{n+k}(x)\}_{n\geq 0}$ and we have

$$\Phi_k(x) D_q \circ D_{q^{-1}} \left(P_n^{[k]}(x)\right) - \Psi_k(x) D_{q^{-1}} \left(P_n^{[k]}(x)\right) = \chi_n^{[k]} P_n^{[k]}(x) \ , \ n \geq 0, \quad (5)$$

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(Kheriji & Maroni, 2002), (M. Ismail, 2009), (Koekoek *et al.*)

Theorem. (AFL & Zeng, 2013) Let $k>0$. If $\{P_n\}_{n\geq 0}$ is q-classical, then there exist Φ (monic) and Ψ with $\deg \Phi \leq 2$ and $\deg \Psi = 1$, such that the elements of $\{P_n\}_{n\geq 0}$ are solutions of the following $2k$-order q-differential equation

$$\mathcal{L}_{k;q}[y](x) := \sum_{\nu=0}^{k} \Lambda_{k,\nu}(x;q)\left(D_{q^{-1}}^{k-\nu} \circ D_q^k \; y\right)(q^{-\nu}x) \;=\; \Xi_n(k;q)\; y(x) \qquad (6)$$

with $y(x) = P_n(x)$ and where

$$\Lambda_{k,\nu}(x;q) = \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} q^{-(k-\nu)} \left(\prod_{\sigma=1}^{\nu} \chi_{\sigma}^{[k-\sigma]}\right) \left(\prod_{\sigma=0}^{k-\nu-1} \underbrace{q^{-\sigma \deg \Phi}\Phi(q^{\sigma}x)}_{\Phi_\sigma(x)}\right) P_{\nu}^{[k-\nu]}(x)$$

for $\nu = 0, 1, \ldots, k$, and

$$\Xi_n(k;q) = \prod_{\sigma=0}^{k-1} \chi_{n-\sigma}^{[\sigma]} \;,\;\; n \geq 0,$$

with $\chi_n^{[k]} = \begin{cases} -[n]_{q^{-1}} q^{-(\deg \Phi)k}\; \Psi'(0) & \text{if} \quad \deg \Phi = 0, 1, \\ [n]_{q^{-1}} q^{-2k-1}\Big([n+2k]_q - (1+q\Psi'(0))\Big) & \text{if} \quad \deg \Phi = 2 \end{cases}$

Higher order q-differential equations

Corollary. (AFL & Zeng, 2013) Let $k>0$. Any q-classical sequence $\{P_n\}_{n\geq 0}$, orthogonal with respect to the weight function $U_q(x)$, fulfills

$$\mathcal{L}_{k;q}[P_n](x) = \Xi_n(k;q)\ P_n(x),\ \ n\geq 0, \tag{7}$$

where

$$\mathcal{L}_{k;q}[y](x) := q^{-k}\big(U_q(x)\big)^{-1} D^k_{q^{-1}}\Big(\ \Big(\prod_{\sigma=0}^{k-1}\Phi_\sigma(x)\Big)\ U_q(x)\big(D^k_q y(x)\big)\Big), \tag{8}$$

with

$$\Xi_n(k;q) = \prod_{\sigma=0}^{k-1}\chi^{[\sigma]}_{n-\sigma}\ ,\ \ n\geq 0,$$

and

$$\chi^{[k]}_n = \begin{cases} -[n]_{q^{-1}}q^{-(\deg\Phi)k}\ \Psi'(0) & \text{if}\quad \deg\Phi = 0,1,\\ [n]_{q^{-1}}q^{-2k-1}\Big([n+2k]_q - (1+q\Psi'(0))\Big) & \text{if}\quad \deg\Phi = 2\end{cases}$$

Higher order q-differential equations

For the cases where $\deg \Phi = 0, 1$ ...

