Isomorphic induced modules and Dynkin diagram automorphims

C. Lecouvey (joint work with J. Guilhot) Tours University

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Basics on representation theory of Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ wih triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha}$$

where $\mathfrak{h}$ is the cartan subalgebra and $R_+ \subset \mathfrak{h}^*_{\mathbb{R}}$ the set of positive roots.

The weight lattice P and the root lattice Q of $\mathfrak{g}$ are realised in $E = \mathfrak{h}^*_{\mathbb{R}}$ equipped with the scalar product $\langle \cdot, \cdot \rangle$.
Let $S \subset R_+$ be the set of simple roots
$W = \langle s_\alpha \mid \alpha \in S \rangle$ is the Weyl group of $\mathfrak{g}$.
Set

$$\overline{C} = \{x \in E \mid \langle x, \alpha \rangle \geq 0 \text{ for any } \alpha \in S\}.$$

We have

$$E = \bigcup_{w \in W} w(\overline{C})$$

The cone of dominant weights of $\mathfrak{g}$ is $P_+ = P \cap \overline{C}$.

$$\{\text{dominant weights } \lambda \in P_+\} \overset{1:1}{\leftrightarrow} \{\text{f.d. irr. rep. of } \mathfrak{g}\}$$
$$\lambda \rightarrow V(\lambda)$$

Levi subalgebra

Consider

- $\overline{S} \subset S$
- $\overline{R} \subset R$ the parabolic root system generated by $\overline{S}$
- $\overline{R}_+ = \overline{R} \cap R_+$ the corresponding set of positive roots.

Let $\overline{\mathfrak{g}} \subset \mathfrak{g}$ be the Levi subalgebra of $\mathfrak{g}$ with set of positive roots $\overline{R}_+$ and triangular decomposition

$$\overline{\mathfrak{g}} = \bigoplus_{\alpha \in \overline{R}_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \overline{R}_+} \mathfrak{g}_{-\alpha}.$$

The algebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ have the same integral weight lattice P. Let

$$\overline{P}_+ = \{\mu \in P \mid \langle \mu, \alpha \rangle \geq 0 \text{ for any } \alpha \in \overline{S}\}.$$

be the cone of dominant weights for $\overline{\mathfrak{g}}$.

Example

For $\mathfrak{g} = \mathfrak{sp}_8$ of type C_4, $S = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, 2\varepsilon_4\}$

$$R_+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq 4\} \cup \{\varepsilon_i + \varepsilon_j \mid 1 < i \leq j \leq 4\}$$

and $P = \mathbb{Z}^4$

$$P_+ = \{x = (x_1, \dots, x_4) \in \mathbb{Z}^4 \mid x_1 \geq \cdots \geq x_4 \geq 0\}.$$

The Levi subalgebra $\overline{\mathfrak{g}} \subset \mathfrak{g}$ such that $\overline{S} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, 2\varepsilon_4\}$

$$\overline{R}_+ = \{\varepsilon_1 - \varepsilon_2\} \cup \{\varepsilon_3 \pm \varepsilon_4\} \cup \{2\varepsilon_3, 2\varepsilon_4\}$$

is isomorphic to

$$\mathfrak{gl}_2 \oplus \mathfrak{sp}_4$$

and

$$\overline{P}_+ = x = (x_1, \dots, x_4) \in \mathbb{Z}^4 \mid x_1 \geq x_2 \text{ and } x_3 \geq x_4 \geq 0\}.$$

Branching coefficients

For any $\lambda \in P_+$ set

$$V(\lambda) \downarrow^{\mathfrak{g}}_{\overline{\mathfrak{g}}} \simeq \bigoplus_{\mu \in \overline{P}_+} \overline{V}(\mu)^{\oplus m_{\mu,\lambda}}.$$

Remark: when $\overline{\mathfrak{g}} = \mathfrak{h}$ we have $\overline{P}_+ = P$ and

$$s_\lambda := \operatorname{char} V(\lambda) = \sum_{\mu \in P} m_{\mu,\lambda} e^{\mu},$$

$$m_{\mu,\lambda} = \dim V(\lambda)_\mu$$

Our problem

Problem: Can we have two identical rows or columns in the infinite matrix

$$M = (m_{\mu,\lambda})_{\mu \in \overline{P}_+, \lambda \in P_+} \ ?$$

Easy for the columns: let $\lambda, \Lambda \in P_+$
if $m_{\mu,\lambda} = m_{\mu,\Lambda}$ for any $\mu \in \overline{P}_+$ we have

$$\begin{array}{c} V(\lambda) \downarrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}} \simeq V(\Lambda) \downarrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}} \\ \Downarrow \\ V(\lambda) \downarrow_{\mathfrak{h}}^{\mathfrak{g}} \simeq V(\Lambda) \downarrow_{\mathfrak{h}}^{\mathfrak{g}} \\ \Downarrow \\ s_\lambda = s_\Lambda \\ \Downarrow \\ \lambda = \Lambda \end{array}$$

Characters of induced modules

Given $\mu \in \overline{P}_+$ we set

$$H_\mu := \mathrm{char} \overline{V}(\mu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}.$$

So

$$H_\mu = \sum_{\lambda \in P_+} m_{\mu,\lambda} s_\lambda.$$

- M has two identical rows labelled by μ and ν if and only if $H_\mu = H_\nu$.
- When $\overline{\mathfrak{g}} = \mathfrak{h}$, we write $h_\mu := H_\mu$.
- We have $h_\mu = h_{w(\mu)}$ for any $\mu \in P$ and any $w \in W$.

