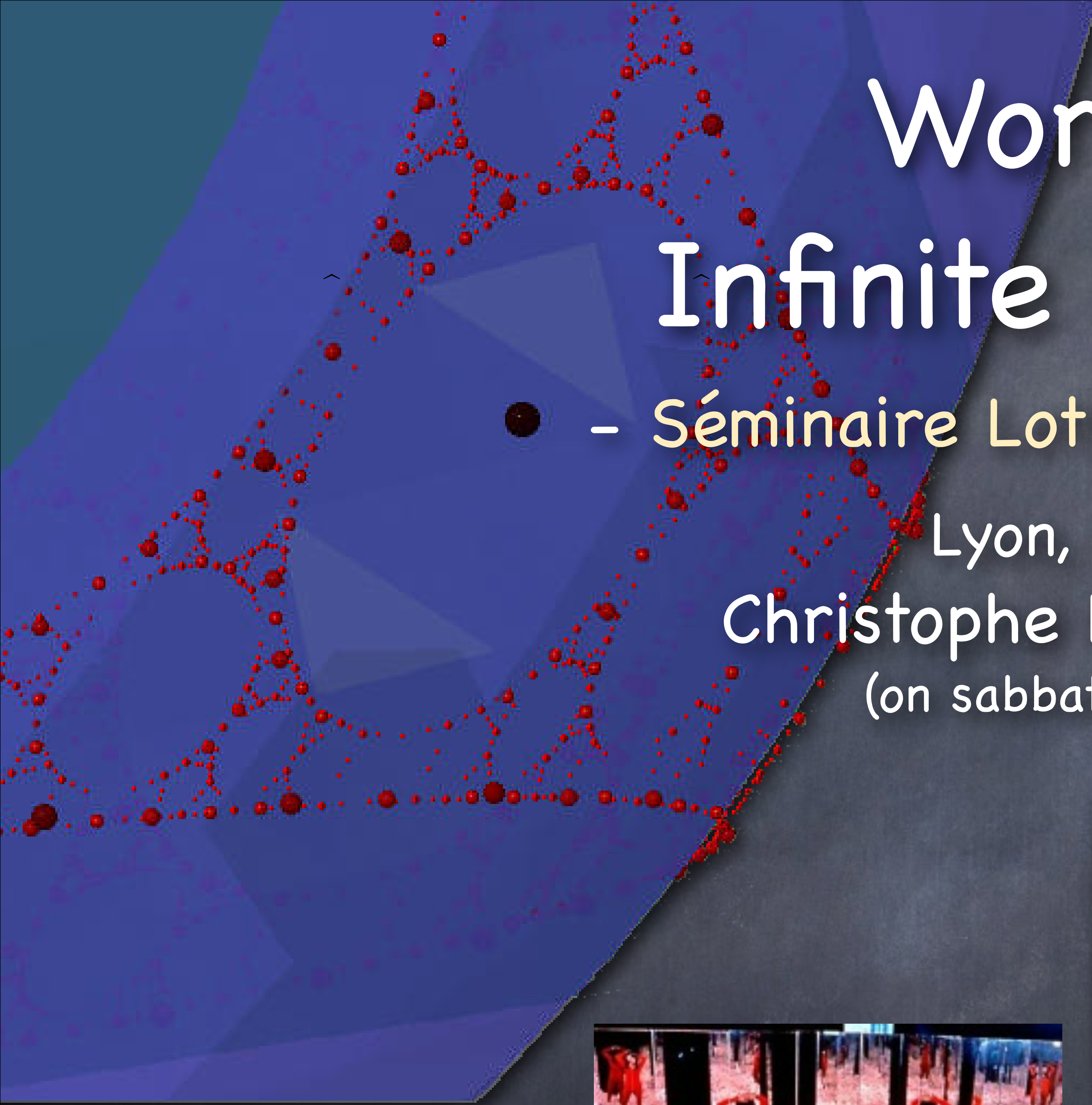

Words and Roots in Infinite Coxeter Groups

- Séminaire Lotharingien de Combinatoire -

Lyon, March 23-26, 2014

Christophe Hohlweg, LaCIM, UQAM
(on sabbatical at IRMA, Strasbourg)

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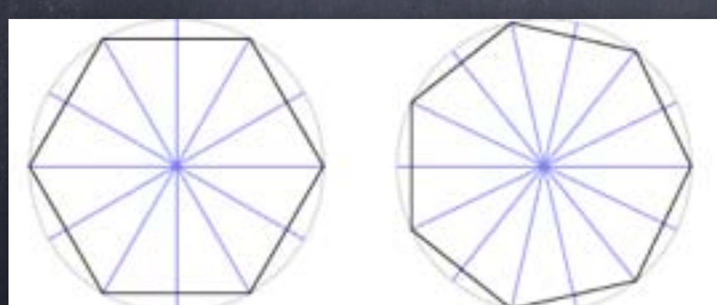
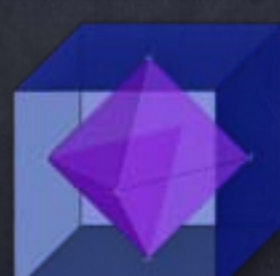
Lecture 1: Coxeter groups & Reflection groups

A bit of history (cf. Bourbaki, Lie groups, Chap. IV-VI)

- Symmetries.
  - Classificat° of regular polygons & polyhedral (cf. Euclid 300BC)
  - Study of regular tilings of the plane and the sphere (Byzantine school, High Middle-age, Kepler ~ 1619)



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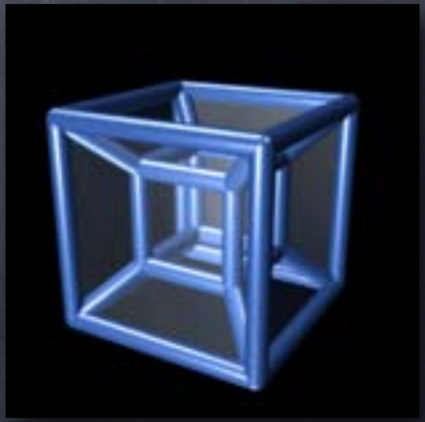
Lecture 1: Coxeter groups & Reflection groups

A bit of history (cf. Bourbaki, Lie groups, Chap. IV-VI)

- 19^{th} - century
  - Study of (discrete groups of) isometries, generated by reflections or not (Möbius ~ 1852, Jordan ~ 1869)
  - Tilings and regular polytopes in high dimension (Schläfli ~ 1850)



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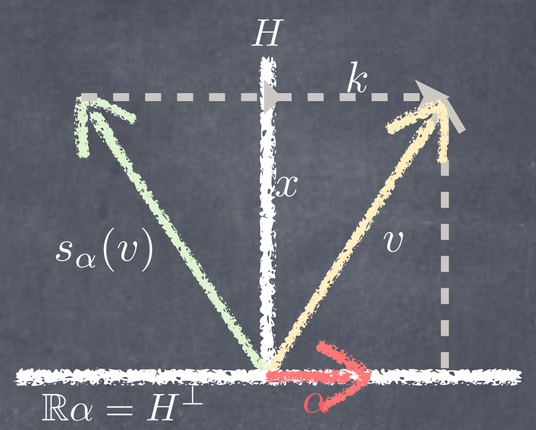
- beginning of 20^{th} - century
  - Classification of discrete subgroups generated by reflections (Cartan, Coxeter, Vinberg, etc…) -> words
  - Lie Theory via root systems (Killing, Cartan, Weyl, Witt, Coxeter, etc…)

Finite Reflection Groups (FRG)



- $(V, \langle\cdot,\cdot\rangle)$ Euclidean space ($\dim V = n$)

  i.e. V $\mathbb{R}-$ vector space, $\langle\cdot,\cdot\rangle$ scalar product,

  $||\cdot||$ associated norm.

- $$\begin{aligned} O(V) &= \{f : V \to V,\ f \text{ isometry}\} \\ &= \{f : V \to V \mid ||f(x)|| = ||x||,\ \forall x \in V\} \\ &\leq \mathrm{GL}(V) \end{aligned}$$ Orthogonal group

- Reflection: $s \in O(V)$ with set of fixed points a hyperplan H.

Properties. A reflection $s \in O(V)$ is uniquely determined:

- by a hyperplan $H = \mathrm{Fix}(s)$;
- or by a nonzero vector $\alpha \in V$ and we write $s_\alpha := s$. "root"

Observe that $\mathbb{R}\alpha = H^\perp$, a line.

Finite Reflection Groups (FRG)

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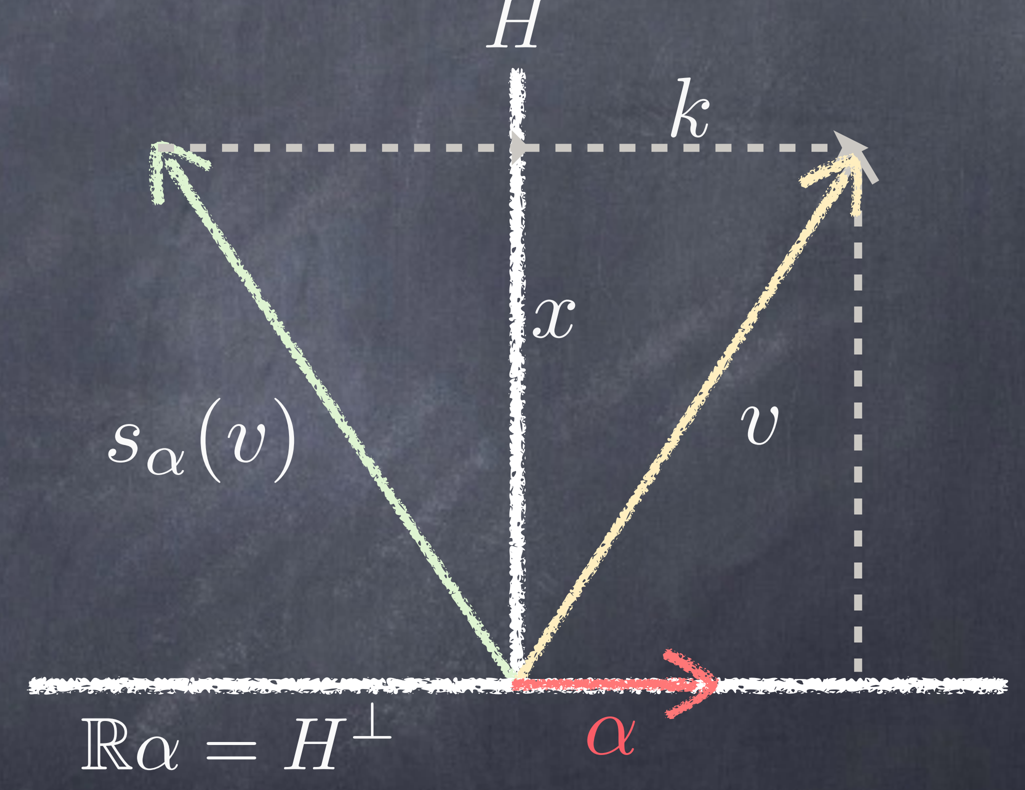
Indeed, for s with $\mathbb{R}\alpha = H^\perp$ we have:

- $s(\mathbb{R}\alpha) = \mathbb{R}\alpha$ and then $s(\alpha) = -\alpha$ (nontrivial isometry);
- for $v = x + k\alpha \in V = H \oplus \mathbb{R}\alpha$

$$s(v) = v - 2k\alpha = v - 2\frac{\langle \alpha, v\rangle}{||\alpha||^2}\alpha$$



Theorem (Cartan-Dieudonné). Any isometry in $O(V)$ is the product of at most $n = \dim V$ reflections.

Finite Reflection Groups (FRG)

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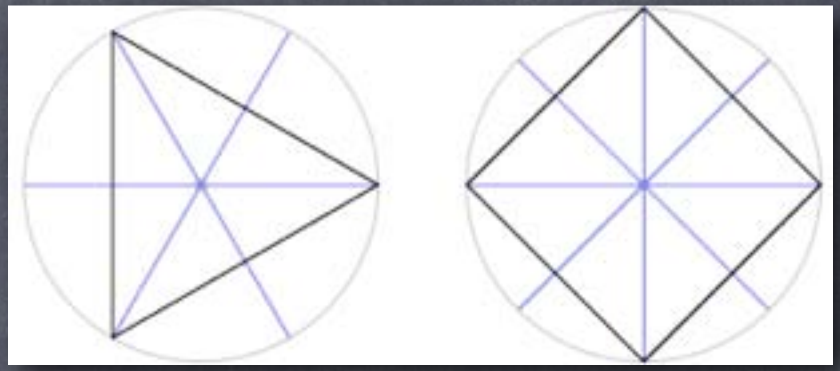
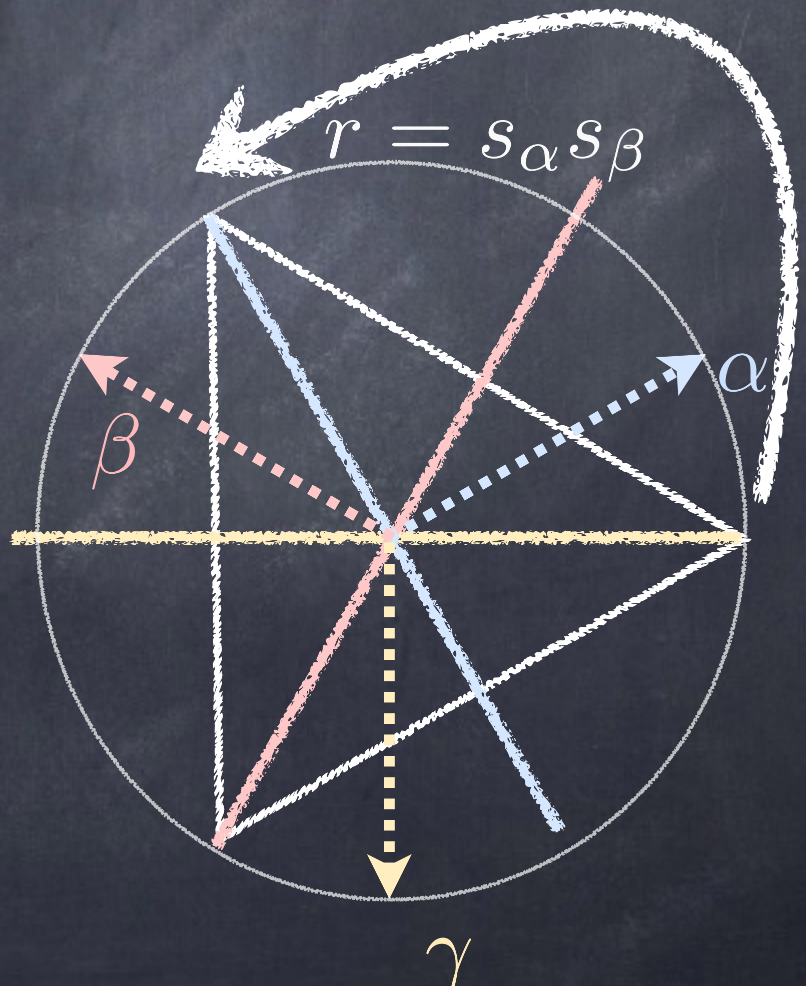


Examples:

- Dihedral groups: V is a plane ($n = 2$), P is a regular polygon with m sides (centred at the origin) and

$\mathcal{D}_m =$ isometry group of P

$$\mathcal{D}_3 = \{s_\alpha, s_\beta, s_\gamma, r, r^2, r^3 = e\}$$
$$= \langle s_\alpha, s_\beta, s_\gamma \rangle$$ is a FRG



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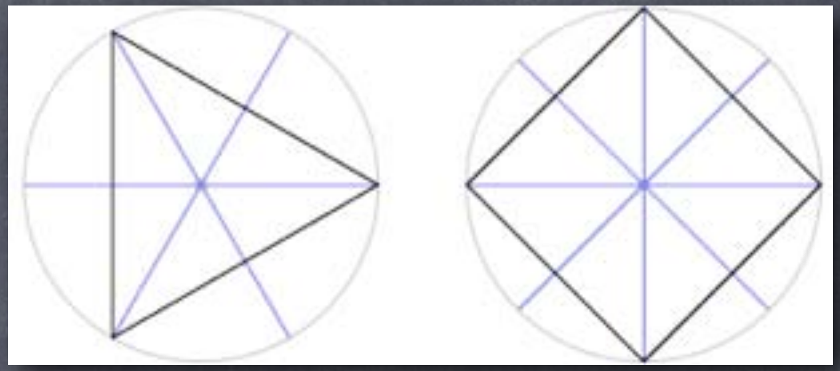

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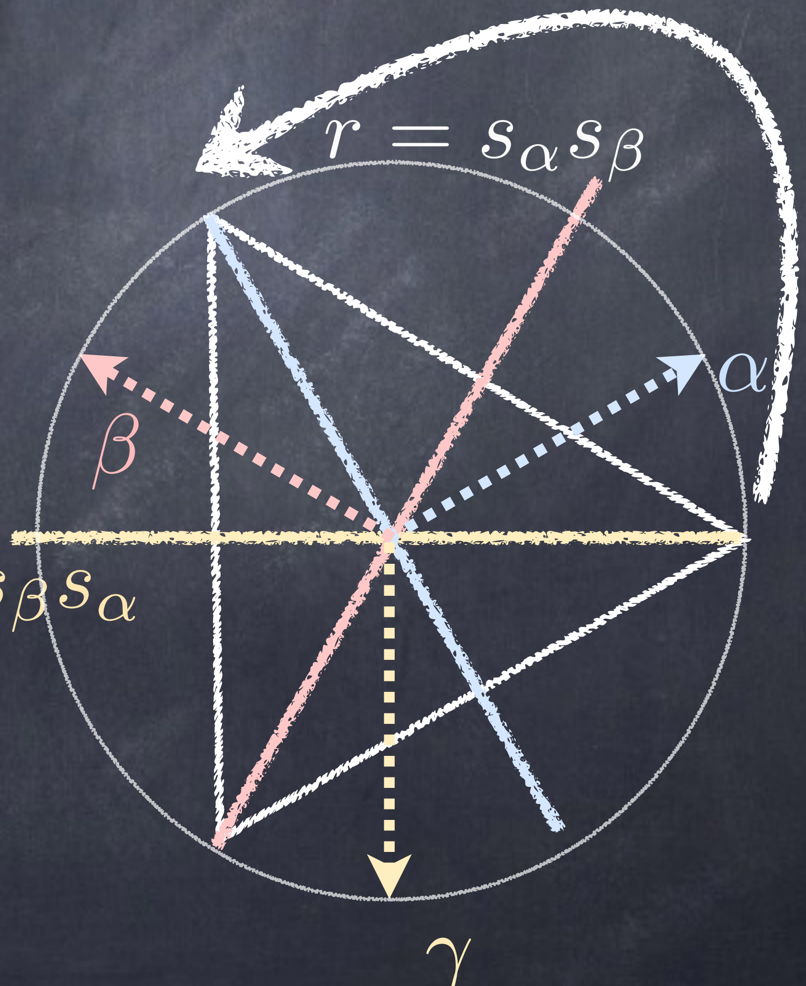

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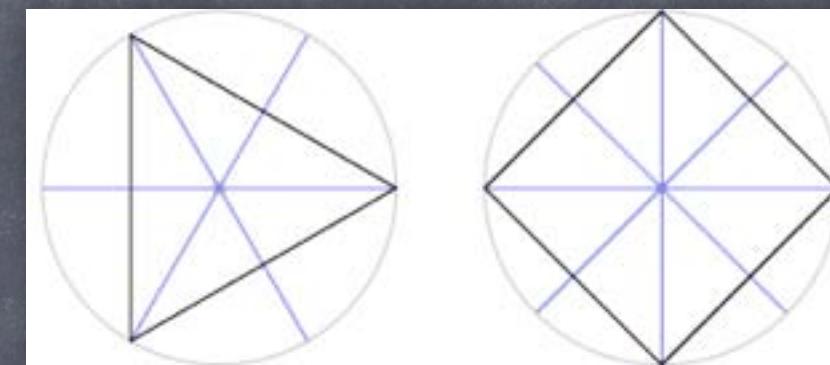

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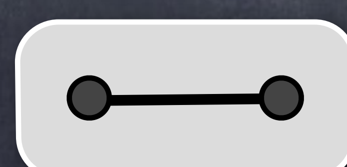
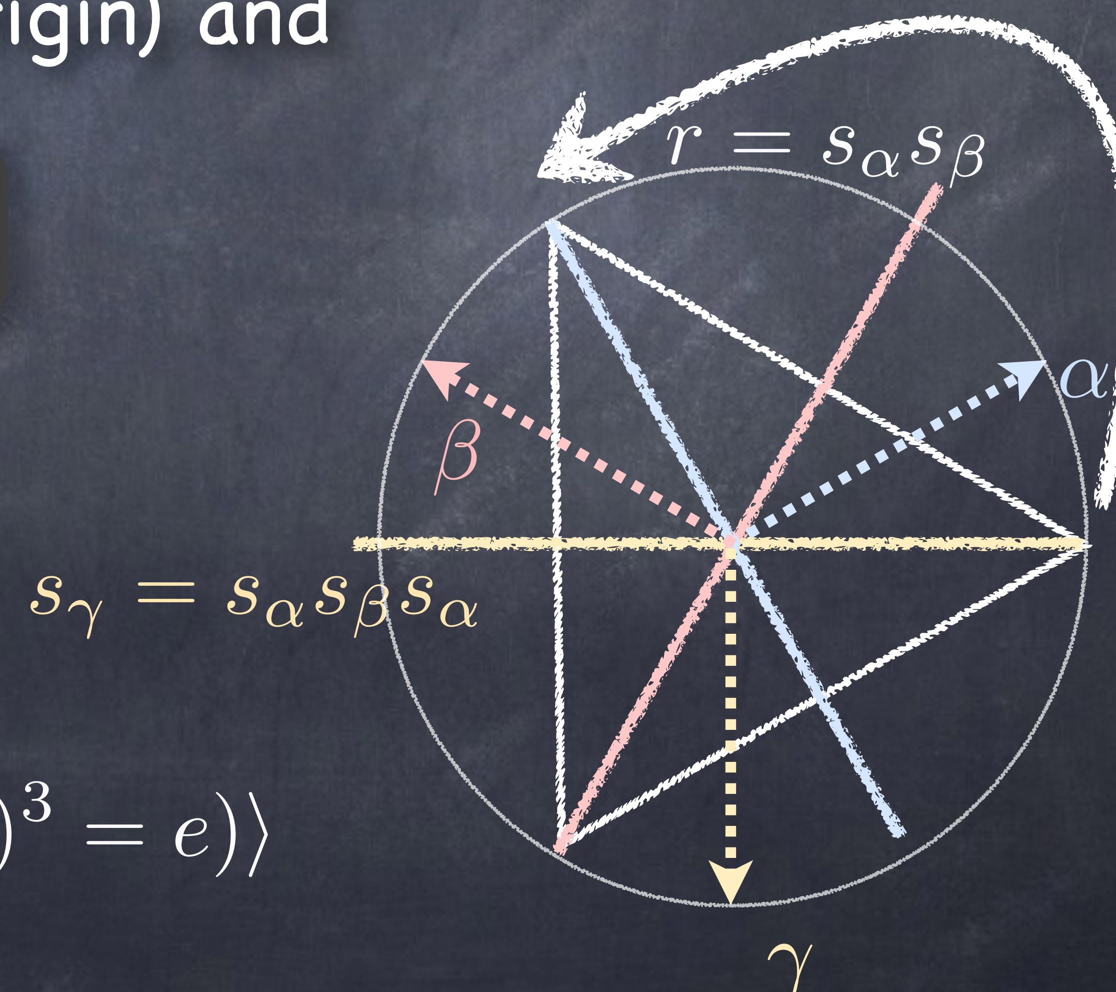
$$= \langle s_\alpha, s_\beta \,|\, s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^3 = e \rangle$$





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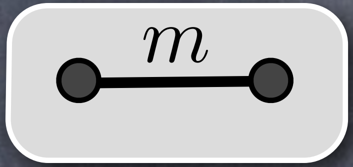
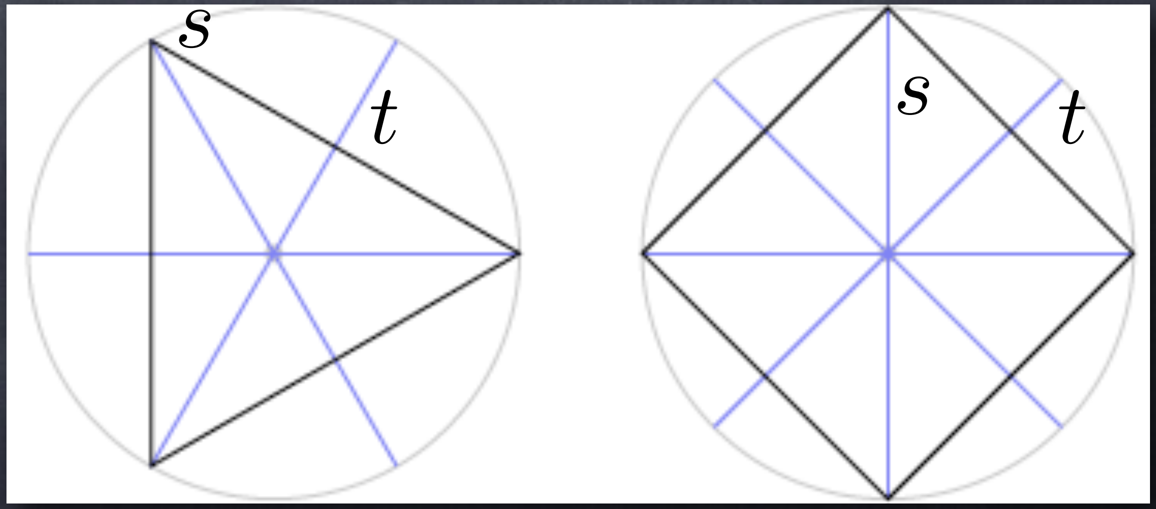


$$= \langle s, t \,|\, s^2 = t^2 = (st)^m = e \rangle$$

where s (resp. t) is the reflection associated to the line passing through a vertex of P (resp. the middle of an adjacent edge).



Finite Reflection Groups (FRG)

Examples:

- Symmetric group: S_n acts on $V = \mathbb{R}^n$ by permutation of the coordinates: $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

  → faithful action: $S_n \leq GL(n)$

A transposition $\tau_{ij} = (i\ j)$ is a reflection with hyperplane $H_{ij} = \{x \in \mathbb{R}^n \,|\, x_i = x_j\}$ or vector $\alpha_{ij} = e_j - e_i$ (i.e. $\tau_{ij} = s_{\alpha_{ij}}$)

→ $S_n = \langle \tau_{ij} \,|\, 1 \leq i < j \leq n \rangle \leq O(\mathbb{R}^n)$ is a FRG

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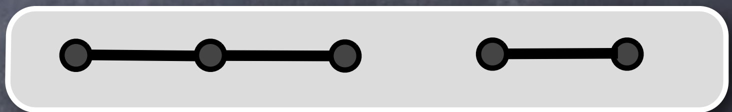
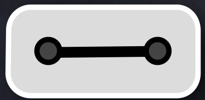
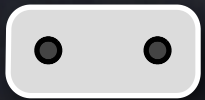
⟶ $S_n=\langle \tau_{ij}\,|\,1\leq i<j\leq n\rangle \leq O(\mathbb{R}^n)$ is a FRG

$=\langle \tau_i = s_{e_{i+1}-e_i}\,|\,1\leq i<n-1\rangle$



where $\tau_i := \tau_{ii+1}$ satisfies $\tau_i^2=(\tau_i\tau_{i+1})^3=(\tau_i\tau_j)^2=e,\ |i-j|>1$

-  (dihedral sg) means $\tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1}$
-  means $\tau_i\tau_j=\tau_j\tau_i$ (they commute)

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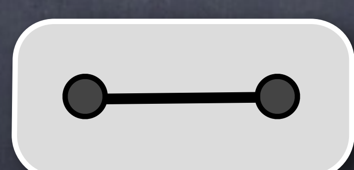
- $W \leq O(V)$ is a FRG i.e. $W = \langle s_\alpha \,|\, \alpha \in A \rangle$ where $A \subseteq V \setminus \{0\}$
(is constituted of same norm vectors for simplification)

Proposition. $\forall w \in O(V),\ \forall \alpha \in V \setminus \{0\},\ w s_\alpha w^{-1} = s_{w(\alpha)}$

- Root system: $\Phi = W(A)$ on which W acts by conjugation

Example:

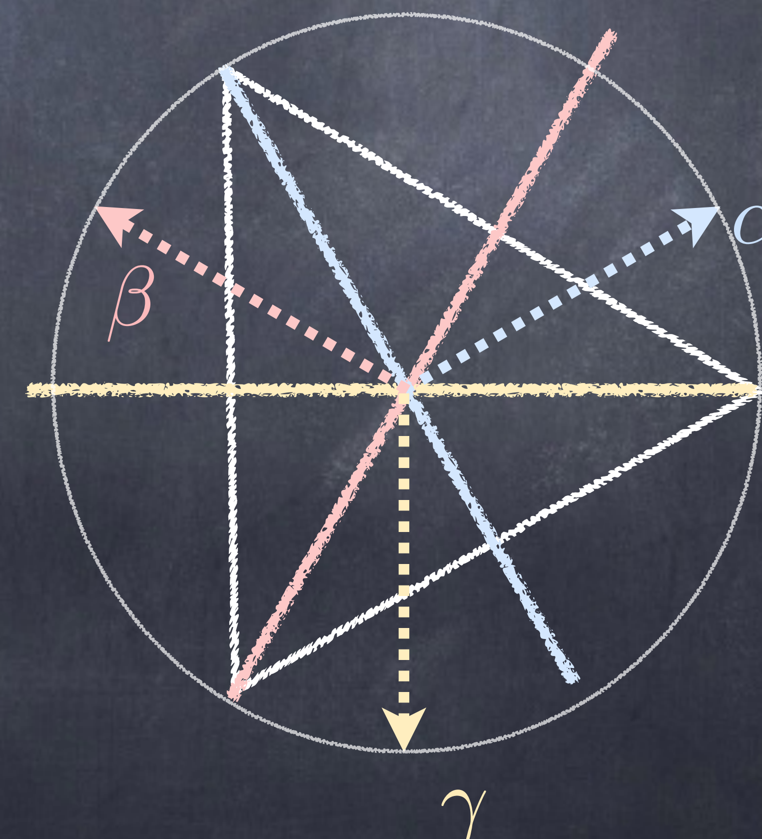
In $\mathcal{D}_3$:



$s := s_\alpha \qquad s_\beta =: t$

$$||\alpha|| = ||\beta|| = 1$$

$$\langle \alpha, \beta \rangle = -\cos\left(\frac{\pi}{3}\right)$$



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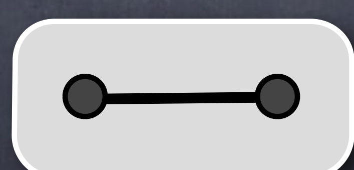

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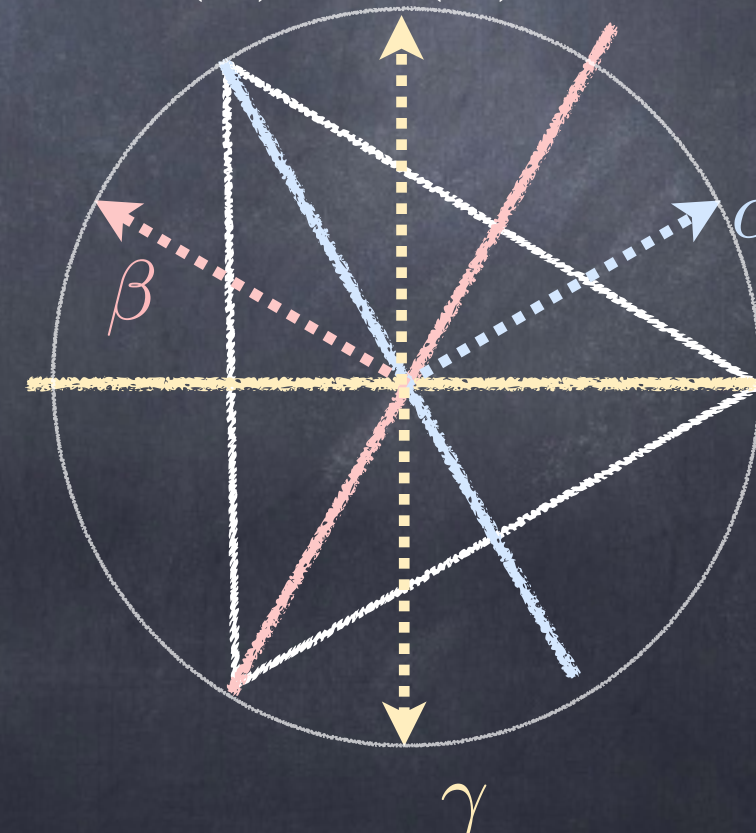
$-\gamma = s(\beta) = t(\alpha) = \alpha + \beta$



$s_\gamma = s_{s(\beta)}$
$= sts$
$= tst$

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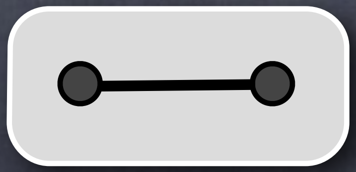

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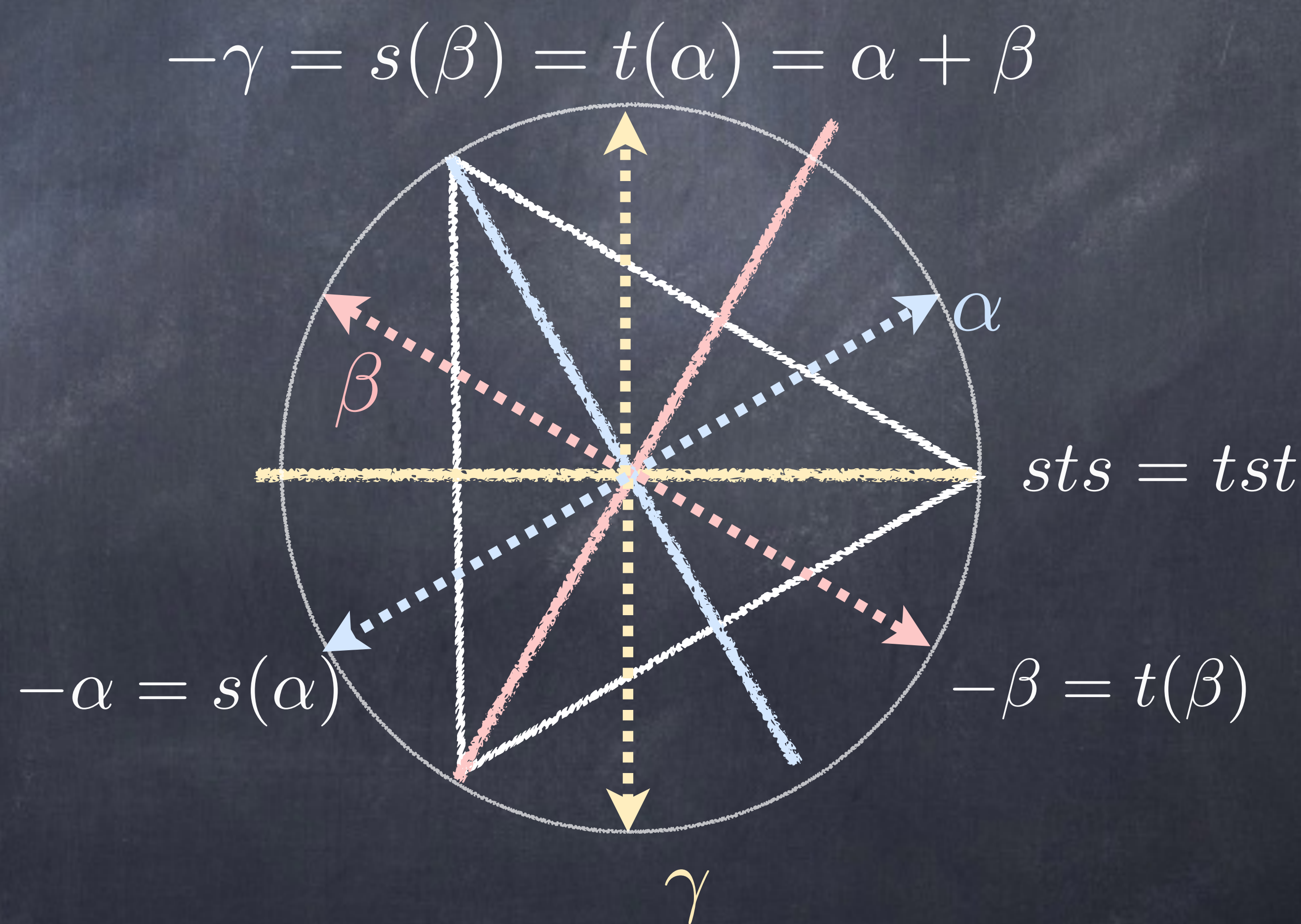

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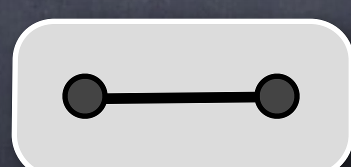

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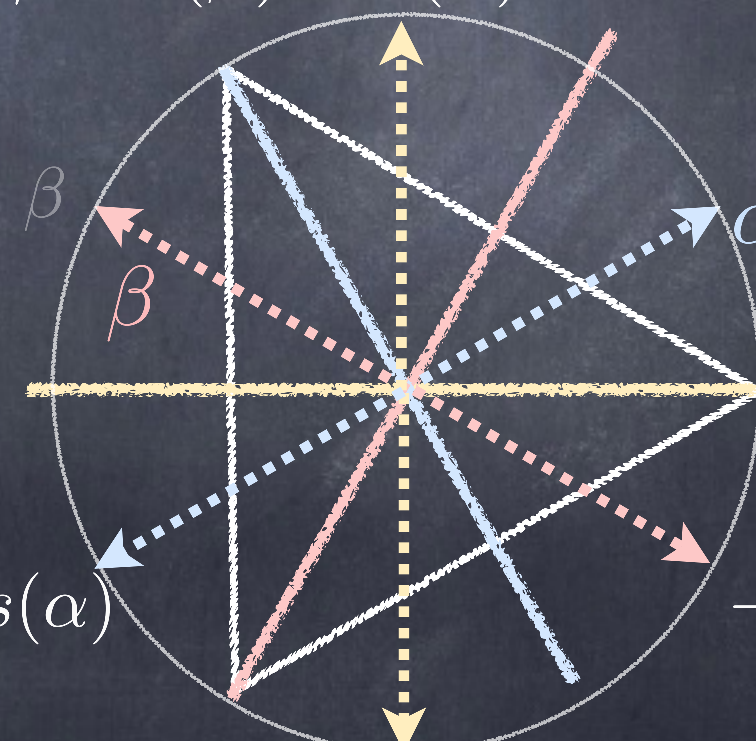

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- Root system: $\Phi = W(A)$ on which W acts by conjugation

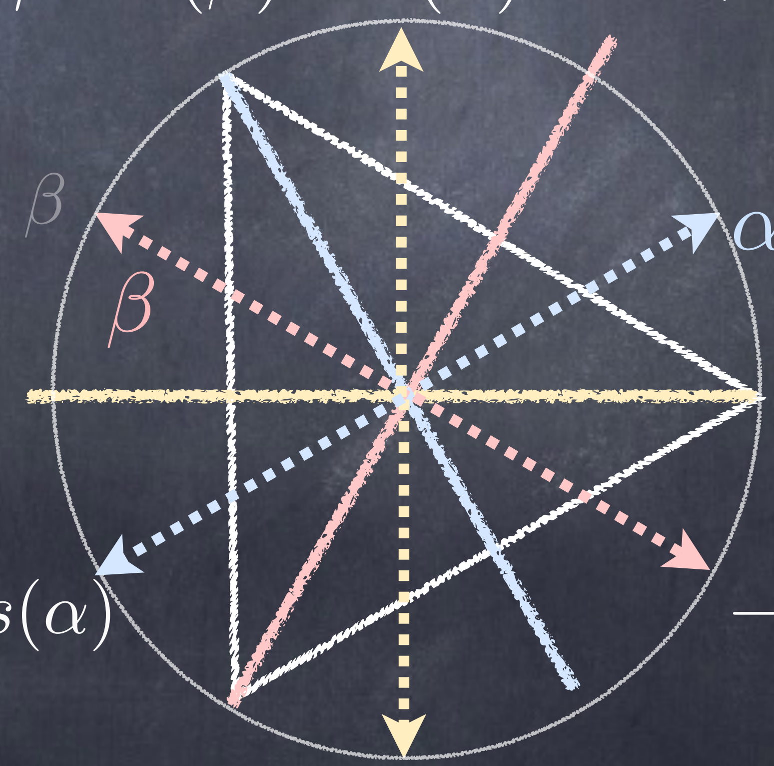
Conclusion: $\mathcal{D}_3$-orbit is

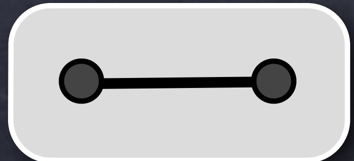
$$\Phi = \{\pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta)\}$$

The positive part is

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$$

The base of $\mathrm{cone}(\Phi^+)$ gives the desired generators s and t.





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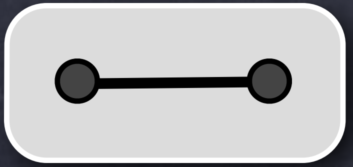
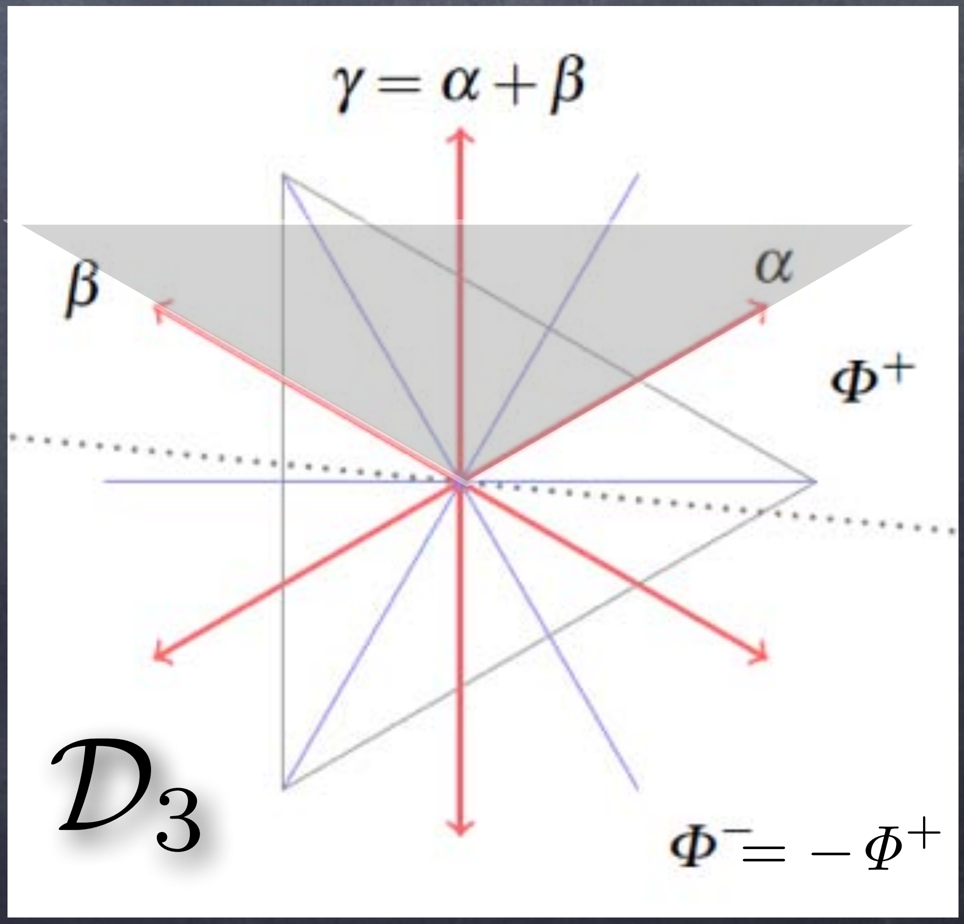
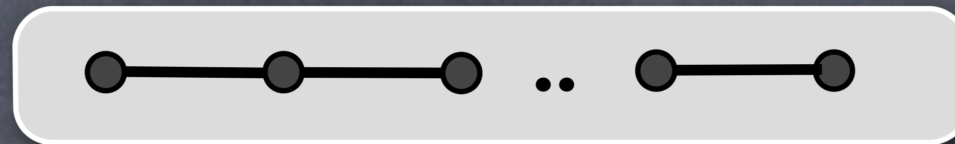




Finite Reflection Groups (FRG)

Example: $\mathcal{S}_n$ is  $(\tau_i = s_{e_{i+1}-e_i})$

Root system: $$\Phi = \{\pm(e_j - e_i) \,|\, 1 \leq i < j \leq n\}$$

The cone of $\Phi^+ = \{e_j - e_i \,|\, 1 \leq i < j \leq n\}$ has as basis

$\Delta = \{e_{i+1} - e_i \,|\, 1 \leq i < n\}$ that corresponds to the generators.

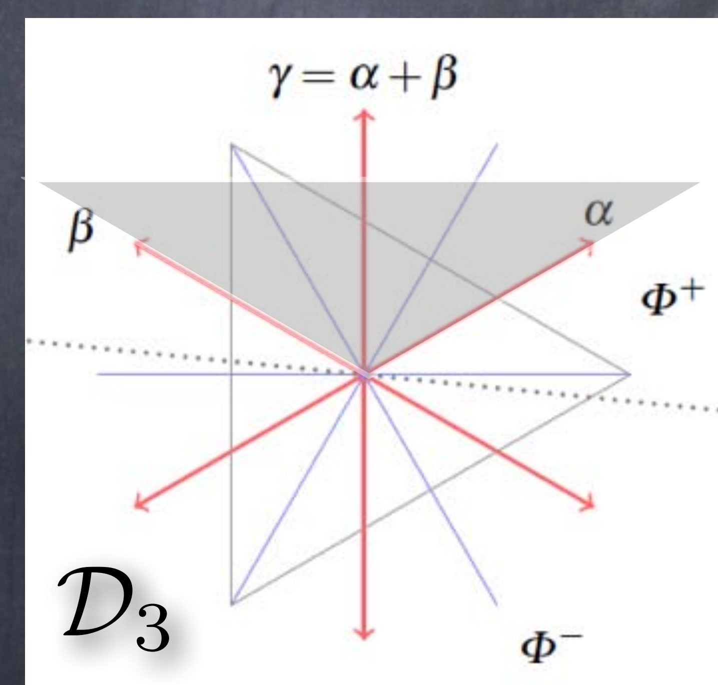
- **Root system:** $\Phi = W(A)$ verifies the following properties

  (i) Φ is finite, nonzero vectors;

  (ii) $s_\alpha(\Phi) = \Phi, \ \forall \alpha \in \Phi$;

  (iii) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}, \ \forall \alpha \in \Phi$.

  and: $W = \langle s_\alpha \,|\, \alpha \in \Phi \rangle$



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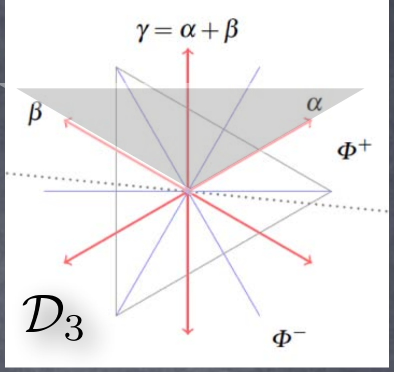


In general:

$W \leq \mathrm{O}(V)$ FRG $\longleftrightarrow$ Φ root system in V

- Separating Φ by a (linear) hyperplane we have:

reflections T $\overset{1:1}{\longleftrightarrow}$ Φ^+ positive roots

s_β $\longleftarrow\!\!|$ β

simple reflections $S \subseteq T$ $\overset{1:1}{\longleftrightarrow}$ Δ basis of $\mathrm{cone}(\Phi^+)$

Theorem. W is generated by $S = \{s_\alpha \,|\, \alpha \in \Delta\}$

Theorem. $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$ where $m_{st} = m_{ts}$ is the order of the rotation st (and $m_{ss} = 1$)

Coxeter groups

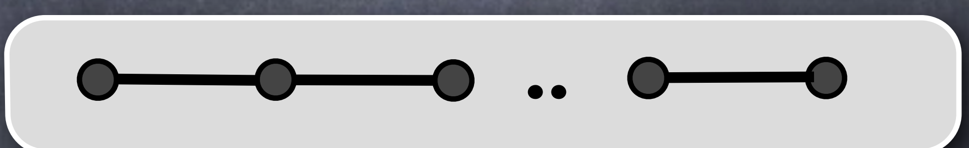
(W, S) Coxeter system of finite rank $|S| < \infty$ i.e.