$$\mathcal{L}_q^k[y](x) = \Big(-[n]_{q^{-1}} \Psi'(0) \Big)^k y(x)$$

$$\mathcal{L}_{k;q}[y](x) = \Big(-q^{\frac{(k-1)}{2}(1-\deg \Phi)} \Psi'(0) \Big)^k \prod_{\sigma=0}^{k-1} \Big([n]_{q^{-1}} - [\sigma]_{q^{-1}} \Big) y(x)$$

whilst $\deg \Phi = 2$ can be written as

$$\mathcal{L}_q^k[y](x) = \Big([n]_{q^{-1}} \Big([n-1]_q - \Psi'(0) \Big) \Big)^k y(x)$$

$$\mathcal{L}_{k;q}[y](x) = q^{-\frac{k(k+1)}{2}} \prod_{\sigma=0}^{k-1} \Big([n]_{q^{-1}} \big(z + [n]_q \big) - [\sigma]_{q^{-1}} \big(z + [\sigma]_q \big) \Big) y(x)$$

Higher order q-differential equations

Theorem. The kth composite power of the operator
$\mathcal{L}_q := \Phi(x) D_q \circ D_{q^{-1}} - \Psi(x) D_{q^{-1}}$ is given by

$$\mathcal{L}_q^k[f](x) = \begin{cases} \displaystyle\sum_{j=0}^{k} S_{q^{-1}}(k,j)\, q^{(\deg\Phi - 1)\frac{j(j-1)}{2}} \left(-\Psi'(0)\right)^{k-j} \mathcal{L}_{j;q}[f](x) & \text{if} \quad \deg\Phi = 0,1, \\ \displaystyle\sum_{j=0}^{k} \mathrm{JS}_k^j(z;q^{-1})\, q^{\frac{j(j+1)}{2}-k} \mathcal{L}_{j;q}[f](x) & \text{if} \quad \deg\Phi = 2, \end{cases}$$

which holds for any $f \in \mathcal{P}$, where $z = -(1 + q\Psi'(0))$.

and reciprocally

Corollary. For any polynomial $f \in \mathcal{P}$ we have

$$\mathcal{L}_{k;q}[f](x) = \begin{cases} q^{(1-\deg\Phi)\frac{k(k-1)}{2}} \displaystyle\sum_{j=0}^{k} c_{q^{-1}}(k,j) \left(-\Psi'(0)\right)^{k-j} \mathcal{L}_q^j[f](x) & \text{if} \quad \deg\Phi = 0,1, \\ \displaystyle\sum_{j=0}^{k} \mathrm{Jc}_k^j(z;q^{-1})\, q^{j-\frac{k(k+1)}{2}} \mathcal{L}_q^j[f](x) & \text{if} \quad \deg\Phi = 2. \end{cases}$$

Higher order q-differential equations

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The q-classical polynomials

$\deg \Phi$	q-**classical MOPS**
0	Al-Salam Carlitz polynomials · Discrete q-Hermite polynomials
1	Big q-Laguerre · q-Meixner · Wall q-polynomials q-Laguerre polynomials · Little q-Laguerre polynomials q-Charlier I polynomials
2 (with double root)	Alternative q-Charlier polynomials · Stieltjes-Wigert q-polynomials
2 (with 2 single roots)	Little q-Jacobi polynomials · q-Charlier II polynomials Generalized Stieltjes-Wigert q-polynomials · Big q-Jacobi Bi-generalized Stieltjes-Wigert q-polynomials

Example 1. The monic Stieltjes-Wigert polynomials

$$P_n(x;q) = \sum_{k=0}^{n} \frac{(-1)^{n+k} q^{k(k+\frac{1}{2})-n(n+\frac{1}{2})}}{(q;q)_{n-k}} x^k \text{ are eigenfunctions of}$$

$$\mathcal{L}_q := x^2 D_q \circ D_{q^{-1}} + (q-1)^{-1}\{x - q^{-3/2}\} D_{q^{-1}}.$$