A sufficient condition to have isomorphic inductions

Theorem

1. *We have for any* $\mu \in \overline{P}_+$

$$H_\mu = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) h_{\mu + \overline{\rho} - \overline{w}(\overline{\rho})}$$

where $\overline{\rho} = \frac{1}{2} \sum_{\alpha \in \overline{R}_+} \alpha$.

2. *Consider* $\mu, \nu \in \overline{P}_+$. *Assume* $u \in W$ *is such that* $\nu = u(\mu)$ *and* $u(\overline{R}_+) = \overline{R}_+$. *Then* $H_\mu = H_\nu$.

Dynkin diagram automorphisms

Lemma

$u \in W$ verifies $u(\overline{R}_+) = \overline{R}_+$ iff $u \in W$ and is a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$.

Example

$\mathfrak{g} = \mathfrak{sp}_{12}$ and $\overline{\mathfrak{g}} = \mathfrak{gl}_3 \oplus \mathfrak{gl}_3$ admits the Dynkin diagrams

$$\circ - \circ - \odot - \circ - \circ \Longleftarrow \odot \quad \text{and} \quad \circ - \circ \qquad \circ - \circ$$

$$\overline{R}_+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_3\} \cup \{\varepsilon_4 - \varepsilon_5, \varepsilon_4 - \varepsilon_6, \varepsilon_5 - \varepsilon_6\}.$$

The signed permutation

$$u = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

is a Dynkin diagram automorphism for $\overline{\mathfrak{g}}$.

The conjecture

Conjecture: Consider $\mu, \nu \in \overline{P}_+$. Then $H_\mu = H_\nu$ iff there exists $u \in W$ such that

1. $u(\mu) = \nu$,
2. $u(\overline{R}_+) = \overline{R}_+$.

Main result

For any $\mu \in \overline{P}_+$, set

$$E_\mu = \mu + \{\overline{\rho} - \overline{w}(\overline{\rho}) \mid \overline{w} \in \overline{W}\}.$$

Theorem

Consider $\mu, \nu \in \overline{P}_+$.

1. *When μ and ν are conjugate under the action of a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$ lying in W, we have $H_\mu = H_\nu$.*
2. *If $H_\mu = H_\nu$, then μ and ν are conjugate under the action of a Dynkin diagram automorphism lying in W in the following cases*
    - *μ and ν belong to the same Weyl chamber of $\mathfrak{g}$ (in which case $\mu = \nu$),*

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    - *$\mathfrak{g} = \mathfrak{gl}_n$,*

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   - *μ and ν belong to the same Weyl chamber of $\mathfrak{g}$ (in which case $\mu = \nu$),*
   - *each set E_μ andr E_ν is entirely contained in a Weyl chamber,*
   - *$\mu + 2\overline{\rho}$ or $\nu + 2\overline{\rho}$ belongs to P_+,*
   - *$\mathfrak{g} = \mathfrak{gl}_n$,*
   - *$\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$ and $\overline{\mathfrak{g}} = \mathfrak{gl}_n$.*

The tensor product setting

Fix $k \in \mathbb{Z}_{>0}$. For any sequence $(\lambda^{(1)}, \ldots, \lambda^{(k)}) \in P_+^k$, set

$$V(\lambda^{(1)}) \otimes \cdots \otimes V(\lambda^{(k)}) \simeq \bigoplus_{\Lambda \in P_+} V(\Lambda)^{\oplus c^{\Lambda}_{\lambda^{(1)},\ldots,\lambda^{(k)}}}$$

and consider the matrix

$$C = (c^{\Lambda}_{\lambda^{(1)},\ldots,\lambda^{(k)}})_{(\lambda^{(1)},\ldots,\lambda^{(k)}) \in P_+^k, \Lambda \in P_+}.$$

Theorem

(Rajan 2004) Two rows of C are equal iff the two associates k-tuples of dominants weights coincide up permutation.

Easy for the columns : consider

$$(\lambda^{(1)}, \ldots, \lambda^{(k)}) = (\Lambda, 0, \ldots, 0).$$

The decomposition numbers setting

Let S_n be the symmetric group of rank n.

$$\{\text{Partitions } \pi \dashv n\} \longleftrightarrow \{\text{f.d. irr. rep. } S(\pi) \text{ over } \mathbb{Q}\}.$$

Let p be a odd prime number.

- $S(\pi)$ does not necessarily remain irreducible over $\mathbb{F}_p$,
- Over $\mathbb{F}_p$ the irreducible are parametrized by the p-regular partitions,
- We have a decomposition number $d_{\pi,\rho}$ $\pi \dashv n$ and $\rho \dashv_{\text{reg}} n$.

Consider the matrix

$$D = (d_{\pi,\rho})_{\pi \dashv n, \rho \dashv_{\text{reg}} n}$$

Theorem

(Wildon 2008) The columns of D are distinct.

Easy for the columns since $d_{\rho,\rho} = 1$ for any $\rho \dashv_{\text{reg}} n$.