- $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$ group
- $m_{ss} = 1$ (s involut°); $m_{st} = m_{ts} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t$

A Coxeter graph Γ is given by:

- vertices S (finite)
- edges 

with $m_{st} \geq 3$ or $m_{st} = \infty$

Examples. Symmetric group $\mathcal{S}_n$ is 

- Dihedral group: $\mathcal{D}_m = \langle s, t \,|\, s^2 = t^2 = (st)^m = e \rangle$;
- Infinite dihedral group: $\mathcal{D}_\infty = \langle s, t \,|\, s^2 = t^2 = e \rangle$;
- Universal Coxeter group: $U_n = \langle a_1, \ldots, a_n \,|\, a_i^2 = e \rangle$

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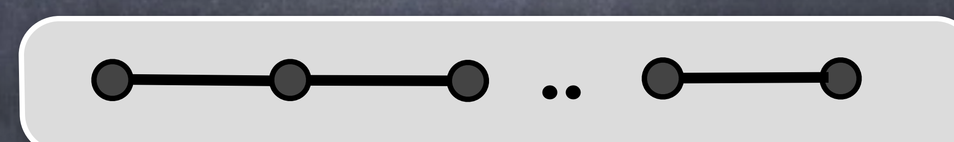
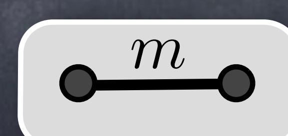
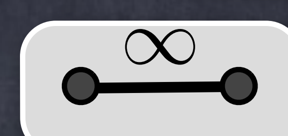
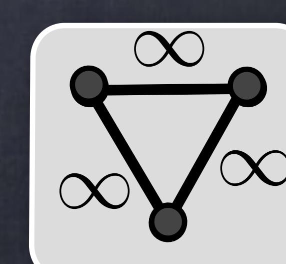
Examples. Symmetric group $\mathcal{S}_n$ is 

- Dihedral group: $\mathcal{D}_m$ is 

or  $(m = 2)$
- Infinite dihedral group: $\mathcal{D}_\infty$ is 

- Universal Coxeter group: $U_n = \langle a_1, \ldots, a_n \mid a_i^2 = e \rangle$

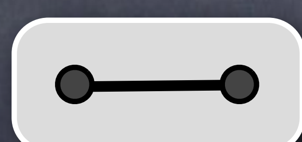


Coxeter groups

- any $w \in W$ is a word in the alphabet S ; $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$
- Length function $\ell : W \to \mathbb{N}$ with $\ell(e) = 0$ and

$$\ell(w) = \min\{k \,|\, w = s_1 s_2 \ldots s_k,\ s_i \in S\}$$

How to study words on S representing w ? Is a word $s_1 s_2 \ldots s_k$ a reduced word for w (i.e. $k = \ell(w)$) ?

Examples. $\mathcal{D}_3$ is  ;

	e	s	t	st	ts	$sts = tst$
ℓ	0	1	1	2	2	3

$\ell(ststs) = 1$ since $ststs = (sts)ts = (tst)ts = t$

Proposition. Let $s \in S$ and $w \in W$, then $\ell(ws) = \ell(w) \pm 1$.

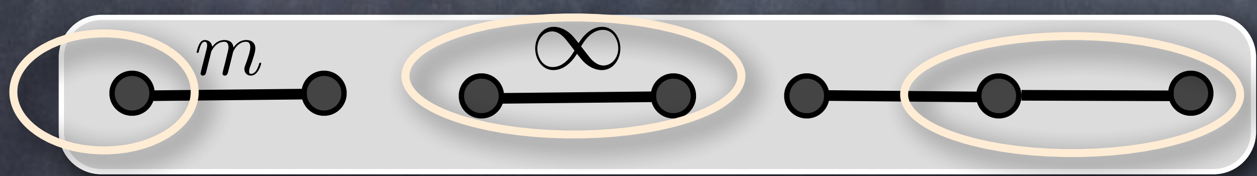
Coxeter groups

- Subgraphs and standard parabolic subgroups

$I \subseteq S \longleftrightarrow \Gamma_I$; (W_I, I) is a Coxeter system

- W is irreducible iff Γ_S is connected

$\mathcal{D}_m \times \mathcal{D}_\infty \times S_4$



$\mathcal{S}_2 \times \mathcal{D}_\infty \times \mathcal{D}_3$

S_n

$W_I \simeq S_3 \times S_2$

Proposition. *If $I_1, \ldots, I_k$ corresponds to the connected components of Γ_I (I may be S), then*

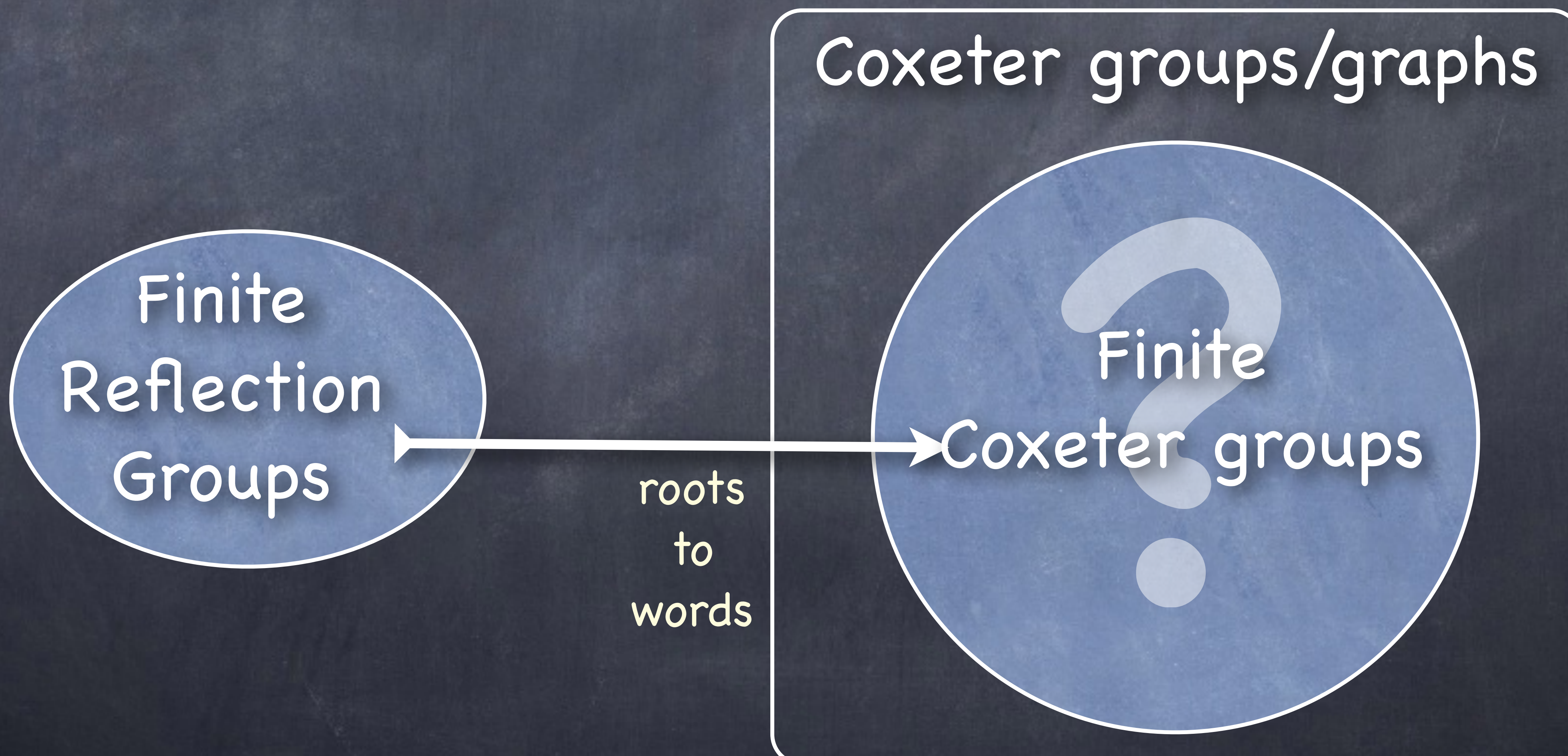
$$W_I \simeq W_{I_1} \times \cdots \times W_{I_k}$$

To study Coxeter groups it is often just necessary to study the irreducible ones. In the following we often consider irreducible Coxeter systems.

Coxeter groups and Reflection groups

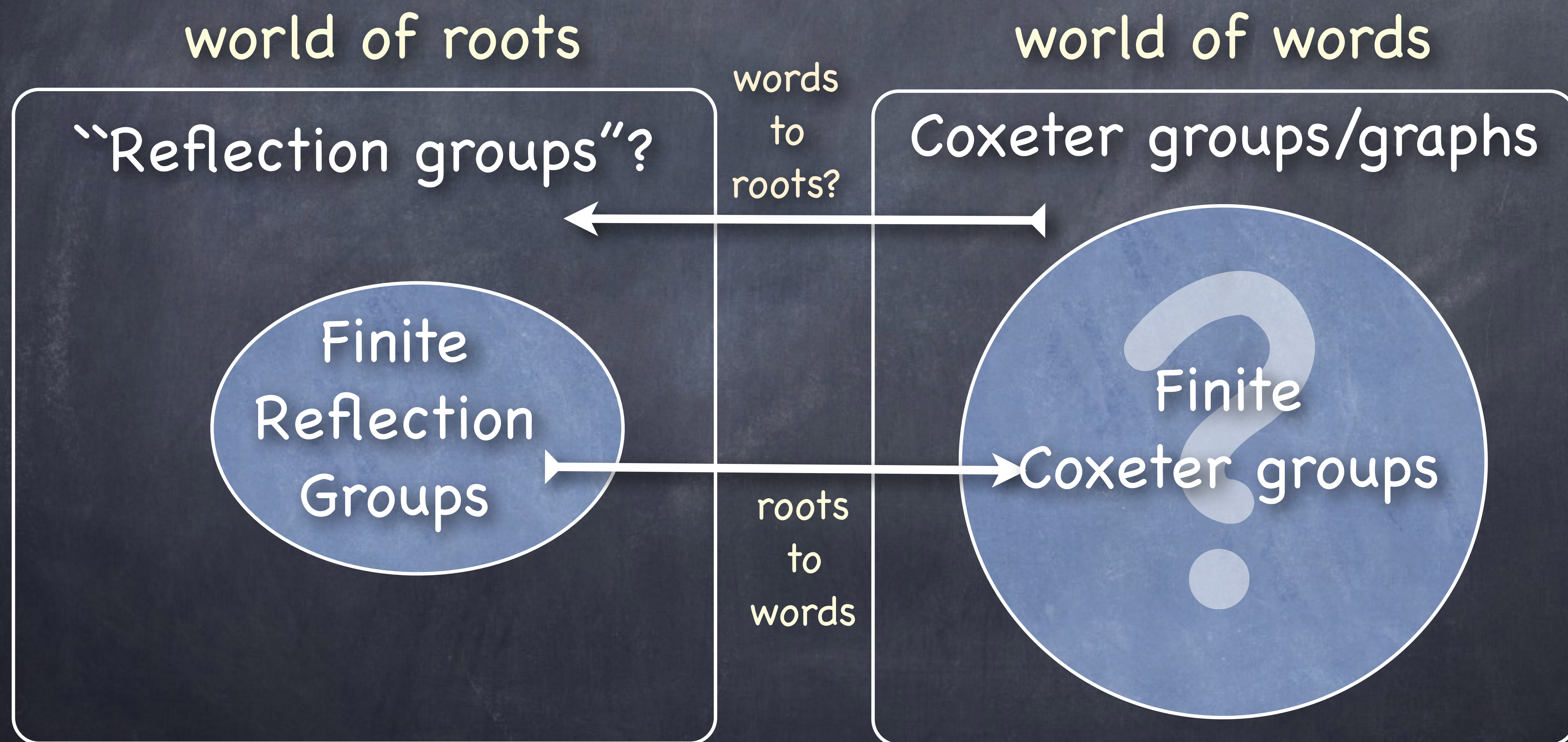
How to find all Coxeter graphs that correspond to Finite Reflection groups (FRG)? to Finite Coxeter groups?

world of roots world of words



Coxeter groups and Reflection groups

How to find all Coxeter graphs that correspond to Finite Reflection groups (FRG)? to Finite Coxeter groups?



Root systems for Coxeter groups ?

An observation

If (W,S) is a Finite Reflection Group with $\Delta \subseteq \Phi^+ \subseteq \Phi$.

- Dihedral (standard) parabolic subgroups: $I = \{s,t\} \subseteq S$
  - $W_I = \langle I \rangle \leq W$ corresponds to the subgraphs:

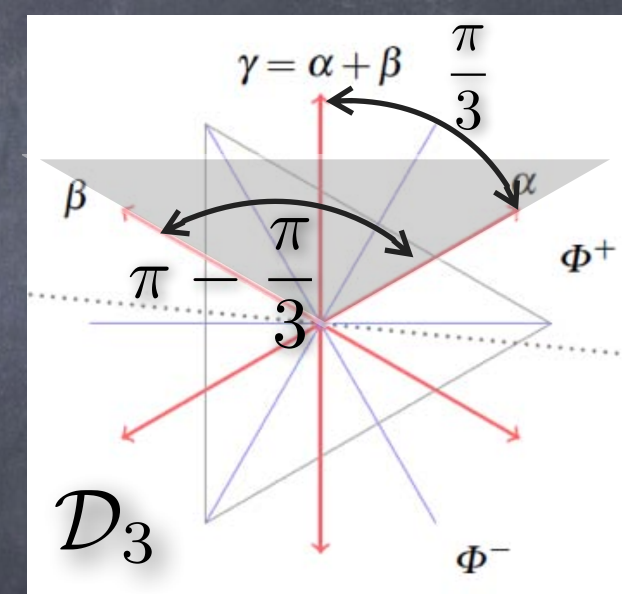
$s = s_\alpha \quad t = s_\beta$

or

$s = s_\alpha \quad t = s_\beta$

  - $W_I = \mathcal{D}_{m_{st}}$ acts on $V_I = \mathrm{span}(\alpha,\beta)$:

$$s_\alpha(\beta) = \beta - 2\langle\alpha,\beta\rangle\alpha$$



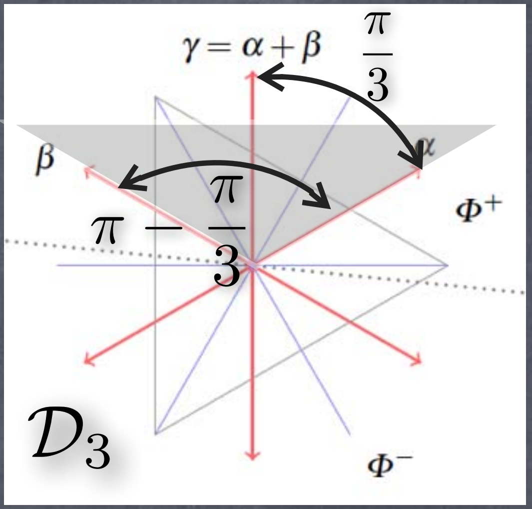
  - We have: $$\langle\alpha,\beta\rangle = -\cos\left(\frac{\pi}{m_{st}}\right)$$

- the scalar product is given on the basis Δ by

$$(\langle\alpha,\beta\rangle)_{\alpha,\beta\in\Delta} = \left(-\cos\left(\frac{\pi}{m_{st}}\right)\right)_{s,t\in S}$$

Geometric representations

Tits classical geometric representation of (W, S)



- (V, B) real quadratic space:
  - basis $\Delta = \{\alpha_s \,|\, s \in S\}$;
  - symmetric bilinear form defined by:

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right), \ (= 1 \text{ if } s = t; \ = -1 \text{ if } m_{st} = \infty)$$

- $W \leq \mathrm{O}_B(V)$ "B-isometry":

$$s(v) = v - 2B(v, \alpha)\alpha, \ s \in S$$

Root system: $\Phi = W(\Delta), \ \Phi^+ = \mathrm{cone}(\Delta) \cap \Phi = -\Phi^-$

Proposition. Let $s \in S$ and $w \in W$, then:

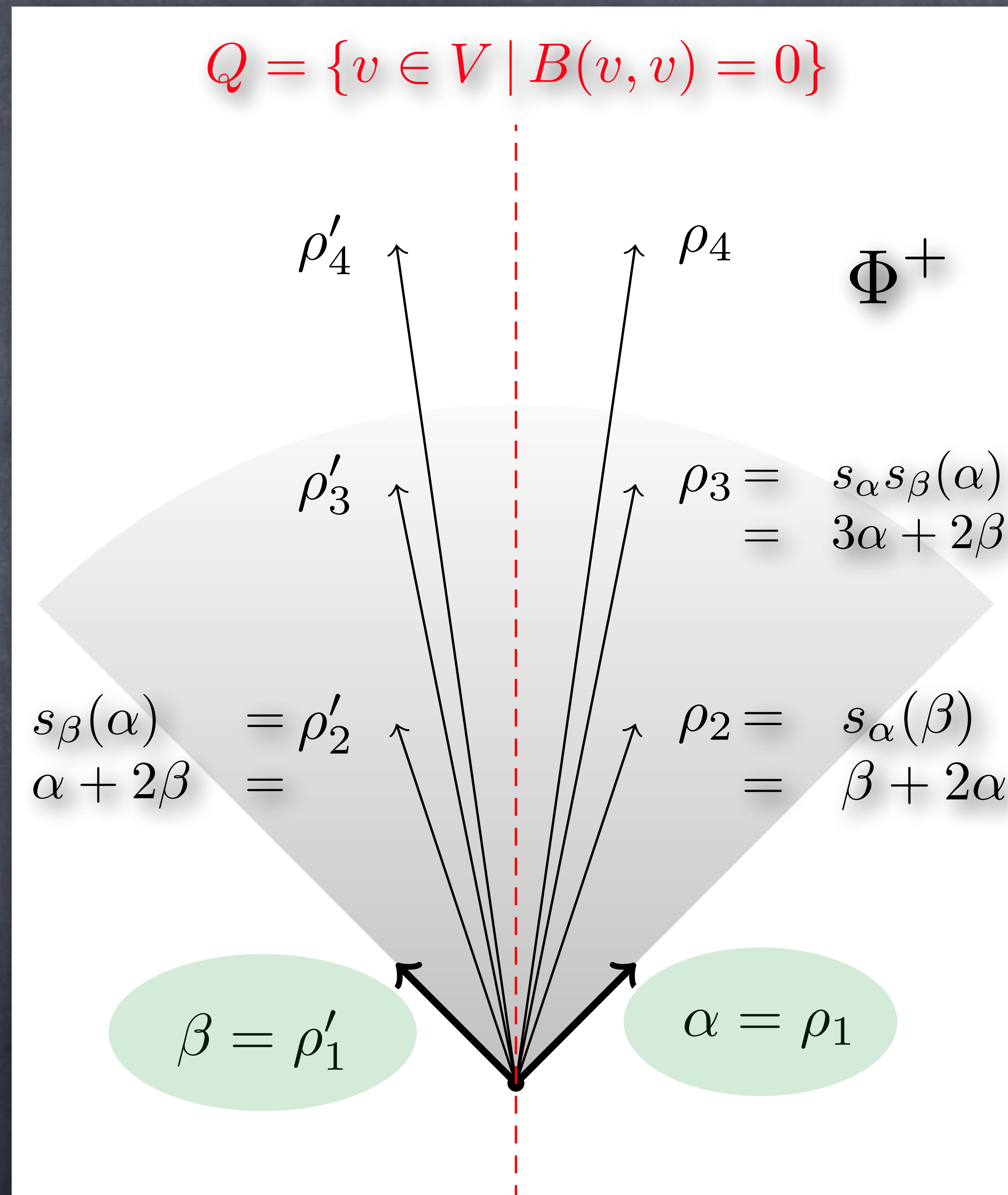
$$\ell(ws) = \ell(w) + 1 \iff w(\alpha_s) \in \Phi^+$$

Geometric representations

$\rho'_n = n\alpha + (n+1)\beta$

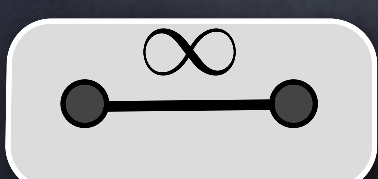
$\rho_n = (n+1)\alpha + n\beta$

Infinite dihedral group



(a) $B(\alpha,\beta) = -1$

$s_\alpha(v) = v - 2B(v,\alpha)\alpha.$



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Geometric representations

Restriction to Reflection subgroups

The isotropic cone of B: $Q = \{v \in V \mid B(v,v) = 0\}$

Root of a B-reflection on V: for $\alpha \notin Q$ and $v \in V$

$s_\alpha(v) = v - 2B(v,\alpha)\alpha$ with $B(\alpha,\alpha) = 1.$

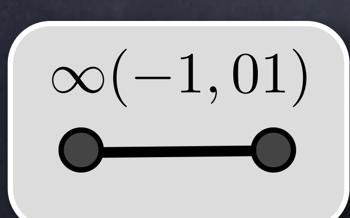
- A reflection subgroup of (W,S) is a subgroup $W_A = \langle s_\alpha \mid \alpha \in A \rangle$ where $A \subseteq \Phi^+$ is finite

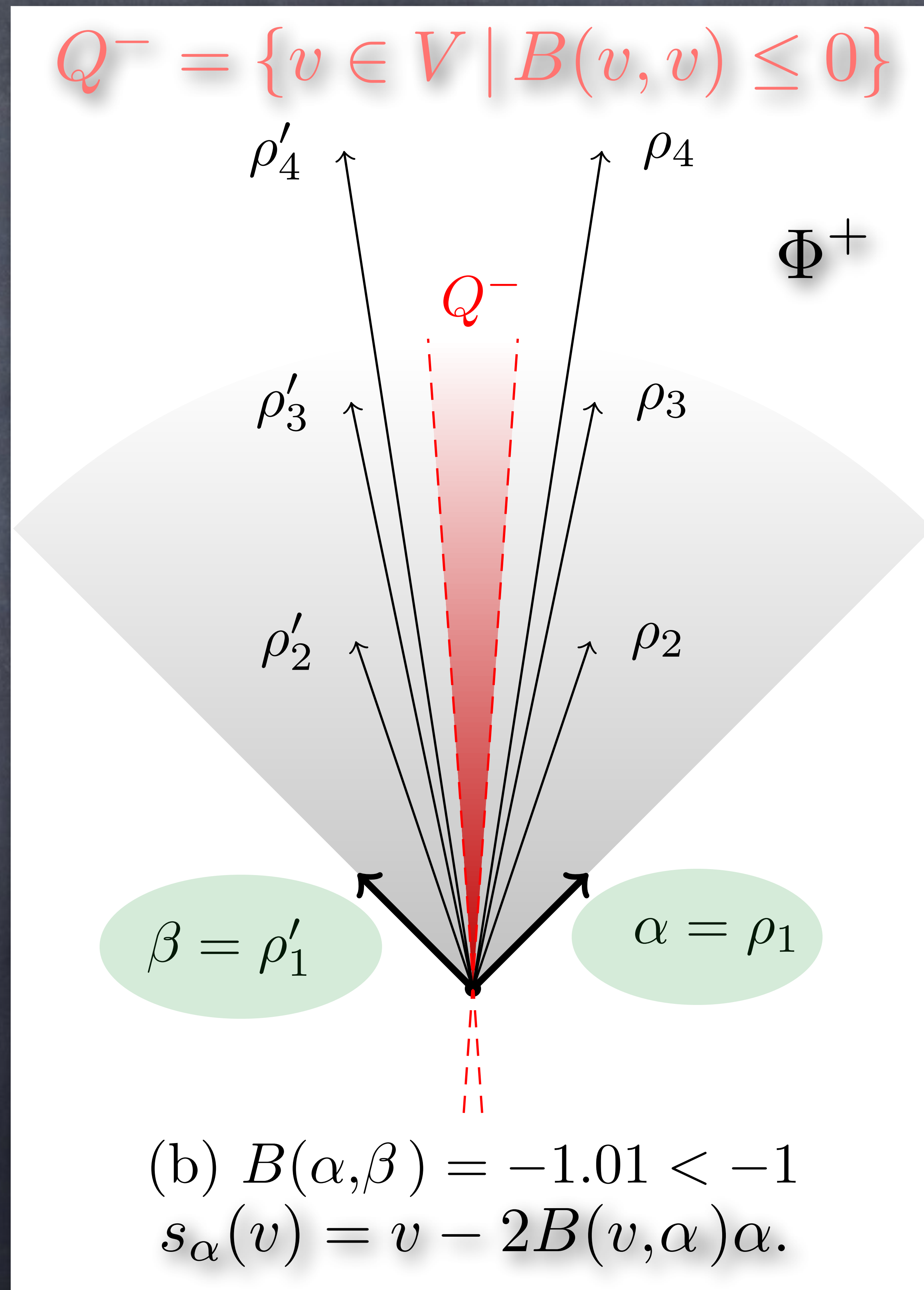
Theorem (Dyer, Deodhar). Let $A \subseteq \Phi^+$, $A' = W_A(A) \cap \Phi^+$ and Δ_A the basis of $\mathrm{cone}(A')$. Then (W_A, S_A) is a Coxeter system, where $S_A = \{s_\alpha \mid \alpha \in \Delta_A\}$.

The restriction of Tits geometric representation to W_A is not necessarily the one for (W_A, S_A)

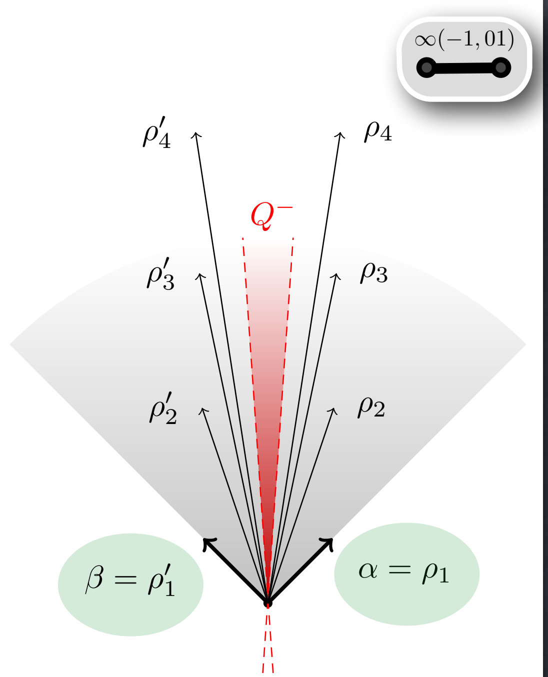
Geometric representations

Infinite dihedral group II





Geometric representations



(b) $B(\alpha,\beta) = -1.01 < -1$

Vinberg geometric representations of (W, S)

- (V, B) real quadratic space and $\Delta \subseteq V$ s.t.
  - $\text{cone}(\Delta) \cap \text{cone}(-\Delta) = \{0\}$;
  - $\Delta = \{\alpha_s \mid s \in S\}$ s.t.

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) & \text{if } m_{st} < \infty \\ a \leq -1 & \text{if } m_{st} = \infty \end{cases}$$

- $W \leq \mathrm{O}_B(V)$ "B-isometry":

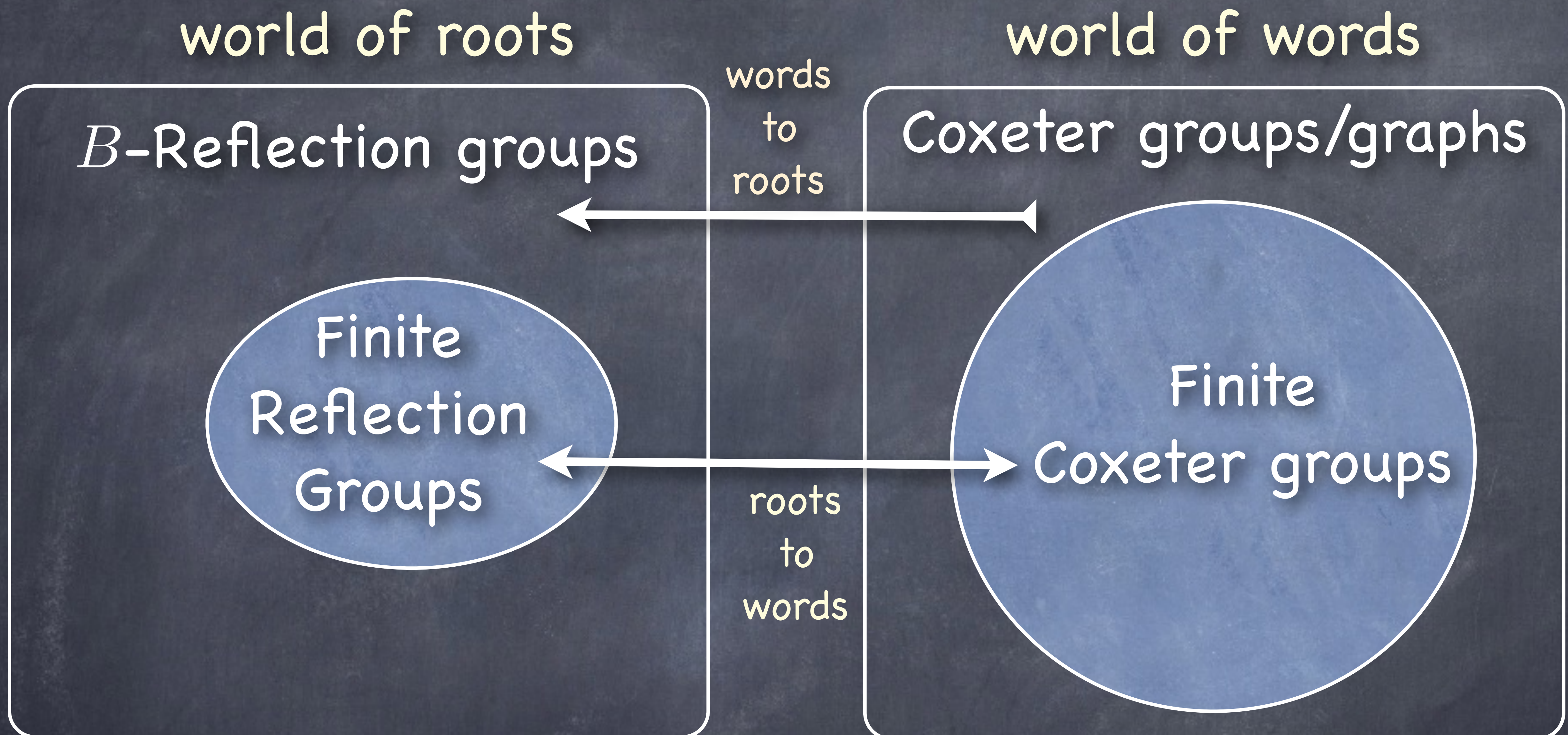
$$s(v) = v - 2B(v,\alpha)\alpha, \ \ s \in S$$

Root system: $\Phi = W(\Delta)$, $\Phi^+ = \text{cone}(\Delta) \cap \Phi = -\Phi^-$

Proposition. Let $s \in S$ and $w \in W$, then:

$$\ell(ws) = \ell(w) + 1 \iff w(\alpha_s) \in \Phi^+$$

Classification of Finite Reflection Groups



Theorem. The following assertions are equivalent:

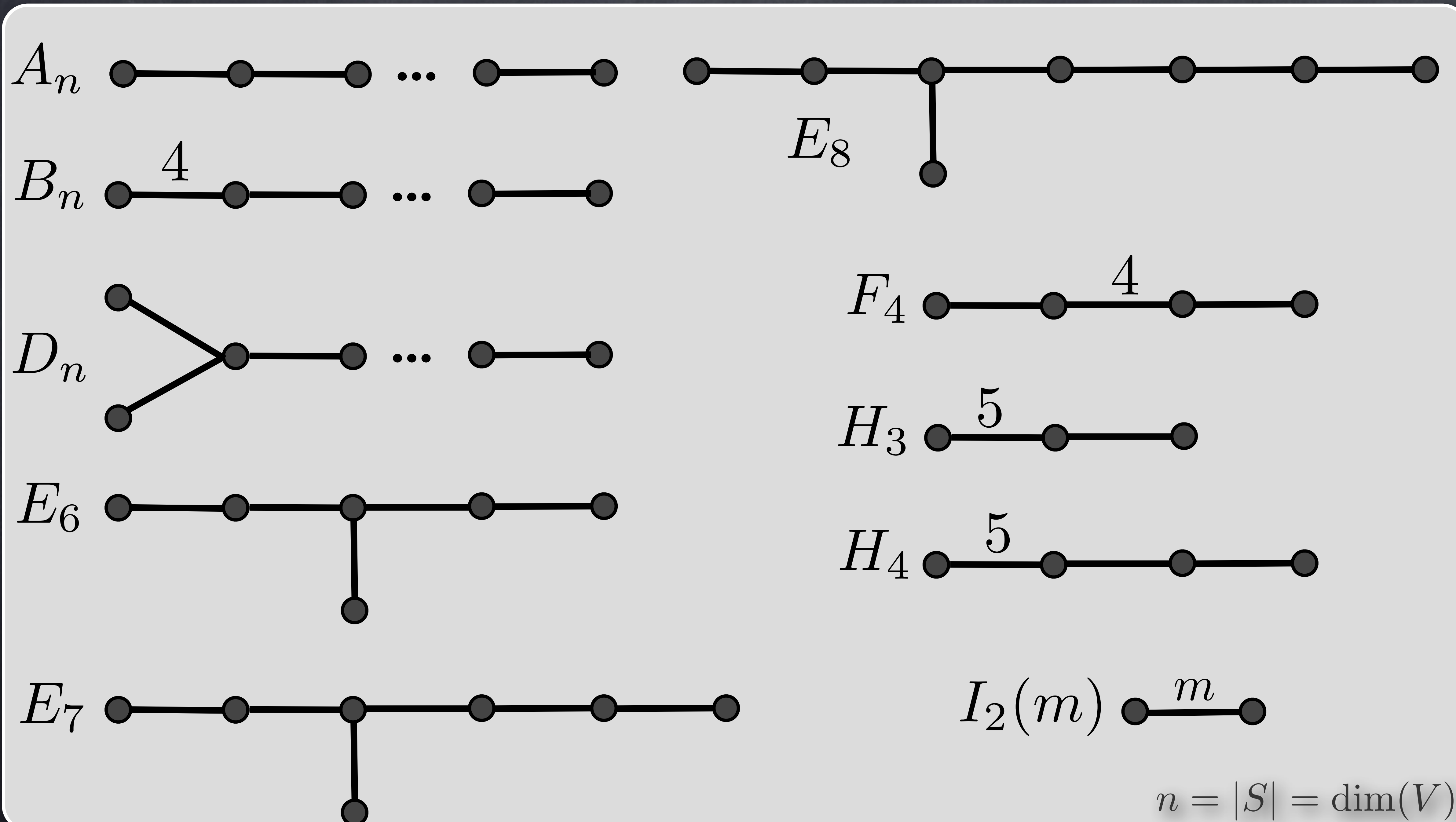
(i) (W, S) is a finite Coxeter system;

(ii) B is a scalar product and $W \leq \mathrm{O}_B(V)$;

(iii) W is a finite reflection group.

Coxeter groups

Theorem. The irreducible FRG are precisely the finite irreducible Coxeter groups. Their graphs are:



Conclusion

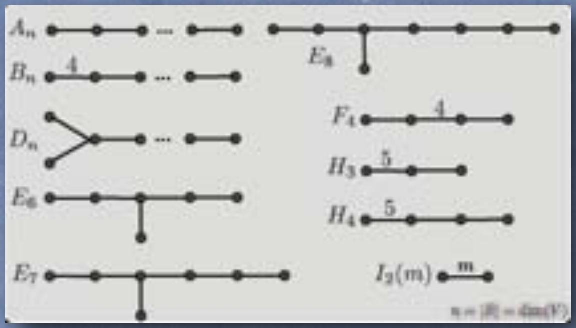
world of roots | words to roots | **world of words**

B-Reflection groups

signature (p, q, r) of B ← Γ_W allowing $\infty (a \leq -1)$

Coxeter groups/graphs

Finite Reflection Groups ↔

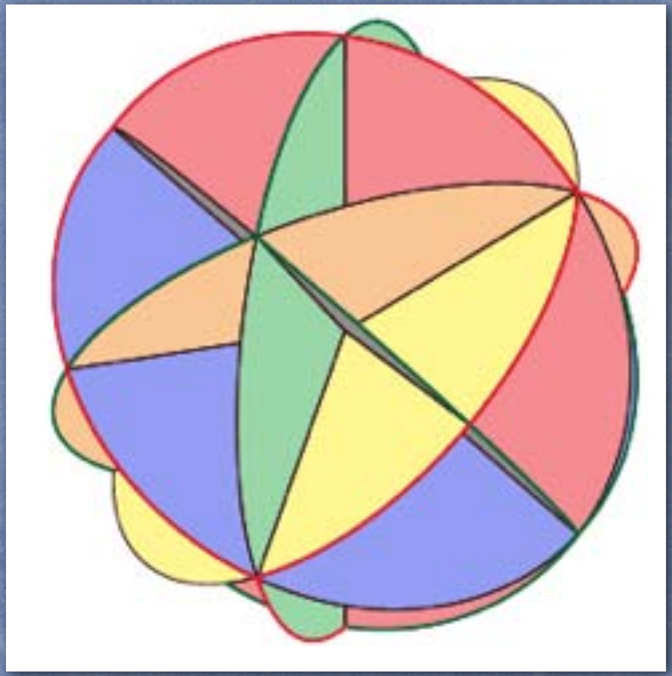


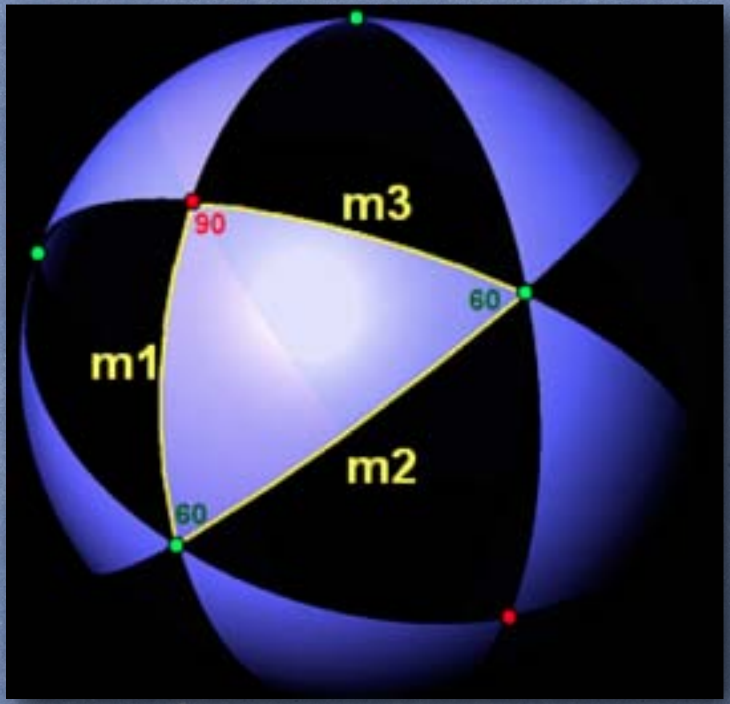
roots to words

Question: Are all B-reflection groups Coxeter groups?

Conclusion

In the spherical, euclidean and hyperbolic case, all finitely generated discrete B-reflection groups are Coxeter groups (models for these geometry exist in V or its dual; `cut' these models by the hyperplanes of reflections)



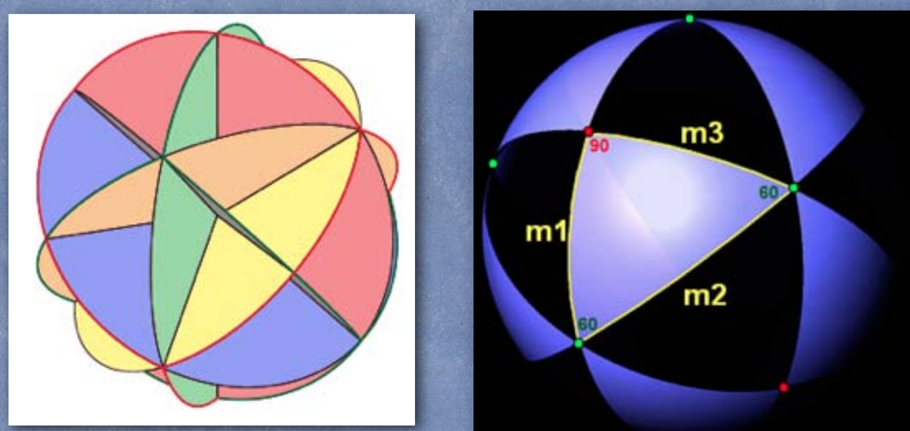


Finite case i.e. B is a scalar product ($V = V^*$): the model is the unit sphere

$$||v||^2 = B(v,v) = 1$$

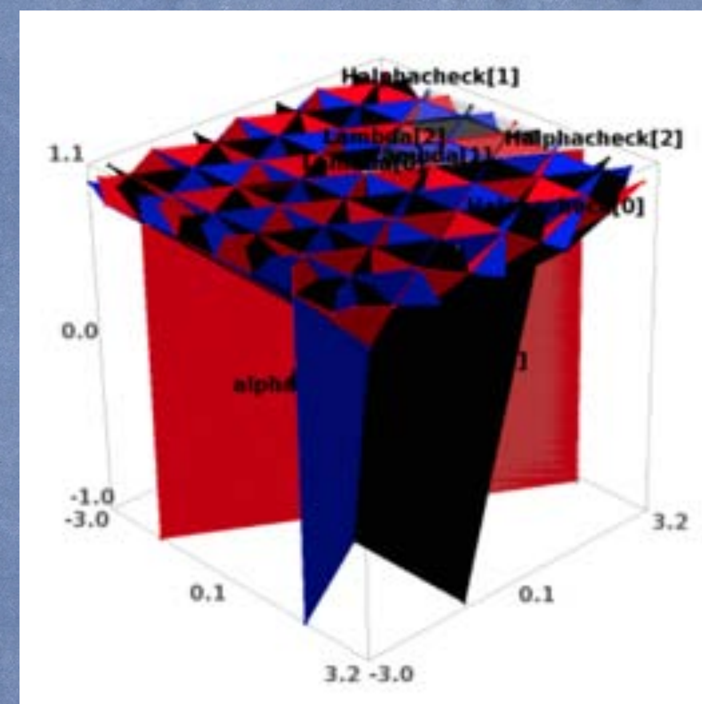
Conclusion

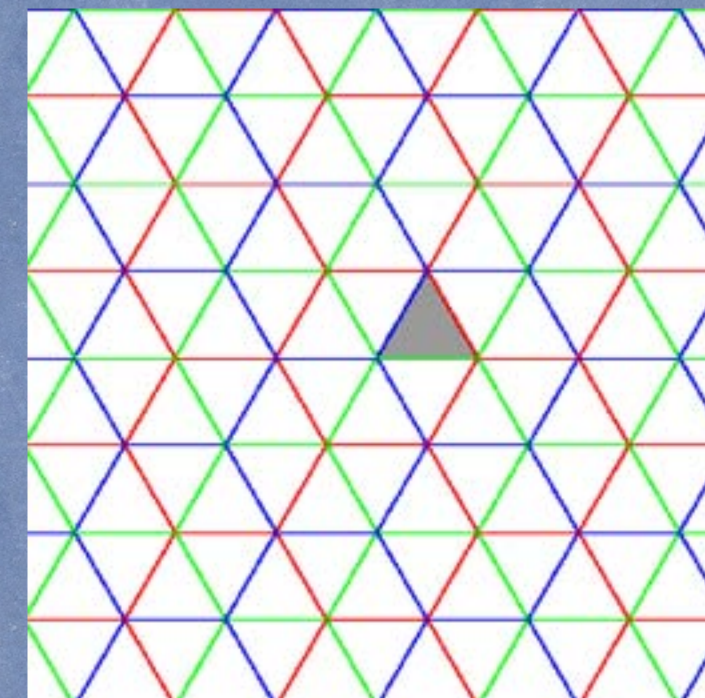
In the spherical, euclidean and hyperbolic case, they are all Coxeter groups (models for these geometry exist in V or its dual; `cut' these models by the hyperplanes of reflections)



Finite case i.e. B is a scalar product

$sgn(B) = (n, 0, 0)$





Affine case i.e. B is positive degenerate. Its radical is a line: $Rad(B) = \{v \in V \mid B(v, \alpha) = 0,\ \forall \alpha \in \Delta\} = \mathbb{R}x$

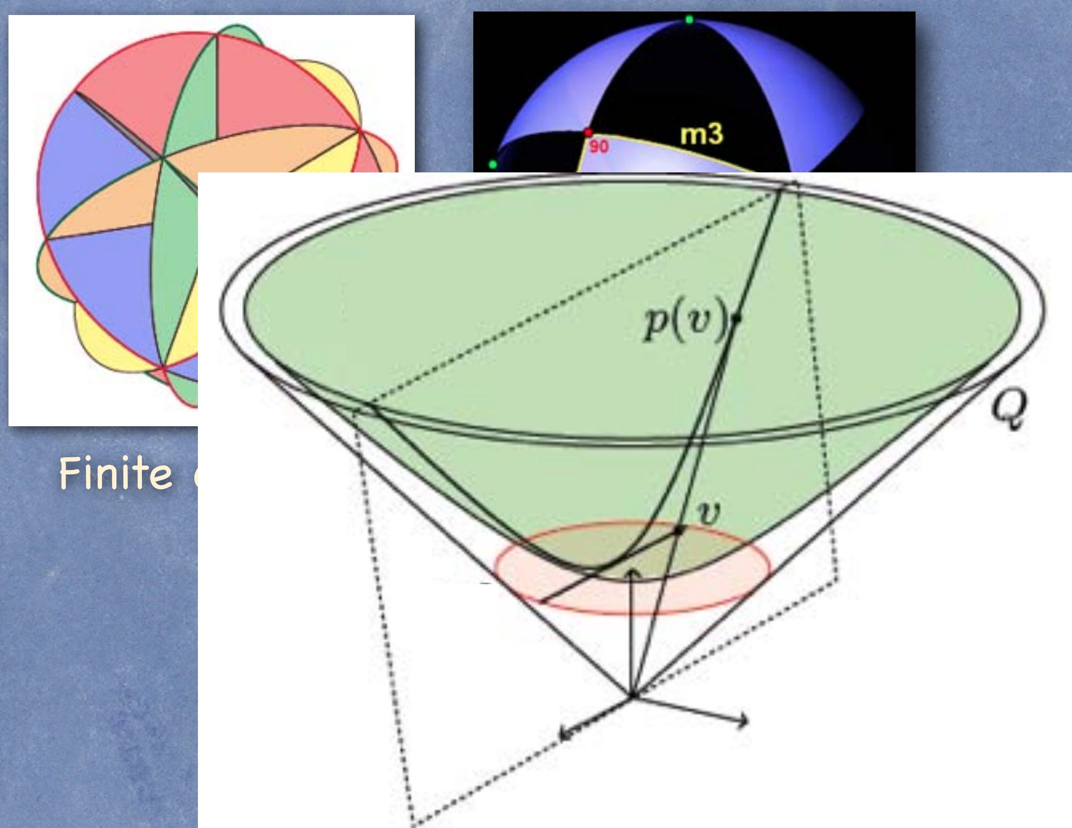
The model is an affine hyperplane in the dual V^*:

$$H = \{\varphi \in V^* \mid \varphi(x) = 1\}$$

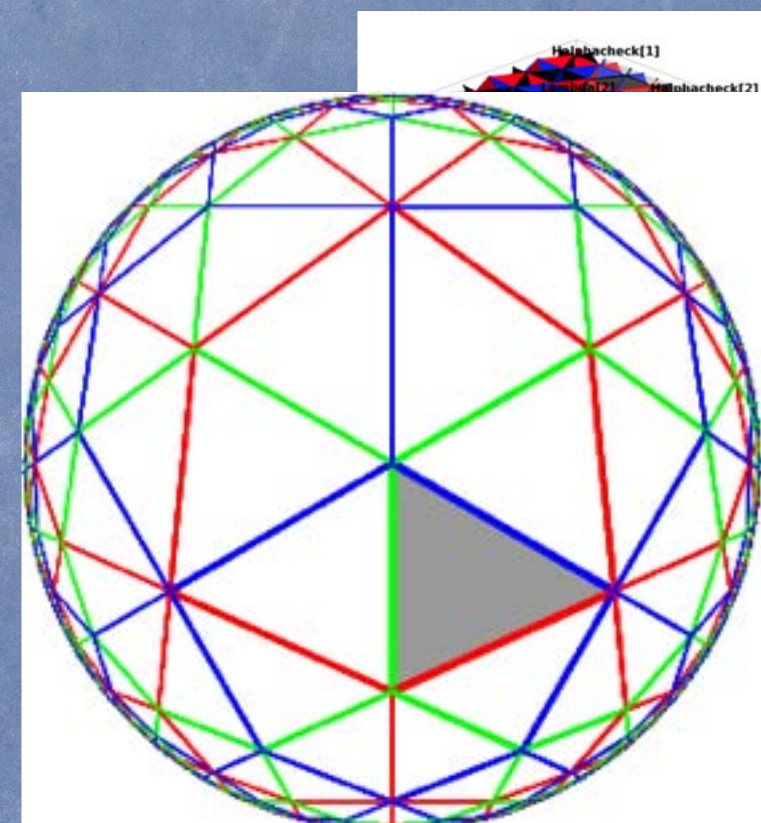
N.B: reflection hyperplanes leave in the dual here.

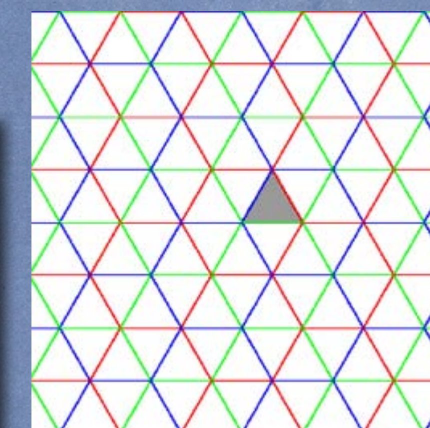
Conclusion

In the spherical, euclidean and hyperbolic case, they are all Coxeter groups (models for these geometry exist in V or its dual; `cut' these models by the hyperplanes of reflections)



Finite





positive degenerate.
$(n-1,0,1)$

Hyperbolic case i.e. $sgn(B) = (n-1,1,0)$ ($V = V^*$). Many models exists: projective (non conformal), hyperboloïd or the ball model

$$H^{n-1} = \{x \in V \mid B(x,x) = -1\}$$

Conclusion

world of roots | **world of words**

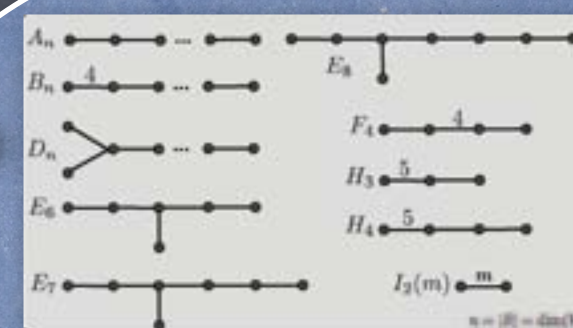
B-Reflection groups

signature (p,q,r) of B ⟵ Γ_W allowing $\infty (a \leq -1)$

words to roots

Coxeter groups/graphs

Finite Reflection Groups ⟷



roots to words

Problem: Let $p,q,r \in \mathbb{N}$, classify all the Coxeter graphs with signature (p,q,r). Count them?