Notice that $P_n^{[k]}(x;q) = q^{-2nk} P_n(q^{2k}x;q)$, $n \in \mathbb{N}_0$.
When $0 < q < 1$

$$\mathcal{L}_{k;q}[y](x) = q^{-k} \exp\left(\frac{\ln^2 x}{2\ln q^{-1}}\right) D_{q^{-1}}^k \left(x^{2k} \exp\left(-\frac{\ln^2 x}{2\ln q^{-1}}\right) \left(D_q^k y(x)\right)\right) .$$

Alternatively,

$$\mathcal{L}_{k;q}[y](x) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} \alpha_{k,\nu:q} x^{2k-2\nu} P_\nu^{[k-\nu]}(x) \left(D_{q^{-1}}^{k-\nu} \circ D_q^k \, y \right)(q^{-\nu}x),$$

$$\mathcal{L}_q^k[f](x) = \sum_{j=0}^{k} \mathrm{JS}_k^j((q-1)^{-1}; q^{-1})\, q^{\frac{j(j+1)}{2}-k} \mathcal{L}_{j;q}[f](x) \ , \ \forall f \in \mathcal{P}.$$

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Alternatively,

$$\mathcal{L}_{k;q}[y](x) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} \alpha_{k,\nu:q} x^{2k-2\nu} P_\nu^{[k-\nu]}(x) \left(D_{q^{-1}}^{k-\nu} \circ D_q^k \; y \right)(q^{-\nu}x),$$

$$\mathcal{L}_q^k[f](x) = \sum_{j=0}^{k} \mathrm{JS}_k^j((q-1)^{-1};q^{-1})\, q^{\frac{j(j+1)}{2}-k} \mathcal{L}_{j;q}[f](x) \;, \; \forall f \in \mathcal{P}.$$

Example 2. The monic Little q-Jacobi polynomials

$$P_n(x;a,b|q) \quad = \quad \frac{(aq;q)_n}{(abq^{n+1};q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^{n-k}(abq^{n+1};q)_k}{(aq;q)_k} q^{\binom{n-k}{2}} x^k, \;\; n \geqslant 0,$$

with $a, b, ab \neq q^{-(n+2)}$, for $n \geqslant 0$,

are eigenfunctions of

$$\mathcal{L}_q := x(x - b^{-1}q^{-1}) D_q \circ D_{q^{-1}} - \left((abq^2(q-1))^{-1} \{(1-abq^2)x + aq - 1\} \right) D_{q^{-1}}$$

as well as of

$$\mathcal{L}_{k;q}[y](x) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} \alpha_{k,\nu:q} \left(\prod_{\sigma=0}^{k-1} x(x - b^{-1}q^{-(\sigma+1)}) \right) P_\nu^{[k-\nu]}(x) \left(D_{q^{-1}}^{k-\nu} \circ D_q^k \, y \right) (q^{-\nu}x)$$

with

$$\alpha_{k,\nu:q} = q^{-(k-\nu)} \left(\prod_{\sigma=1}^{\nu} [\sigma]_{q^{-1}} q^{-2k+\sigma} \left([2k-\sigma]_q + \frac{1}{q-1} \left(1 - (abq)^{-1} \right) \right) \right).$$

Notice that $P_n^{[k]}(x;a,b|q) = P_n(x;aq^k,bq^k|q), \quad n \in \mathbb{N}_0.$

Example 2. Little q-Jacobi polynomials (cont.)

In particular, when $0 < q < 1$, $b \in]-\infty, 1[- \{0\}$ and $a := q^{\alpha-1}$ with $\alpha > 0$,

$$\mathcal{L}_{k;q}[y](x)$$
$$= q^{-k}\big(x^{\alpha-1}\frac{(qx;q)_\infty}{(bqx;q)_\infty}\big)^{-1} D^k_{q^{-1}}\Big(\Big(\prod_{\sigma=0}^{k-1} x(x - b^{-1}q^{-(\sigma+1)})\Big)\ x^{\alpha-1}\frac{(qx;q)_\infty}{(bqx;q)_\infty}\big(D^k_q y(x)\big)\Big)\ .$$

In any case, we always have

$$\mathcal{L}^k_q[f](x) = \sum_{j=0}^{k} \mathrm{JS}^j_k(z; q^{-1})\, q^{\frac{j(j+1)}{2}-k}\, \mathcal{L}_{j;q}[f](x),$$

where $z = -(1 + q\Psi'(0)) = \frac{1}{q-1}\left(1 - (abq)^{-1}\right)$.

Example 2. Little q-Jacobi polynomials (cont.)

In particular, when $0 < q < 1$, $b \in]-\infty, 1[- \{0\}$ and $a := q^{\alpha-1}$ with $\alpha > 0$,

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q-Jacobi Stirling numbers

$$\left\{\prod_{i=0}^{n-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right)\right\}_{n\geq 0} \quad \longleftrightarrow \quad \{x^n\}_{n\geq 0}$$

Definition.

$$\prod_{i=0}^{n-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right)=\sum_{k=0}^{n}(-1)^{n-k}\,\mathrm{Jc}_n^k(z;q)\,x^k,\ \ n\geqslant 0,$$

$$x^n=\sum_{k=0}^{n}\mathrm{JS}_n^k(z;q)\prod_{i=0}^{k-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right),\ \ n\geqslant 0.$$

They satisfy the triangular relations

$$\mathrm{Jc}_{n+1}^{k+1}(z;q)=\mathrm{Jc}_n^k(z;q)+[n]_q\left(z+[n]_{q^{-1}}\right)\mathrm{Jc}_n^{k+1}(z;q),\ 0\leq k\leq n,$$

$$\mathrm{JS}_{n+1}^{k+1}(z;q)=\mathrm{JS}_n^k(z;q))+[k+1]_q\left(z+[k+1]_{q^{-1}}\right)\mathrm{JS}_n^{k+1}(z;q),\ 0\leq k\leq n,$$

with $\mathrm{Jc}_n^k(z;q)=\mathrm{JS}_n^k(z;q)=0,\ \ \text{if } k\notin\{1,\ldots,n\}$, and $\mathrm{Jc}_0^0(z;q)=\mathrm{JS}_0^0(z;q)=1,\ n\geq 0.$

q-Jacobi Stirling numbers

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$$x^n=\sum_{k=0}^{n}\mathrm{JS}_n^k(z;q)\prod_{i=0}^{k-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right),\ \ n\geqslant 0.$$

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$$\mathrm{Jc}_{n+1}^{k+1}(z;q)=\mathrm{Jc}_n^k(z;q)+[n]_q\left(z+[n]_{q^{-1}}\right)\mathrm{Jc}_n^{k+1}(z;q),\ \ 0\leq k\leq n,$$

$$\mathrm{JS}_{n+1}^{k+1}(z;q)=\mathrm{JS}_n^k(z;q))+[k+1]_q\left(z+[k+1]_{q^{-1}}\right)\mathrm{JS}_n^{k+1}(z;q),\ \ 0\leq k\leq n,$$

with $\mathrm{Jc}_n^k(z;q)=\mathrm{JS}_n^k(z;q)=0,\quad \text{if } k\notin\{1,\ldots,n\}$, and
$\mathrm{Jc}_0^0(z;q)=\mathrm{JS}_0^0(z;q)=1,\ n\geq 0.$

q-Jacobi Stirling numbers

$$\left\{\prod_{i=0}^{n-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right)\right\}_{n\geq 0} \longleftrightarrow \{x^n\}_{n\geq 0}$$

Definition.

$$\prod_{i=0}^{n-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right)=\sum_{k=0}^{n}(-1)^{n-k}\,\mathrm{Jc}_n^k(z;q)\,x^k,\ \ n\geqslant 0,$$

$$x^n=\sum_{k=0}^{n}\mathrm{JS}_n^k(z;q)\prod_{i=0}^{k-1}\left(x-[i]_q\left(z+[i]_{q^{-1}}\right)\right),\ \ n\geqslant 0.$$