N.B.: Known for $(n,0,0)$ - FRG -; $(n-1,0,1)$ - affine type - and partially for $(n-1,1,0)$ - ``weakly hyperbolic" type

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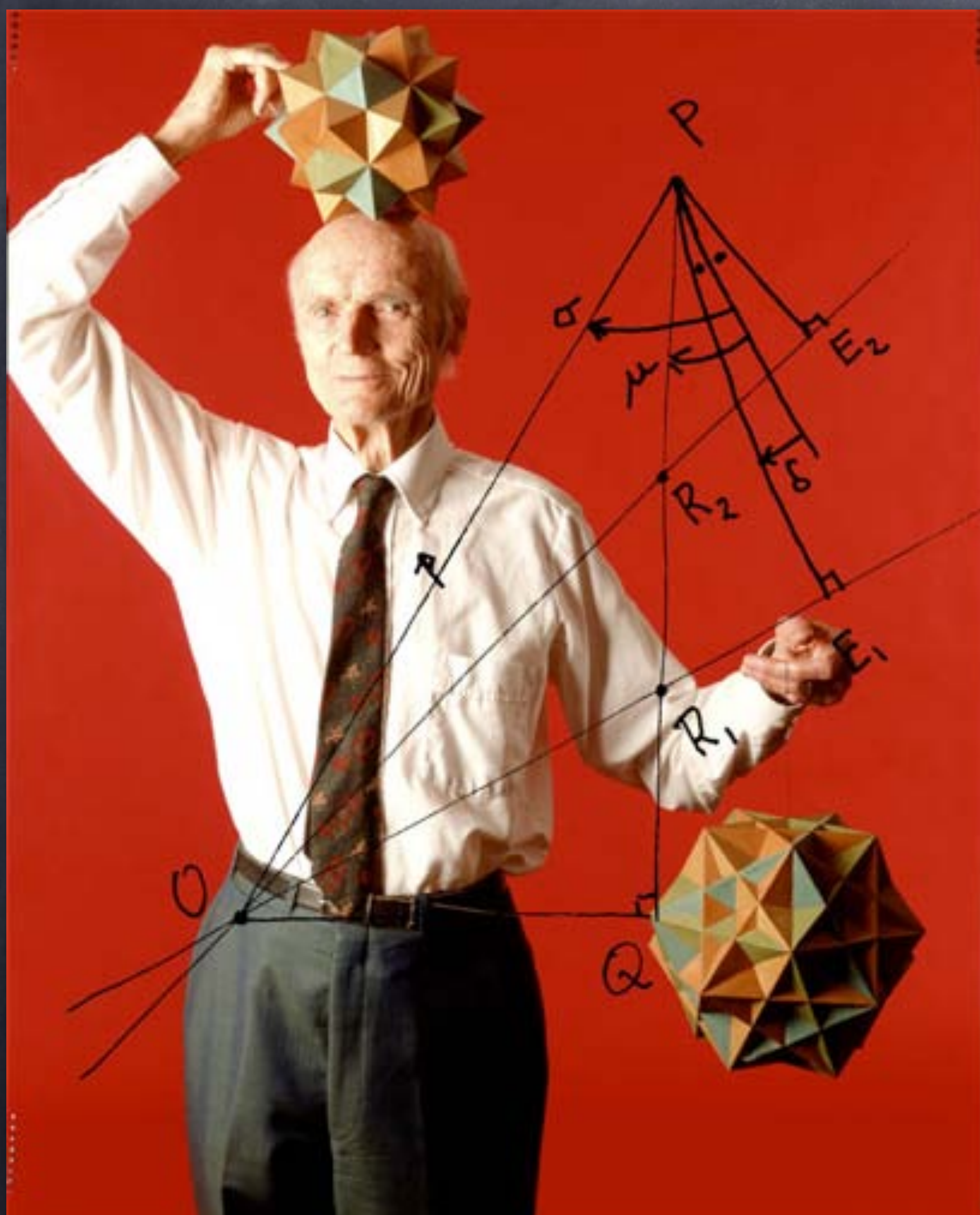
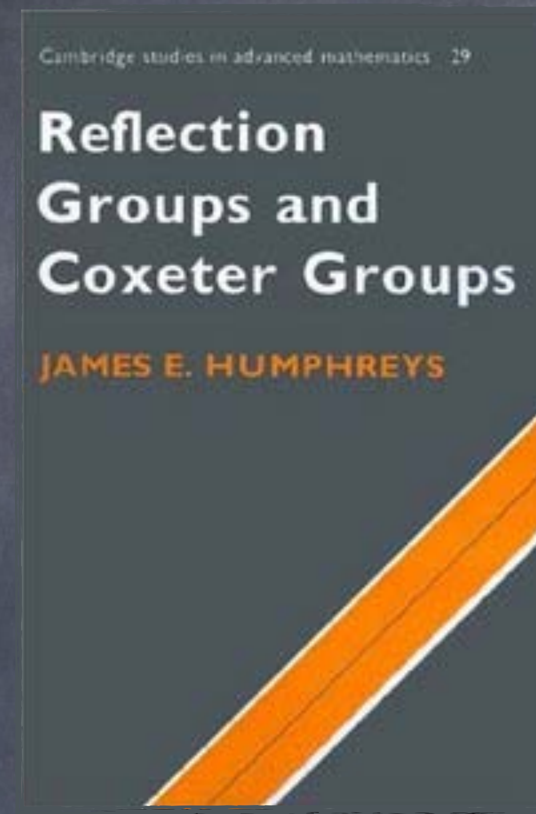
Selected biblio of Part 1 ...

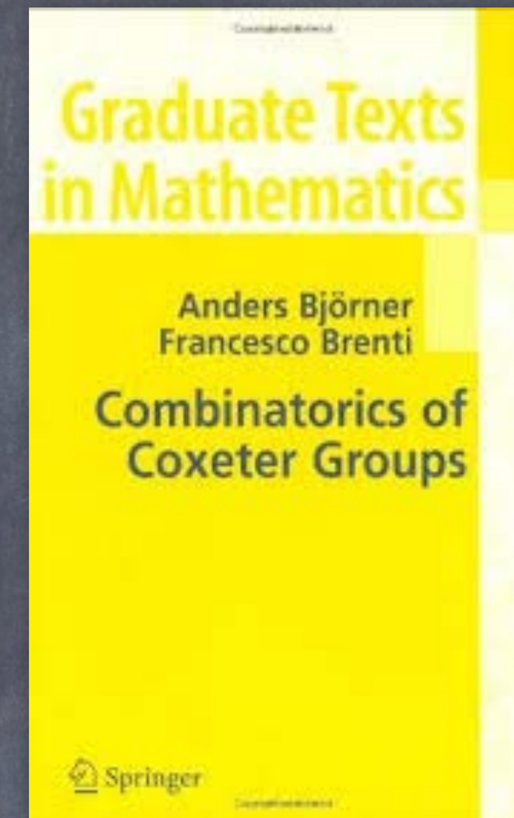
Donald Coxeter

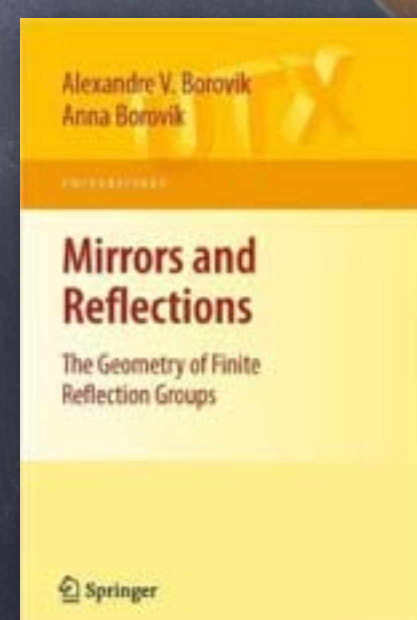
(London 1907, Toronto 2003)
Professor at University of Toronto
(1936-2003)

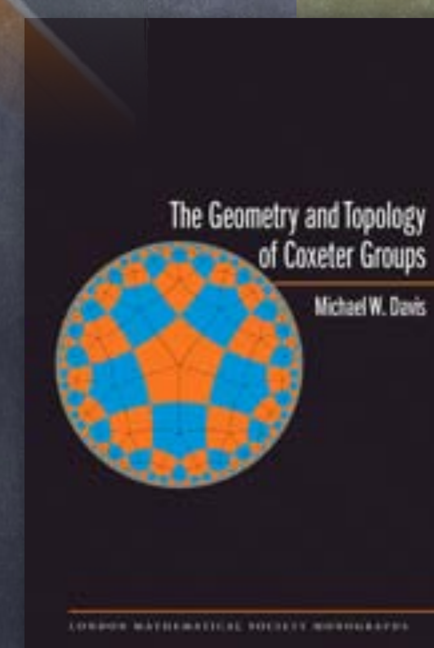


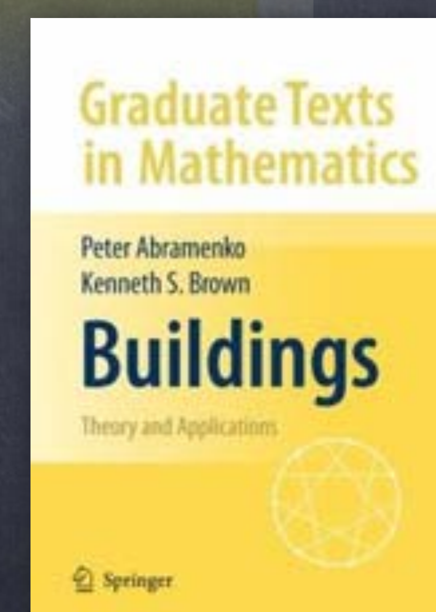
Mandatory photo credit. Mathematics genius Donald Coxeter is the subject of a public talk by journalist Siobhan Roberts, *The Man Who Saved Geometry*, on Sunday, July 31. Photo courtesy of The Banff Centre.











Lecture 2: Weak order and roots

In the last episode

world of roots

world of words

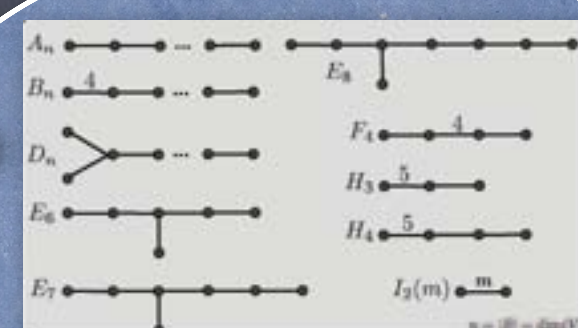
words to roots

B-Reflection groups

signature (p,q,r) of B ⟵ Γ_W allowing $\infty (a \leq -1)$

Coxeter groups/graphs

$W = \langle S \mid (st)^{m_{st}} = e \rangle$

Finite Reflection Groups ⟷ 

roots to words

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Weak order and reduced words

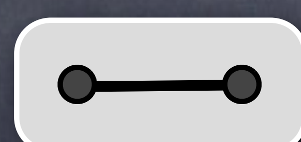
(W,S) Coxeter system of finite rank $|S|<\infty$

- any $w \in W$ is a word in the alphabet S ; $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$

- Length function $\ell : W \to \mathbb{N}$ with $\ell(e) = 0$ and

$$\ell(w) = \min\{k \,|\, w = s_1 s_2 \ldots s_k,\ s_i \in S\}$$

How to study words on S representing w ? Is a word $s_1 s_2 \ldots s_k$ a reduced word for w (i.e. $k = \ell(w)$) ?

Examples. $\mathcal{D}_3$ is  ;

	e	s	t	st	ts	$sts = tst$
ℓ	0	1	1	2	2	3

$\ell(ststs) = 1$ since $ststs = (sts)ts = (tst)ts = t$

Proposition. Let $s \in S$ and $w \in W$, then $\ell(ws) = \ell(w) \pm 1$.

Weak order and reduced words

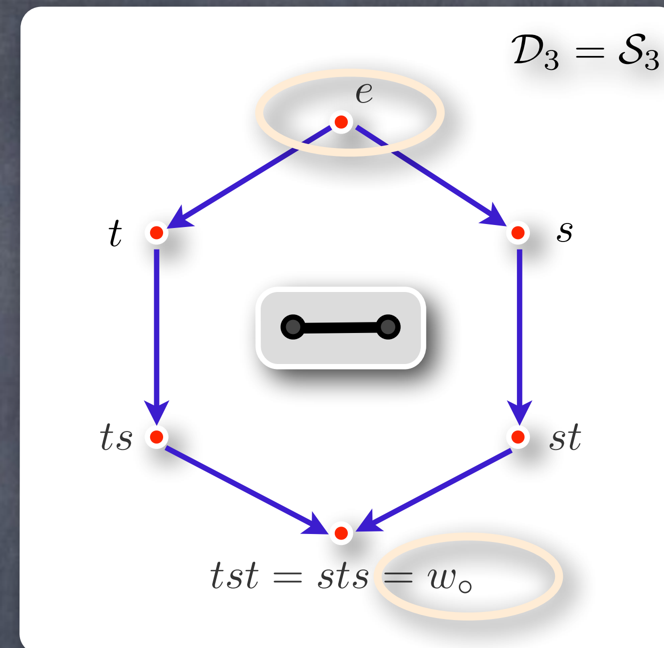
Cayley graph of $W = \langle S \rangle$ i.e.

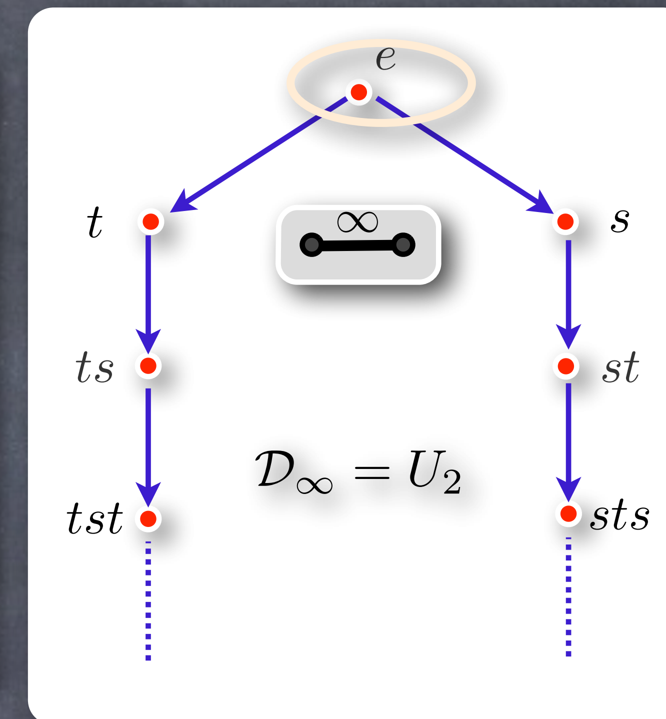
- vertices W
- edges $w \overset{s}{\text{——}} ws \quad (s \in S)$

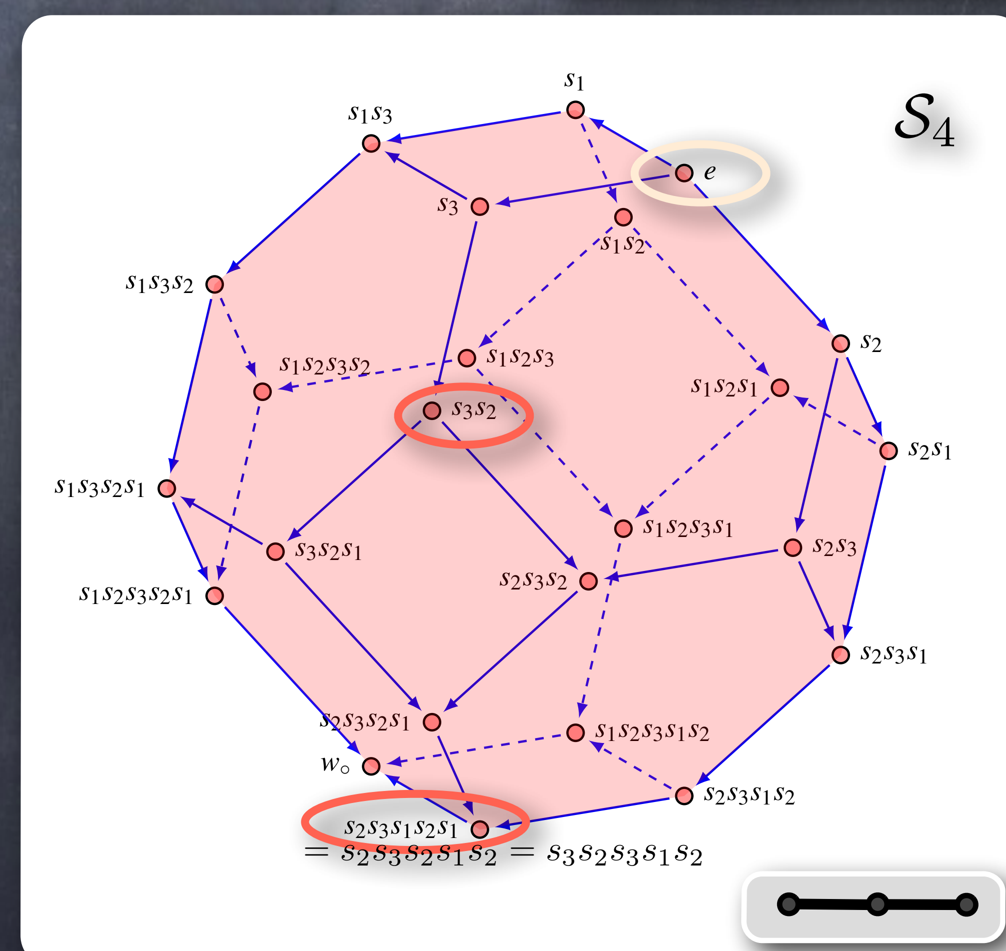
is naturally oriented by the (right) weak order:

$w < ws$ if $\ell(w) < \ell(ws)$

write: $w \xrightarrow{s} ws$







Fact: (a) $u \le w$ iff a reduced word of u is a prefix of a red. word of w.

(b) reduced words of w corresp. to maximal chains in the interval $[e, w]$.

(c) Chain property: if $u \le w$ with $\ell(u) + 1 < \ell(w)$ then:

$$\exists v \in W,\ u \lneq v \lneq w$$

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Weak order and reduced words

Theorem (Björner). The weak order is a complete meet-semilattice. In particular $u \wedge v = \inf(u,v)$, $\forall u,v \in W$, exists.

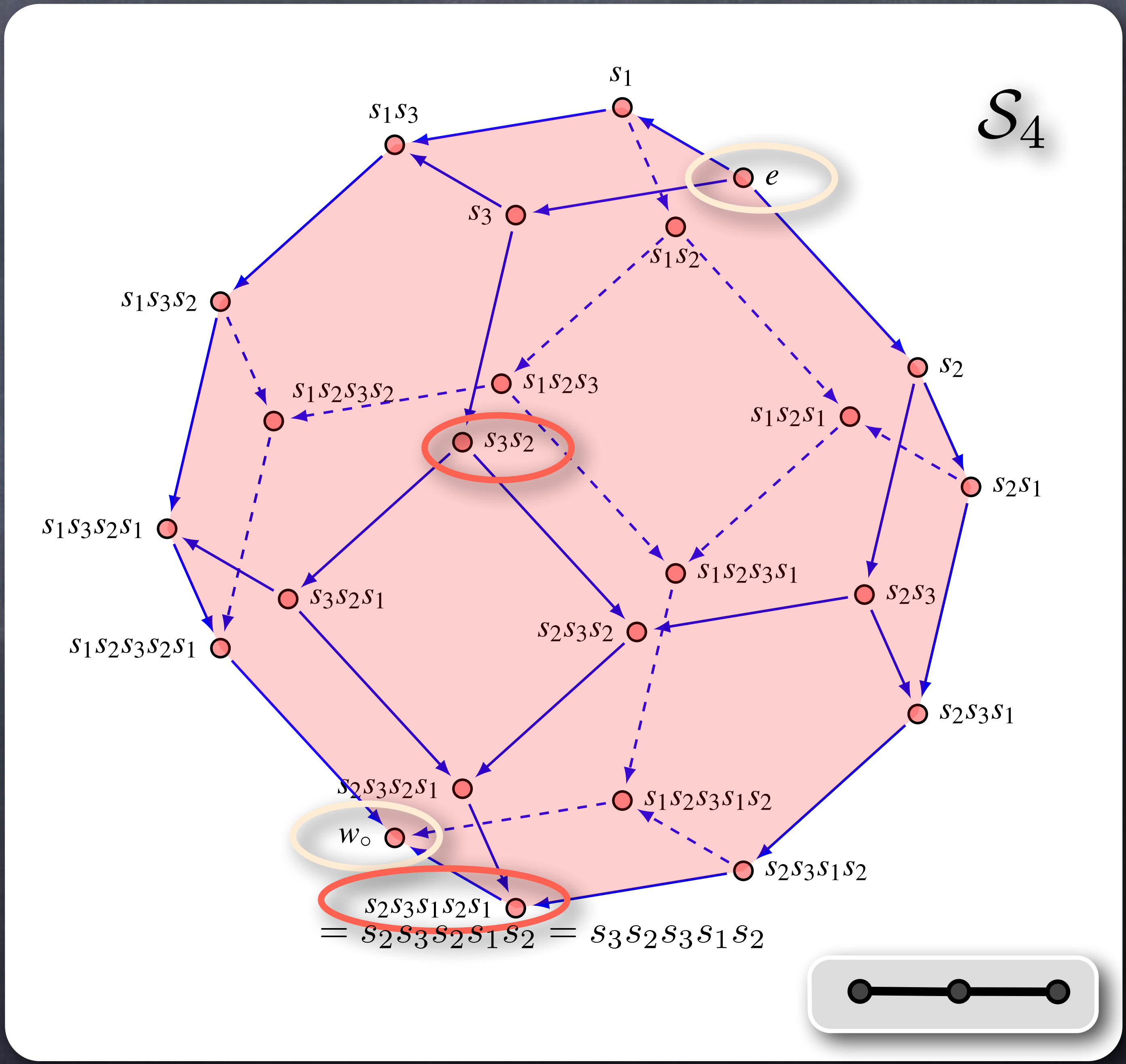
Proposition. Assume W is finite, then:

(i) there is a unique $w_\circ \in W$ such that: $u \leq w_\circ$, $\forall u \in W$.

(ii) the map $w \mapsto w_\circ w$ is a poset antiautomorphism.

(iii) the weak order is a complete lattice. In part., $u \vee v = \sup(u,v)$ exists.

(iv) $u \wedge v = w_\circ(w_\circ u \vee w_\circ v)$



Weak order & Generalized Associahedra

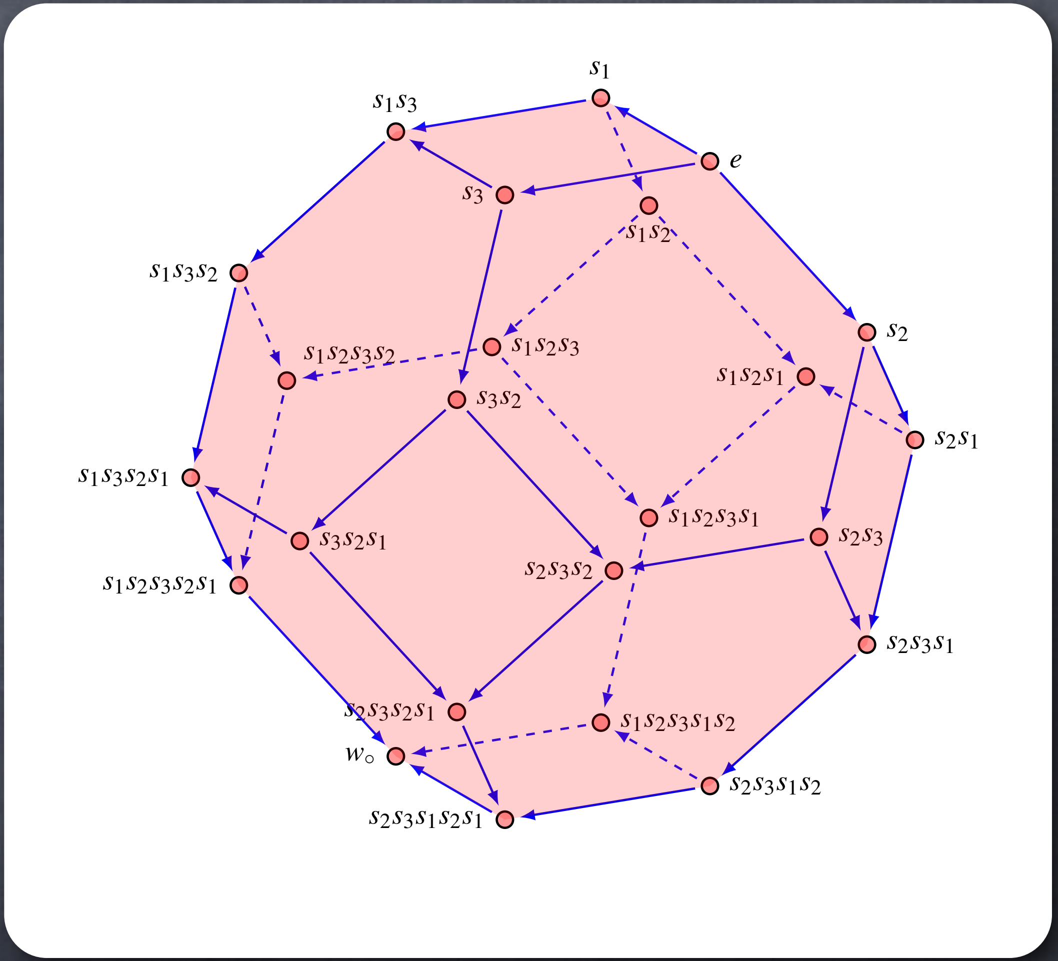
(W, S) finite Coxeter system, so $W \leq \mathrm{O}(V)$

Permutahedra

- Δ simple system;
- $S = \{s_\alpha \,|\, \alpha \in \Delta\}$;
- Choose $\boldsymbol{a}$ generic i.e.

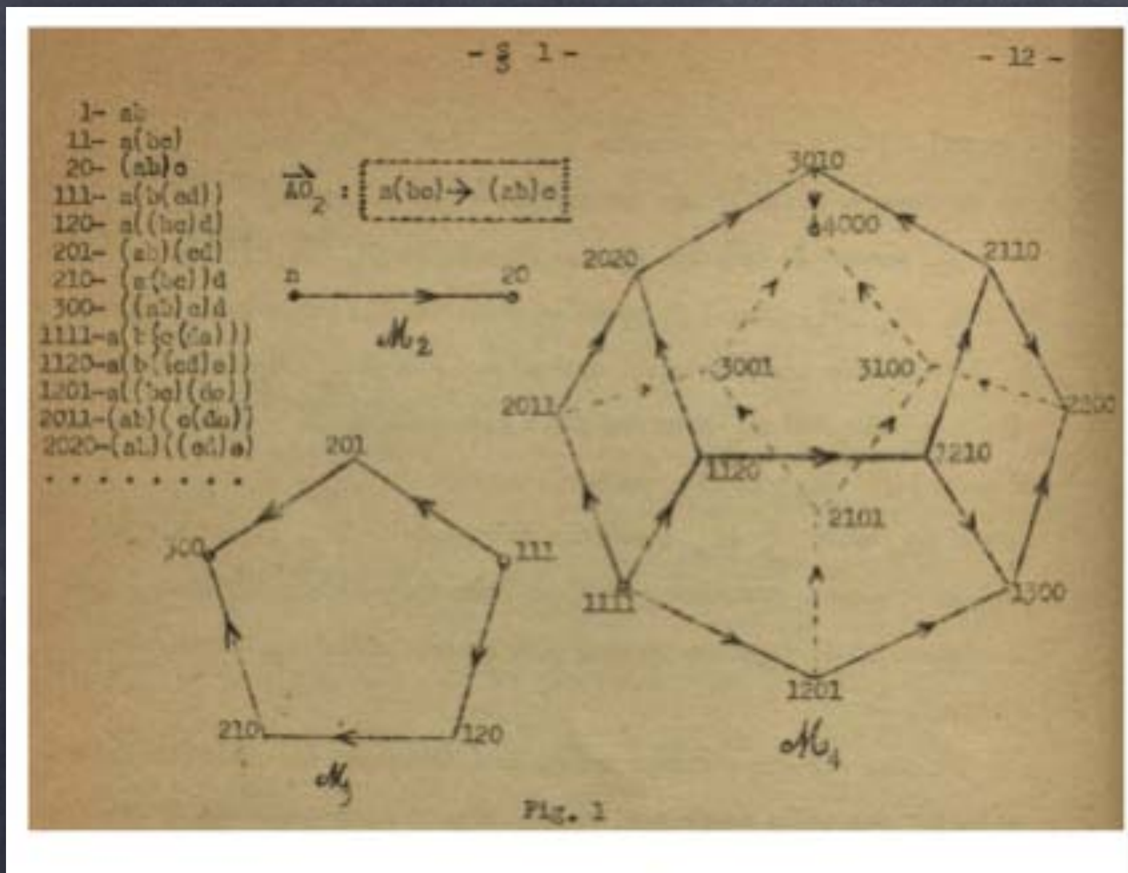
$$\langle \boldsymbol{a}, \alpha \rangle > 0, \ \forall \alpha \in \Delta$$

$$\mathrm{Perm}^{\boldsymbol{a}}(W) = \mathrm{conv}\,\{w(\boldsymbol{a}) \,|\, w \in W\}$$



Proposition. $\mathrm{Perm}^{\boldsymbol{a}}(W)$ is a simple polytope whose oriented 1-skeleton is the graph of the (right) weak order.

Building Generalized Associahedra



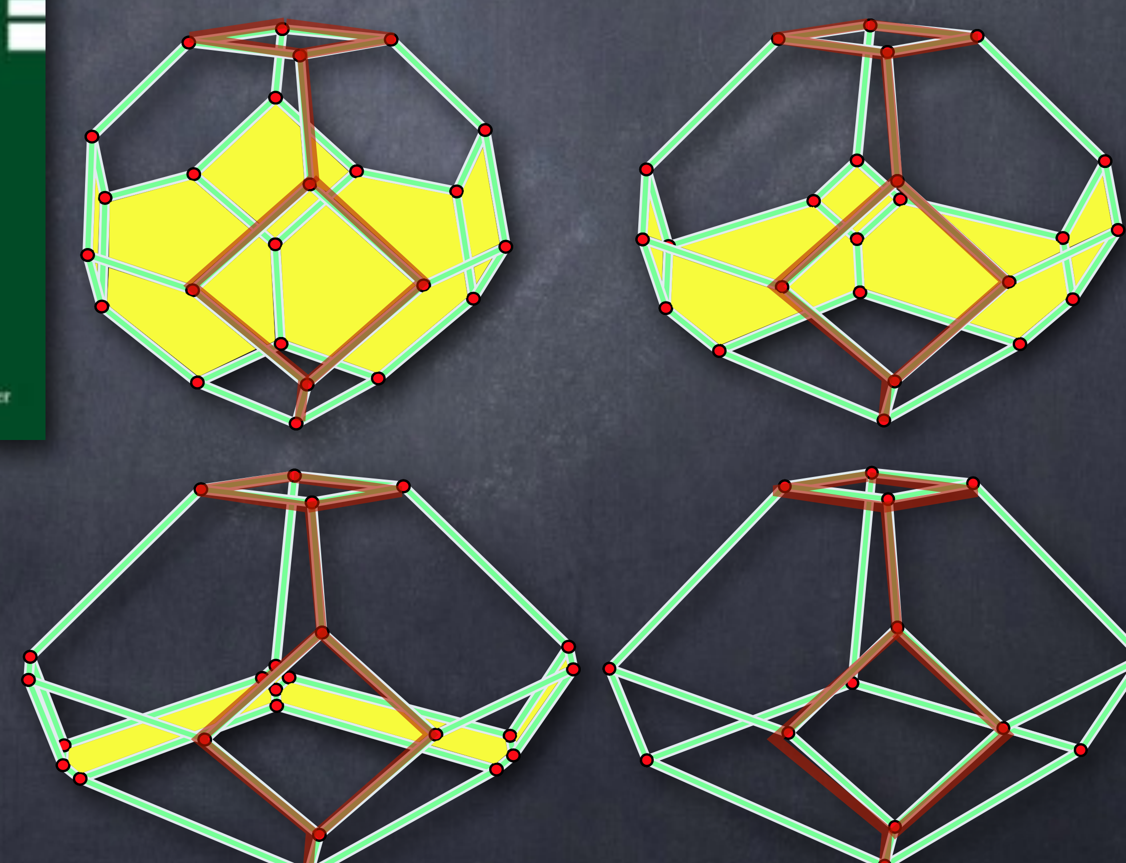
Tamari's associahedron

Associahedra (lattices/complexes):

- Lattice (Tamari, 1951)
- Cell complex (Stasheff, 1963)
- Cluster complex (Fomin-Zelevinsky, 2003)
- Cambrian lattices (Reading 2007, 2007)and more ...

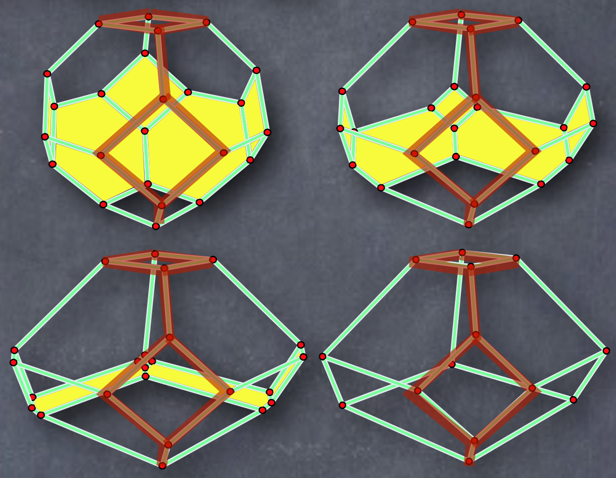
Associahedra (Convex polytopes):

- Type A (Haiman 1984, Lee, Loday, ...)
- Type B - cyclohedra (Bott-Taubes 1994, ...)
- Weyl groups (Chapoton-Fomin-Zelevinsky, 2003)
- from permutahedra of finite Coxeter groups (CH-Lange-Thomas 2011, ...)

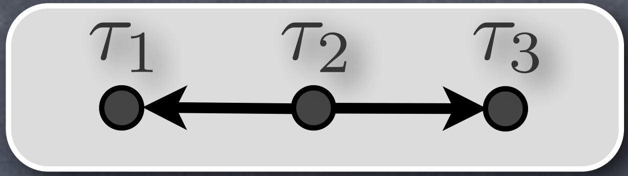


Building Generalized Associahedra

Hohlweg, C. Lange, H. Thomas (2009)



- Data: $\mathrm{Perm}^{\boldsymbol{a}}(W)$ and an orientation of Γ_W

$W = S_4$ 

- c Coxeter element associated to this orientation i.e product without repetition of all the simple reflections;

$$c = \tau_2\tau_3\tau_1$$

- $c_{(I)}$ subword with letters $I \subseteq S$

$$I = \{\tau_1, \tau_2\} \subseteq S \Rightarrow c_{(I)} = \tau_2\tau_1$$

- c - word of $w_\circ$: $\boldsymbol{w_o}(c) = c_{(K_1)}c_{(K_2)}\dots c_{(K_p)}$ reduced expression s.t. $S \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p \neq \emptyset$

$$\boldsymbol{w_o}(\tau_1\tau_2\tau_3) = \tau_1\tau_2\tau_3.\tau_1\tau_2.\tau_1 = c_{(S)}c_{(\{\tau_1,\tau_2\})}c_{(\{\tau_1\})}$$

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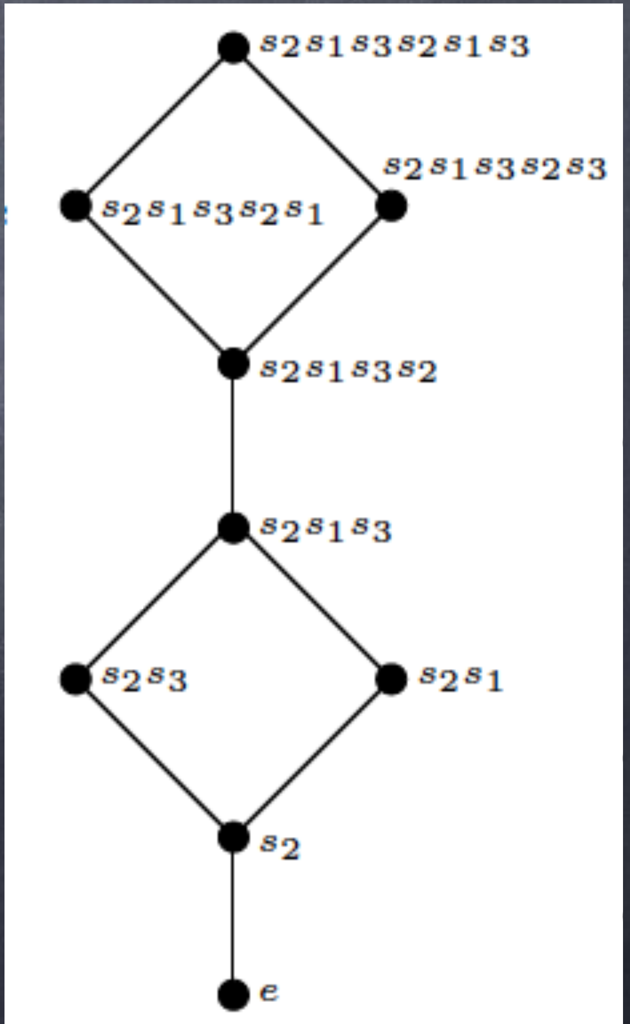
$$\boldsymbol{w_o}(\tau_2\tau_3\tau_1) = \tau_2\tau_3\tau_1.\tau_2\tau_3\tau_1 = c_{(S)}c_{(S)}.$$

- c - singletons are the prefixes of $\boldsymbol{w_o}(c)$up to commutations

e,	$\tau_2\tau_3$,	$\tau_2\tau_3\tau_1\tau_2\tau_3$,
τ_2,	$\tau_2\tau_3\tau_1$,	$\tau_2\tau_3\tau_1\tau_2\tau_1$, and
$\tau_2\tau_1$,	$\tau_2\tau_3\tau_1\tau_2$,	$w_o = \tau_2\tau_1\tau_3\tau_2\tau_1\tau_3$.



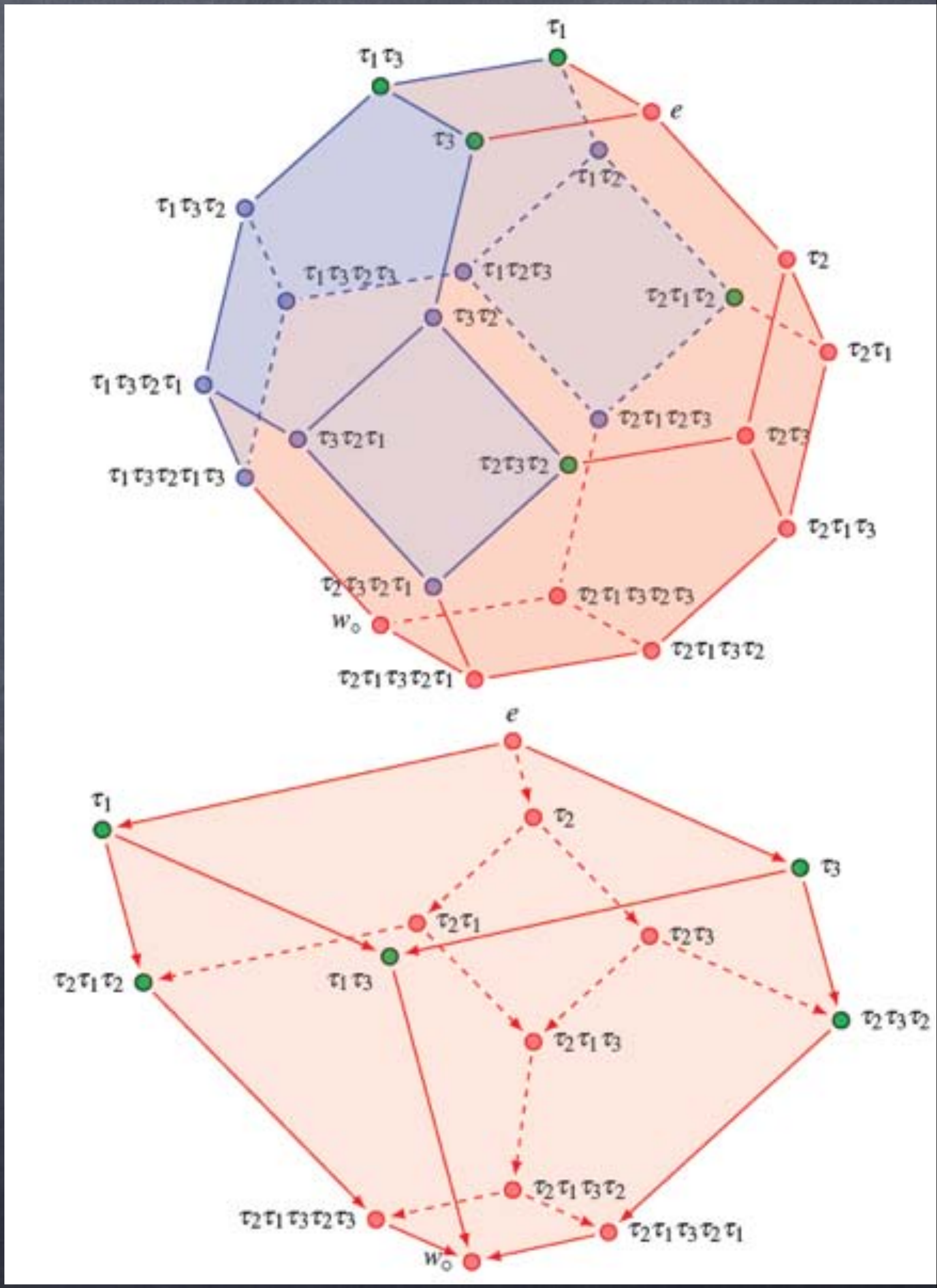
$c = s_2s_1s_3$

Proposition. c - singletons form a distributive sublattice of the weak order.