They satisfy the triangular relations

$$\mathrm{Jc}_{n+1}^{k+1}(z;q)=\mathrm{Jc}_n^k(z;q)+[n]_q\left(z+[n]_{q^{-1}}\right)\mathrm{Jc}_n^{k+1}(z;q),\ \ 0\leq k\leq n,$$

$$\mathrm{JS}_{n+1}^{k+1}(z;q)=\mathrm{JS}_n^k(z;q))+[k+1]_q\left(z+[k+1]_{q^{-1}}\right)\mathrm{JS}_n^{k+1}(z;q),\ \ 0\leq k\leq n,$$

with $\mathrm{Jc}_n^k(z;q)=\mathrm{JS}_n^k(z;q)=0,\ \ \text{if } k\notin\{1,\dots,n\}$, and
$\mathrm{Jc}_0^0(z;q)=\mathrm{JS}_0^0(z;q)=1,\ n\geq 0.$

First q-Jacobi-Stirling numbers

Some value of the q-Jacobi-Stirling numbers of first kind are as follows :

$$\mathrm{Jc}_n^1(z;q) = \prod_{k=1}^{n-1}[k]_q(z+[k]_{q^{-1}}), \quad Jc_n^{(n)}(z,q) = 1,$$

$$\mathrm{Jc}_3^2(z;q) = (3+q+q^{-1}) + (2+q)z,$$

$$\mathrm{Jc}_4^2(z;q) = (q^{-3}+5\,q^{-2}+11\,q^{-1}+q^3+11\,q+5\,q^2+15)$$
$$+\left(2\,q^3+14\,q+8\,q^2+2\,q^{-2}+7\,q^{-1}+15\right)z+\left(4\,q+q^3+3+3\,q^2\right)z^2,$$

$$\mathrm{Jc}_4^3(z;q) = \left(3q^{-1}+6+q^2+3q+q^{-2}\right) + (3+2q+q^2)z.$$

Some values of the q-Jacobi-Stirling numbers of second kind are as follows :

$$\mathrm{JS}_n^1(z;q) = (1+z)^{n-1}, \quad JS_n^{(n)}(z;q) = 1,$$

$$\mathrm{JS}_3^2(z;q) = (3+q+q^{-1}) + (2+q)z,$$

$$\mathrm{JS}_4^2(z;q) = (9+q^{-2}+q^2+5q+5q^{-1}) + (11+3q^{-1}+2q^2+8q)z + (3q+3+q^2)z^2$$

Other properties of the q-Jacobi-Stirling numbers

$$\mathrm{JS}_n^j(z;q) = \sum_{r=0}^{j} (-1)^{j-r} \frac{q^{-\binom{r}{2}-r(j-r)} \big([r]_q([r]_{q^{-1}}+z)\big)^n}{[r]_q![j-r]_q! \prod_{0\le k\le j,\ k\ne r} (z+[k+r]_{q^{-1}})}.$$

and

$$\prod_{i=1}^{k} \frac{x}{1-[i]_q([i]_{q^{-1}}+z)x} = \sum_{n\ge k} \mathrm{JS}_n^k(z;q)x^n.$$

Theorem. Let n, k be positive integers with $n \ge k$. The Jacobi-Stirling numbers $\mathrm{JS}_n^k(z,q)$ and $\mathrm{Jc}_n^k(z,q)$ are polynomials in z of degree $n-k$ with coefficients in $\mathbb{N}[q,q^{-1}]$. Moreover, if

$$\mathrm{JS}_n^k(z;q) = a_{n,k}^{(0)}(q) + a_{n,k}^{(1)}(q)z + \cdots + a_{n,k}^{(n-k)}(q)z^{n-k},$$
$$\mathrm{Jc}_n^k(z;q) = b_{n,k}^{(0)}(q) + b_{n,k}^{(1)}(q)z + \cdots + b_{n,k}^{(n-k)}(q)z^{n-k},$$

then

$$a_{n,k}^{(n-k)} = S_q(n,k), \quad b_{n,k}^{(n-k)} = c_q(n,k).$$

Other properties of the q-Jacobi-Stirling numbers

$$\mathsf{JS}_n^j(z;q) = \sum_{r=0}^{j} (-1)^{j-r} \frac{q^{-\binom{r}{2}-r(j-r)}\big([r]_q([r]_{q^{-1}}+z)\big)^n}{[r]_q![j-r]_q! \prod_{0\le k\le j,\ k\neq r} (z+[k+r]_{q^{-1}})}.$$

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then

$$a_{n,k}^{(n-k)} = S_q(n,k), \quad b_{n,k}^{(n-k)} = c_q(n,k).$$

Combinatorial interpretation of the q-Jacobi-Stirling numbers

Let $[n]_2 = \{1_1, 1_2, \ldots, n_1, n_2\}$.

Definition. A Jacobi-Stirling k-partition of $[n]_2$ is a partition of $[n]_2$ into $k+1$ subsets $B_0, B_1, \ldots B_k$ of $[n]_2$ satisfying the following conditions :

1. there is a zero block B_0, which may be empty and cannot contain both copies of any $i \in [n]$,
2. $\forall j \in [k]$, each nonzero block B_j is not empty and contains the two copies of its smallest element and does not contain both copies of any other number.

Example.

- $\pi = \{\{2_2, 5_1\}_0, \{1_1, 1_2, 2_1\}, \{3_1, 3_2, 4_2\}, \{4_1, 5_2\}\}$ is **not** a Jacobi-Stirling 3-partition of $[5]_2$,
- $\pi' = \{\{2_2, 5_1\}_0, \{1_1, 1_2, 2_1\}, \{3_1, 3_2\}, \{4_1, 4_2, 5_2\}\}$ is a Jacobi-Stirling 3-partition of $[5]_2$.

We order the blocks of a partition in increasing order of their minimal elements. By convention, the zero block is at the first position.

Combinatorial interpretation of the q-Jacobi-Stirling numbers (cont.)

Definition.

- An inversion of type 1 of π is a pair (b_1, B_j), where $b_1 \in B_i$ for some i $(1 \le i < j)$ and $b_1 > c_1$ for some $c_1 \in B_j$.
- An inversion of type 2 of π is a pair (b_2, B_j), where $b_2 \in B_i$ for some i $(0 \le i < j)$ and $b_2 > c_2$ for some $c_2 \in B_j$ and $b_1 \notin B_j$, where a_i means integer a with subscript $i = 1, 2$.
- Let $\mathrm{inv}_i(\pi)$ be the number of inversions of π of type $i = 1, 2$ and set $\mathrm{inv}(\pi) = \mathrm{inv}_2(\pi) - \mathrm{inv}_1(\pi)$.

Let $\Pi(n, k, i)$ denote the set of Jacobi-Stirling k-partitions of $[n]_2$ such that the zero-block contains i numbers with subscript 1.

Theorem. For any positive integers n and k and $0 \le i \le n - k$ we have

$$a_{n,k}^{(i)}(q) = \sum_{\pi \in \Pi(n,k,i)} q^{\mathrm{inv}(\pi)},$$

where

$$\mathrm{JS}_n^k(z;q) = a_{n,k}^{(0)}(q) + a_{n,k}^{(1)}(q)z + \cdots + a_{n,k}^{(n-k)}(q)z^{n-k},$$

Combinatorial interpretation of the q-Jacobi-Stirling numbers (cont.)

Example.

JS 2-partitions of $[3]_2$	inv_1	inv_2	inv
$\{\}_0,\ \{1_1,1_2,3_2\},\ \{2_1,2_2,3_1\}$	0	0	0
$\{\}_0,\ \{1_1,1_2,3_1\},\ \{2_1,2_2,3_2\}$	1	0	-1
$\{3_2\}_0,\ \{1_1,1_2,3_1\},\ \{2_1,2_2\}$	1	1	0
$\{3_2\}_0,\ \{1_1,1_2\},\ \{2_1,2_2,3_1\}$	0	1	1
$\{2_2\}_0,\ \{1_1,1_2,2_1\},\ \{3_1,3_2\}$	0	0	0
$\{2_1\}_0,\ \{1_1,1_2,2_2\},\ \{3_1,3_2\}$	0	0	0
$\{3_1\}_0,\ \{1_1,1_2,3_2\},\ \{2_1,2_2\}$	0	1	1
$\{3_1\}_0,\ \{1_1,1_2\},\ \{2_1,2_2,3_2\}$	0	0	0