Building Generalized Associahedra

Hohlweg, C. Lange, H. Thomas (2009)

- c - generalized associahedron is the polytope $\mathrm{Asso}_c^{\boldsymbol{a}}(W)$ obtained from $\mathrm{Perm}^{\boldsymbol{a}}(W)$ by keeping only the facets containing a c - singleton

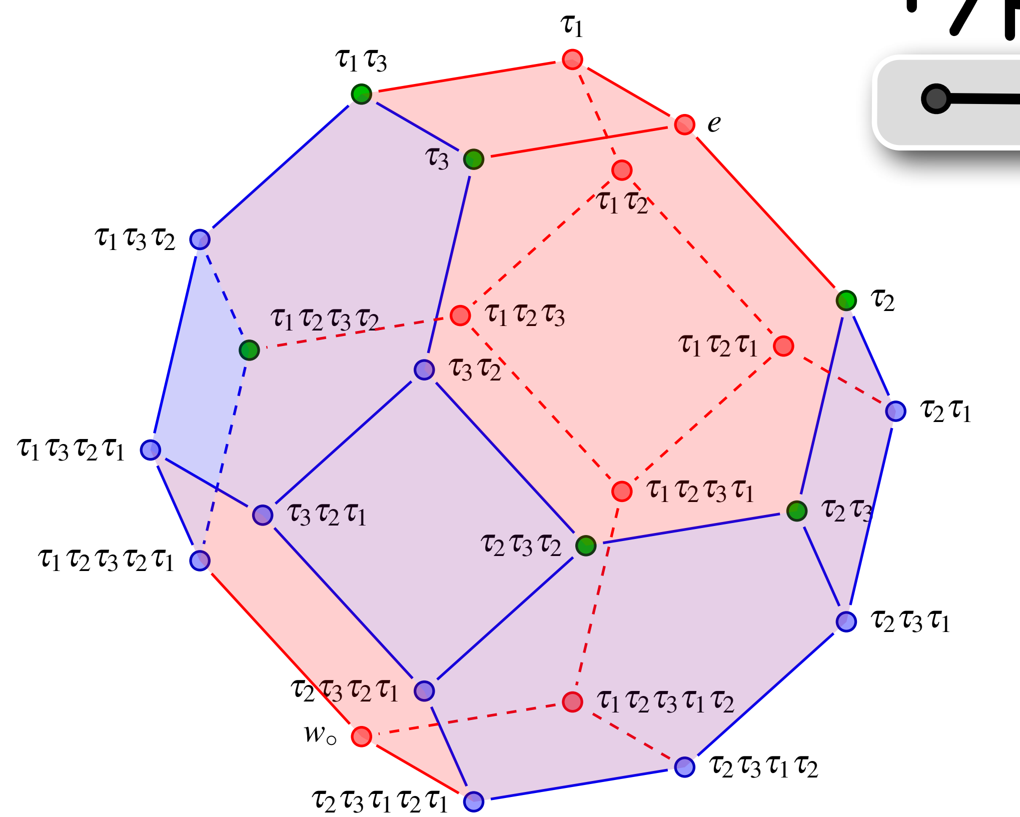


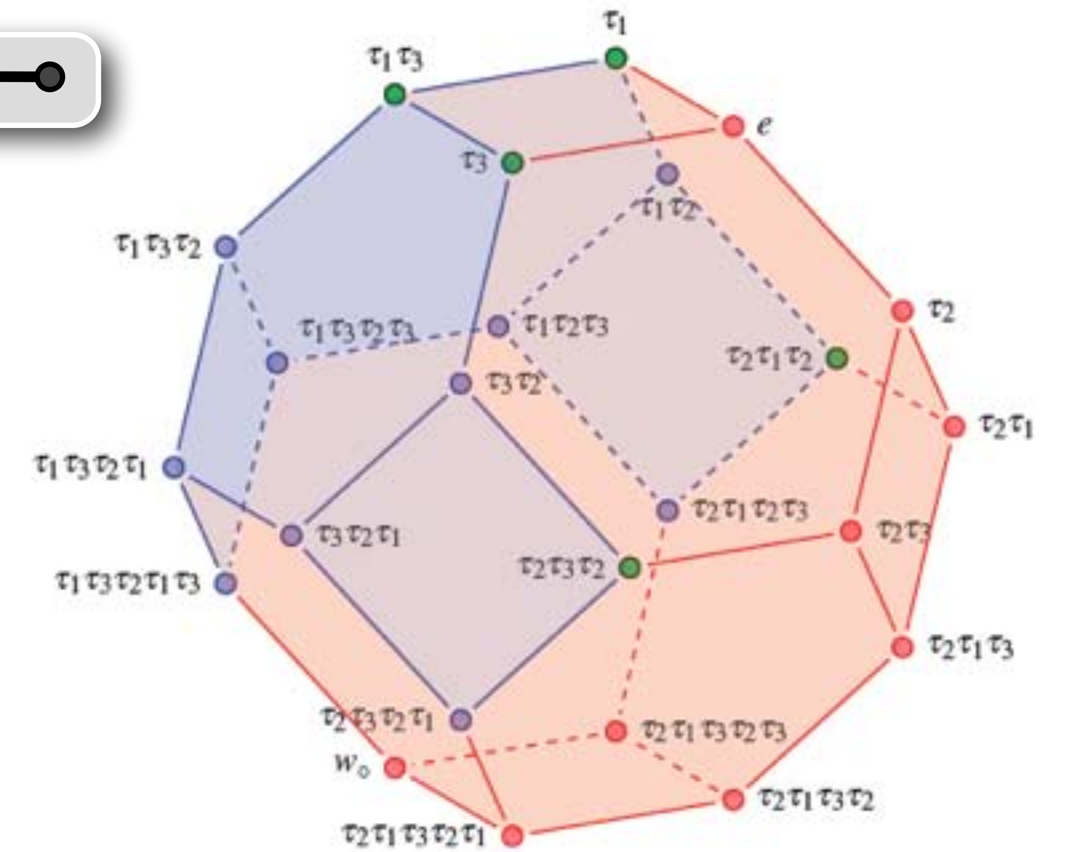
Theorem. The 1-skeleton of

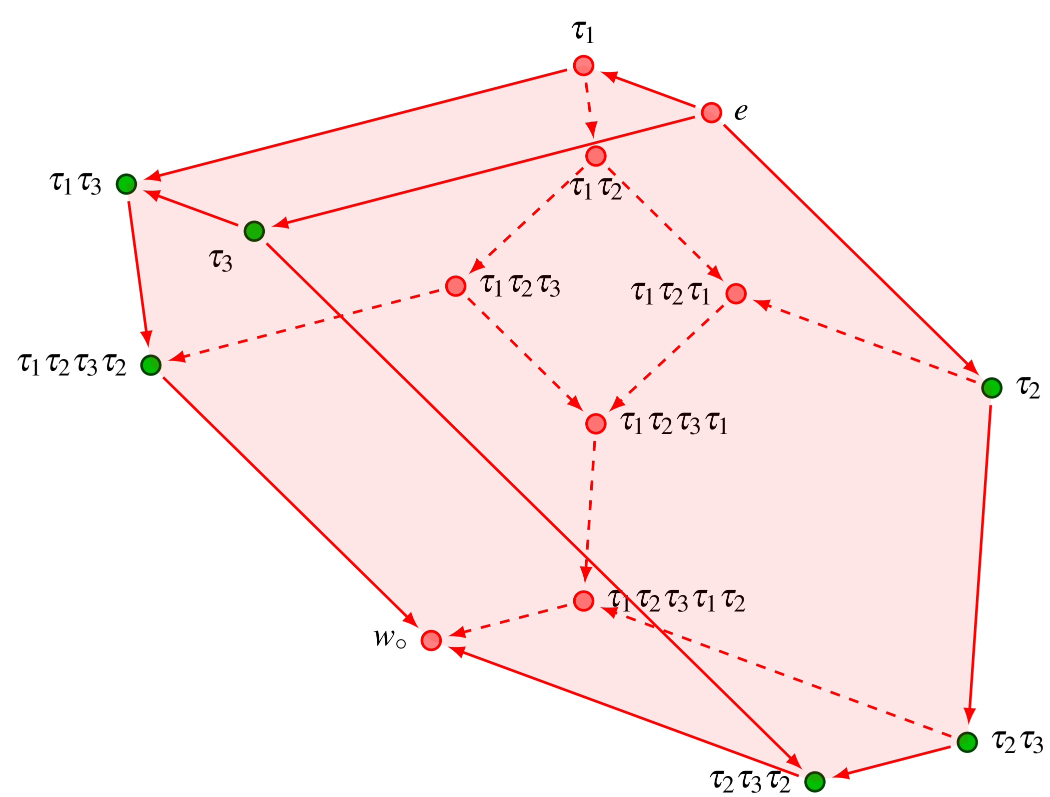
$$\mathrm{Asso}_c^{\boldsymbol{a}}(W)$$

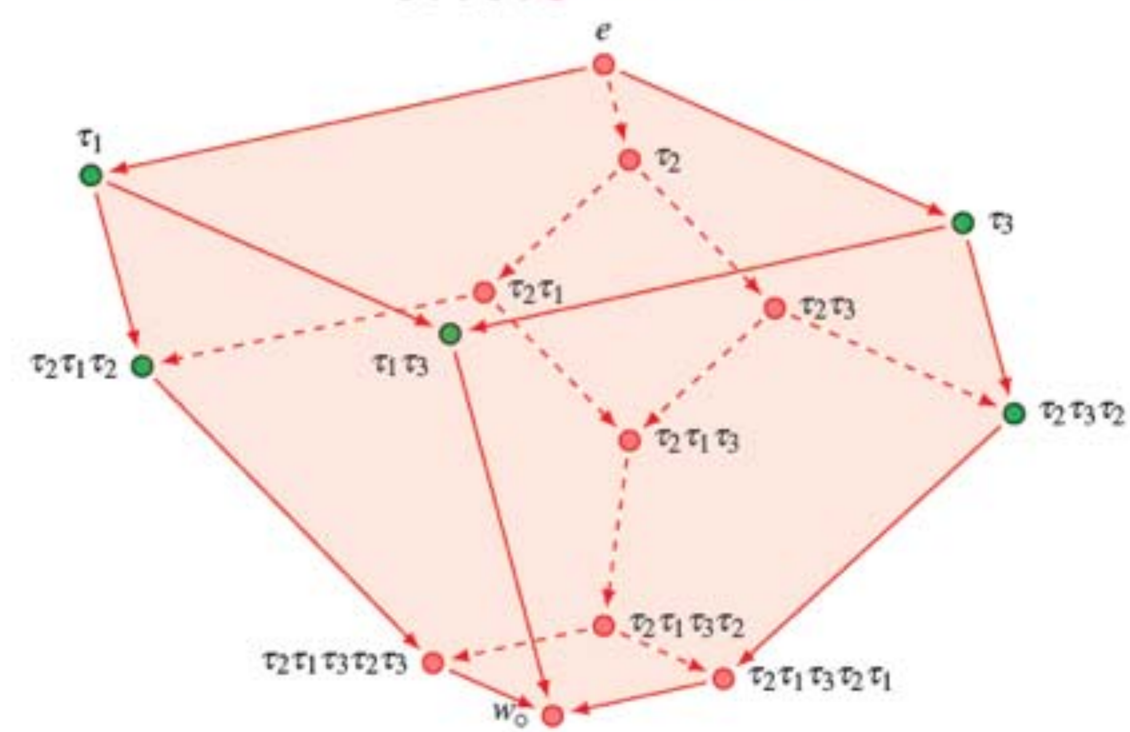
is N. Reading's c- Cambrian lattice; its normal fan is the corresponding Cambrian fan studied in detailed by N. Reading & D. Speyer. The facets are labelled by almost positive roots

Type A



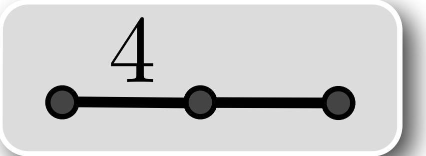






Christophe Hohlweg, 2013

Type B



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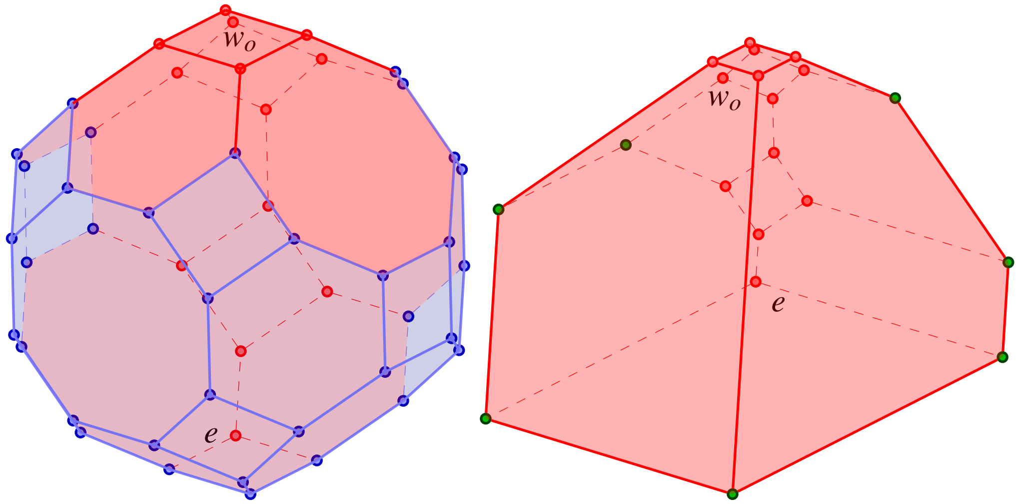
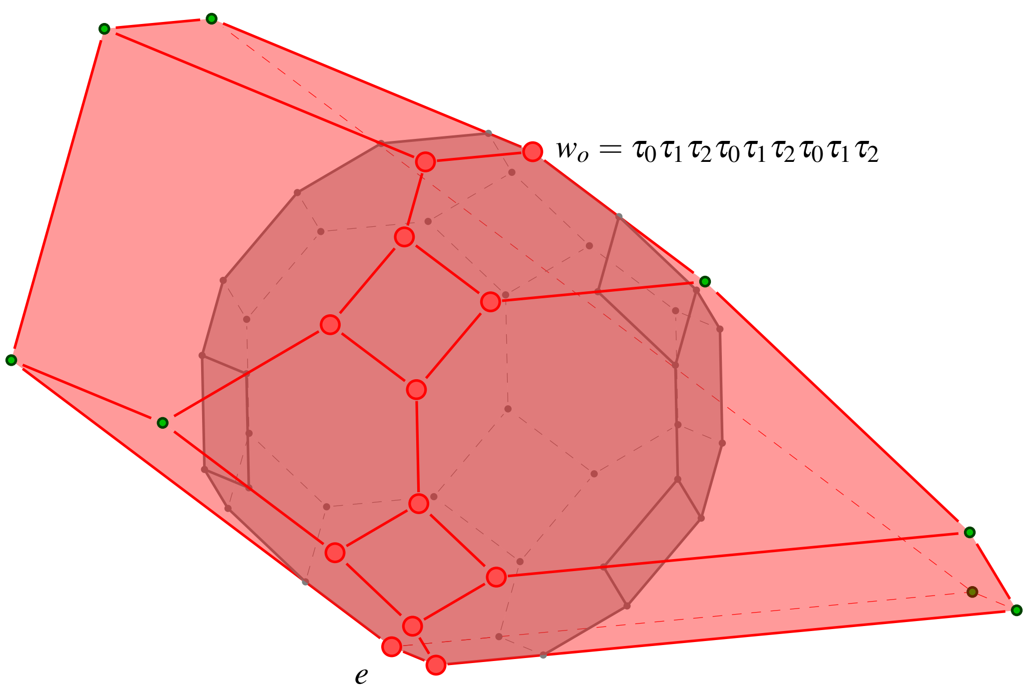
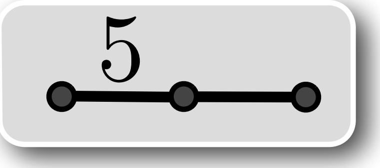


w_o
e
w_o
e



$w_o = \tau_0\tau_1\tau_2\tau_0\tau_1\tau_2\tau_0\tau_1\tau_2$
e

Type H



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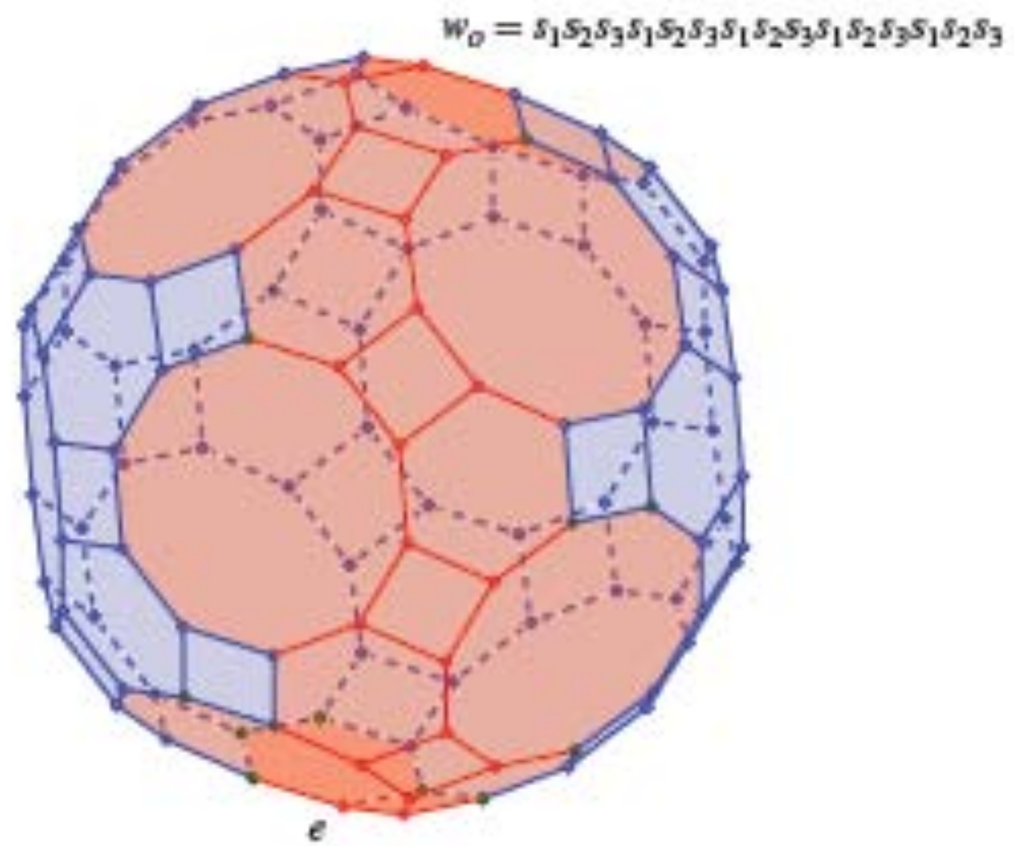
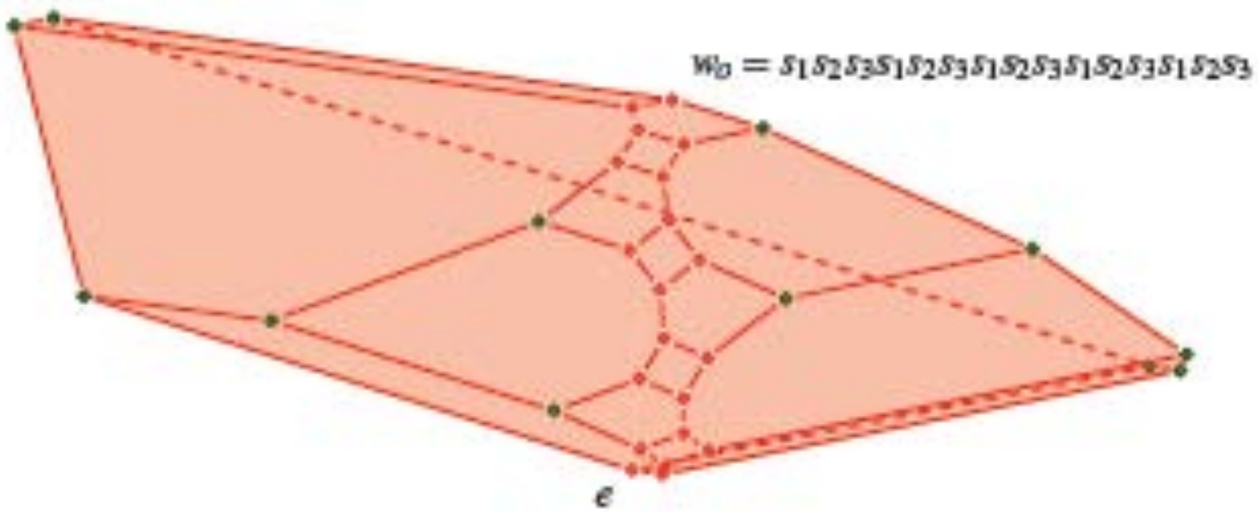


$w_o = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3$
e



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e

Selected developements on the subject

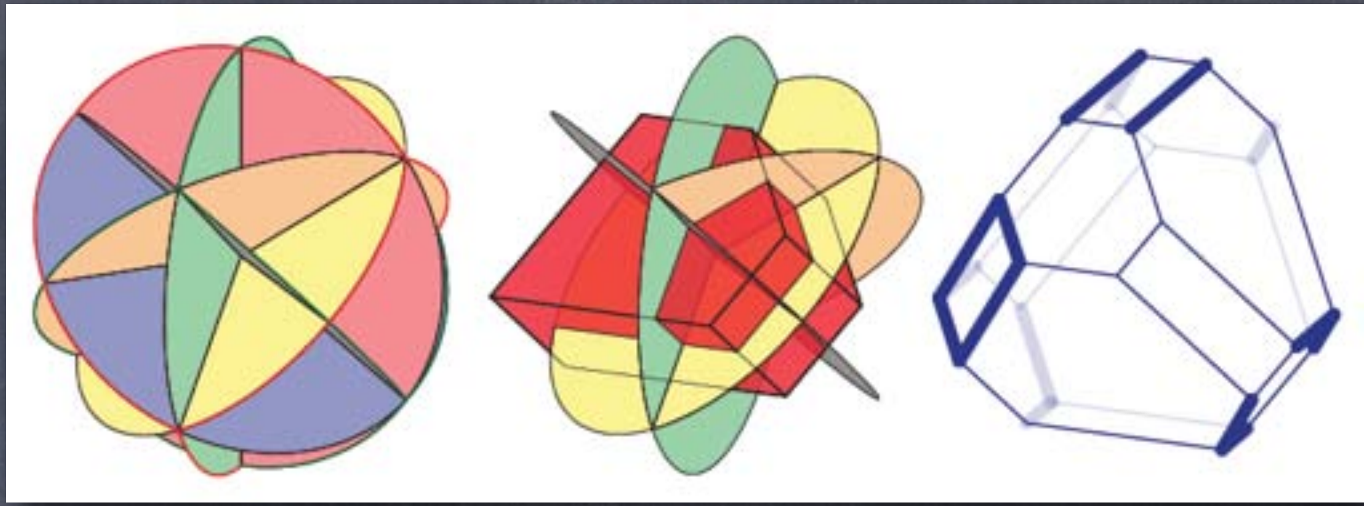
- Convex hull of the vertices: brick polytopes. Barycenter identical to the permutahedron:

©Pilaud-Stump

V. Pilaud and C. Stump:

1. Brick polytopes of spherical subword complexes: A new approach to generalized associahedra (2012)

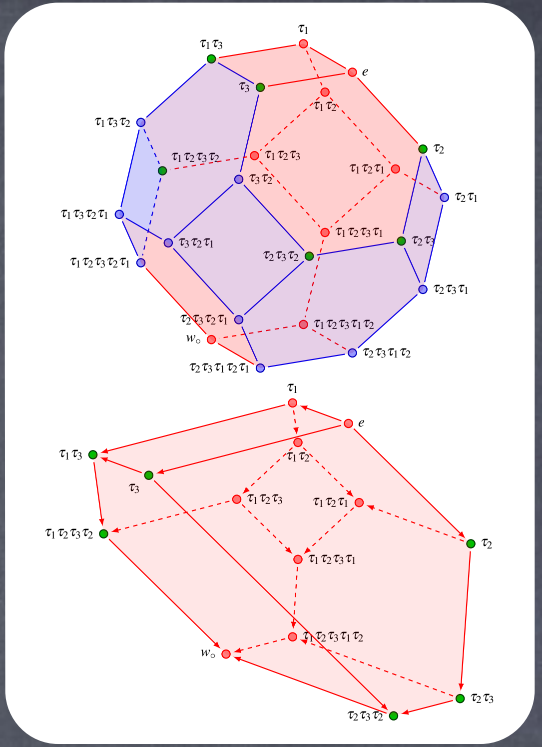
2. Vertex barycenter of generalized associahedra (2012)



- Classification of isometry classes in term of the lattices of $c-$singletons (N. Bergeron, Hohlweg, C. Lange, H. Thomas, 2009)
- Recovering the corresponding cluster algebra:

S. Stella, Polyhedral models for generalized associahedra via Coxeter elements (2013)

Weak order: a combinatorial model



Cambrian (semi)lattices/fans in finite case &
Generalized associahedra in finite case

Initial section of reflection orders and KL-polynomials (M. Dyer):
combinatorial formulas for KL-polynomials (F. Brenti, M. Dyer).

A combinatorial model for cambrian lattices/generalized associahedra in infinite case, or twisted Bruhat order and KL-polynomials (M. Dyer)? Is it possible to «enlarge» Coxeter groups to have a weak order that is a complete lattice?

Weak order and root system

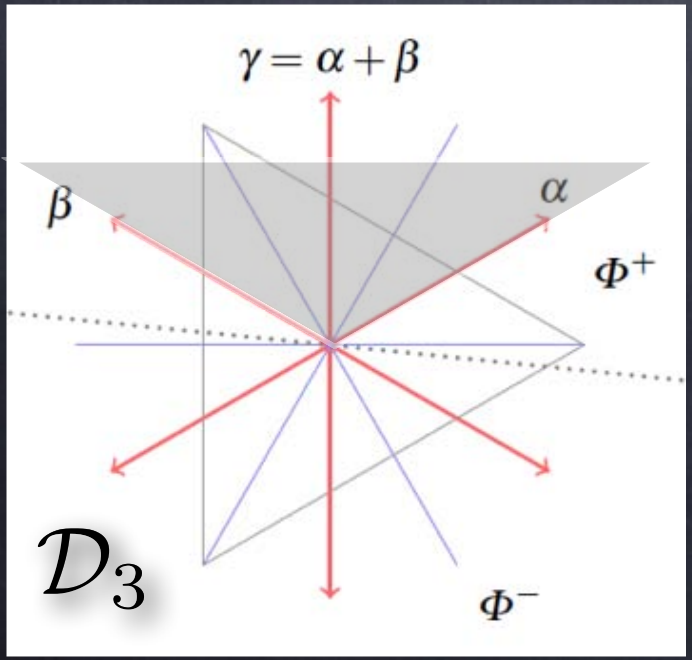
Geometric representations of (W, S)

- (V, B) real quadratic space and $\Delta \subseteq V$ s.t.
  - $\mathrm{cone}(\Delta) \cap \mathrm{cone}(-\Delta) = \{0\}$;
  - $\Delta = \{\alpha_s \,|\, s \in S\}$ s.t.

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) & \text{if } m_{st} < \infty \\ a \leq -1 & \text{if } m_{st} = \infty \end{cases}$$

- $W \leq \mathrm{O}_B(V)$: $\quad s(v) = v - 2B(v, \alpha)\alpha, \;\; s \in S$

Root system: $\Phi = W(\Delta), \;\; \Phi^+ = \mathrm{cone}(\Delta) \cap \Phi = -\Phi^-$



	e	$s = s_\alpha$	$t = s_\beta$	st	ts	$sts = tst$
ℓ	0	1	1	2	2	3
α	α					
β	β					
γ	γ					

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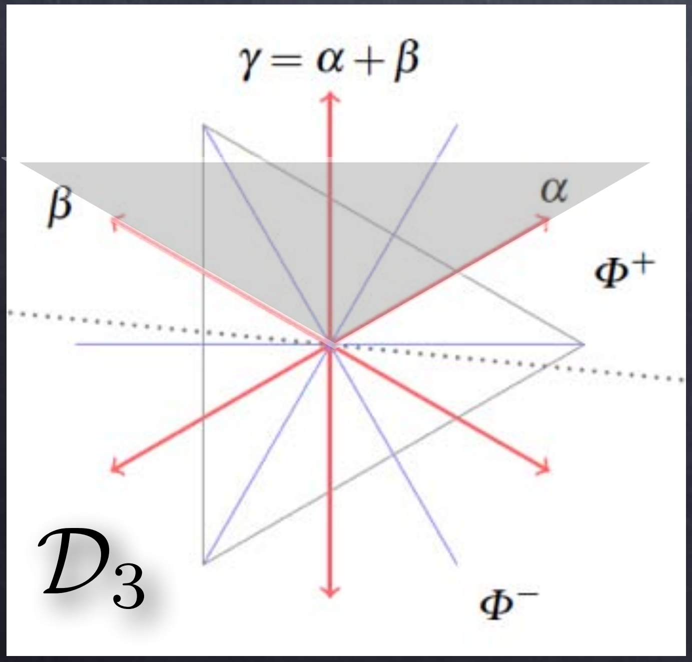

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α	α	$-\alpha$	γ			
β	β	γ	$-\beta$			
γ	γ	β	α			

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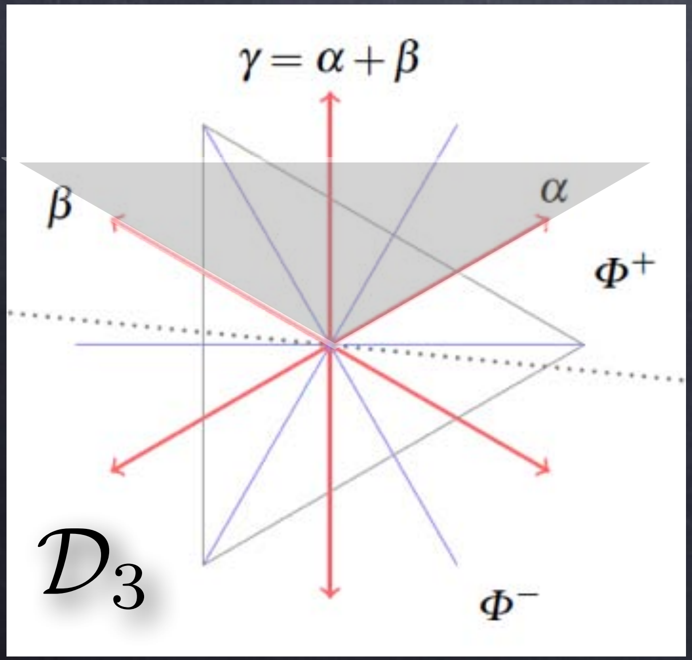

	e	$s = s_\alpha$	$t = s_\beta$	st	ts	$sts = tst$
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β	β	γ	$-\beta$	$-\gamma$	α	$-\alpha$
γ	γ	β	α	$-\alpha$	$-\beta$	$-\gamma$

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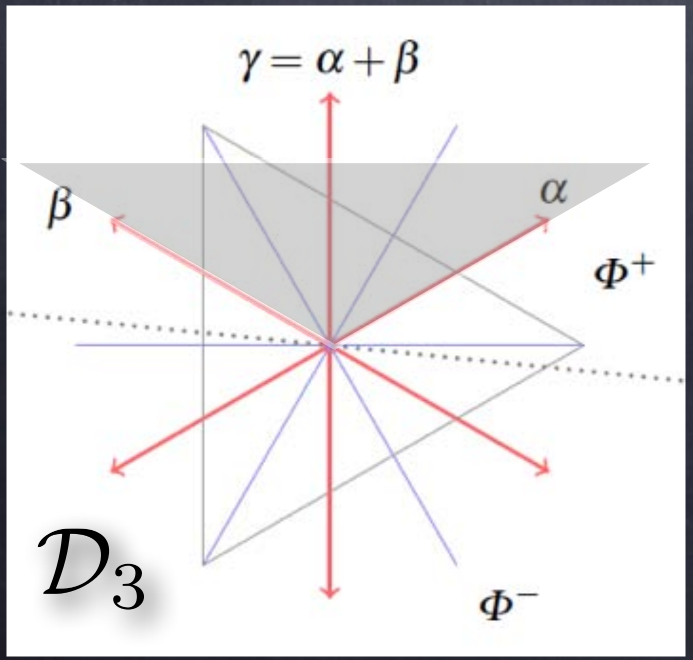

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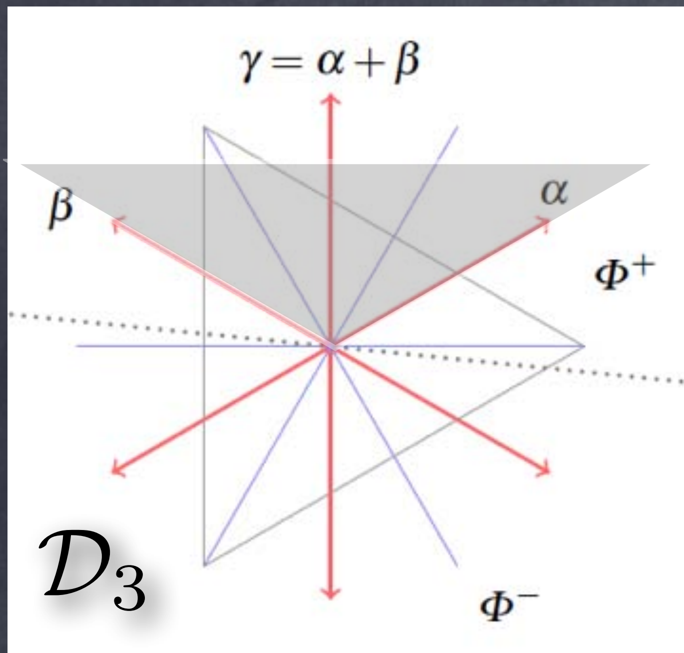
β	β	γ	$-\beta$	$-\gamma$	α	$-\alpha$
γ	γ	β	α	$-\alpha$	$-\beta$	$-\gamma$

$$\ell(w) = |\{\nu \in \Phi^+ \,|\, w(\nu) \in \Phi^-\}|$$

Weak order and root system



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β	β	γ	$-\beta$	$-\gamma$	α	$-\alpha$
γ	γ	β	α	$-\alpha$	$-\beta$	$-\gamma$

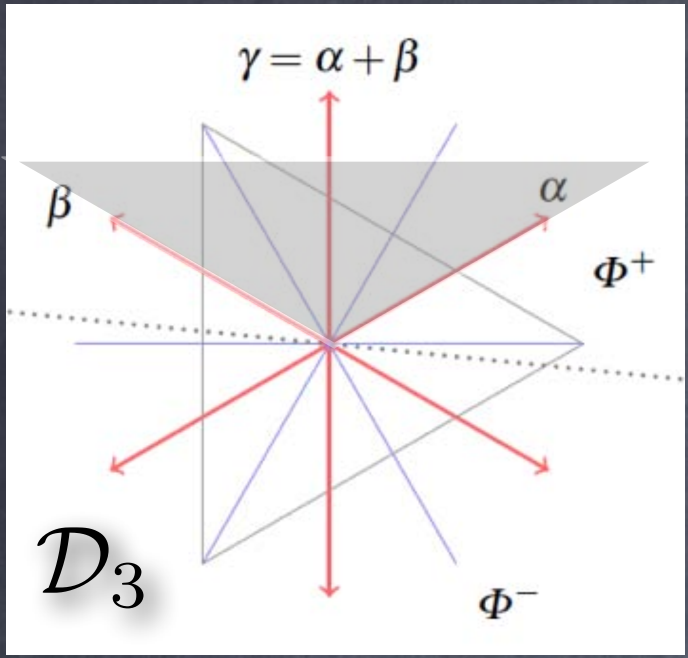
Definition. The inversion set of $w \in W$ is

$$\mathrm{inv}(w) = \Phi^+ \cap w^{-1}(\Phi^-) = \{\nu \in \Phi^+ \,|\, w(\nu) \in \Phi^-\}$$

- If $W = S_n$ then those "are" the natural inversion.

$$\mathrm{inv}(\sigma) = \{e_j - e_i \,|\, 1 \leq i < j \leq n,\ e_{\sigma(j)} - e_{\sigma(i)} \in \Phi^-\}$$

Weak order and root system



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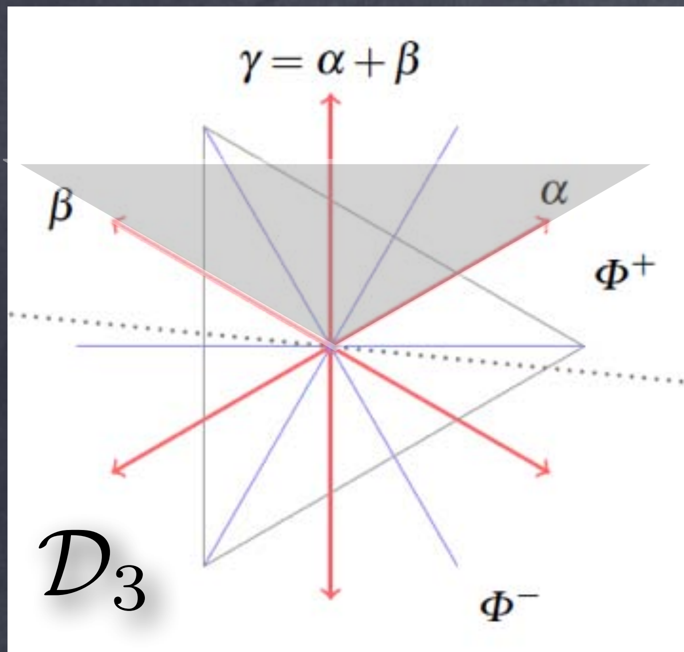
Weak order and root system



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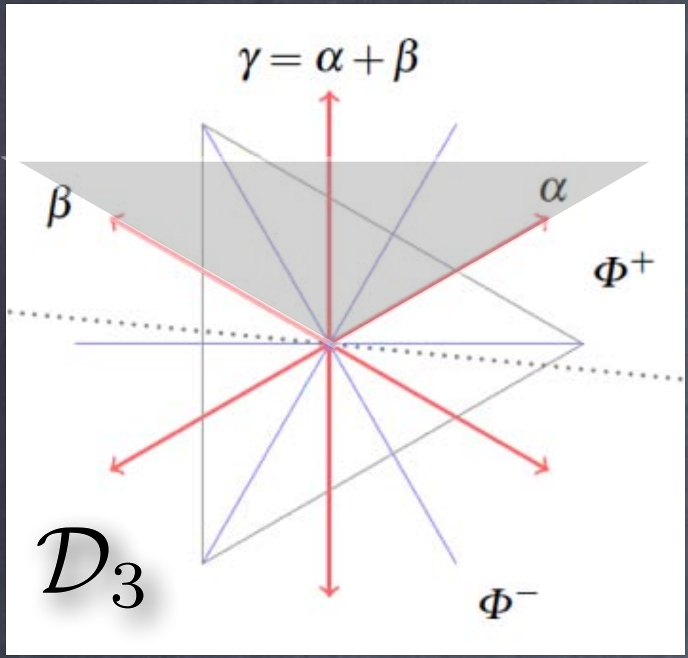
$$\mathrm{inv}(\sigma) \simeq \{(i,j) \mid 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}$$

Christophe Hohlweg, 2014

Weak order and root system



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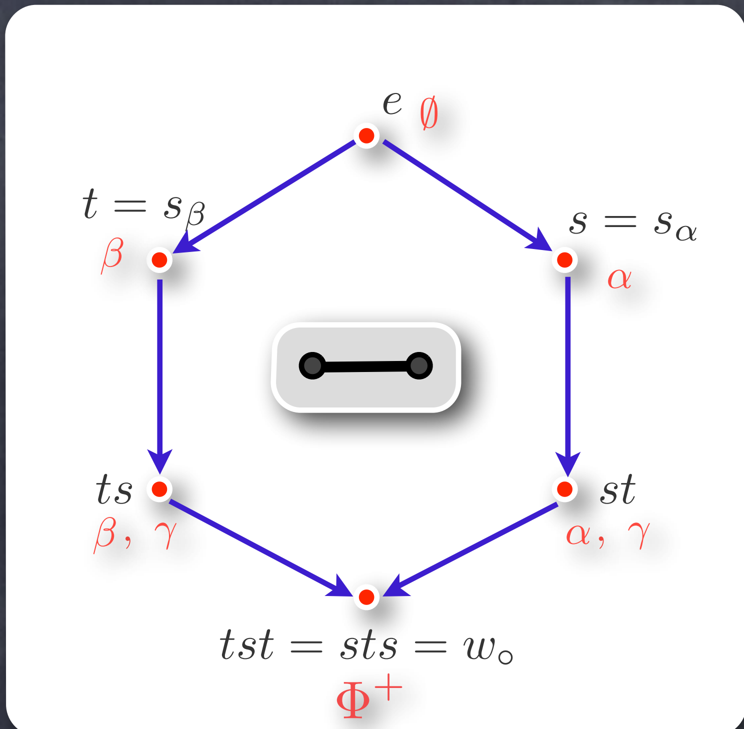
Proposition. Let $w = s_1 s_2 \dots s_k$ be a reduced word, then:

$$N(w) := \mathrm{inv}(w^{-1}) = \{\alpha_1, s_1(\alpha_2), \cdots, s_1 \dots s_{k-1}(\alpha_k)\}$$

In particular: $|N(w)| = |\mathrm{inv}(w)| = \ell(w)$

Weak order and root system



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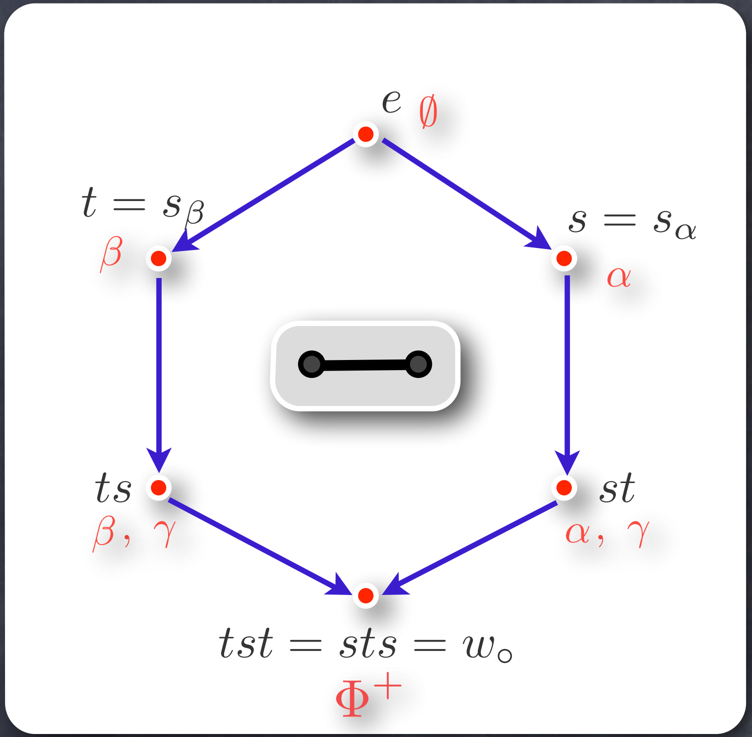

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β	β	γ	$-\beta$	$-\gamma$	α	$-\alpha$
γ	γ	β	α	$-\alpha$	$-\beta$	$-\gamma$

Definition. The **inversion set** of $w \in W$ is

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In particular: $|N(w)| = |\mathrm{inv}(w)| = \ell(w)$

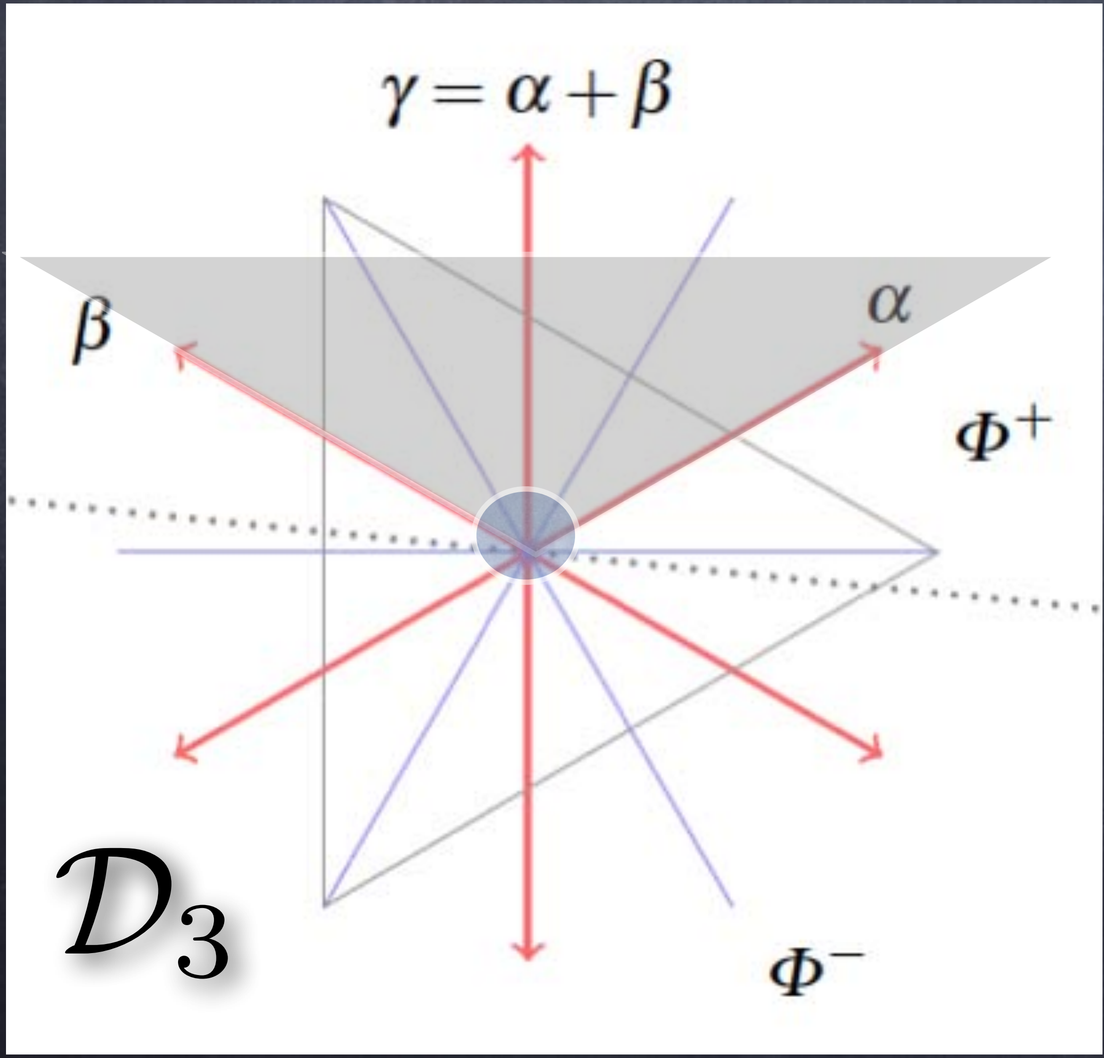
What is $\mathrm{Im}(N)$?

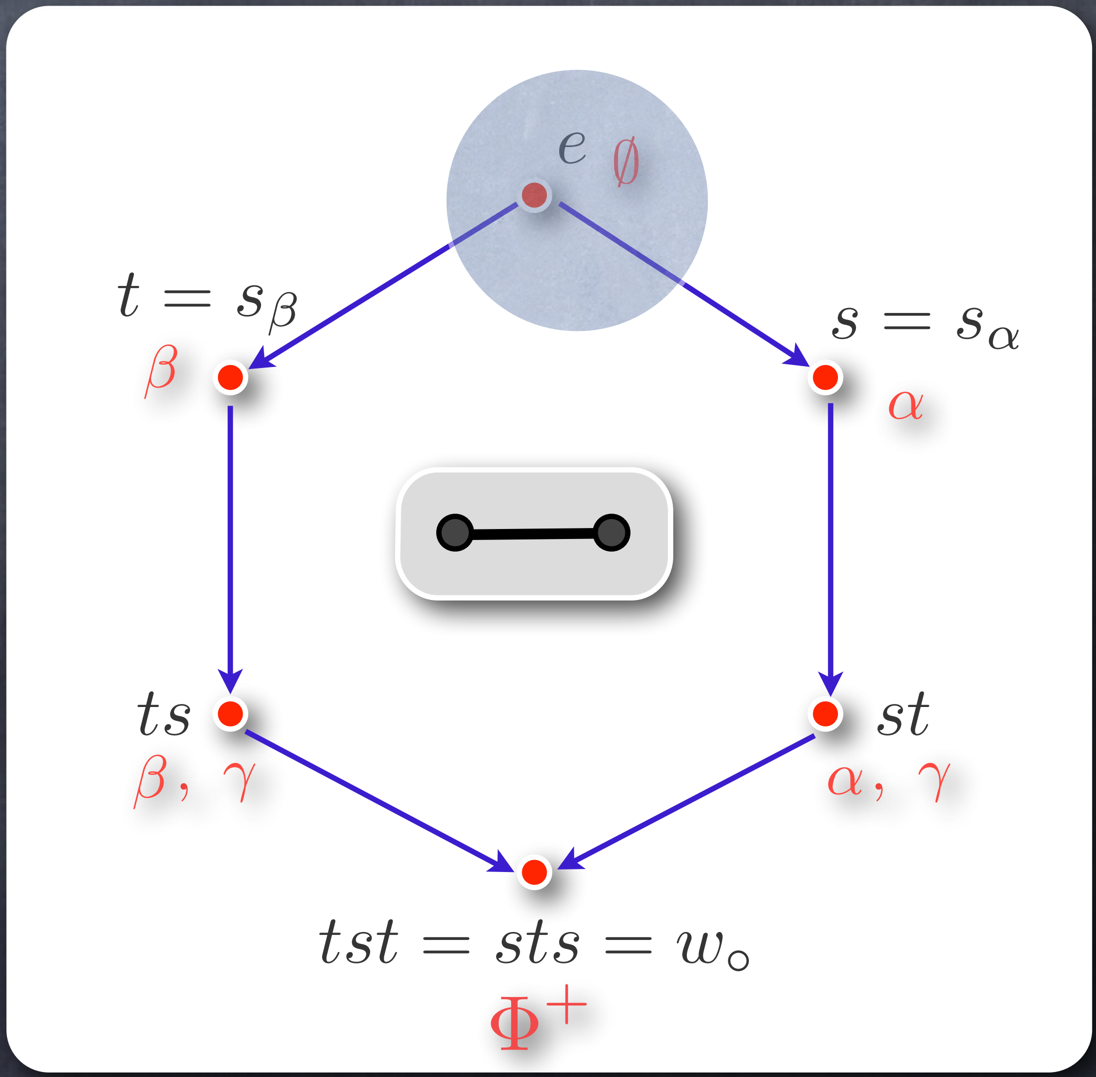
Proposition. The map $N : (W, \leq) \to (\mathcal{P}(\Phi^+), \subseteq)$ is an injective morphism of posets.

Weak order and biclosed sets

- $A \subseteq \Phi^+$ is closed if for all $\alpha, \beta \in A$, $\mathrm{cone}(\alpha,\beta) \cap \Phi \subseteq A$;
- $A \subseteq \Phi^+$ is biclosed if A, $A^c := \Phi^+ \setminus A$ are closed.
- $\mathcal{B}(W) = \{\text{biclosed sets}\}$; $\mathcal{B}_0(W) = \{A \subseteq \mathcal{B}(W) \mid |A| < \infty\}$

Proposition. $N : (W, \leq) \to (\mathcal{B}_0(W), \subseteq)$ is a poset isomorphism and $N(w_\circ) = \Phi^+$ if W is finite.





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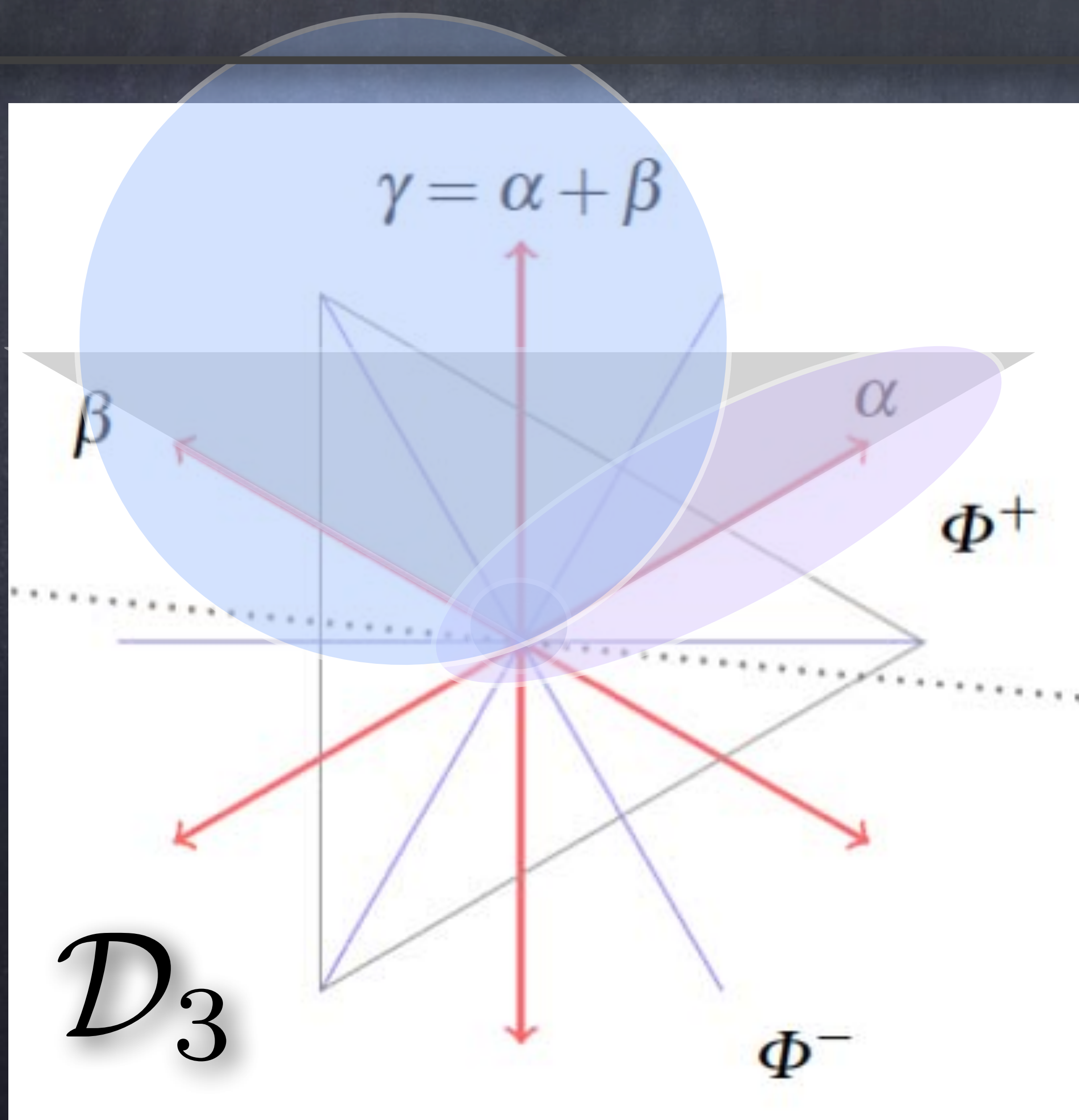

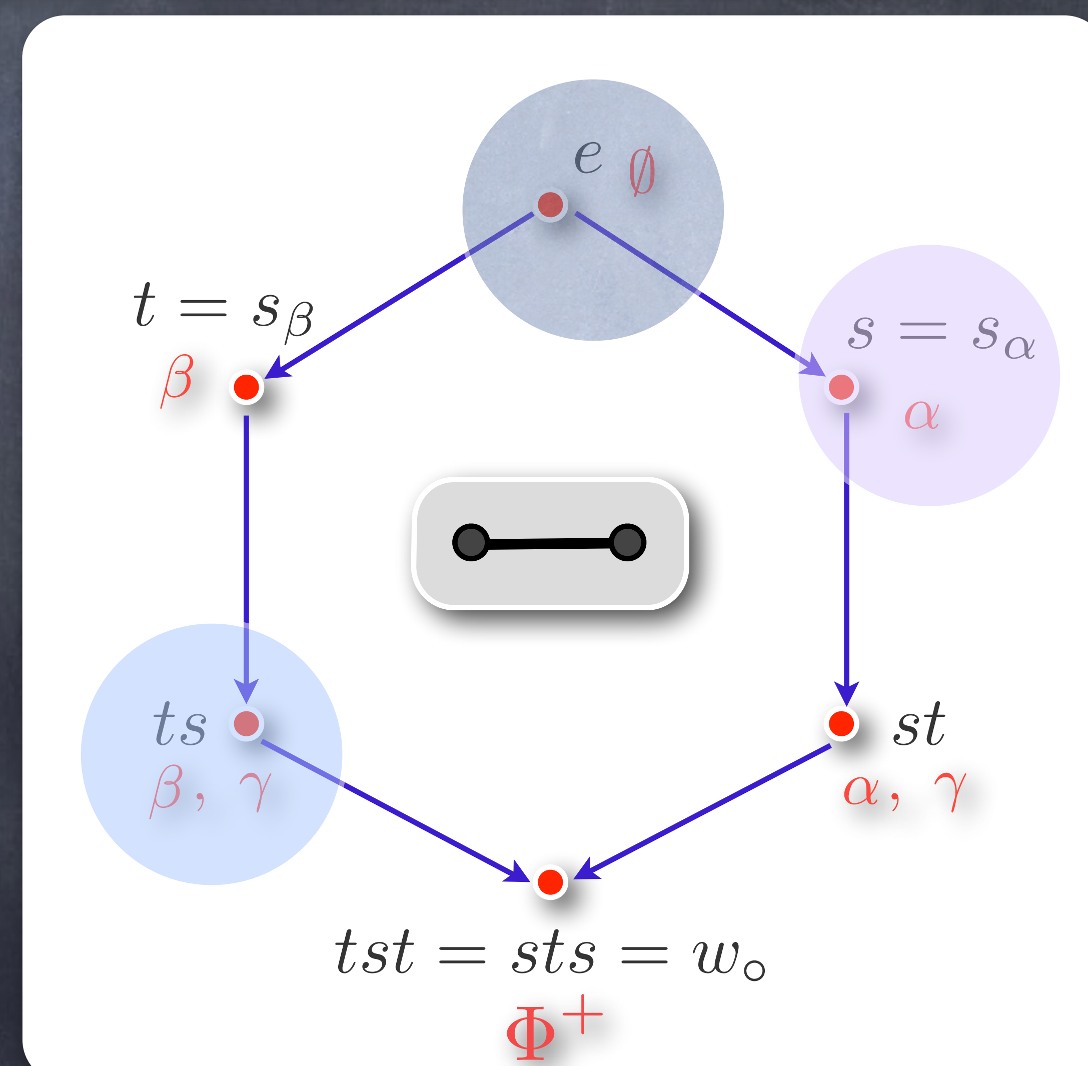

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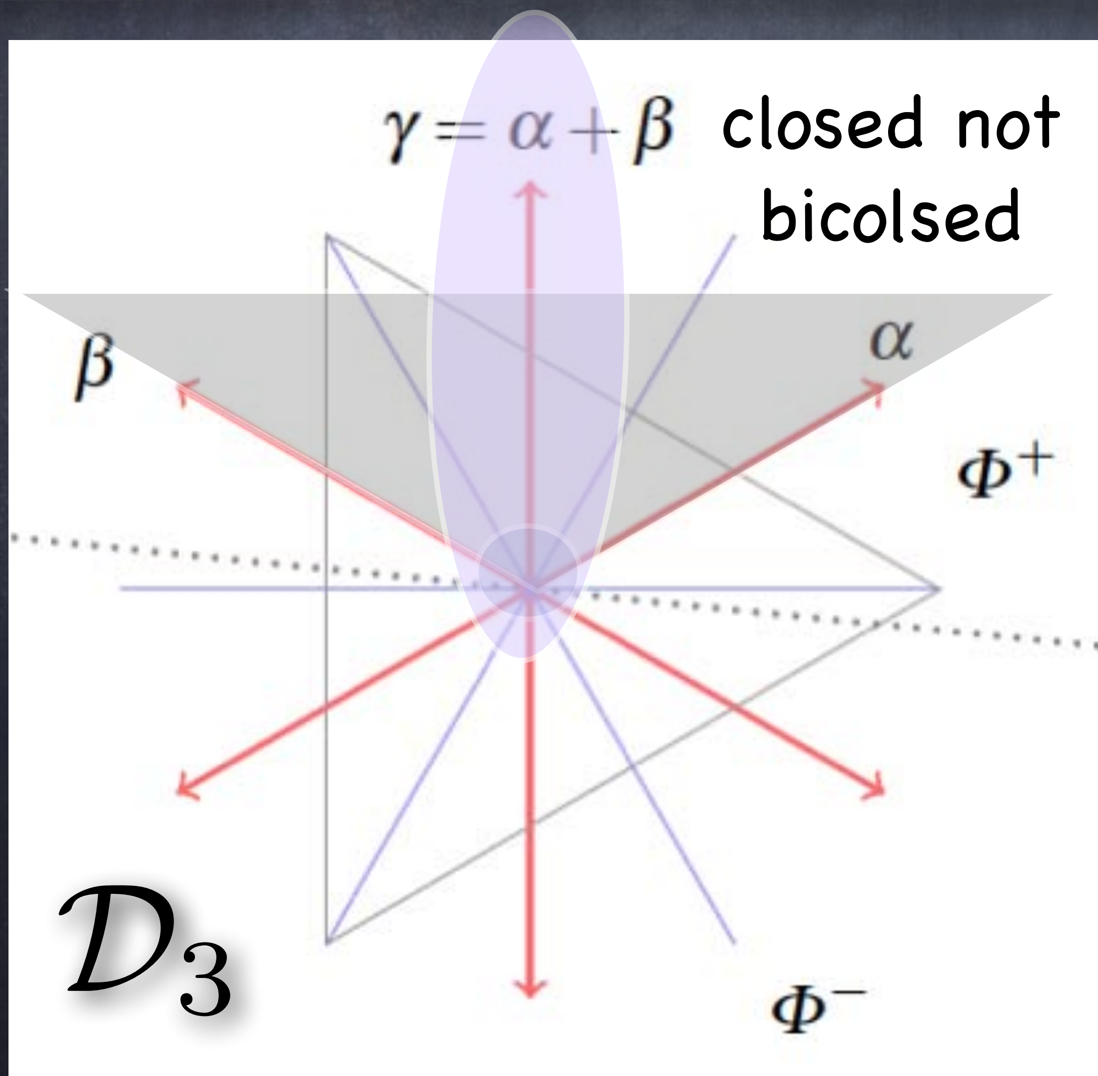

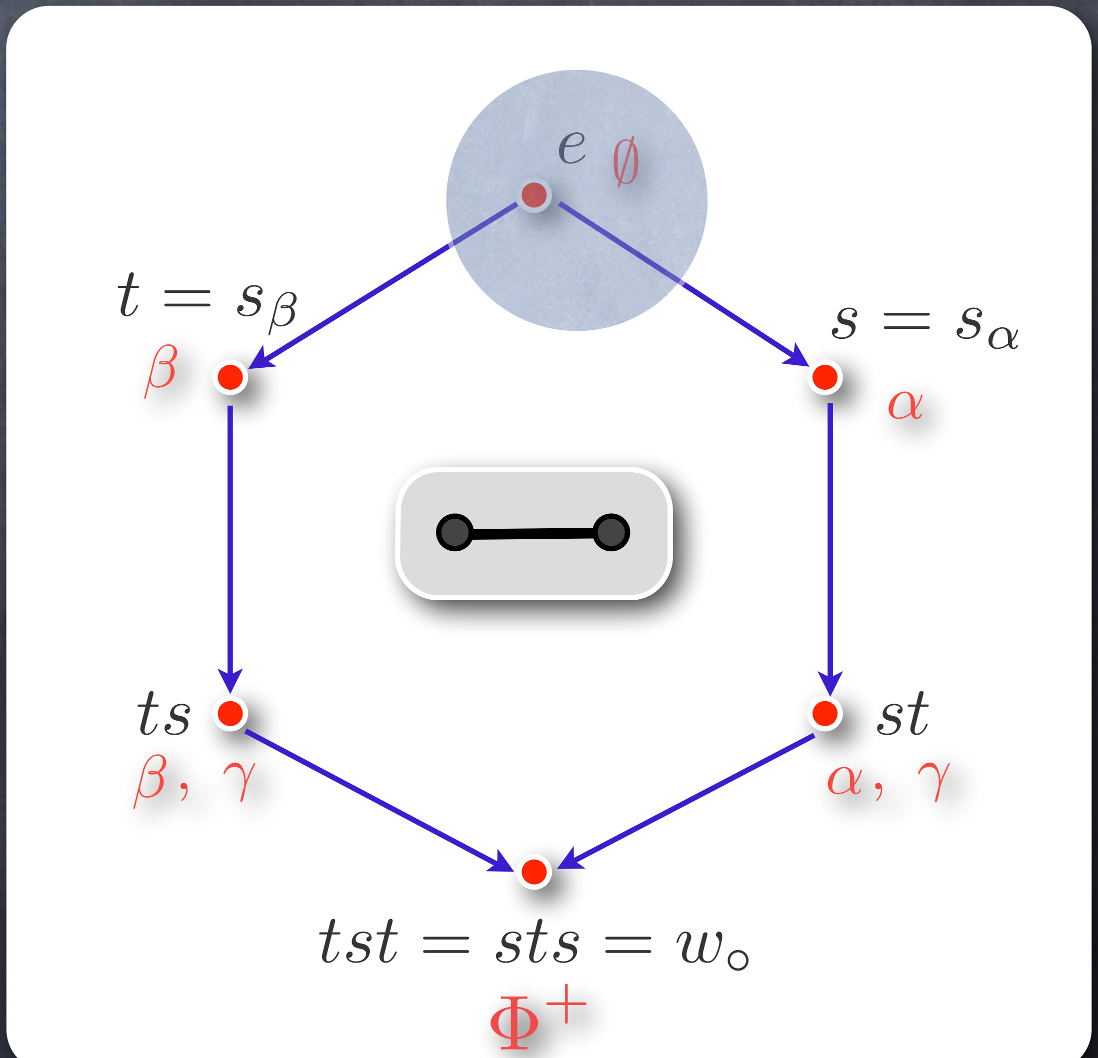

Weak order and biclosed sets

- $A \subseteq \Phi^+$ is **closed** if for all $\alpha, \beta \in A$, $\operatorname{cone}(\alpha,\beta) \cap \Phi \subseteq A$;
- $A \subseteq \Phi^+$ is **biclosed** if A, $A^c := \Phi^+ \setminus A$ are closed.
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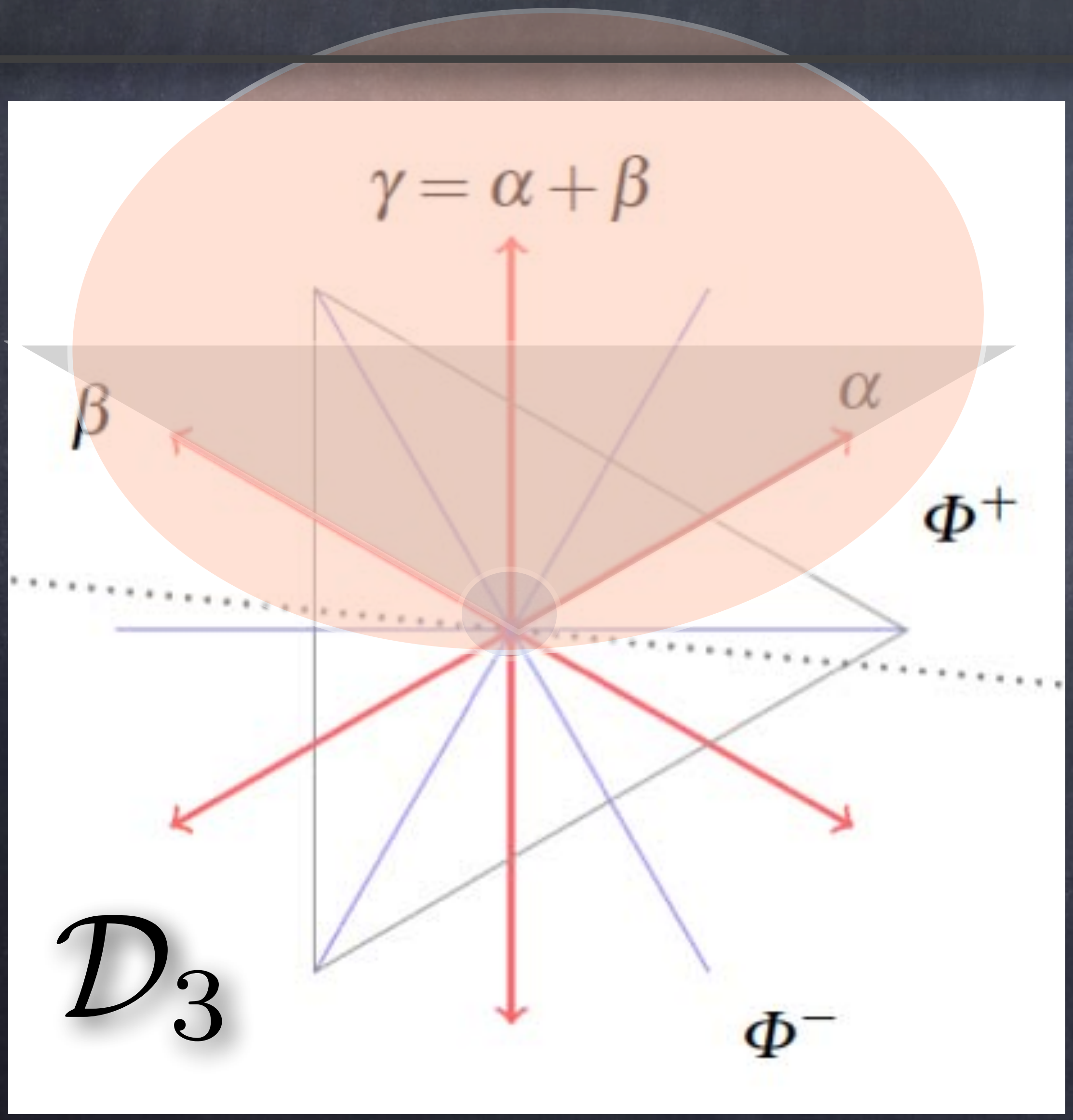

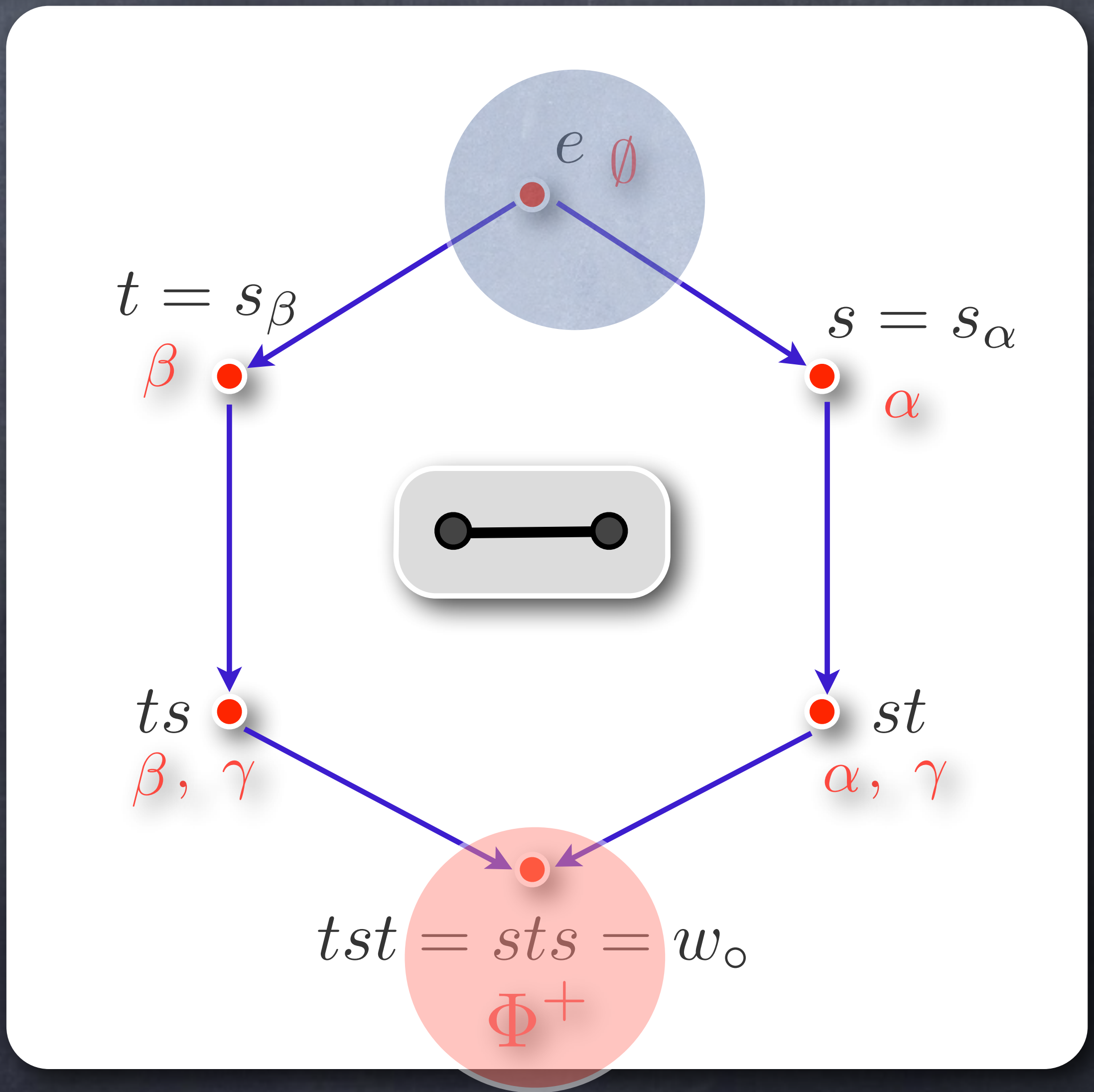

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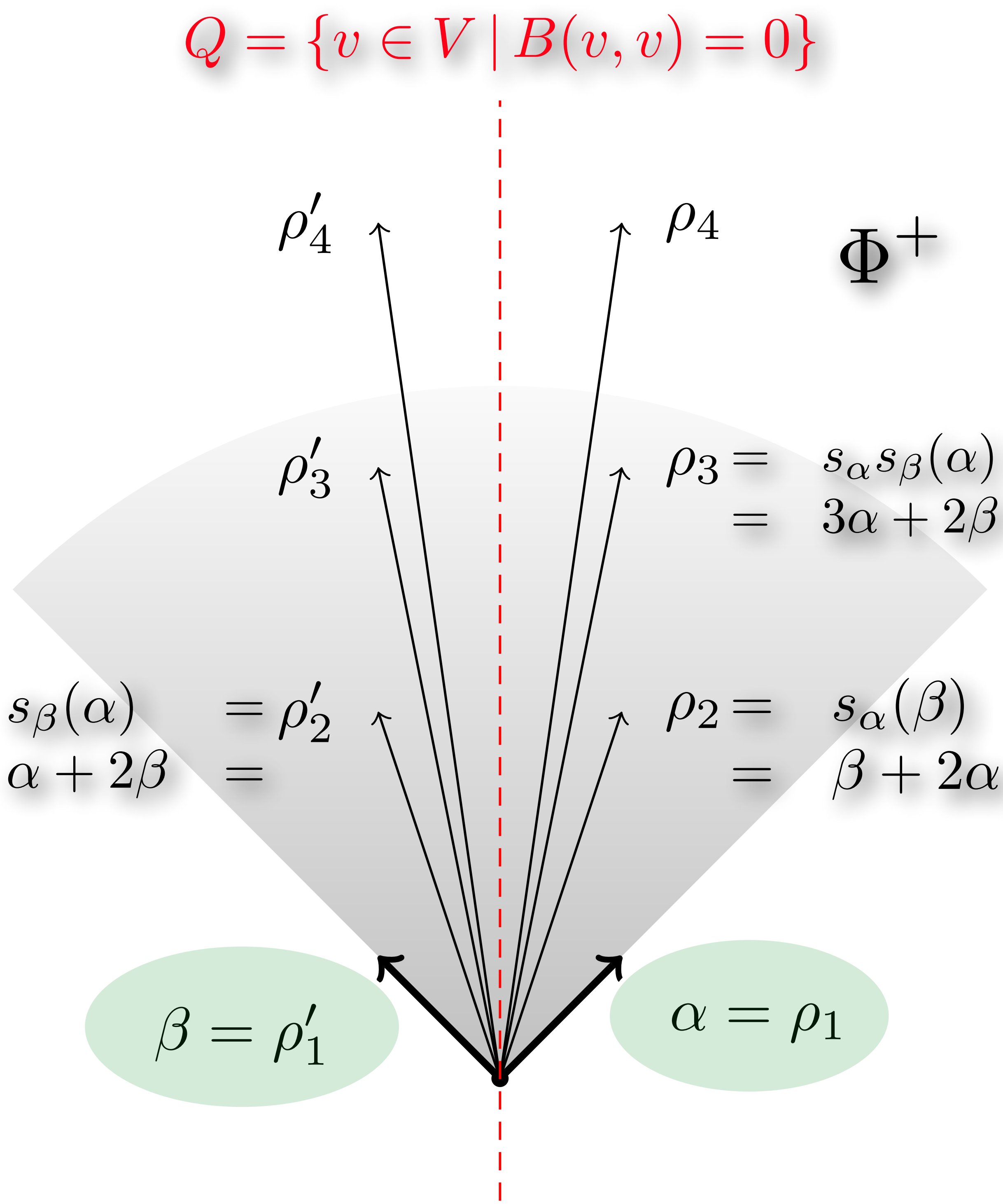
Inverse map (recursive construction)

$\exists \alpha \in \Delta \cap A$, $s_\alpha(A \setminus \{\alpha\})$ is finite biclosed and

$$A = \{\alpha\} \sqcup s_\alpha(A \setminus \{\alpha\})$$

$$w_A = s_\alpha w_{s_\alpha(A\setminus\{\alpha\})}$$

$\rho'_n = n\alpha + (n+1)\beta \qquad \rho_n = (n+1)\alpha + n\beta$

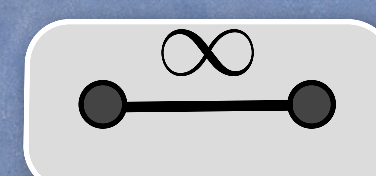


(a) $B(\alpha,\beta) = -1$

$$s_\alpha(v) = v - 2B(v,\alpha)\alpha.$$

The biclosed are:

- the finite ones;
- their complements;
- and two infinite ones: the left and right side of Q!



Weak order and biclosed sets

world of words

Chain property: if $u \leq w$ with $\ell(u) + 1 < \ell(w)$ then:

$$\exists v \in W,\ u \lneq v \lneq w$$

If W is finite, then:

(i) a unique $w_\circ \in W$ s.t

$$u \leq w_\circ,\ \forall u \in W$$

(ii) $w \mapsto w_\circ w$ is a poset antiautomorphism.

(iii) the weak order is a complete lattice.

(iv) $u \wedge v = w_\circ(w_\circ u \vee w_\circ v)$

$\overset{N}{\longleftrightarrow}$

world of roots

Chain property: if $A \subseteq B$ finite biclosed with $|B \setminus A| > 1$ then:

$$\exists C \in \mathcal{B}_0,\ A \subsetneq C \subsetneq B$$

If W is finite, then:

(i) $N(w_\circ) = \Phi^+$ and

$$A \subseteq \Phi^+,\ \forall A \in \mathcal{B} = \mathcal{B}_0$$

(ii) $A \mapsto A^c$ is a poset antiautomorphism.

(iii) the weak order is a complete lattice.

(iv) $A \wedge B = (A^c \vee B^c)^c$

Weak order and biclosed sets

Conjectures (M. Dyer, 2011).
(a) chain property: if $A \subseteq B$ are biclosed and $|B \setminus A| > 1$ then there is $C \in \mathcal{B}$ s.t. $A \subsetneq C \subsetneq B$.

(b) $(\mathcal{B}, \subseteq)$ is a complete lattice (with minimal element $\emptyset$ and maximal element Φ^+).

- $\vee \neq \cup$; $\wedge \neq \cap$ so how to understand them geometrically?
- if $\vee$ exists then

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world of roots

Chain property: if $A \subseteq B$ finite biclosed with $|B \setminus A| > 1$ then:

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(ii) $A \mapsto A^c$ is a poset antiautomorphism.
(iii) the weak order is a complete lattice.
(iv) $A \wedge B = (A^c \vee B^c)^c$

Weak order and Bruhat order

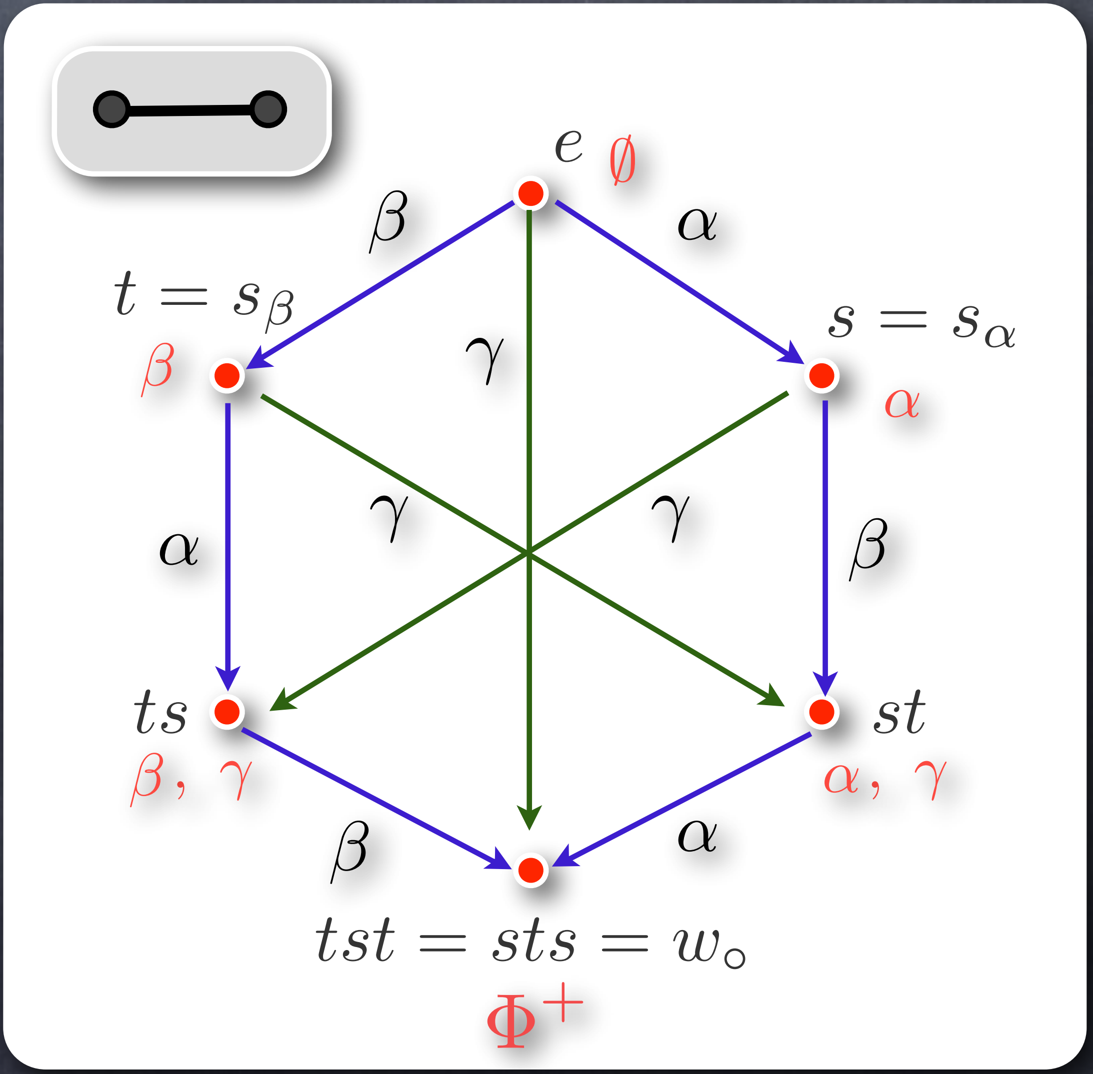
Set of reflections: $T = \bigcup_{w \in W} wSw^{-1} = \{s_\beta \,|\, \beta \in \Phi^+\}$

Bruhat order: transitive closure of $w \leq_B wt$ if $\ell(w) < \ell(wt)$

Bruhat graph of $W = \langle S \rangle$

- vertices W
- edges $w \xrightarrow{\beta} ws_\beta$

Weak order implies Bruhat order.



Weak order and Bruhat order

A- path: path starting with e in the Bruhat graph and indexed by elements in $A \cup B$.

Exemple. $A = \{\alpha, \gamma\}$:

$$e \to w_\circ = s_\gamma$$

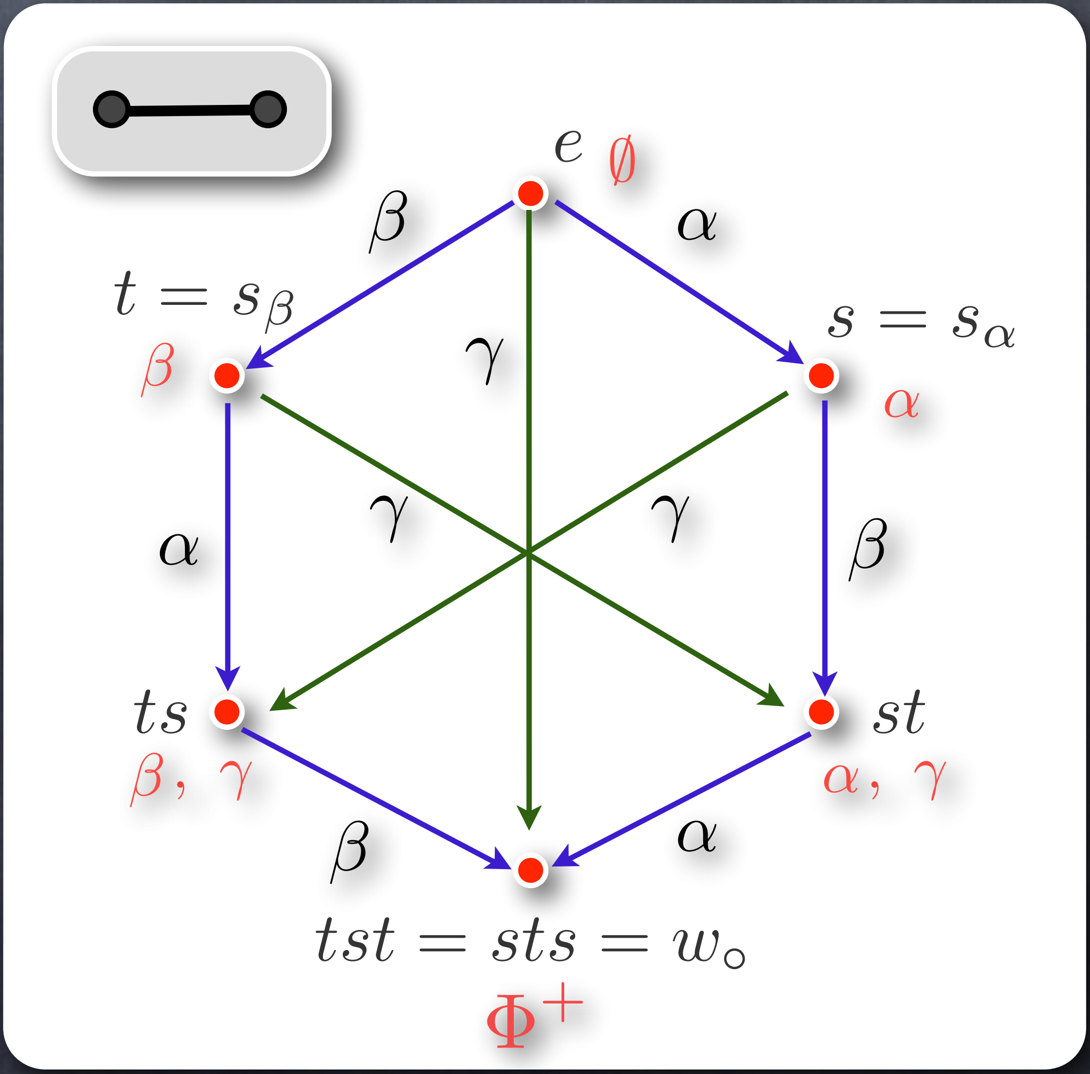
$$e \to s \to ts$$

B-closure of $A \subseteq \Phi^+$: $\overline{A} = \{\beta \in \Phi^+ \mid s_\beta \text{ is in a } A - \text{path}\}$

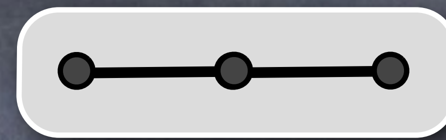
Conjecture (M. Dyer).
Let A, B be biclosed sets, then

$$A \vee B = \overline{A \cup B}$$

This conjecture is open even in finite cases!



Weak order and Bruhat order

Another example: (W, S) is 

$$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1 + \alpha_2\}; \quad s_{\alpha_1+\alpha_2} = \tau_1\tau_2\tau_1$$
$$B = N(\tau_3) = \{\alpha_3\}$$
$$A \cup B = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2\}$$

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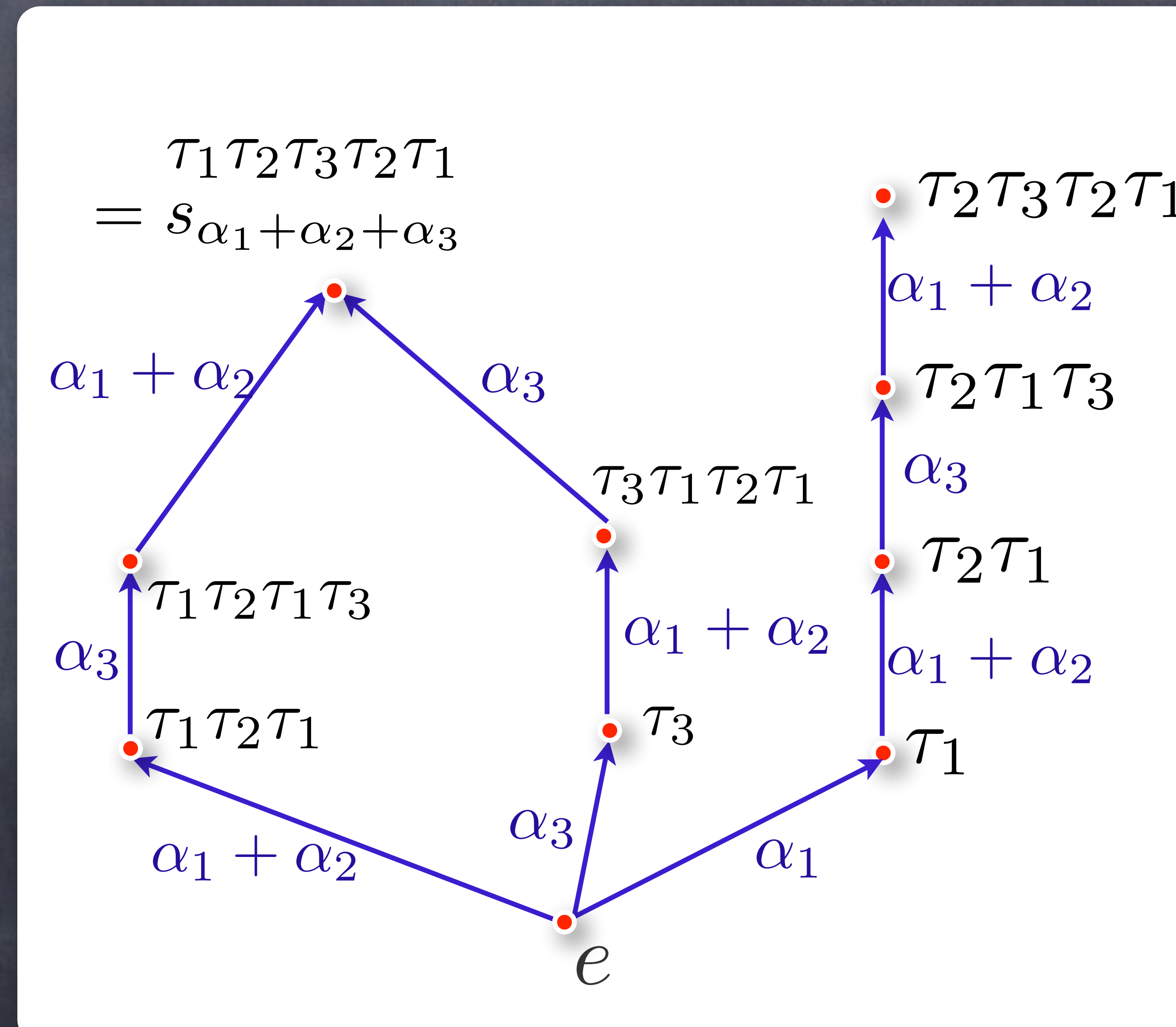
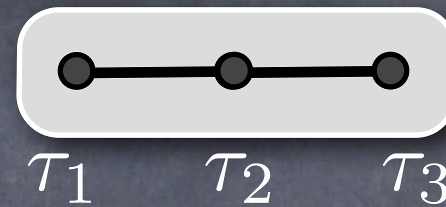


Graph of $A \cup B$ paths

Weak order and Bruhat order

Another example: (W,S) is 

$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1+\alpha_2\}; \quad s_{\alpha_1+\alpha_2} = \tau_1\tau_2\tau_1$

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$$\tau_1\tau_2 \vee \tau_3 = \tau_1\tau_3\tau_2\tau_3$$

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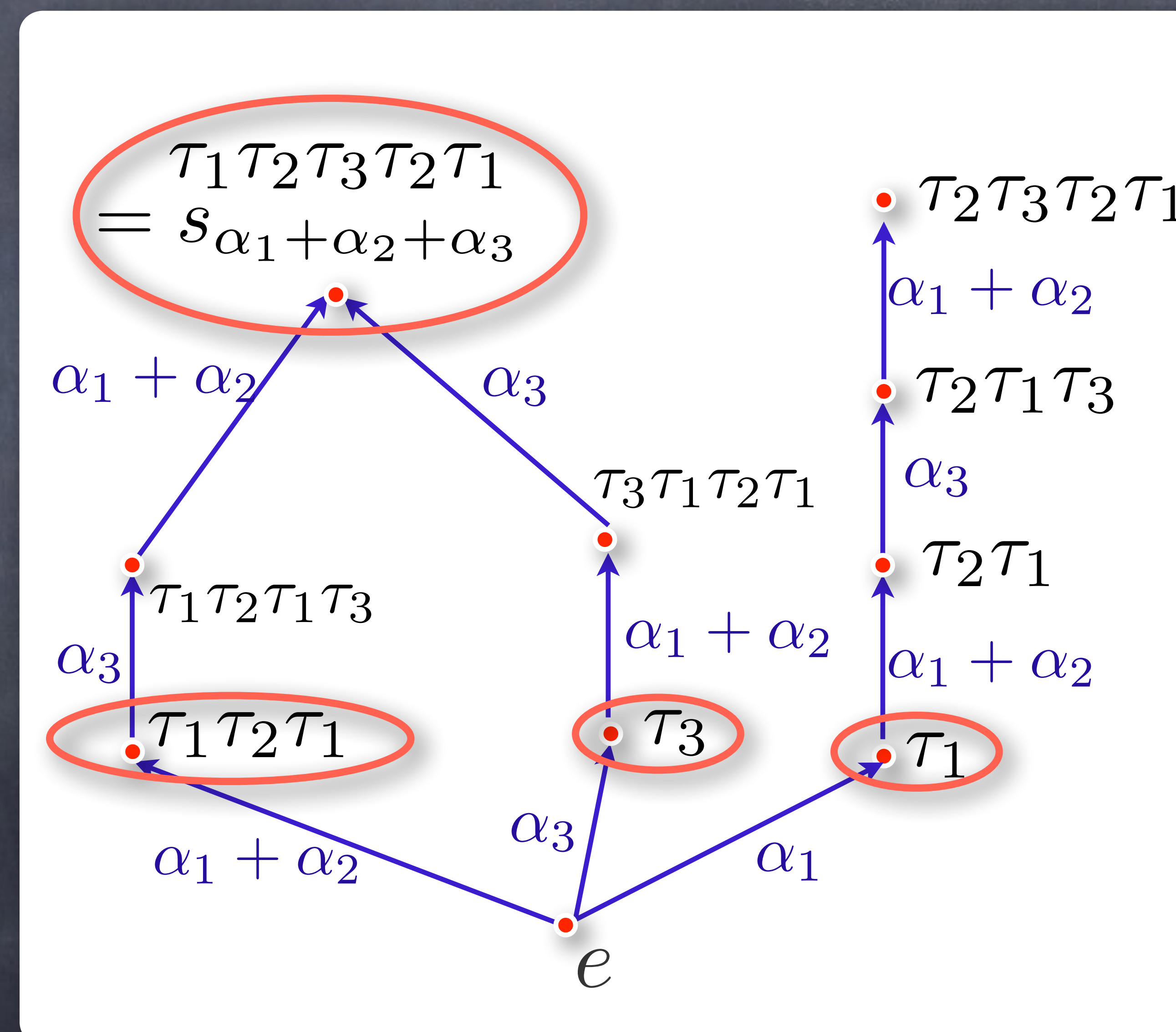


$$\overline{A \cup B} = \{\alpha_1, \alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\} = N(\tau_1\tau_3\tau_2\tau_3)$$

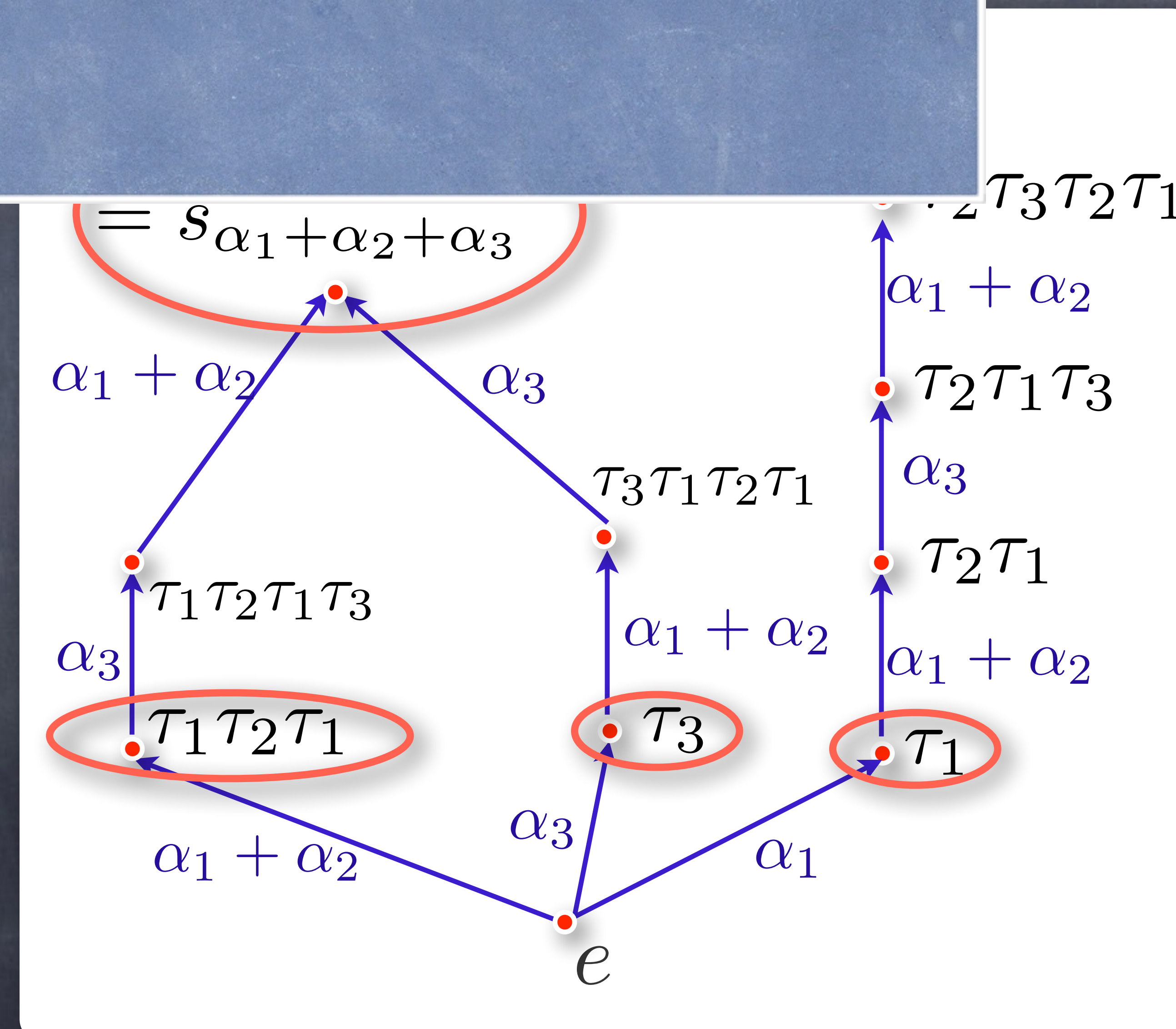
Ar

A

B

$A \cup$

Another way to interpret the join?



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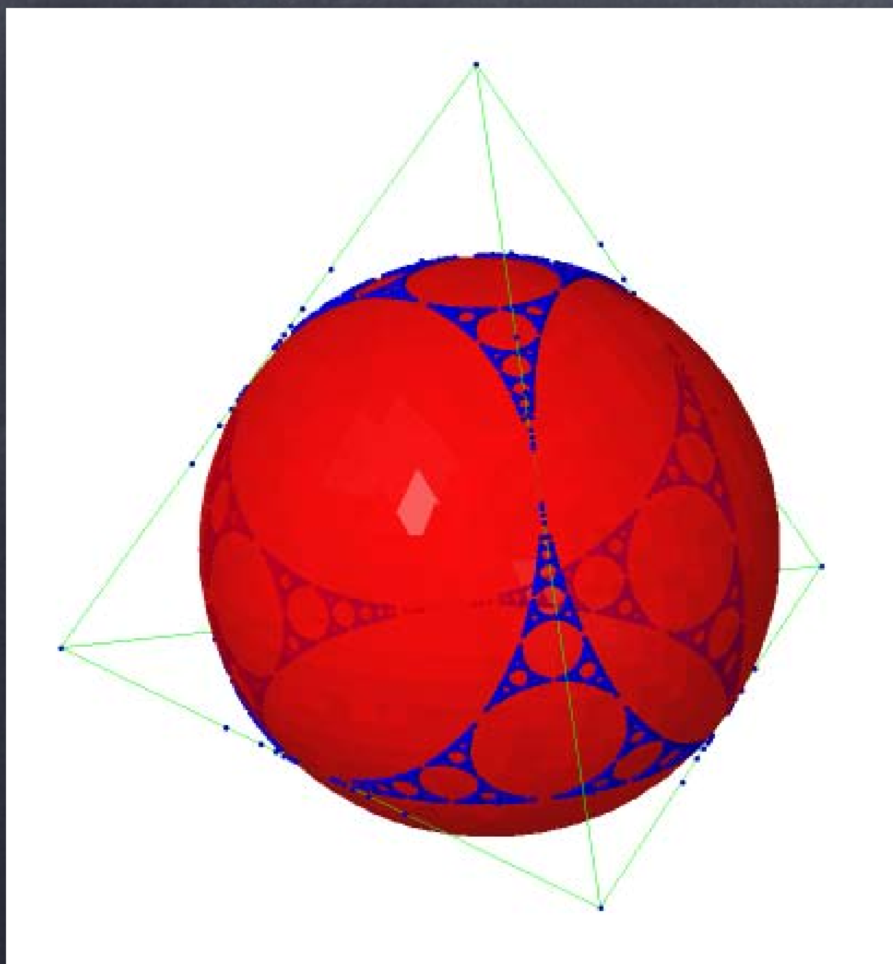
Another way to interpret the join?