Thus, $\displaystyle\sum_{\pi\in\Pi(3,2,0)} q^{\mathrm{inv}(\pi)} = 3+q+q^{-1}$ and $\displaystyle\sum_{\pi\in\Pi(3,2,1)} q^{\mathrm{inv}(\pi)} = 2+q.$

Combinatorial interpretation of the q-Jacobi-Stirling numbers (1st kind)

For a permutation σ of $[n]$ and for $j \in [n]$.
Let $\mathrm{Orb}_\sigma(j) := \{\sigma^\ell(j) : \ell \geq 1\}$ the orbit of j and
Let $\min(\sigma) = \{j \in [n] : j = \min(\mathrm{Orb}_\sigma(j) \cap [n])\}$ (set of its positive cyclic minima)

Definition. Given a word $w = w(1) \ldots w(\ell)$ on the finite alphabet $[n]$, a letter $w(j)$ is a *record* of w if $w(k) > w(j)$ for every $k \in \{1, \ldots, j-1\}$.
By $\mathrm{rec}(w)$ we mean the number of records of w and $\mathrm{rec}_0(w) = \mathrm{rec}(w) - 1$.

Example. If $w = \mathbf{5}7\mathbf{4}86\mathbf{2}3\mathbf{1}9$, then the records are $5, 4, 2, 1$ and $\mathrm{rec}(w) = 4$.

Definition. Let $\mathcal{P}(n, k, i)$ be the set of all pairs (σ, τ) such that σ is a permutation of $[n]_0$, τ is a permutation of $[n]$, both having k cycles and such that

i) 1 and 0 are in the same cycle in σ ;
ii) among their nonzero entries, σ and τ have the same cycle minima ;
iii) $\mathrm{rec}_0(w) = i$, where $w = \sigma(0)\sigma^2(0) \ldots \sigma^l(0)$ with $\sigma^{l+1}(0) = 0$.

Combinatorial interpretation of the q-Jacobi-Stirling numbers (1st kind)

For a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $[n]$ and each $i = 1, 2, \ldots, n$, let $k := k(i)$ be the smallest integer $k \geq 1$ such that $\sigma^{-k}(i) \leq i$.

Let $\text{B-code}(\sigma) := (b_1, b_2, \ldots, b_n)$ with $b_i := \sigma^{-k(i)}(i)$ $(1 \leq i \leq n)$..

$\text{Sor}(\sigma) = \sum_{i=1}^{n}(i - b_i)$, : the sorting index for permutation σ of $[n]$

$\text{Sor}_0(\sigma) = \sum_{i=1}^{n}(i - b_i')$, : the modified sorting index for a permutation σ of $[n]_0$

Here, $b_i' = b_i$ if $\sigma^{-1}(i) \neq 0$ and $b_i' = i$ if $\sigma^{-1}(i) = 0$.

Finally, for any pair (σ, τ) in $\mathcal{P}(n, k, i)$ we define the statistic

$$\text{Sor}(\sigma, \tau) = \text{Sor}(\tau) - \text{Sor}_0(\sigma).$$

Theorem. We have $b_{n,k}^{(i)}(q) = \sum_{(\sigma,\tau)\in\mathcal{P}(n,k,i)} q^{\text{Sor}(\sigma,\tau)}$. where

$$\mathrm{Jc}_n^k(z; q) = b_{n,k}^{(0)}(q) + b_{n,k}^{(1)}(q)z + \cdots + b_{n,k}^{(n-k)}(q)z^{n-k}$$

Combinatorial interpretation of the q-Jacobi-Stirling numbers 1st kind (cont.)

Example.