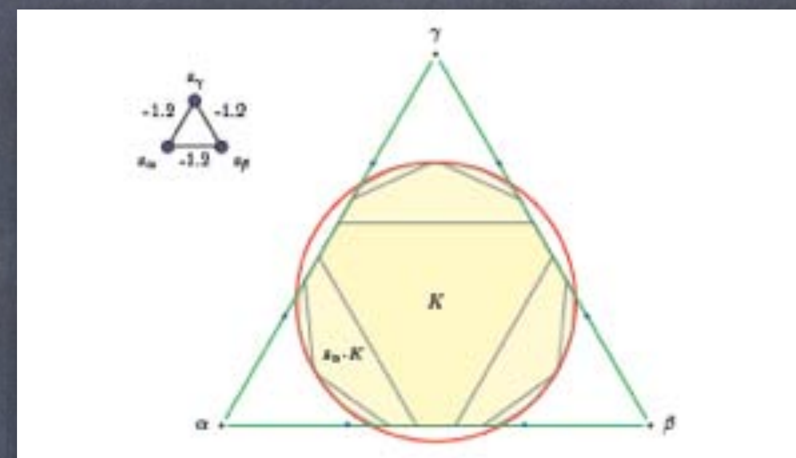
END of Part 2

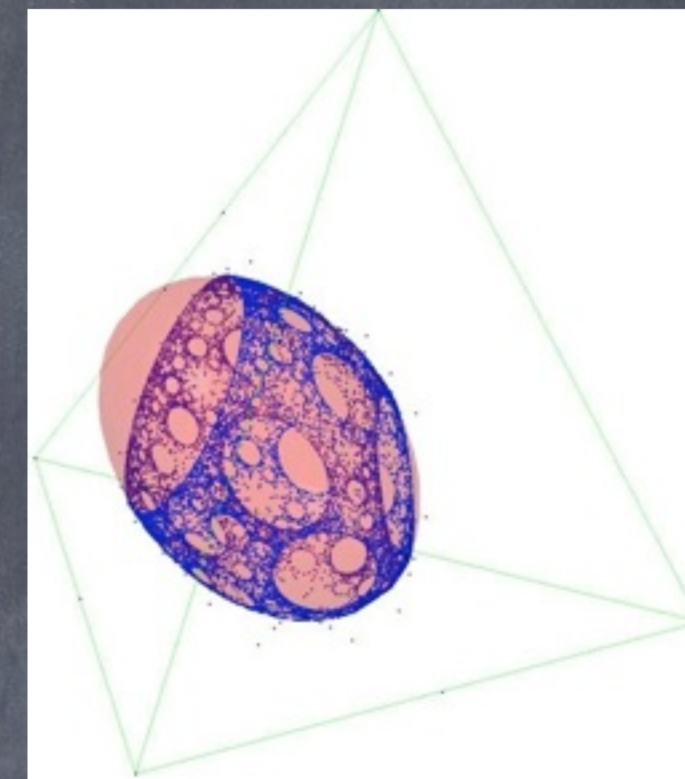
- to be continued in Part 3 -

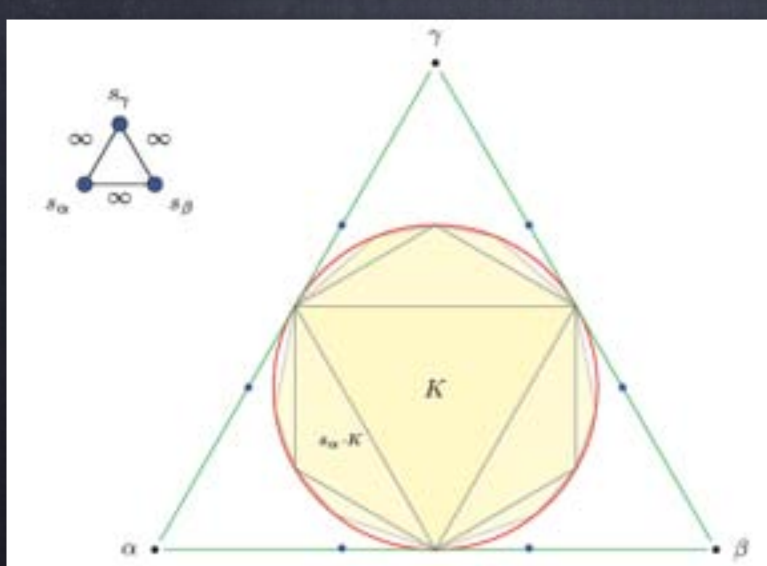
$$A \cup B = \{\alpha_1, \alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\} = N(\tau_1\tau_3\tau_2\tau_3)$$

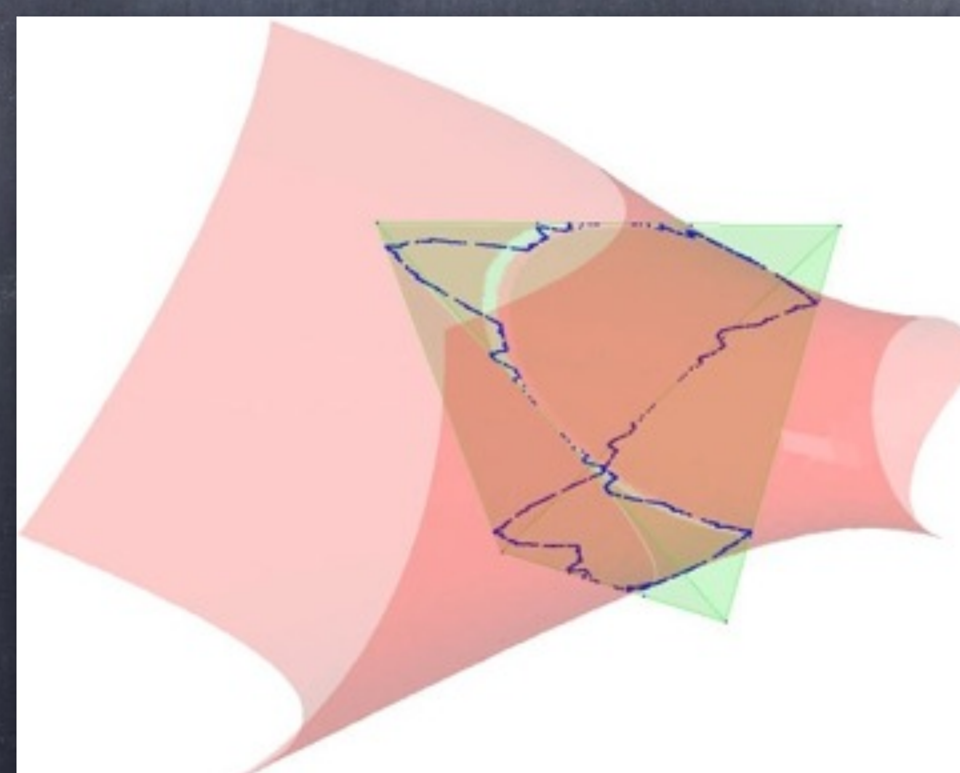
Lecture 3: Words & infinite root systems

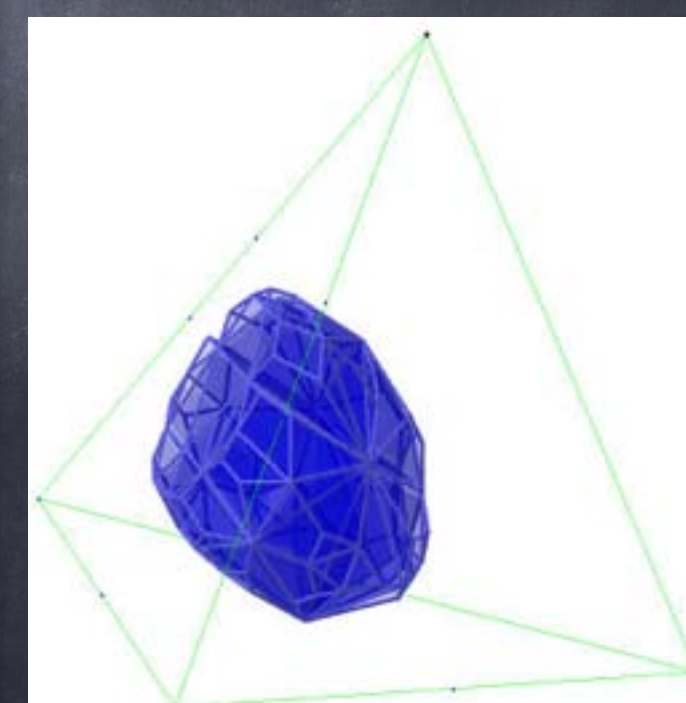












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In the last episode

world of roots

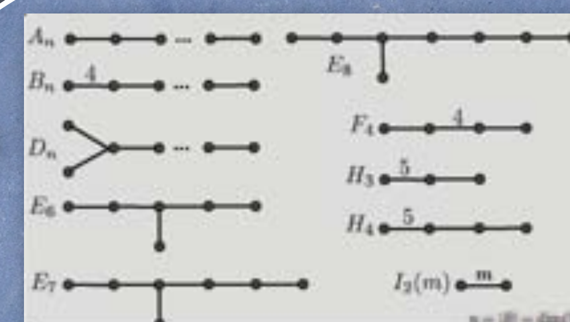
B-Reflection groups

signature (p,q,r) of B ← Γ_W allowing $\infty (a \le -1)$

Finite Reflection Groups ↔

world of words

Coxeter groups/graphs



words to roots

roots to words

The Cayley graph of (W,S) is naturally oriented by the (right) weak order: $w < ws$ if $\ell(w) < \ell(ws)$.

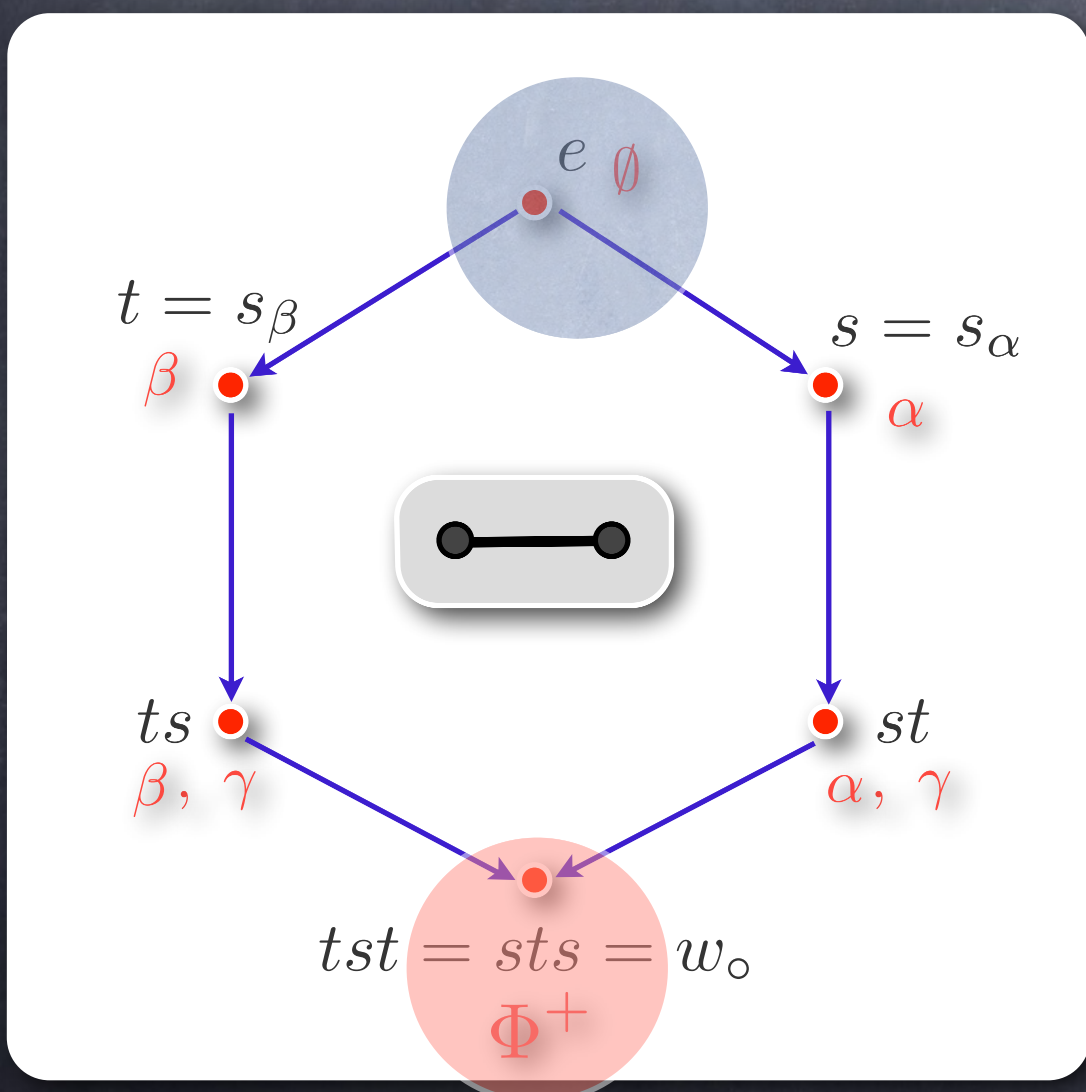
The weak order is a complete meet-semilattice and

$$u \le v \iff N(u) \subseteq N(v); \quad N(u) = \Phi^+ \cap u(\Phi^-)$$

In the last episode

Finite and infinite biclosed sets

$$\mathrm{Im}(N) = \mathcal{B}_0 \subseteq \mathcal{B}$$



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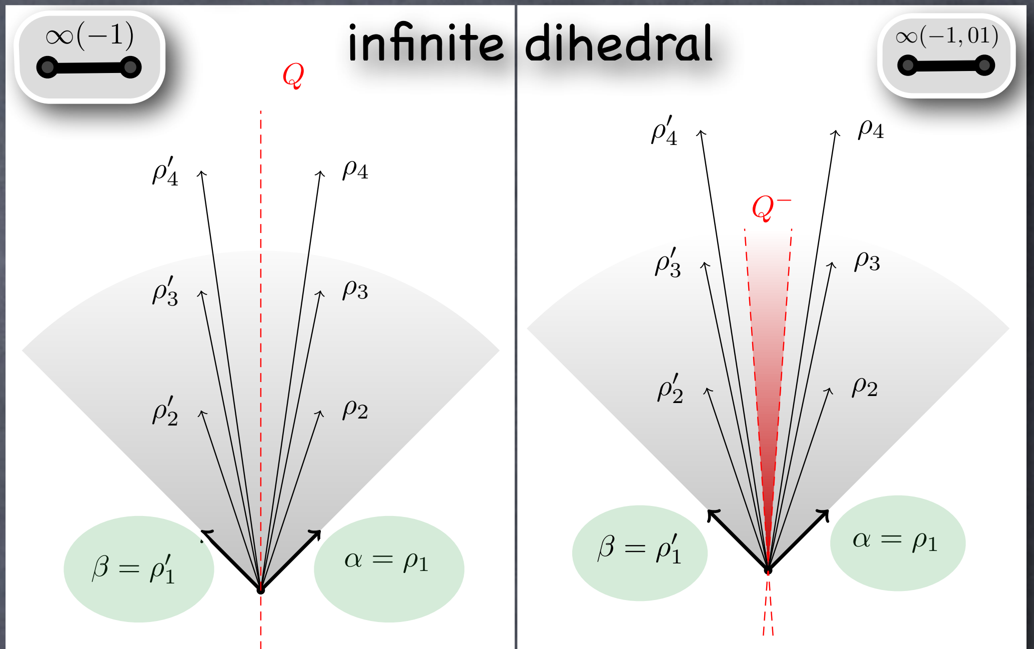
Christophe Hohlweg, 2014

A Projective view of root systems



(a) $B(\alpha,\beta) = -1$

(b) $B(\alpha,\beta) = -1.01 < -1$

Geometric representations of (W, S)

- (V, B) real quadratic space and $\Delta \subseteq V$ s.t.
  - $\operatorname{cone}(\Delta) \cap \operatorname{cone}(-\Delta) = \{0\}$;
  - $\Delta = \{\alpha_s \mid s \in S\}$ s.t.

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) & \text{if } m_{st} < \infty \\ a \leq -1 & \text{if } m_{st} = \infty \end{cases}$$

- $W \leq \mathrm{O}_B(V)$: $\quad s(v) = v - 2B(v,\alpha)\alpha, \;\; s \in S$

Root system: $\Phi = W(\Delta),\;\; \Phi^+ = \operatorname{cone}(\Delta) \cap \Phi = -\Phi^-$

A Projective view of root systems

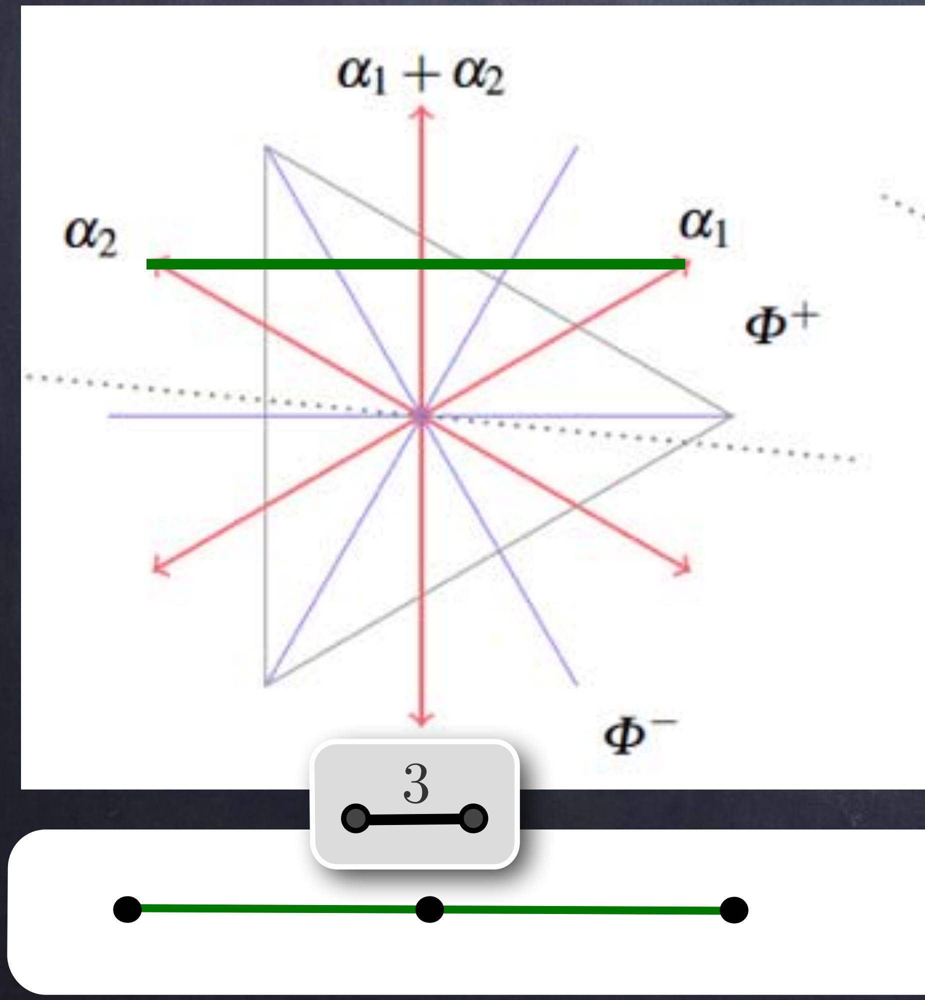
'Cut' $\mathrm{cone}(\Delta)$ by an affine hyperplane: $V_1 = \{v \in V \mid \sum_{\alpha\in\Delta} v_\alpha = 1\}$

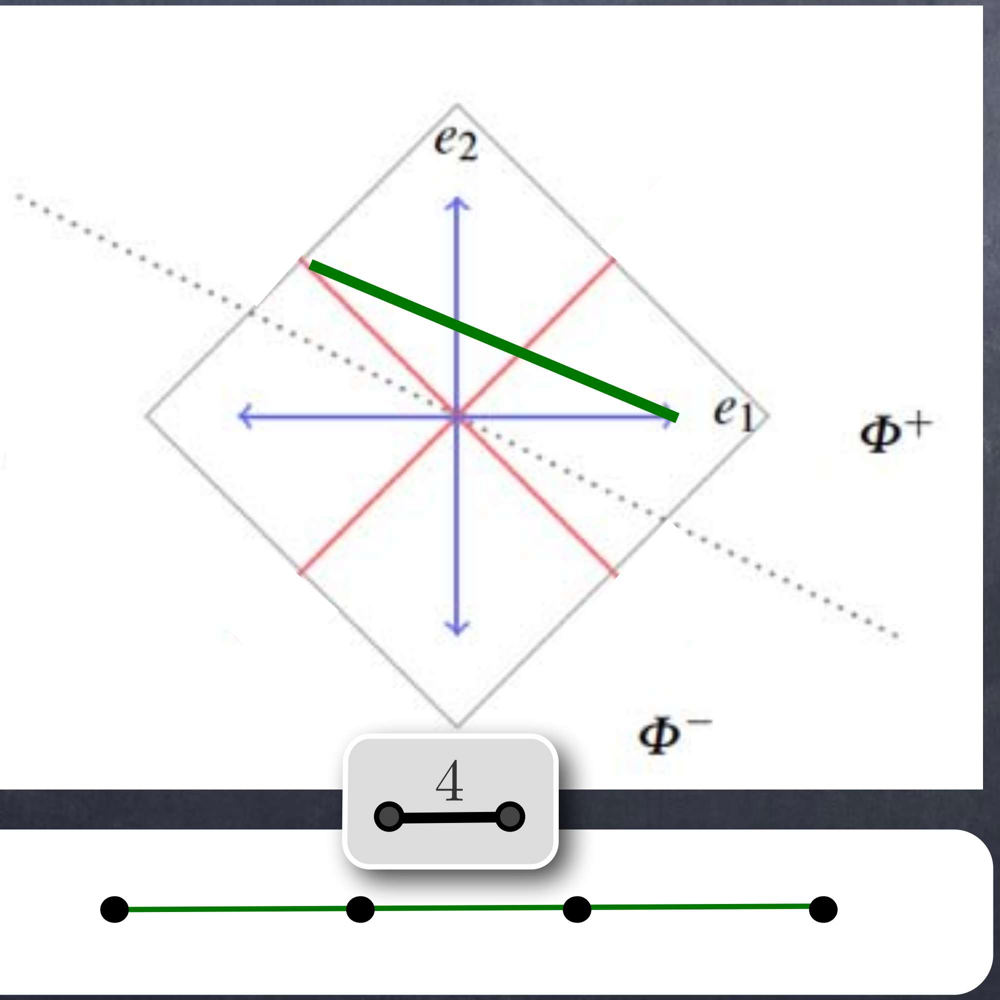
Normalized roots: $\widehat{\rho} := \rho / \sum_{\alpha\in\Delta} \rho_\alpha$ in $\widehat{\Phi} := \bigcup_{\rho\in\Phi} \mathbb{R}\rho \cap V_1$

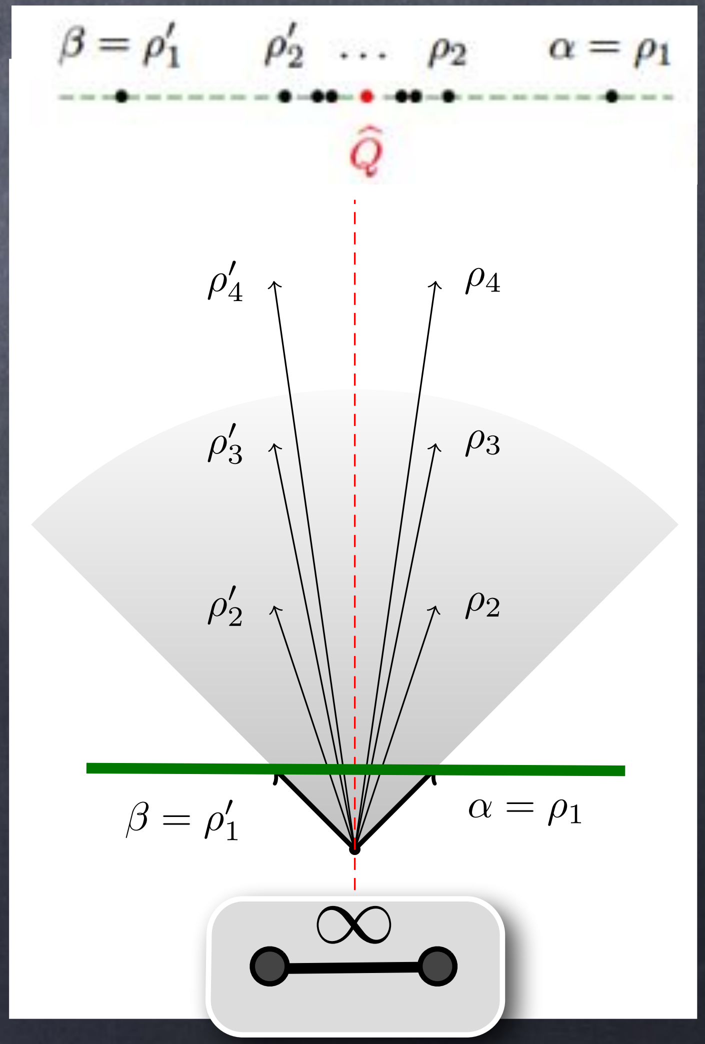
Action of W on $\widehat{\Phi}$: $w \cdot \hat{\rho} = \widehat{w(\rho)}$

Normalized isotropic cone: $\widehat{Q} := Q \cap V_1$

- Rank 2 root systems







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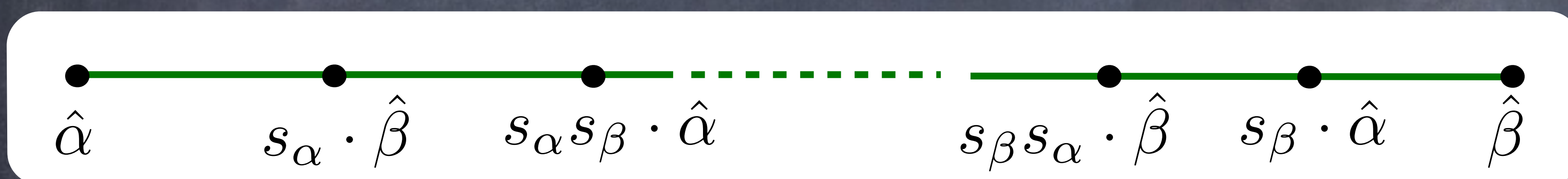
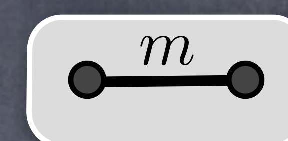
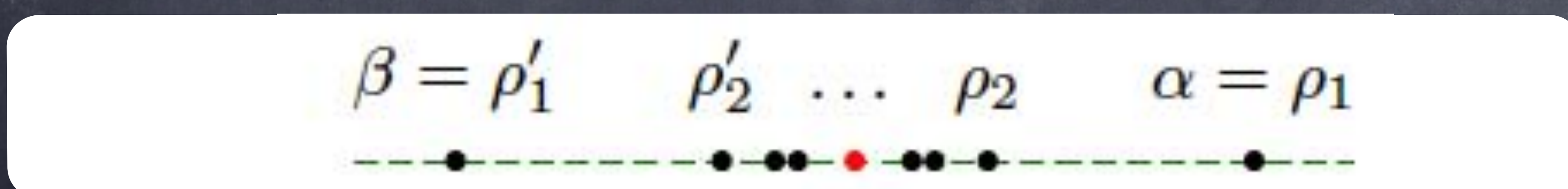
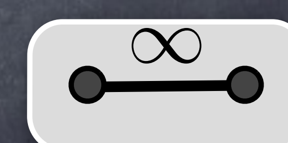
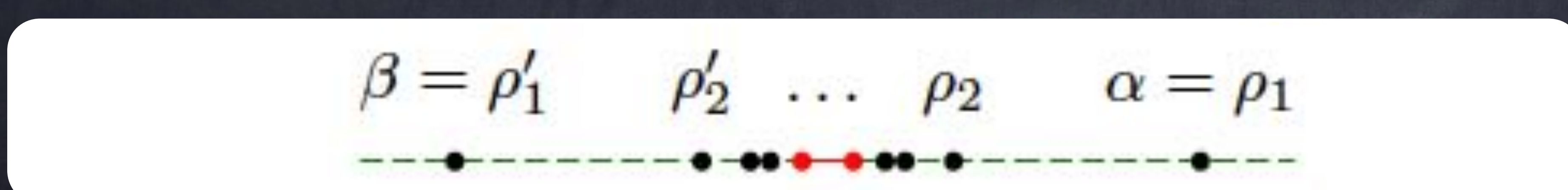


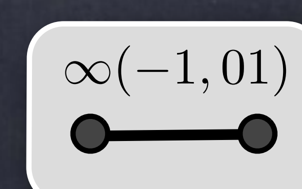






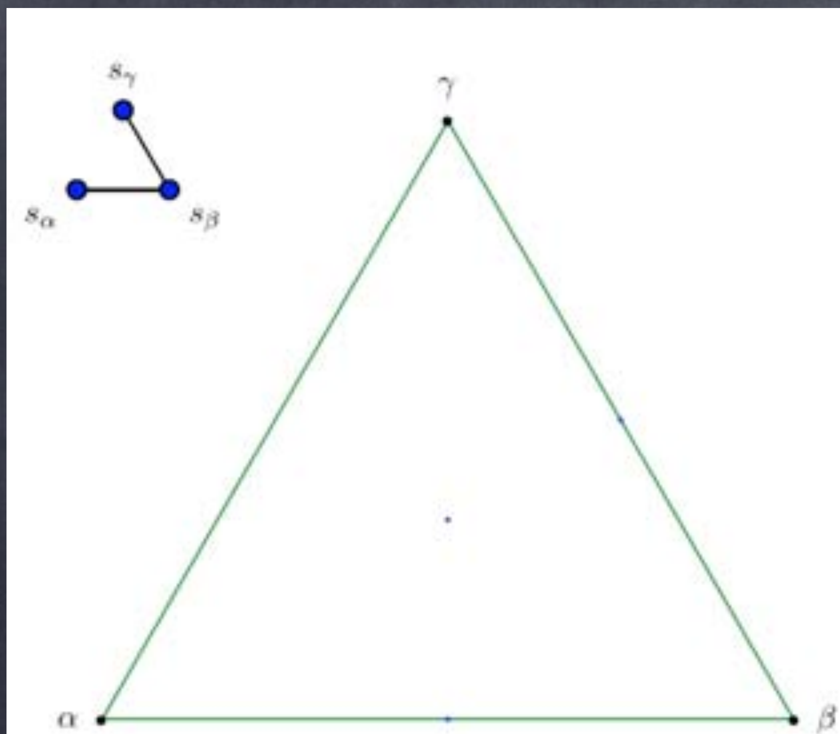
$\rho'_n = n\alpha + (n+1)\beta \qquad \rho_n = (n+1)\alpha + n\beta$

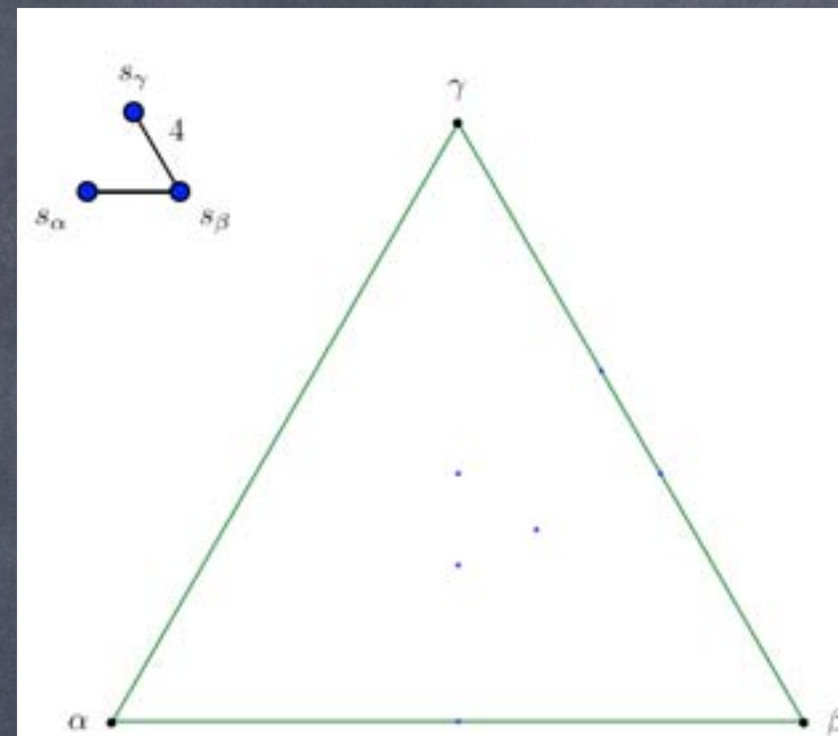


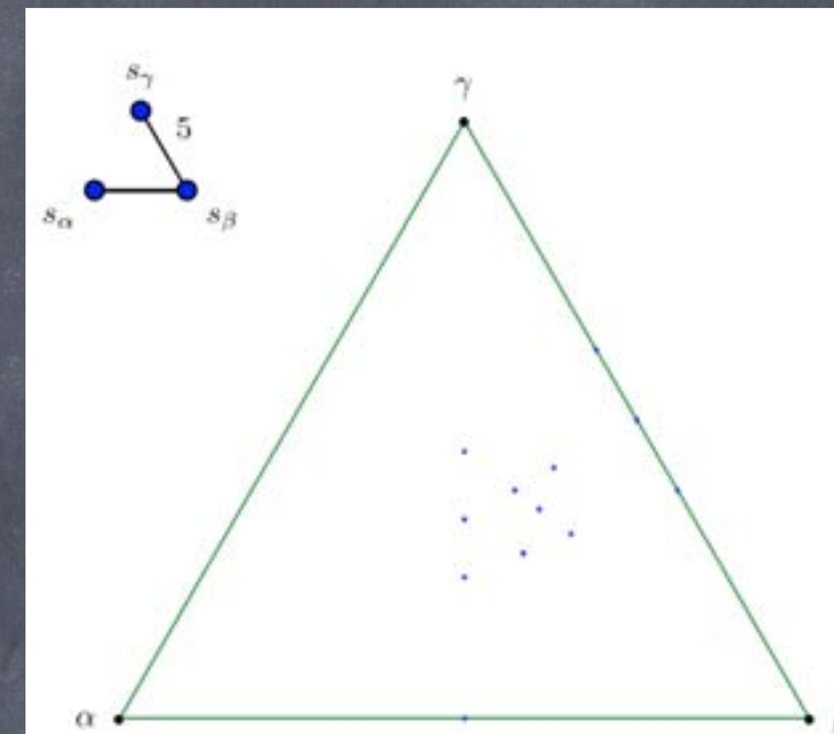


A Projective view of root systems

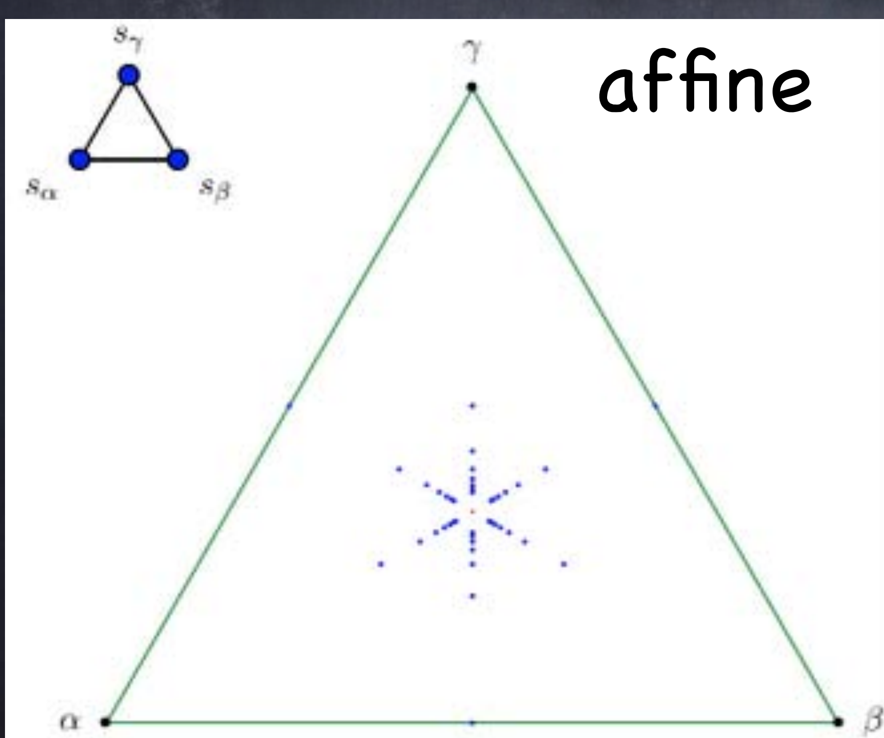
- Rank 3 root systems

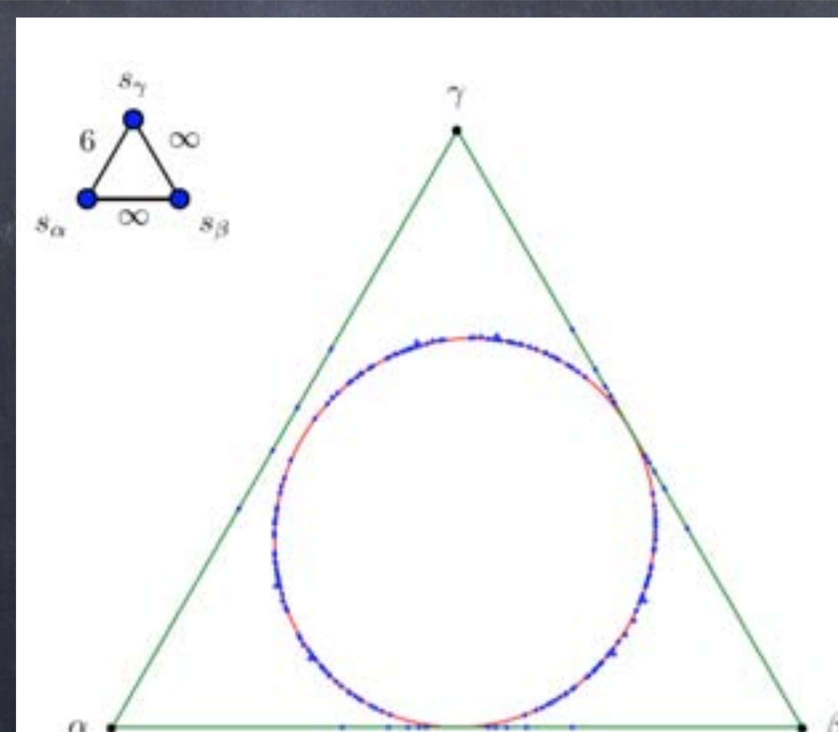


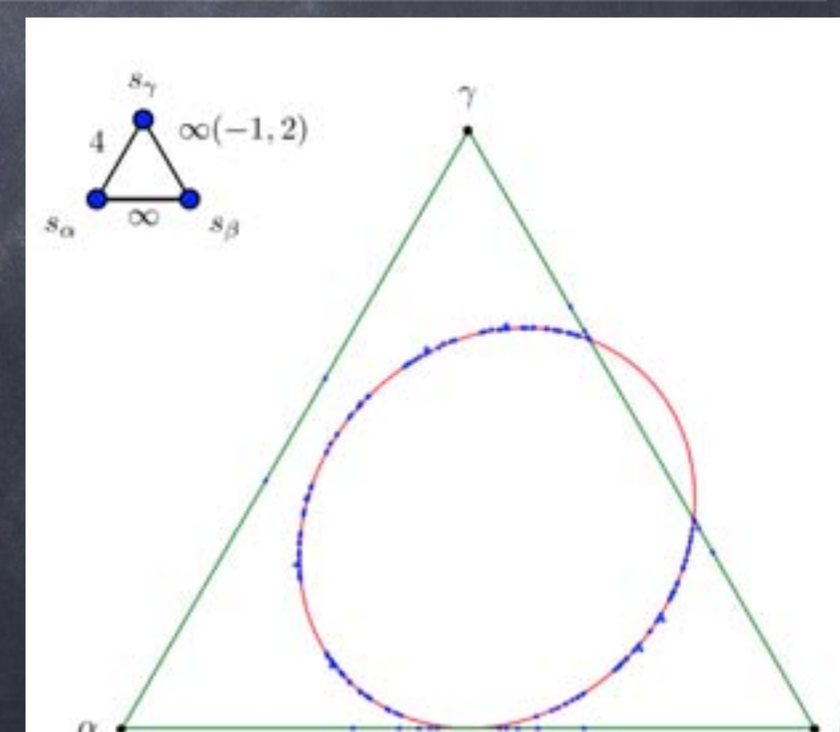


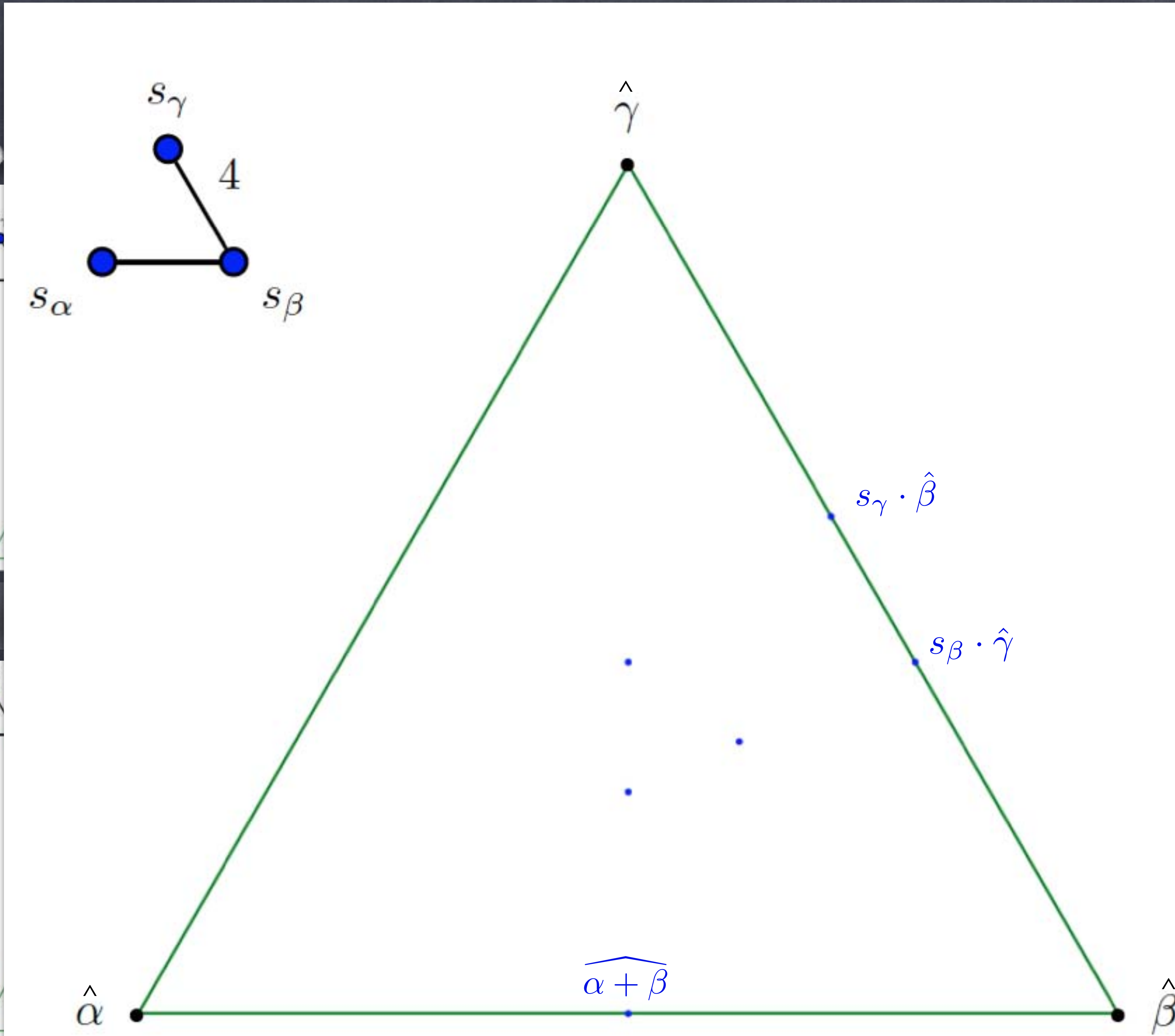


A dihedral subgroup group is infinite iff the associated line cuts Q

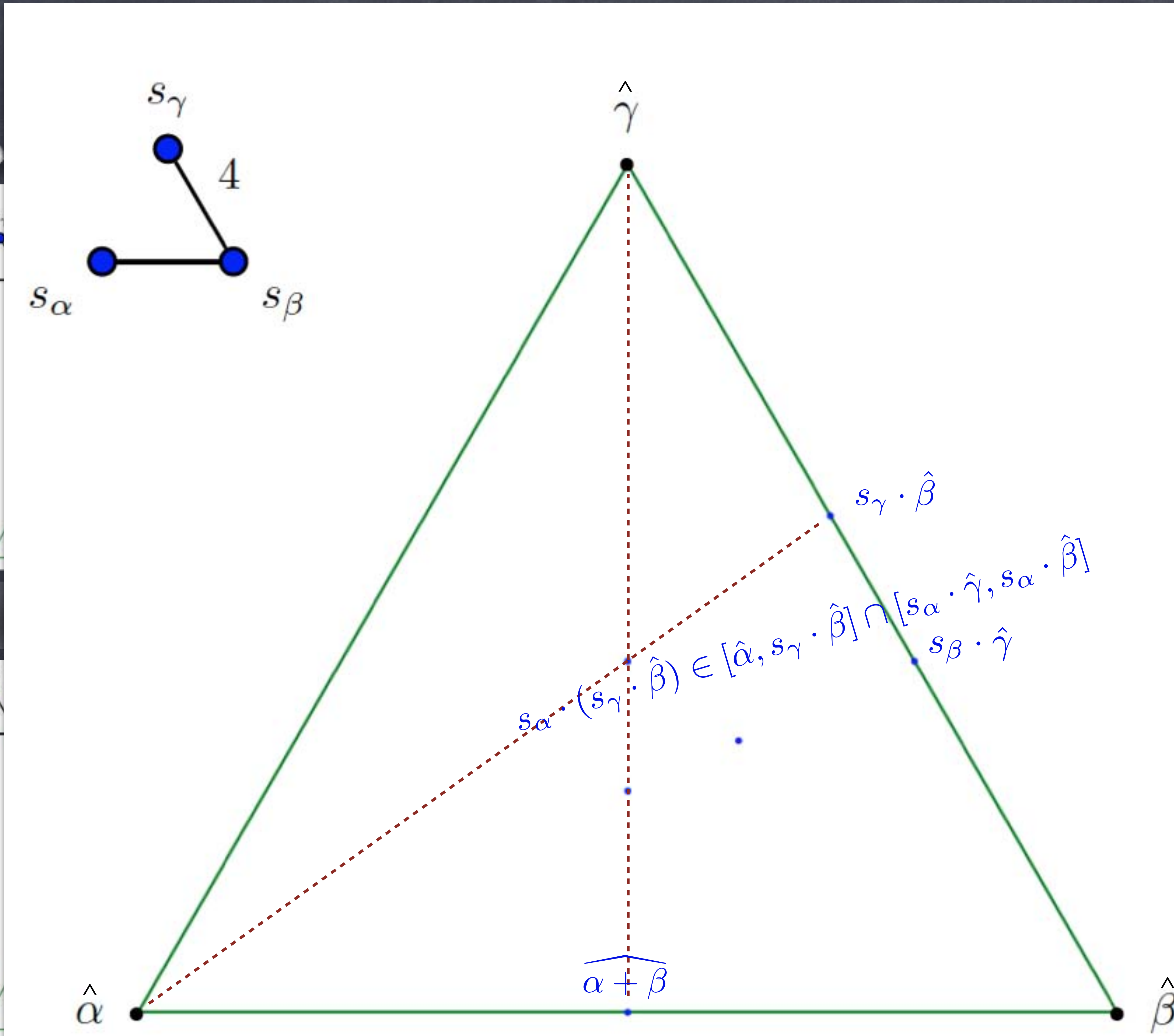


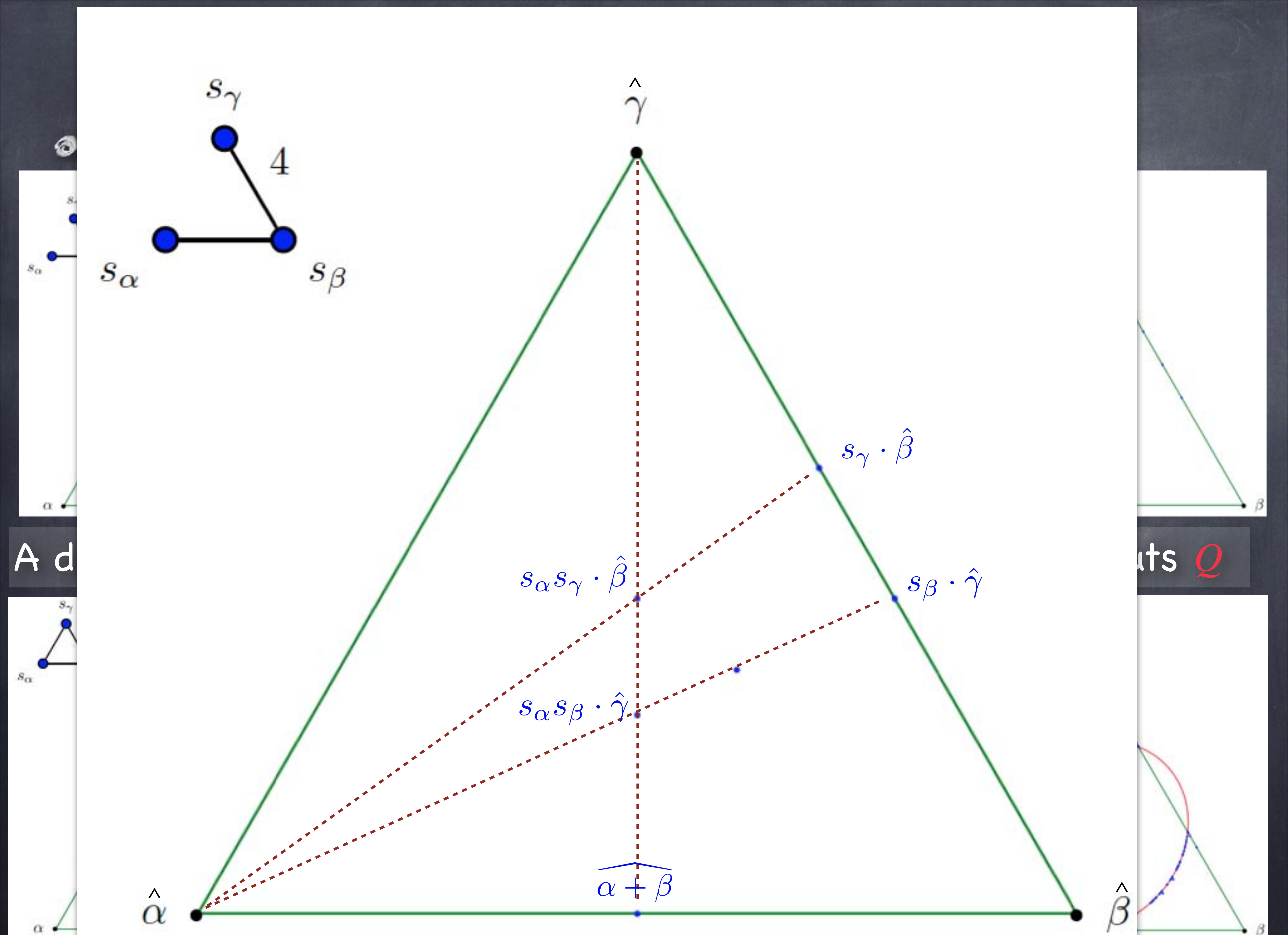





s_γ
4
s_α
s_β
$\hat{\gamma}$
$s_\gamma \cdot \hat{\beta}$
$s_\beta \cdot \hat{\gamma}$
$\widehat{\alpha + \beta}$
$\hat{\alpha}$
$\hat{\beta}$