(σ, τ)	$\mathrm{rec}_0(\sigma)$	B_0-code σ	B-code τ	$\mathrm{Sor}(\tau)$	$\mathrm{Sor}_0(\sigma)$	$\mathrm{Sor}(\sigma, \tau)$
(0 1)(2 3), (1)(23)	0	(1,2,2)	(1,2,2)	1	1	0
(0 1)(2 3), (13)(2)	0	(1,2,2)	(1,2,1)	2	1	1
(0 1 2) (3), (1 2)(3)	0	(1,1,3)	(1,1,3)	1	1	0
(0 1 3) (2), (13)(2)	0	(1,2,1)	(1,2,1)	2	2	0
(0 1 3))(2), (1)(23)	0	(1,2,1)	(1,2,2)	1	2	-1
(0 3 1)(2), (1)(23)	1	(0,2,3)	(1,2,2)	1	1	0
(0 3 1)(2), (1)(23)	1	(0,2,3)	(1,1,3)	1	1	0
(0 2 1)(3), (1)(23)	1	(0,2,3)	(1,2,1)	2	1	1

Thus,

$$\sum_{(\sigma,\tau)\in\mathcal{P}(3,2,0)} q^{\mathrm{Sor}(\sigma,\tau)} = 3 + q + q^{-1}, \qquad \sum_{(\sigma,\tau)\in\mathcal{P}(3,2,1)} q^{\mathrm{Sor}(\sigma,\tau)} = 2 + q.$$

Symmetric generalisation of Jacobi-Stirling numbers

Consider the pair of connection coefficients $\{(S_{z,w}(n,k), s_{z,w}(n,k))\}_{n\geq k\geq 0}$ satisfying

$$x^n = \sum_{k=0}^{n} S_{z,w}(n,k) \prod_{i=0}^{k-1}(x-(i+z)(i+w)), \tag{9}$$

$$\prod_{i=0}^{n-1}(x-(i+z)(i+w)) = \sum_{k=0}^{n} s_{z,w}(n,k)x^k. \tag{10}$$

It is readily seen that we have the following recurrence relation

$$S_{z,w}(n+1,k+1) = S_{z,w}(n,k) + (z+k+1)(w+k+1)S_{z,w}(n,k+1), \tag{11}$$

$$s_{z,w}(n+1,k+1) = s_{z,w}(n,k) - (z+n)(w+n)s_{z,w}(n,k+1), \tag{12}$$

with $S_{z,w}(n,k) = s_{z,w}(n,k) = 0, \quad \text{if } k \notin \{1,\dots,n\}$, and $S_{z,w}(0,0) = s_{z,w}(0,0) = 1,\ n \geq 0$.

Symmetric generalisation of Jacobi-Stirling numbers

Definition. A double signed k-partition of $[n]_2 = \{1_1, 1_2, \dots, n_1, n_2\}$ is a partition of $[n]_2$ into $k+2$ subsets $(B_0, B_0', B_1, \dots, B_k)$ such that

1. there are two distinguishable zero blocks B_0 and B_0', any of which may be empty ;
2. there are k indistinguishable nonzero blocks, all nonempty, each of which contains both copies of its smallest element and does not contain both copies of any other number ;
3. each zero block does not contain both copies of any number and B_0' may contain only numbers with subscript 2.

$\Pi(n,k)$: the set of double signed k-partitions of $[n]_2$
$s(\pi)$ with $\pi \in \Pi(n,k)$: the number of integers with subscript 1 in B_0 of π.
$t(\pi)$ with $\pi \in \Pi(n,k)$: the number of integers with subscript 2 in B_0' of π.

Theorem. The polynomial $S_{z,w}(n,k)$ is the enumerative polynomial of $\Pi(n,k)$ with z enumerating the numbers with subscript 1 in B_0 and w enumerating the numbers with subscript 2 in B_0', i.e.,

$$S_{z,w}(n,k) = \sum_{\pi \in \Pi(n,k)} z^{s(\pi)} w^{t(\pi)}.$$