A d ... ts Q


s_γ
4
s_α
s_β
$\hat{\gamma}$
$s_\gamma \cdot \hat{\beta}$
$s_\alpha \cdot (s_\gamma \cdot \hat{\beta}) \in [\hat{\alpha}, s_\gamma \cdot \hat{\beta}] \cap [s_\alpha \cdot \hat{\gamma}, s_\alpha \cdot \hat{\beta}]$
$s_\beta \cdot \hat{\gamma}$
$\widehat{\alpha + \beta}$
$\hat{\alpha}$
$\hat{\beta}$



s_γ
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$s_\gamma \cdot \hat{\beta}$
$s_\alpha s_\gamma \cdot \hat{\beta}$
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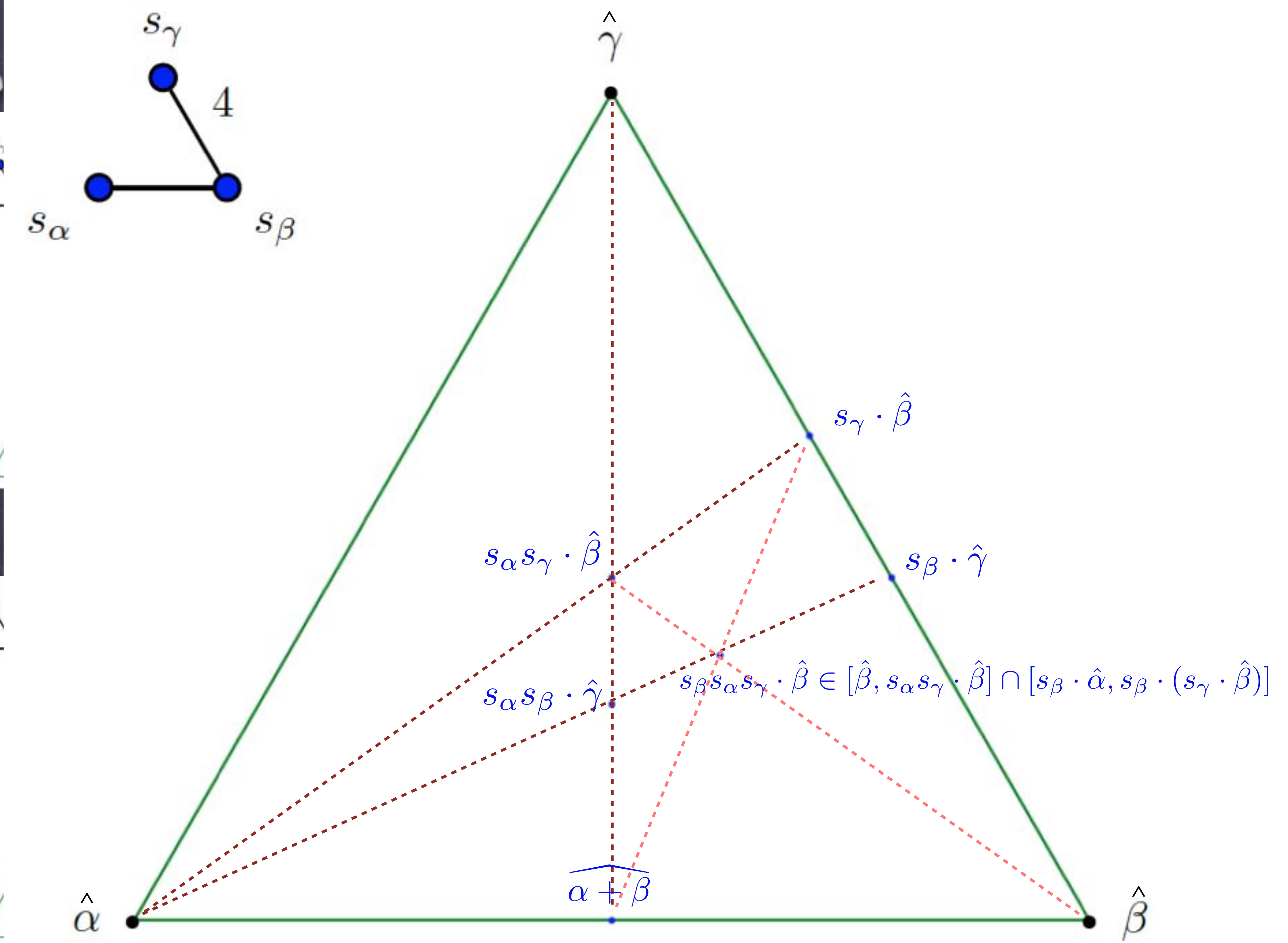
ts Q



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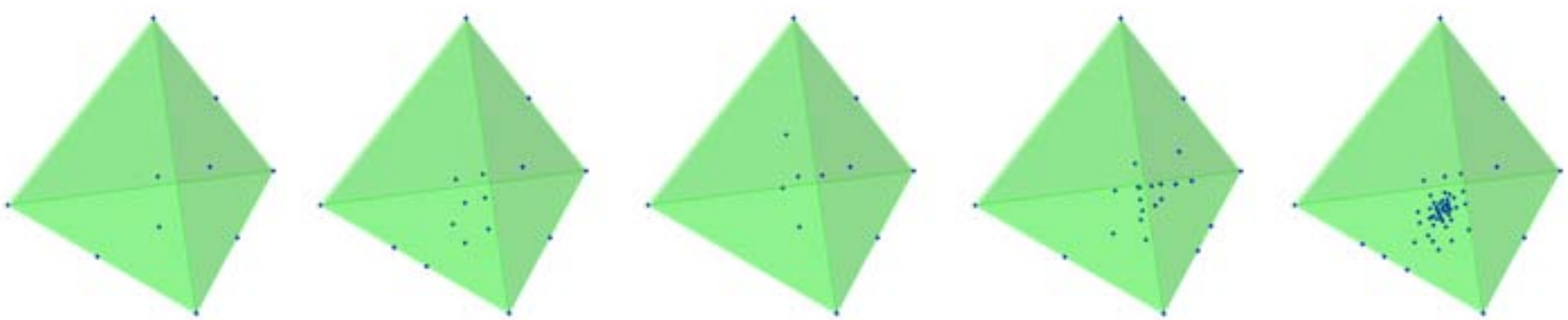
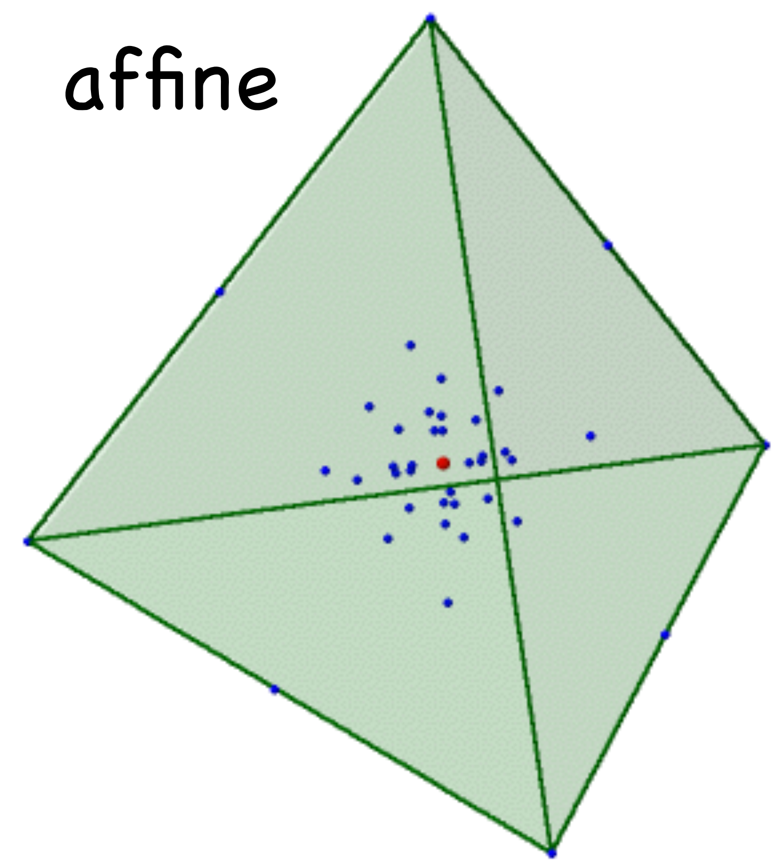
A Projective view of root systems

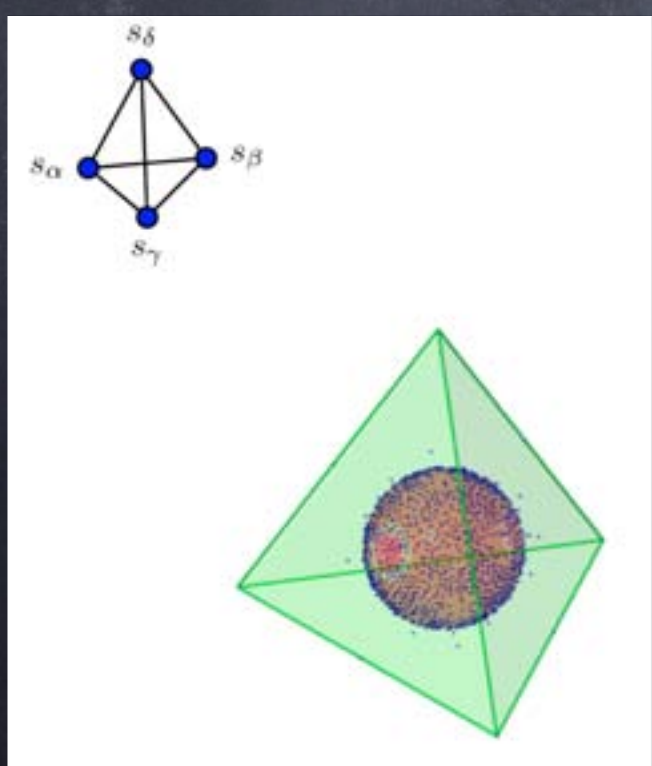
- Rank 4 root systems

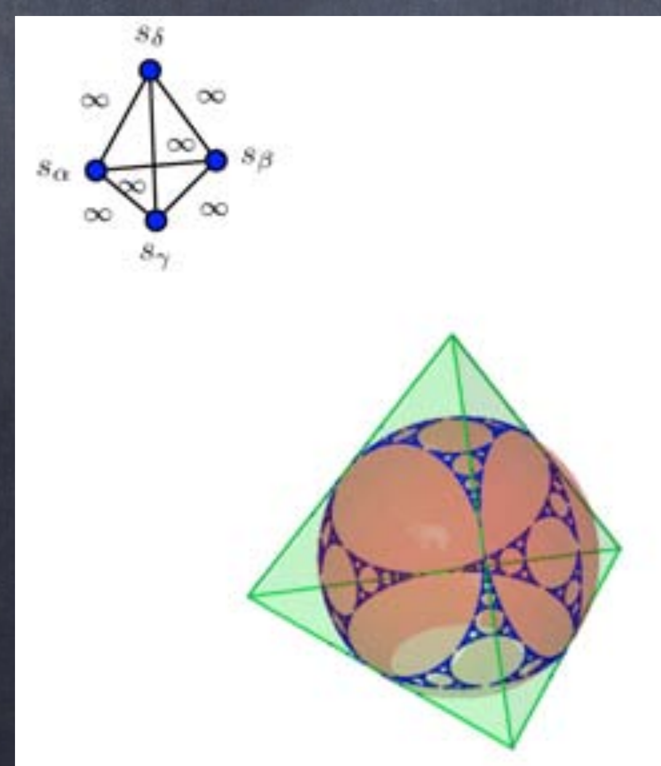
finite



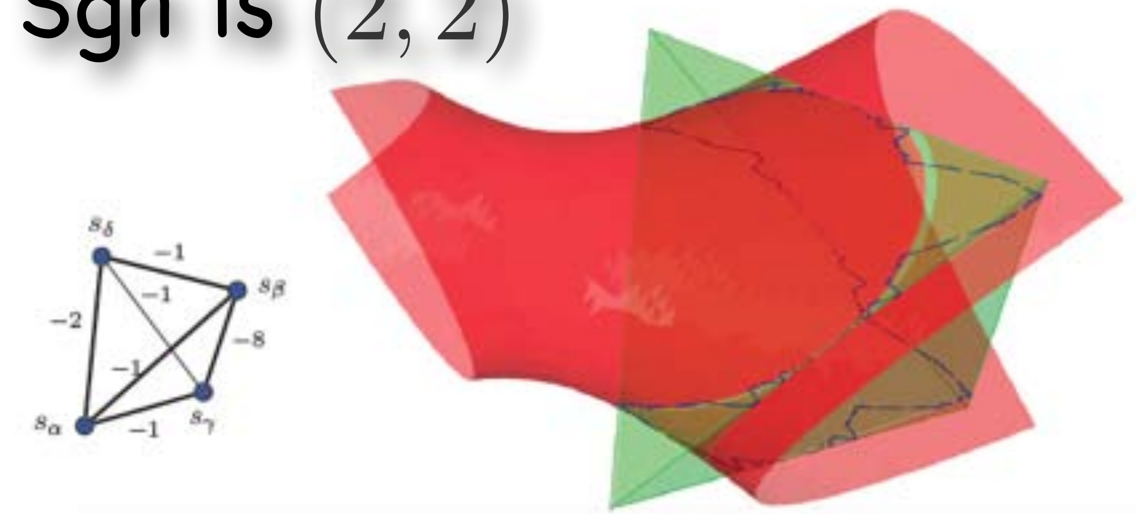
affine





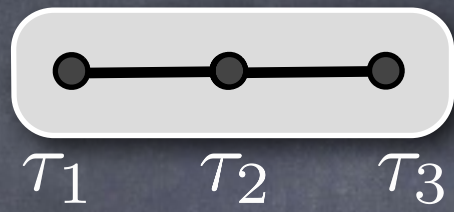


Sgn is $(2, 2)$



(weakly) hyperbolic

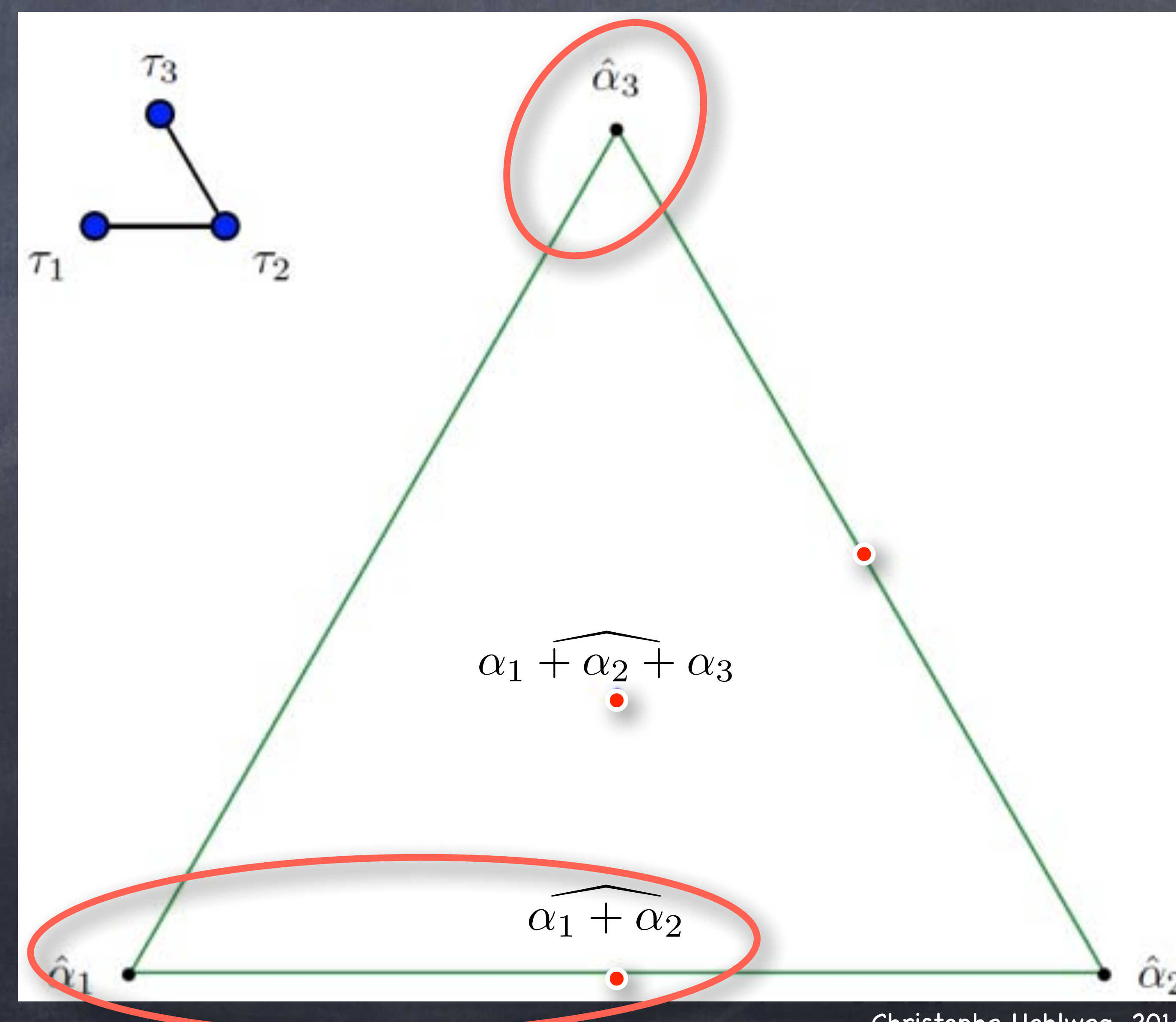
Join in finite Coxeter groups

Example: (W, S) is 

$$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1 + \alpha_2\} \;\; ; \quad B = N(\tau_3) = \{\alpha_3\}$$

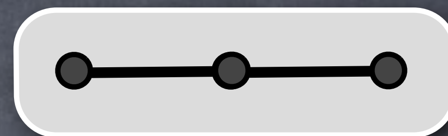
$$\tau_1\tau_2 \vee \tau_3 = \tau_1\tau_3\tau_2\tau_3 \quad ; N(\tau_1\tau_3\tau_2\tau_3) = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$$

$$A \cup B = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2\}$$



Christophe Hohlweg, 2014

Join in finite Coxeter groups

Example: (W, S) is 

$\tau_1 \quad \tau_2 \quad \tau_3$

$$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1 + \alpha_2\} \;\; ; \quad B = N(\tau_3) = \{\alpha_3\}$$

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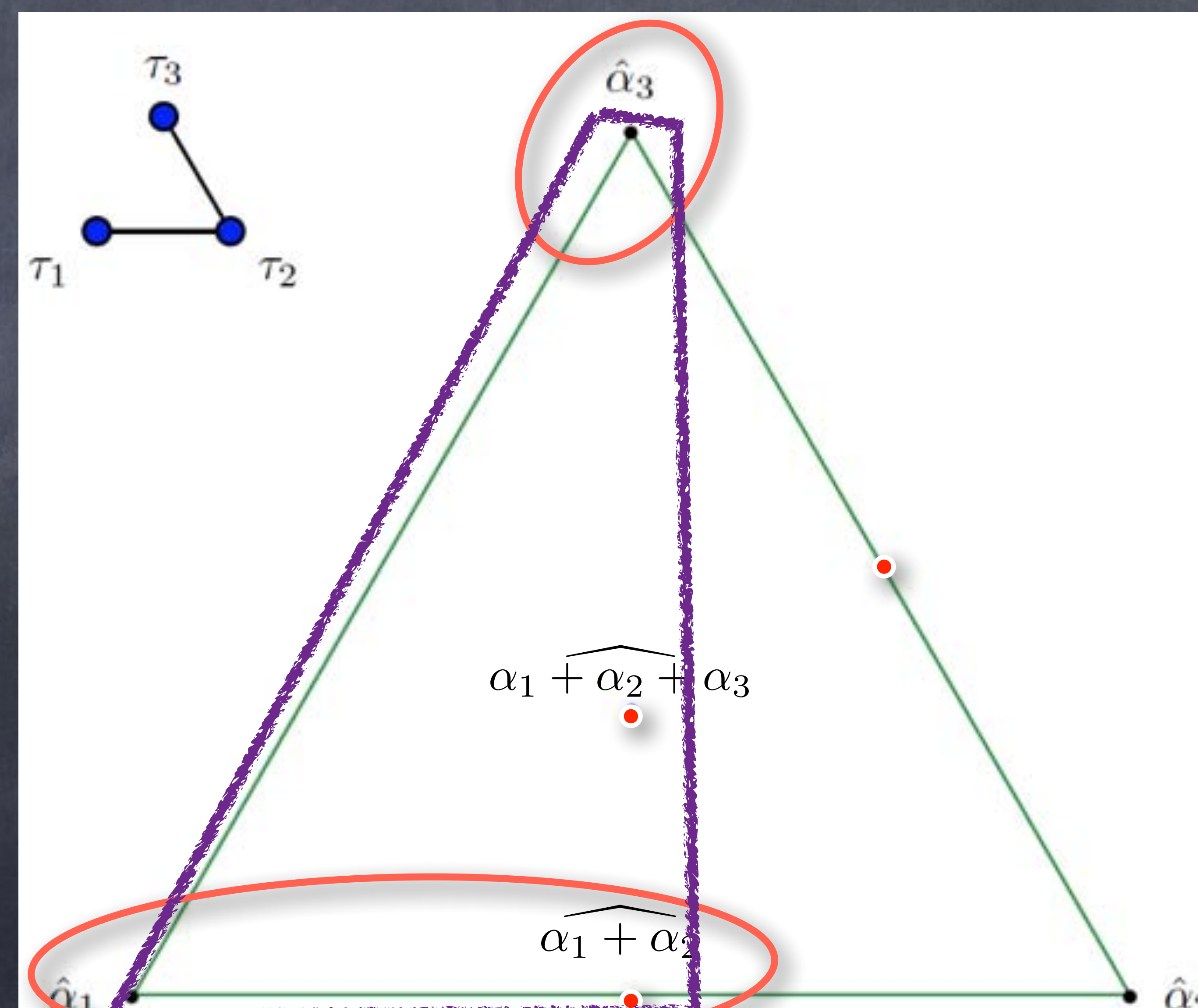
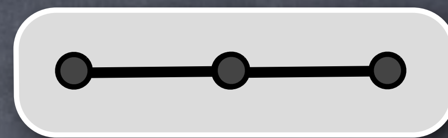
$$A \cup B = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2\}$$

$$\hat{A} \vee \hat{B} = \mathrm{conv}(\hat{A} \cup \hat{B}) \cap \hat{\Phi}$$



Join in finite Coxeter groups

Example: (W, S) is 

$\tau_1 \quad \tau_2 \quad \tau_3$

$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1 + \alpha_2\}$; $B = N(\tau_3) = \{\alpha_3\}$

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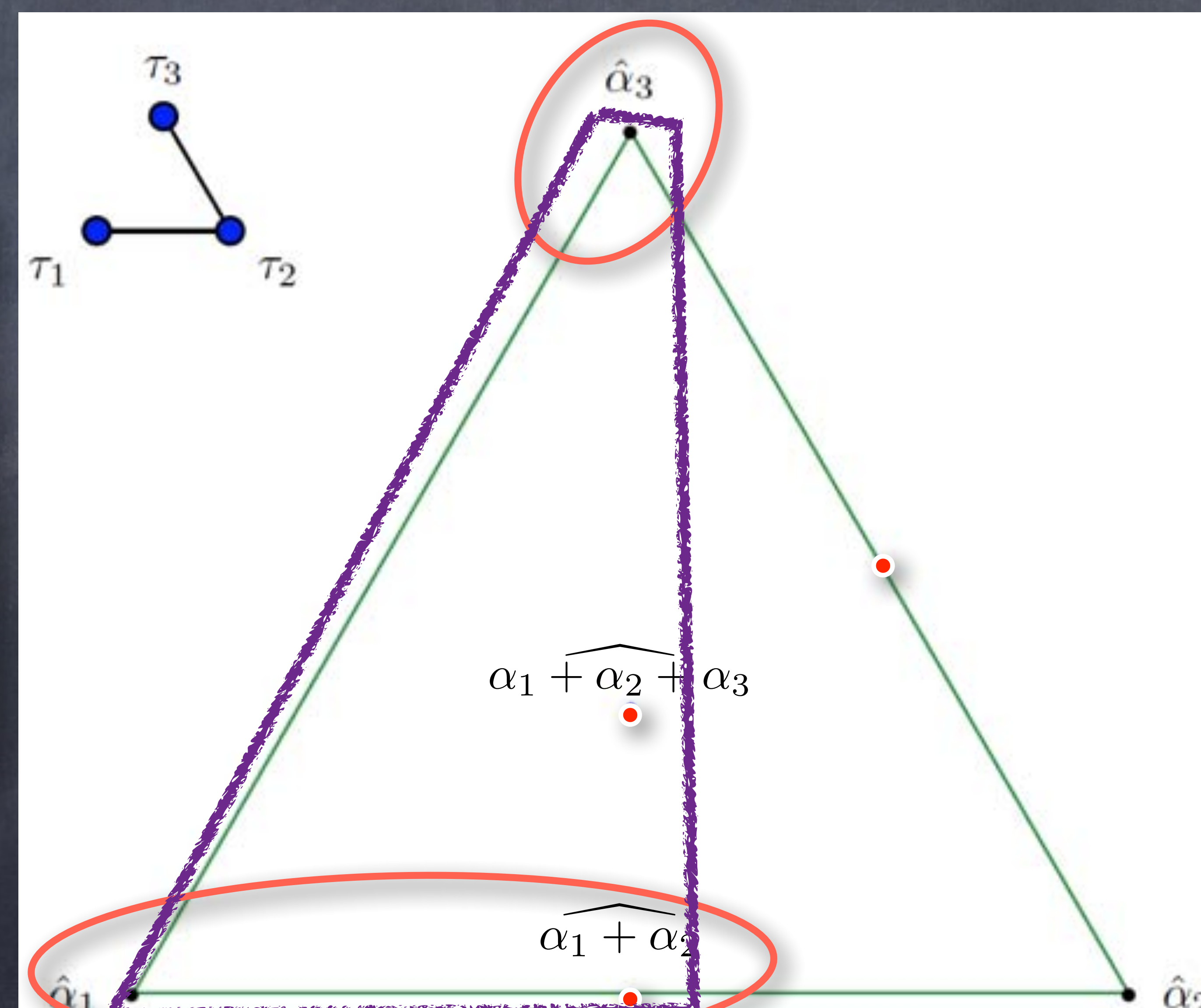


Proposition (CH, Labbé).
Let A, B be biclosed sets in a finite Coxeter group, then

$$\hat{A} \vee \hat{B} = \mathrm{conv}(\hat{A} \cup \hat{B}) \cap \hat{\Phi}$$

Ex

No true in general: the convex hull of the union of biclosed is not biclosed in general

(counterexample in rank 4).

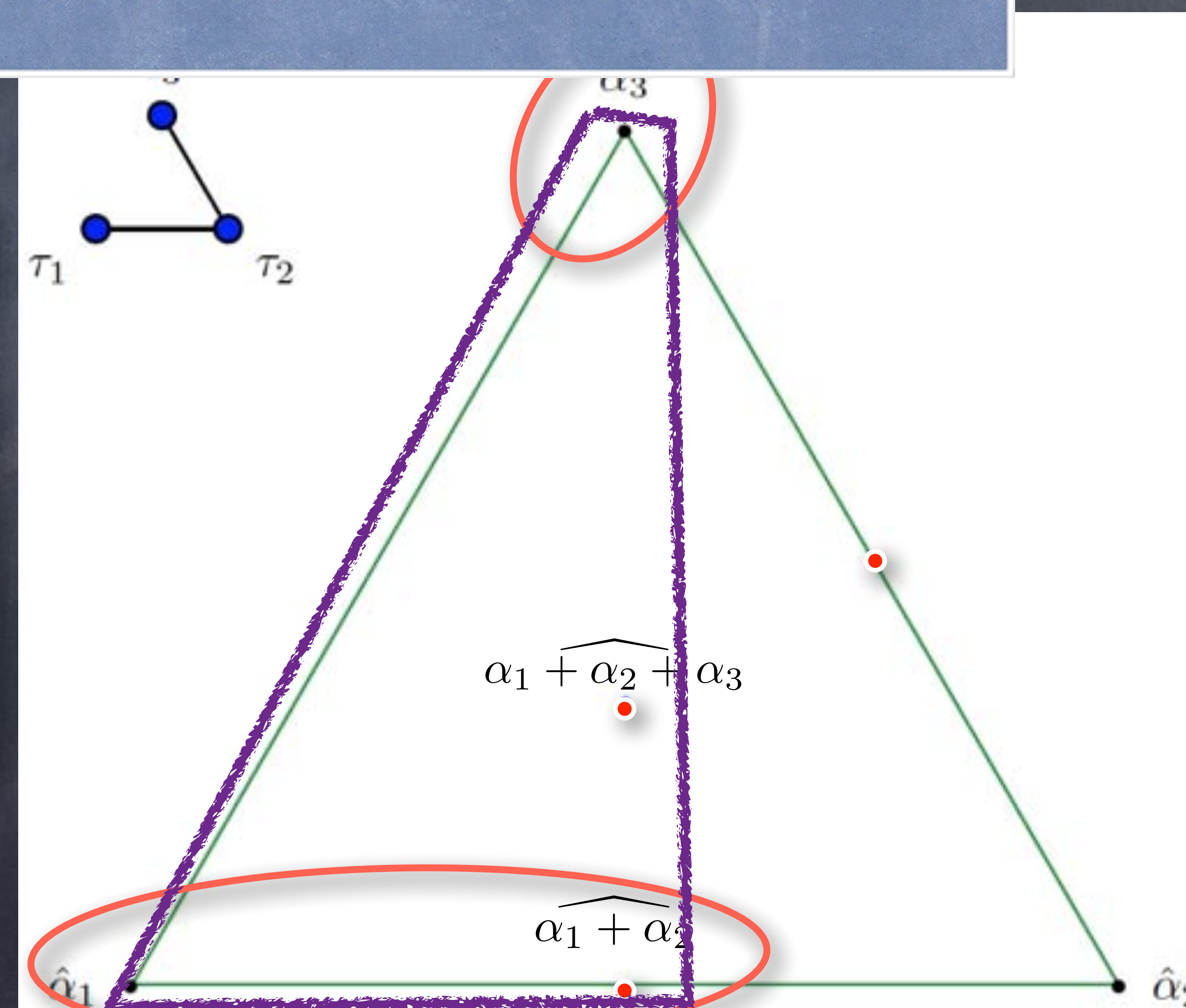
A = … $_3\}$

$\tau_1\tau_2$ … $\alpha_3\}$

$A \cup$ …

Proposition (CH, Labbé).
Let A, B be biclosed sets in a finite Coxeter group, then

$$\hat{A} \vee \hat{B} = \mathrm{conv}(\hat{A} \cup \hat{B}) \cap \hat{\Phi}$$



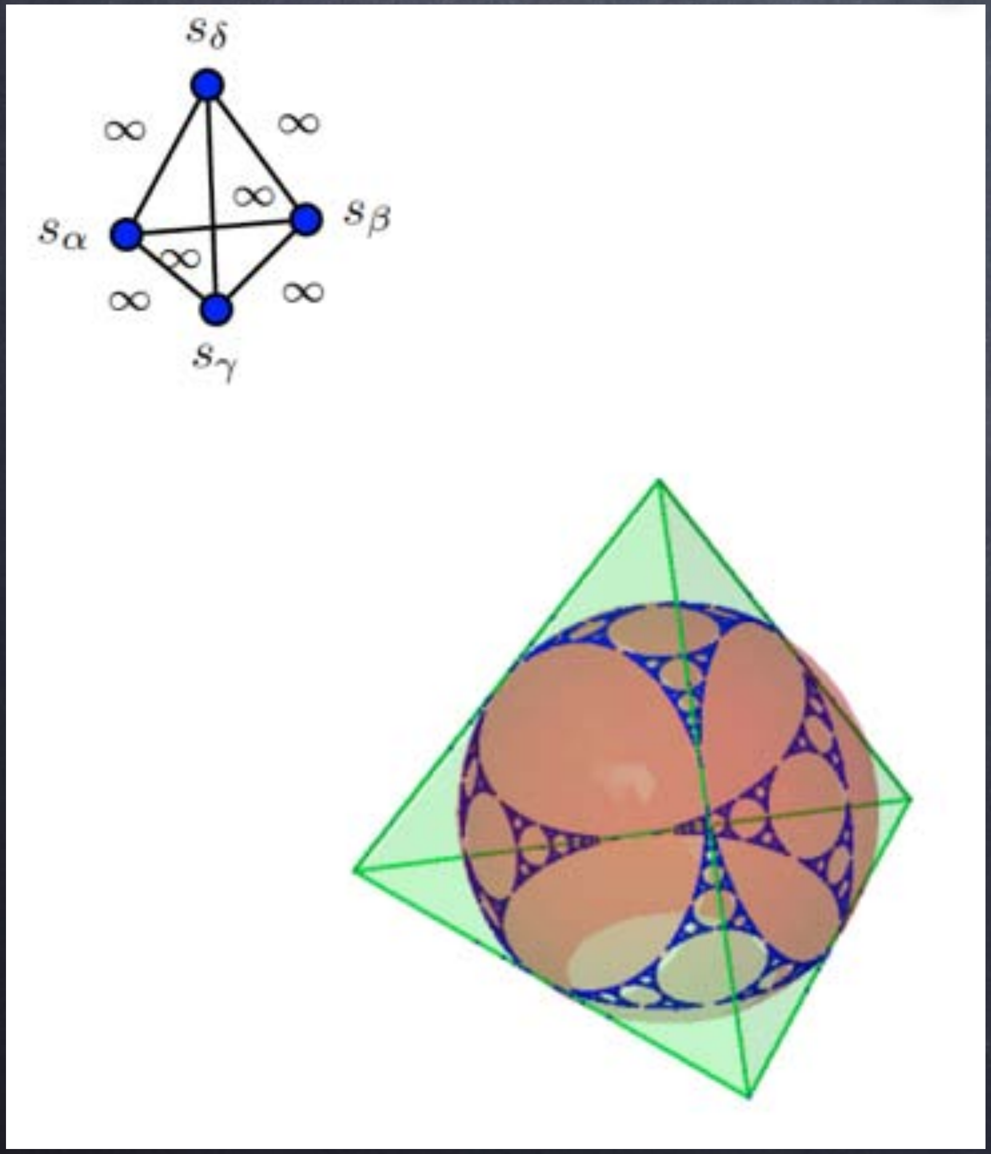
Limit roots

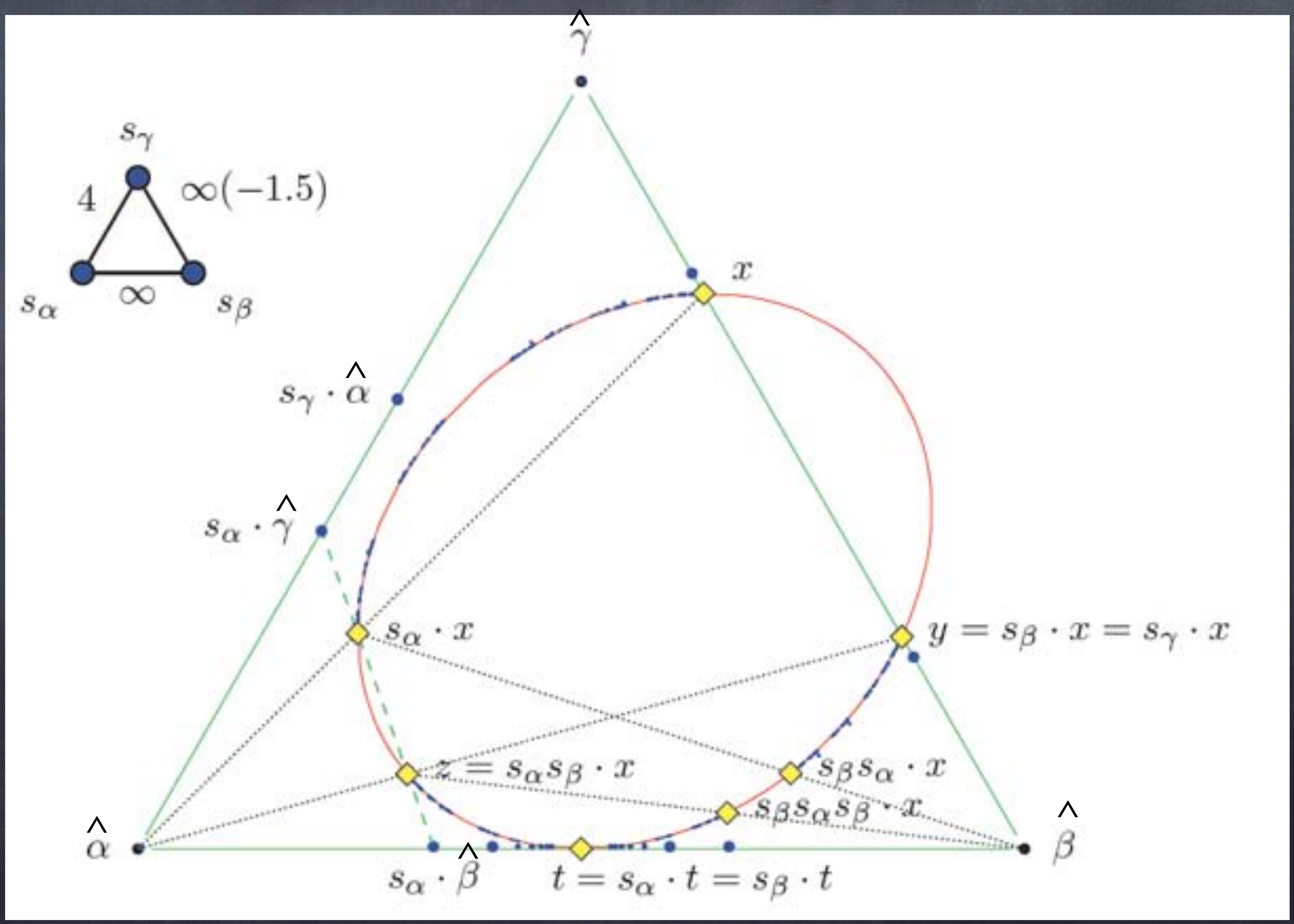
Limit roots (CH, Labbé, Ripoll): the set of limit roots is:

$$E(\Phi) = \mathrm{Acc}(\widehat{\Phi}) \subseteq Q \cap \mathrm{conv}(\Delta)$$

- Action of W on $\widehat{\Phi} \sqcup E$: given on E by $\widehat{Q} \cap L(\alpha, x) = \{x, s_\alpha \cdot x\}$

Remark. $E = \hat{Q}$ is a singleton in the case of affine root system.



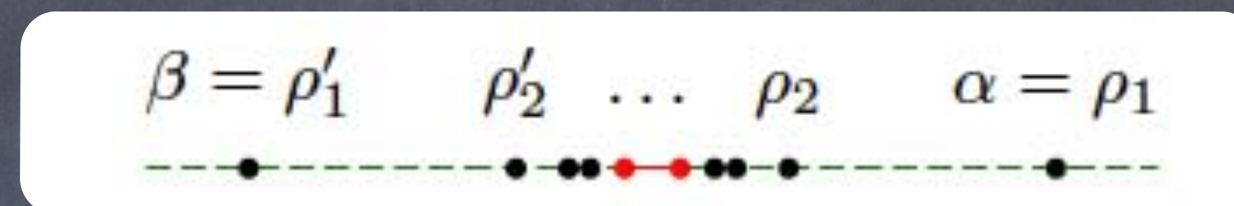


Limit roots

Dihedral reflection subgroups: $W' = \langle s_\rho, s_\gamma \rangle$, $\rho,\gamma \in \Phi^+$

Associated root system: $\Phi' = W'(\{\rho,\gamma\})$

Observation: $E(\Phi') = \widehat{Q} \cap L(\widehat{\rho}, \widehat{\gamma})$



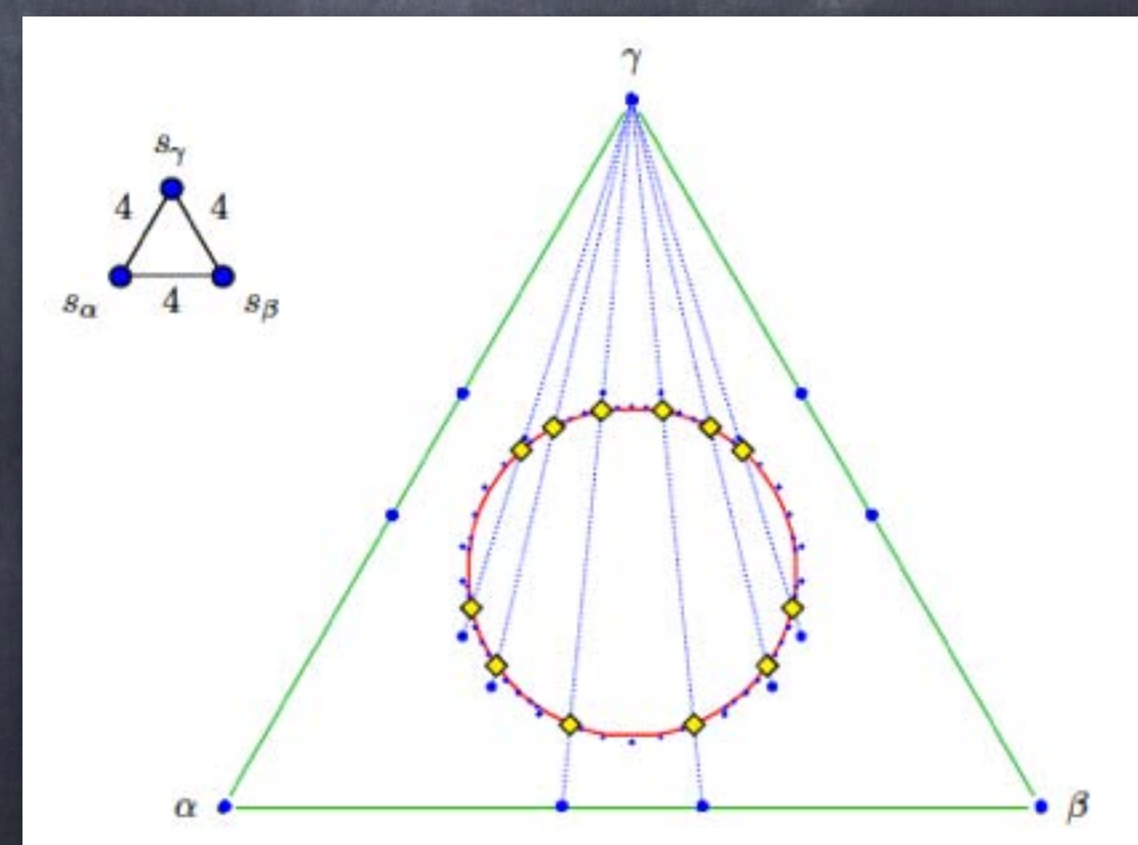
Limits of roots of dihedral reflection subgroups:

- $E_2 = W \cdot E_2^\circ$ where

$$E_2^\circ := \bigcup_{\substack{\alpha\in\Delta \\ \rho\in\Phi^+}} L(\alpha, \widehat{\rho}) \cap \widehat{Q}$$

Theorem (CH, Labbé, Ripoll 2012)

The sets E_2 and E_2° are dense in $E(\Phi)$.



Limit roots

Theorem (Dyer, CH, Ripoll 2013)
The closure of $W \cdot x$ is dense in $E(\Phi)$ for $x \in E(\Phi)$

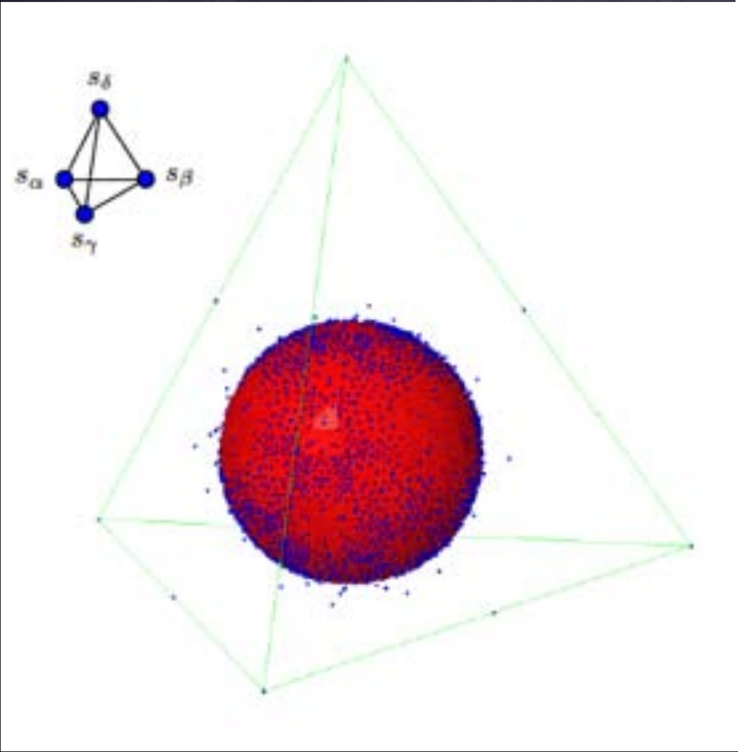
Theorem (Dyer, CH, Ripoll, 2013)

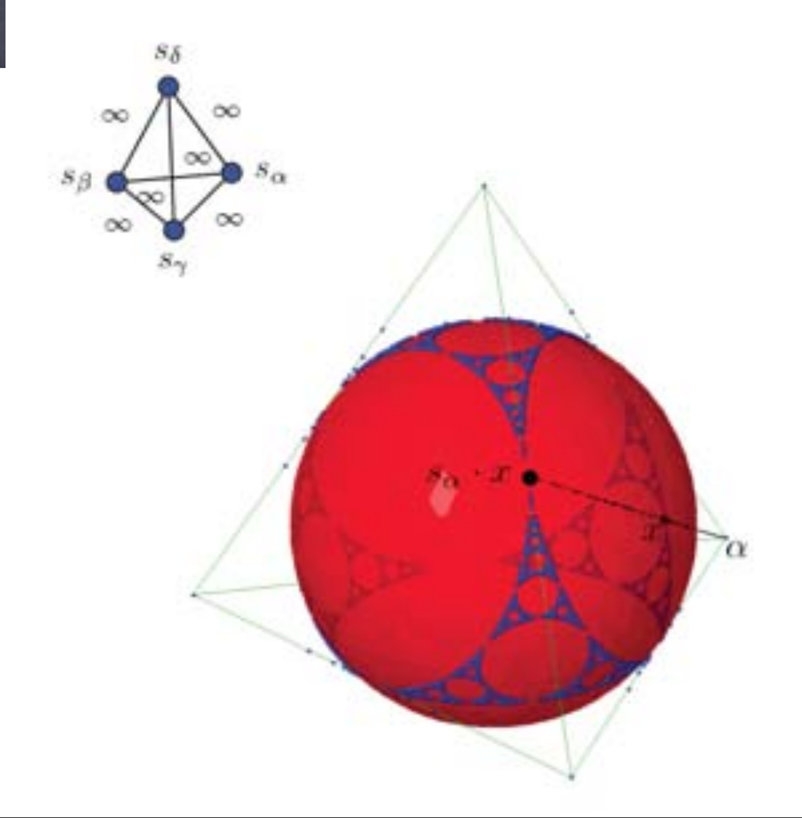
$$E = \hat{Q} \iff \hat{Q} \subseteq \mathrm{conv}(\Delta)$$

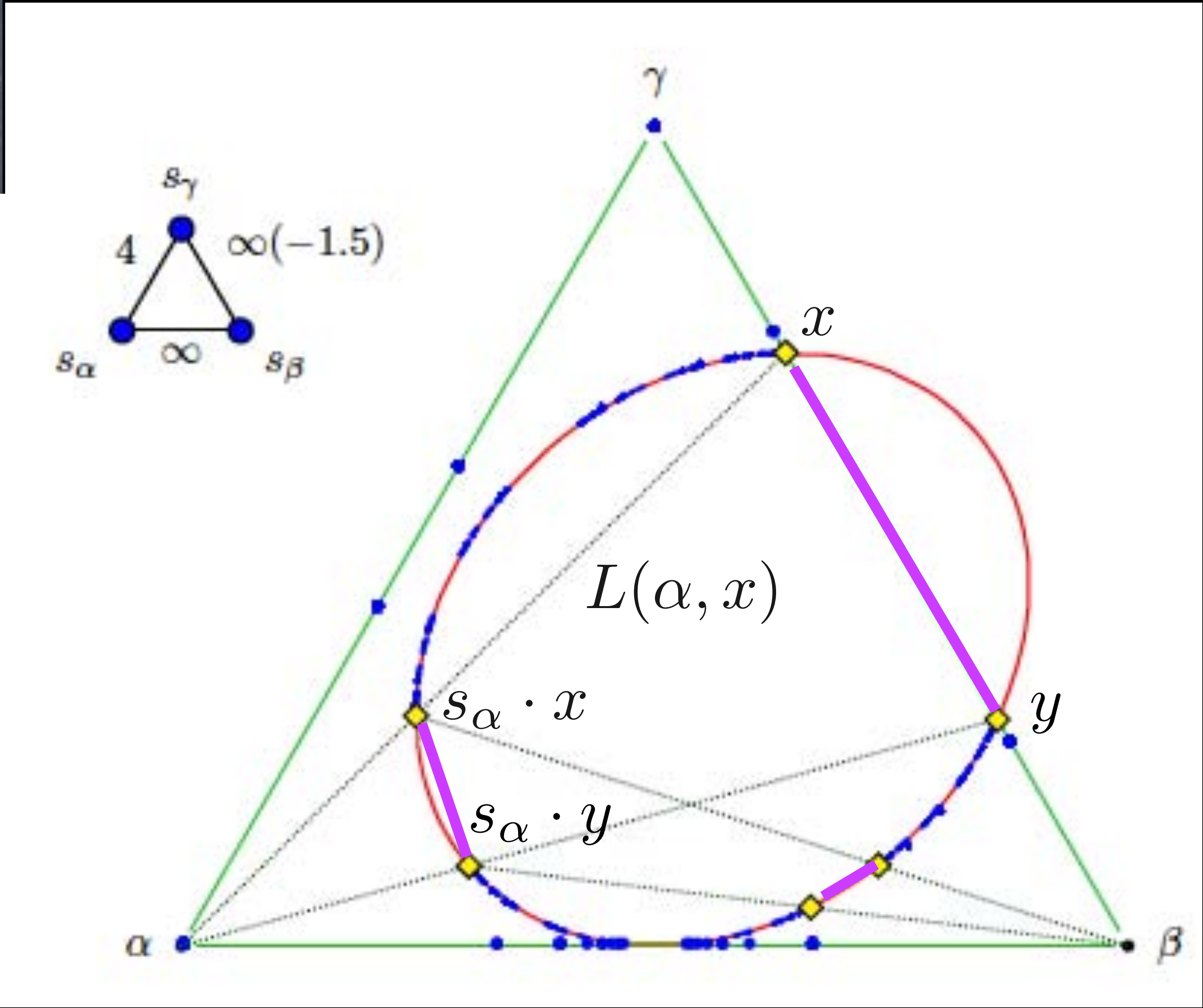
Morever, in this case,

$$\mathrm{sgn}(B) = (n, 1, 0)$$

Corollary (Dyer, CH, Ripoll, 2013)
A first fractal Phenomenon.



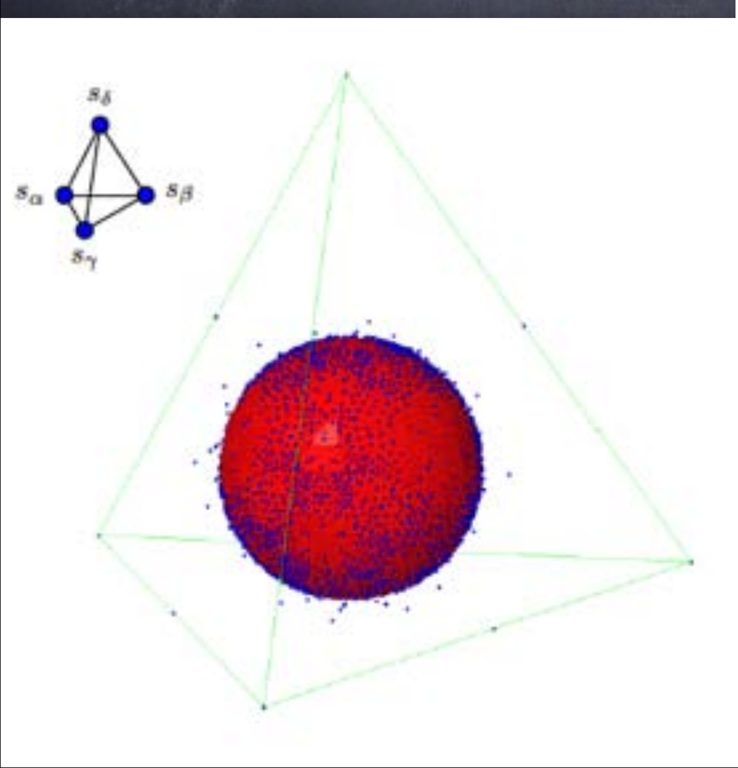


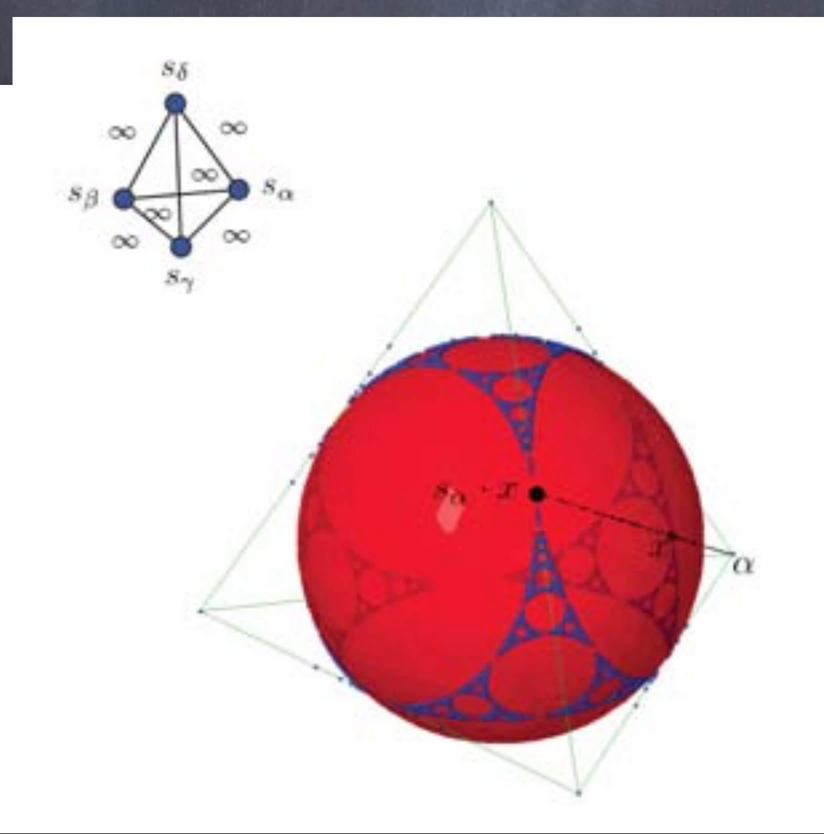


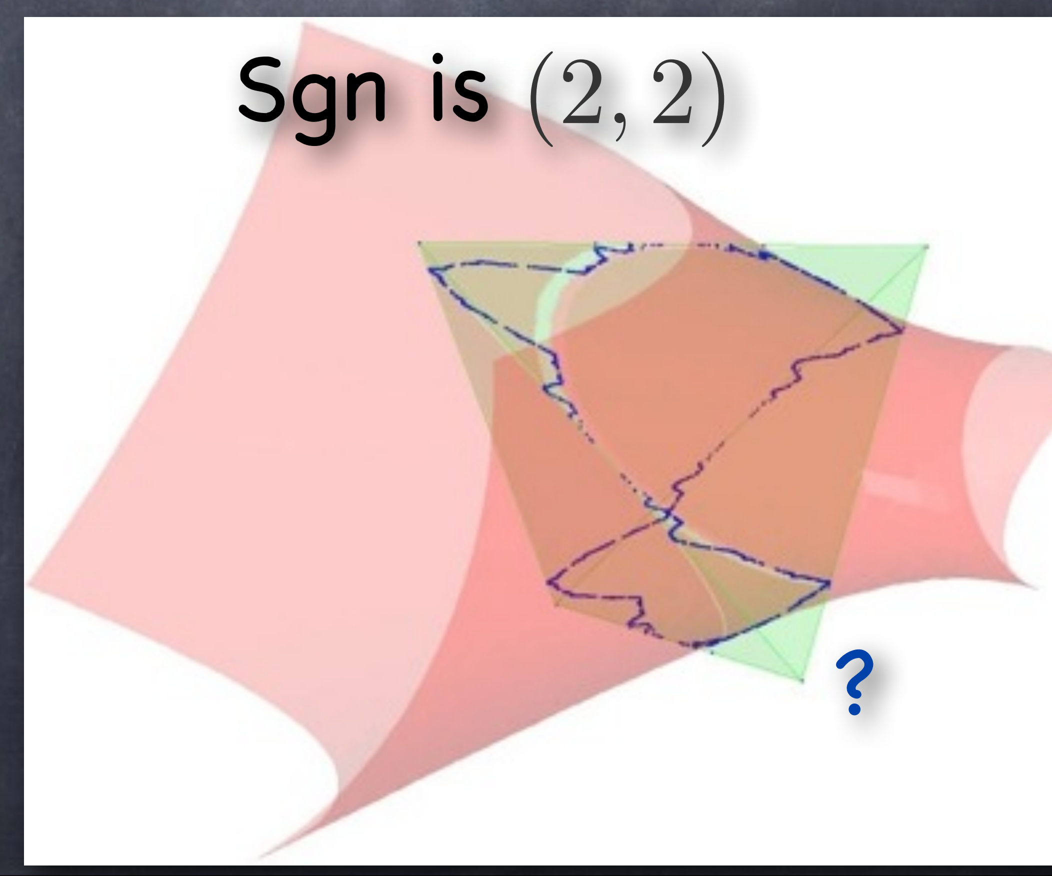
Limit roots

Theorem (Dyer, CH, Ripoll 2013) For irreducible root of signature $(n, 1, 0)$ we have: $E = \mathrm{conv}(E) \cap Q$

Problem (second fractal phenomenon): is it true for other indefinite types?







Limit roots

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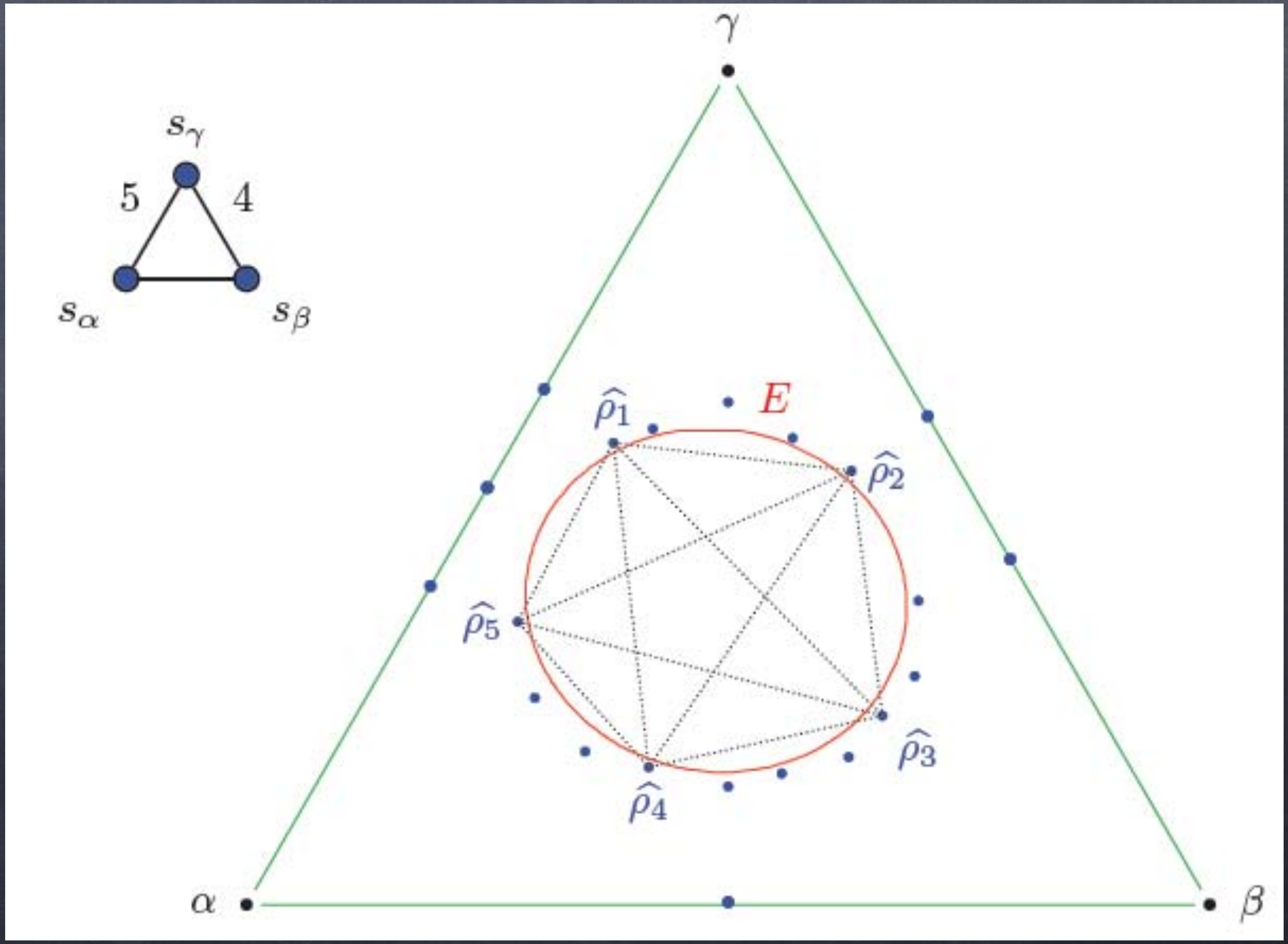
Theorem (Dyer, CH, Ripoll)
Action on E faithful if irreducible not affine nor finite of rank > 2.

Proof. Difficult. Main ingredient: one can approximate E with arbitrary precision with the sets of limit roots of universal Coxeter subgroups



Limit roots

Limit ro

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Theorem (

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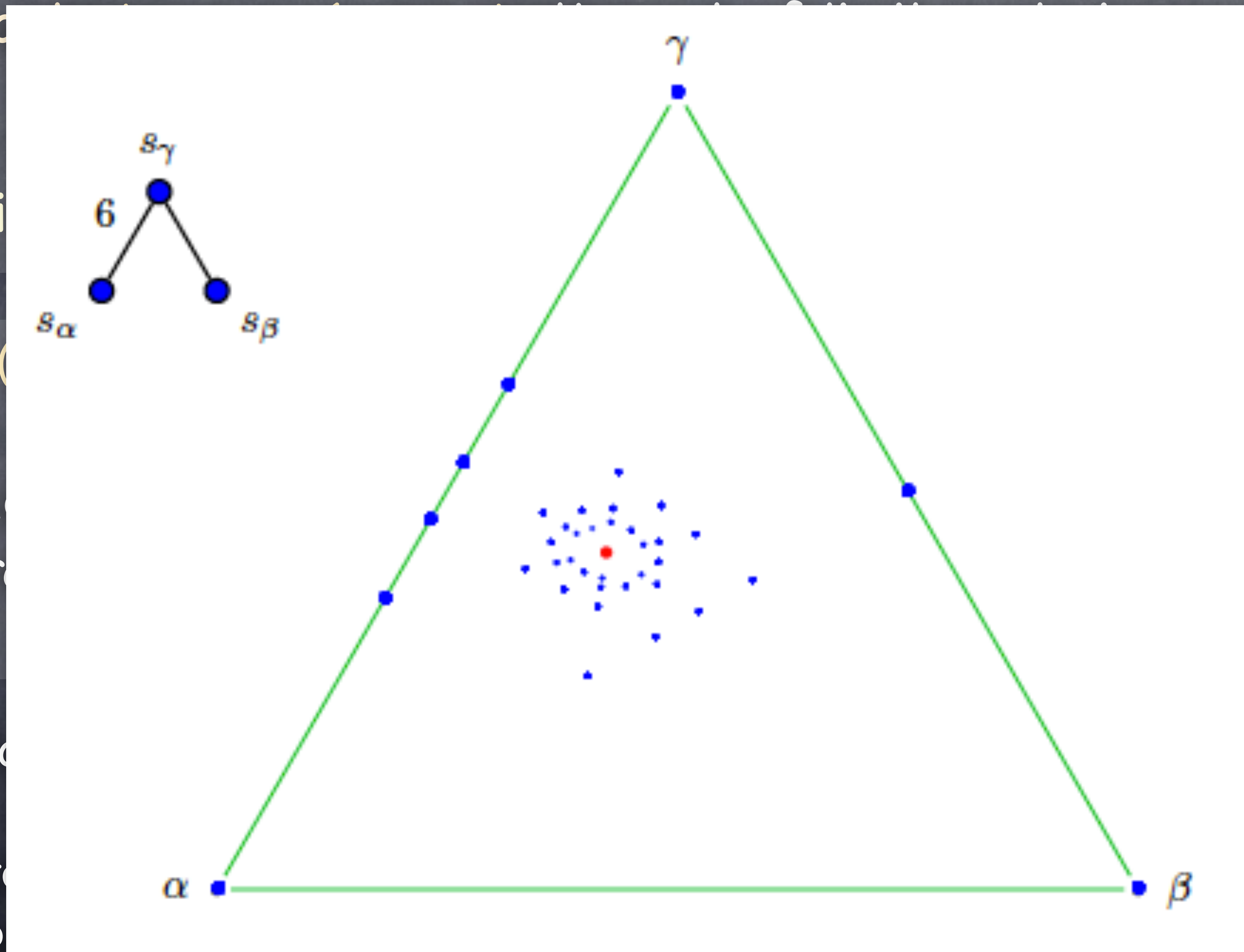

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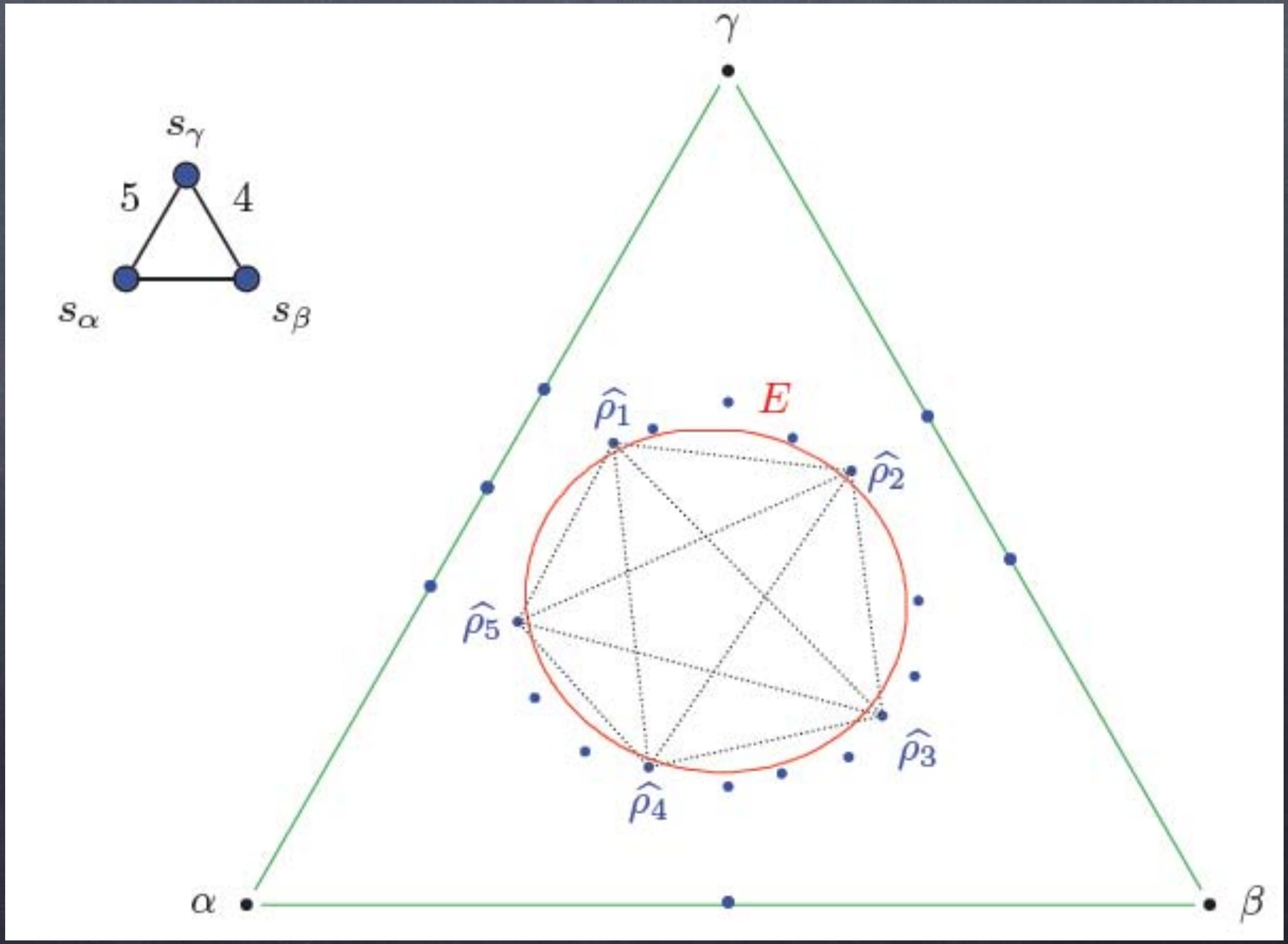


Limit roots and imaginary cone

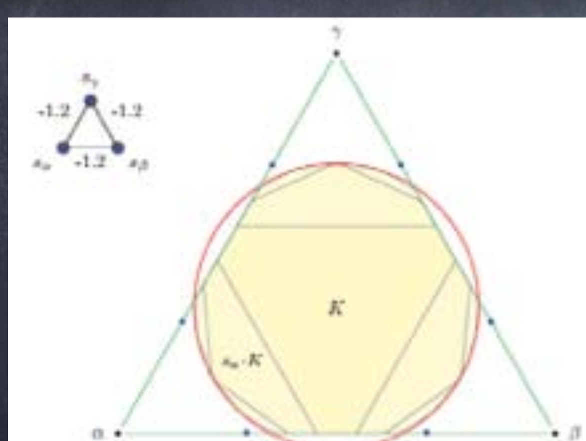
Tiling of $\mathrm{conv}(E)$

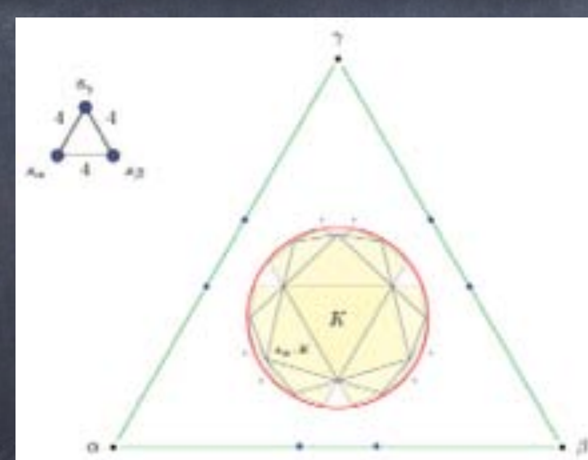
Assume the root system to be not finite nor affine

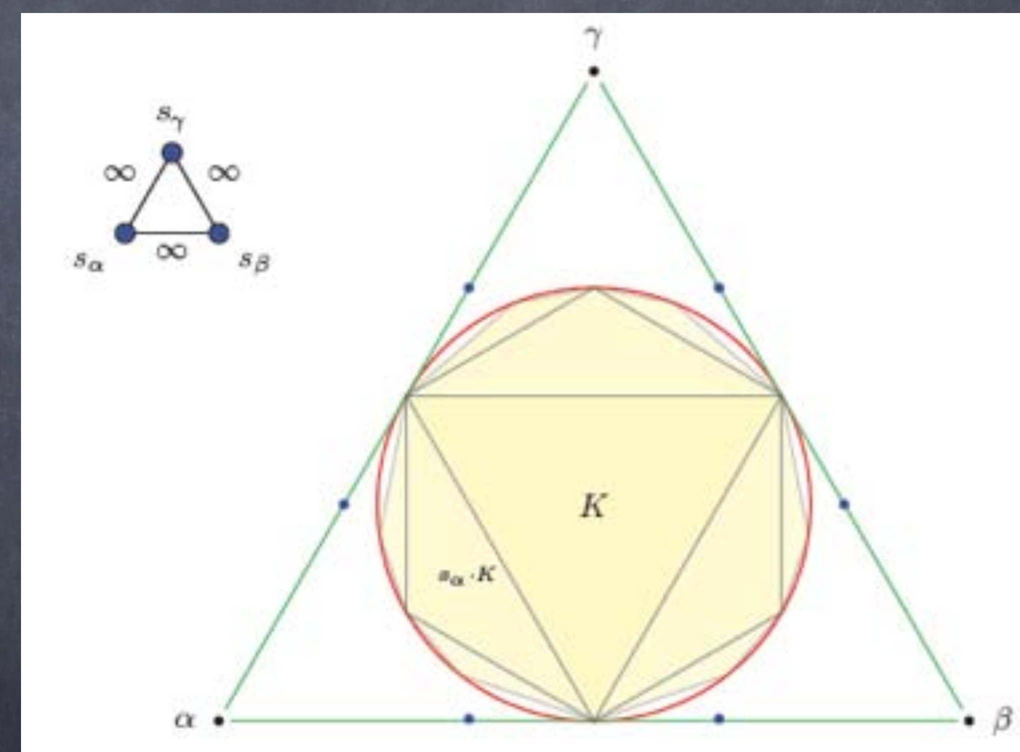
- Imaginary convex set $\mathcal{I}$ is the $W-$orbit of the polytope

$$K = \{v \in \mathrm{conv}(\Delta) \,|\, B(v,\alpha) \leq 0, \forall \alpha \in \Delta\}$$

Theorem (Dyer, 2012). $\overline{\mathcal{I}} = \mathrm{conv}(E)$



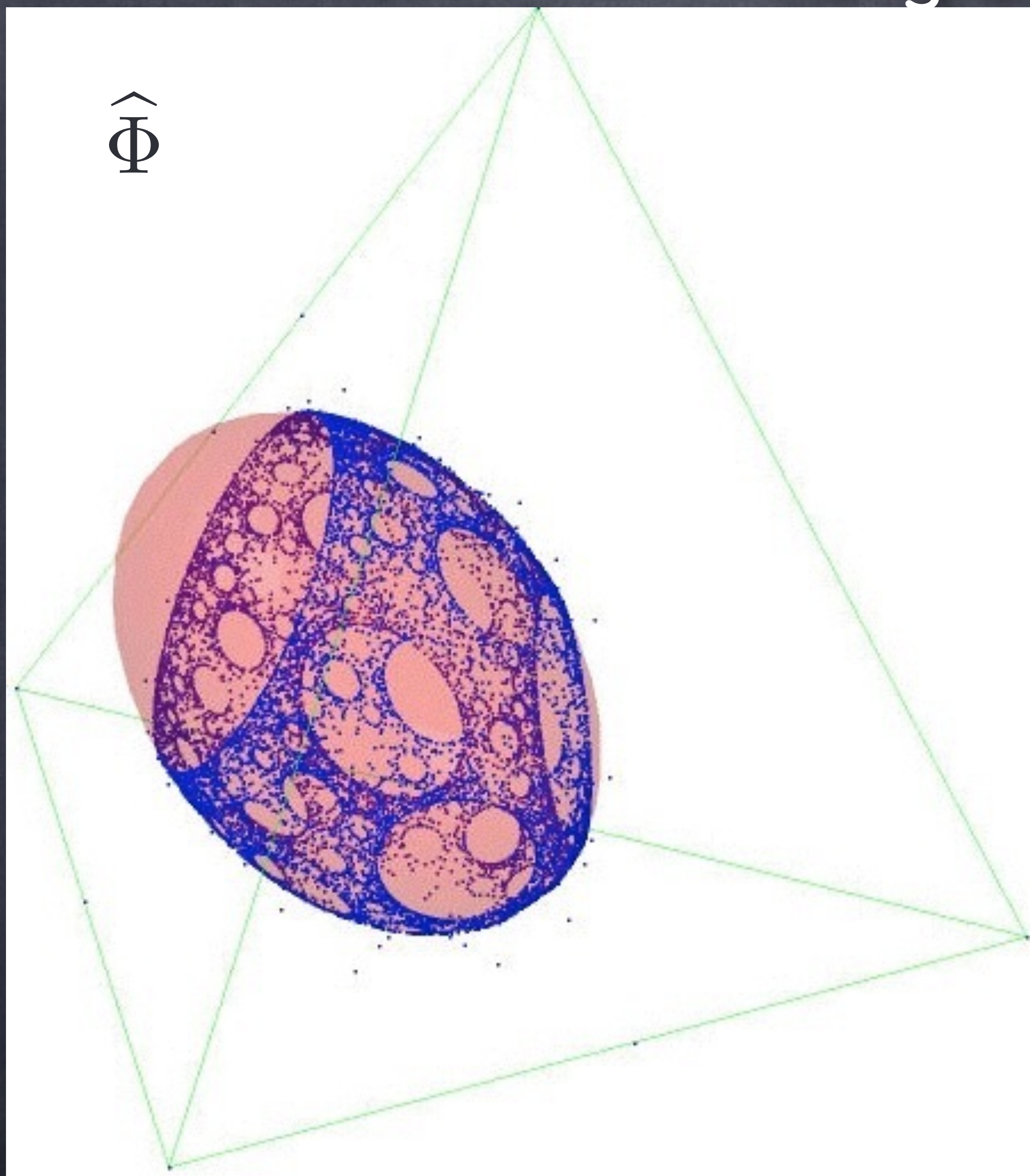


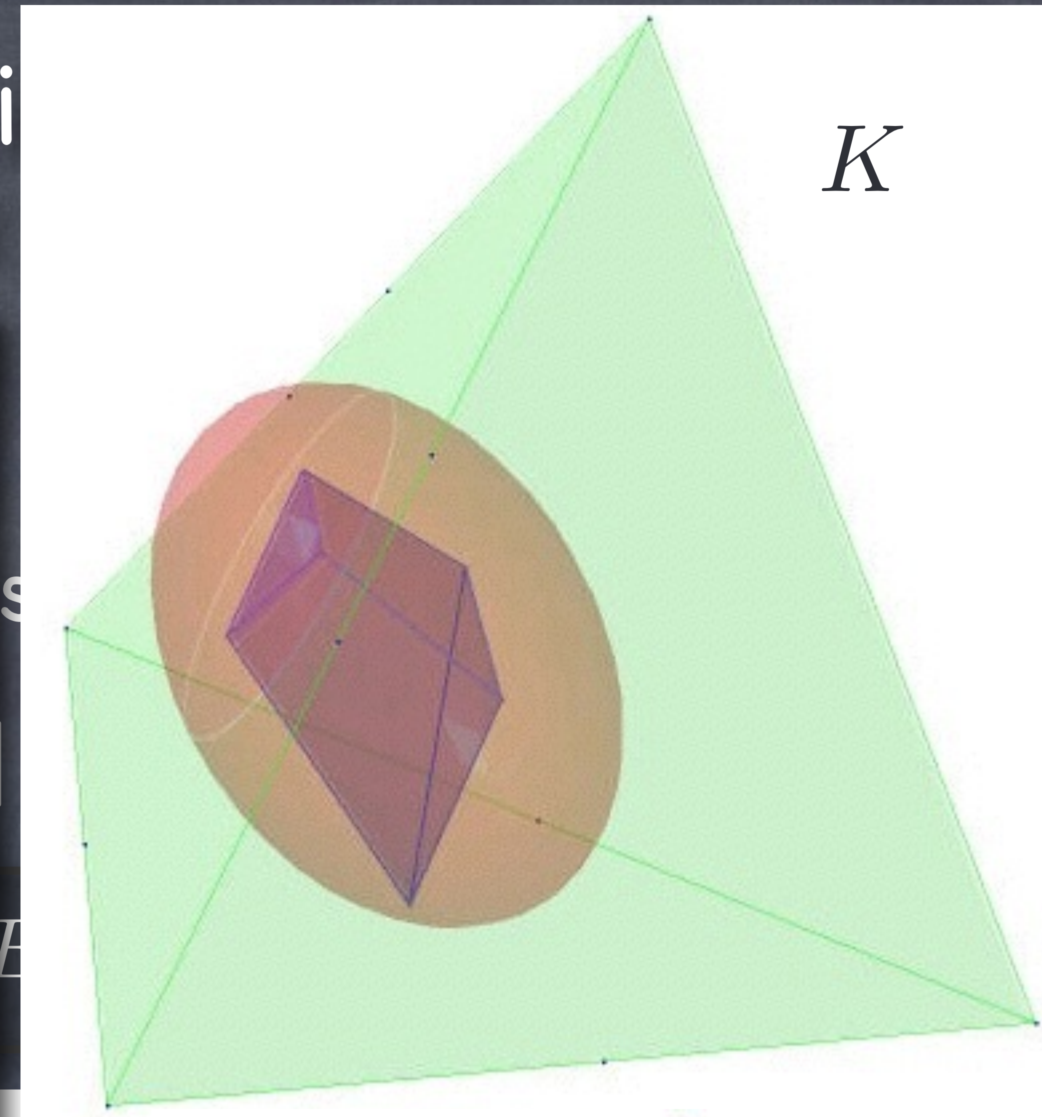


Proposition (Dyer, CH, Ripoll 2013). The action of W on E extends to an action of W on $\mathrm{conv}(E)$. So W acts on $\widehat{\Phi} \sqcup \mathrm{conv}(E)$

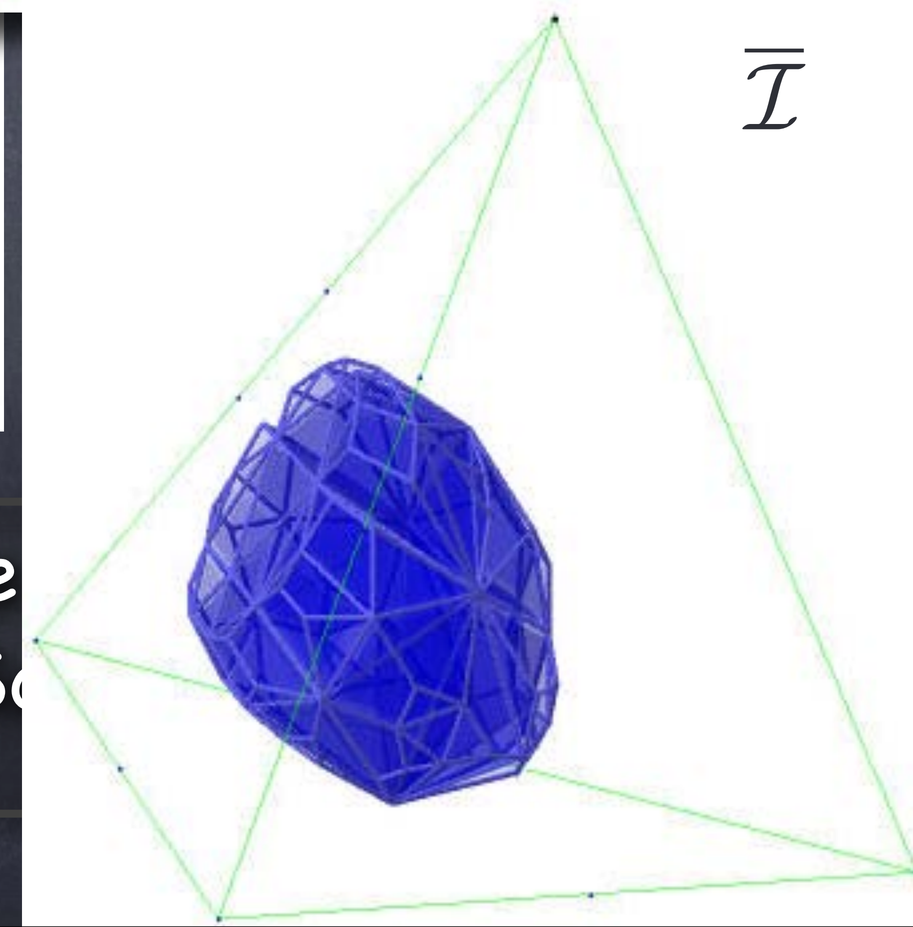
Limit roots and i…

Tiling of …





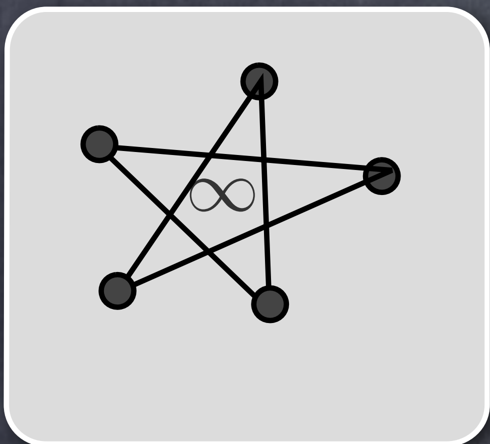
…ope



Proposition (Dyer, CH, Ripoll 2013). The … nds to an action of W on $\mathrm{conv}(E)$. S… (E)

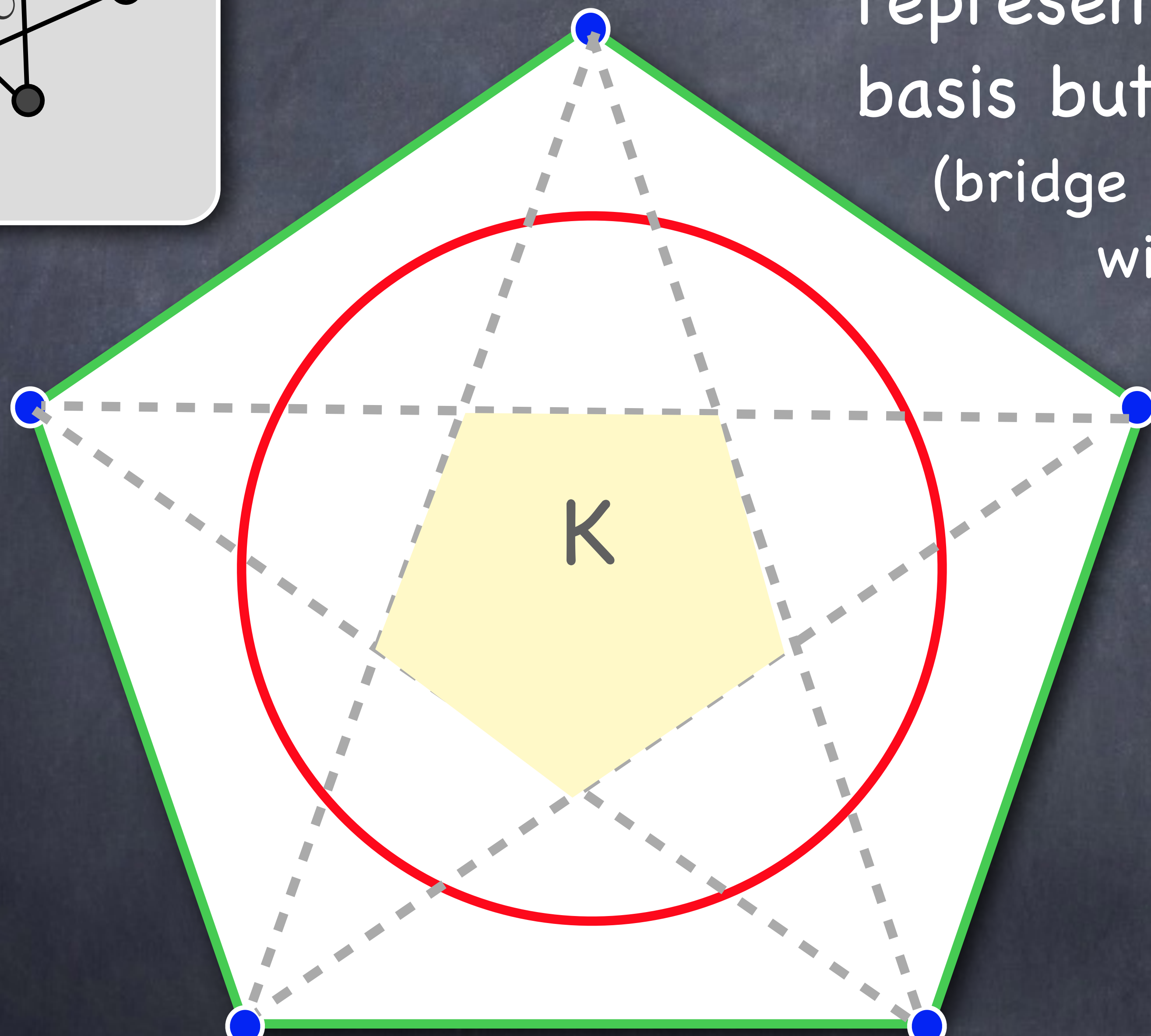
Limit roots and imaginary cone

Tiling of $\mathrm{conv}(E)$



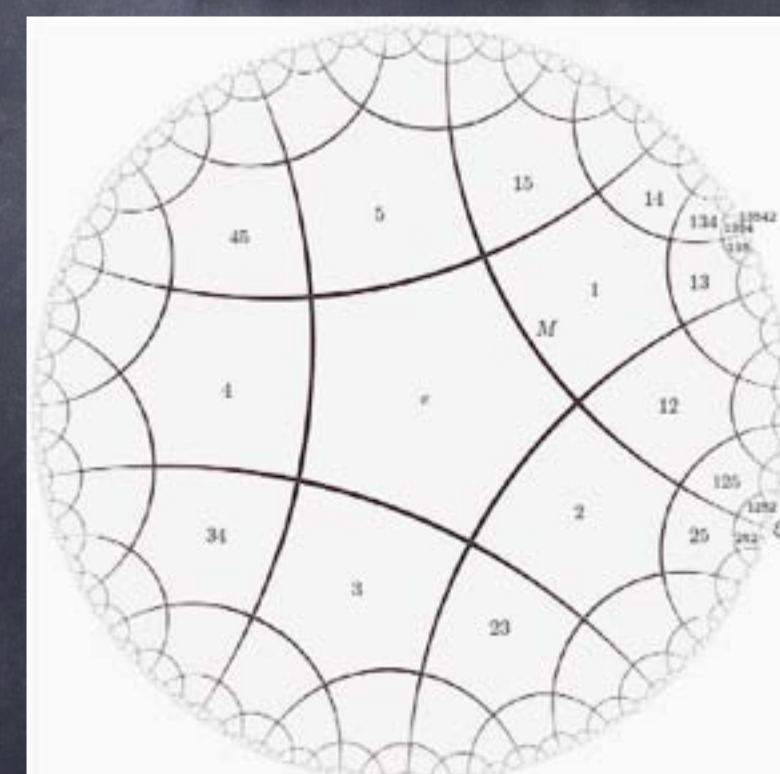
Here a rank 5 Coxeter group is represented in dim 3 Δ is not a basis but is positively independent.

(bridge with hyperbolic geometry, work with JP Préaux & V. Ripoll)



Roots and imaginary convex body model

Ball model



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Biconvex sets and biclosed sets (CH & JP Labbé)

Biconvex sets. Let $A \subseteq \Phi^+$.

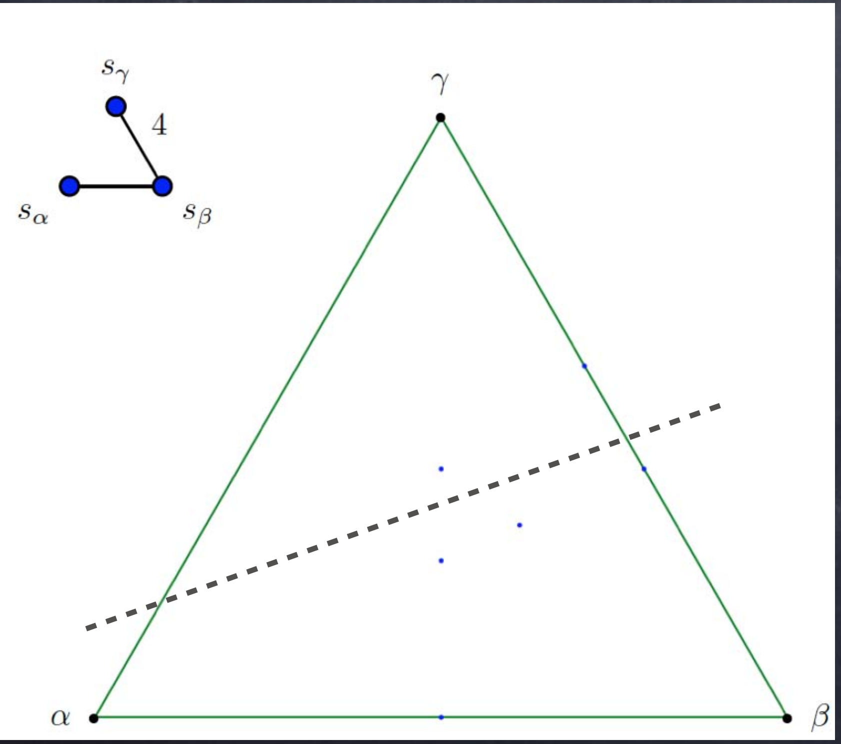
- A is convex if $\operatorname{conv}(\hat{A}) \cap \hat{\Phi} = \hat{A}$;
- A is biconvex if A and A^c are convex;
- A is separable if $\operatorname{conv}(\hat{A}) \cap \operatorname{conv}(\hat{A}^c) = \emptyset$

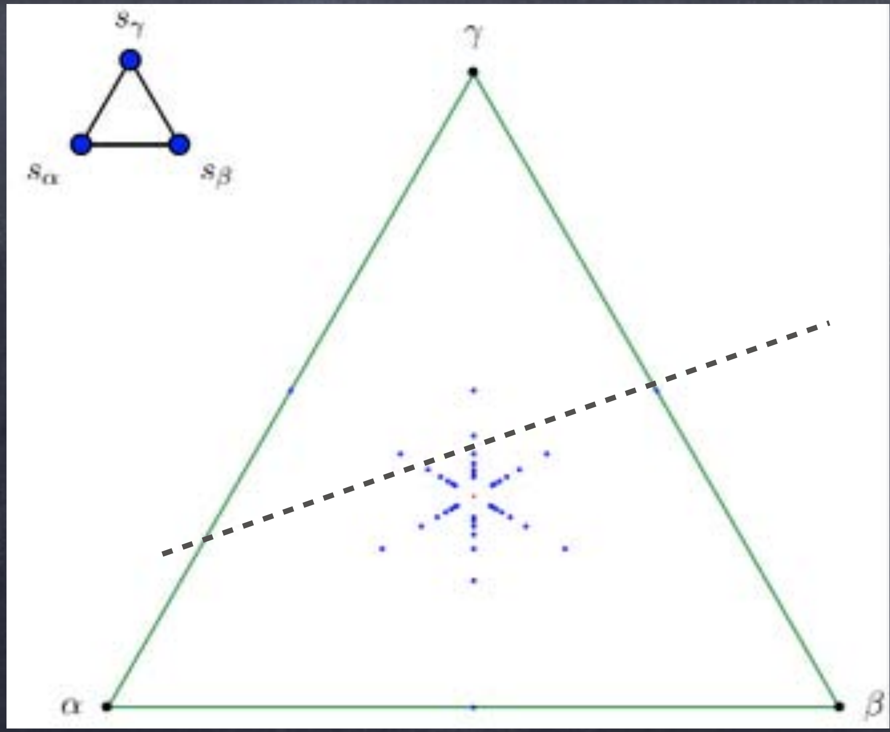
Proposition. Let $A \subseteq \Phi^+$.

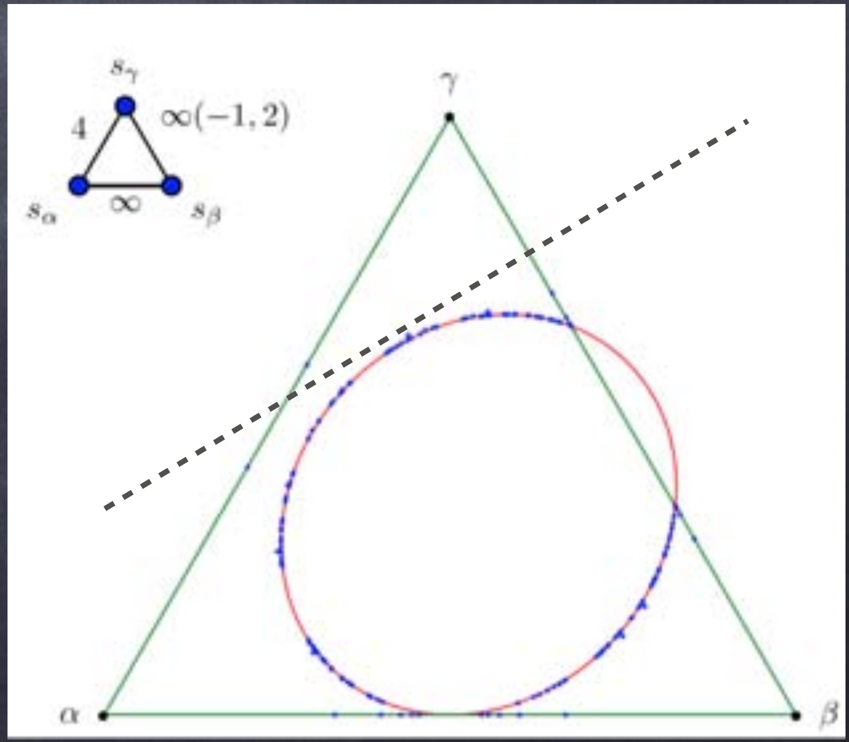
i) A is closed iff $[\hat{\alpha}, \hat{\beta}] \cap \hat{\Phi} \subseteq \hat{A}, \ \forall \alpha, \beta \in A$;

ii) separable $\Longrightarrow$ (bi)convex $\Longrightarrow$ (bi)closed

iii) A is finite biclosed iff finite separable iff $A = N(w), \ w \in W$







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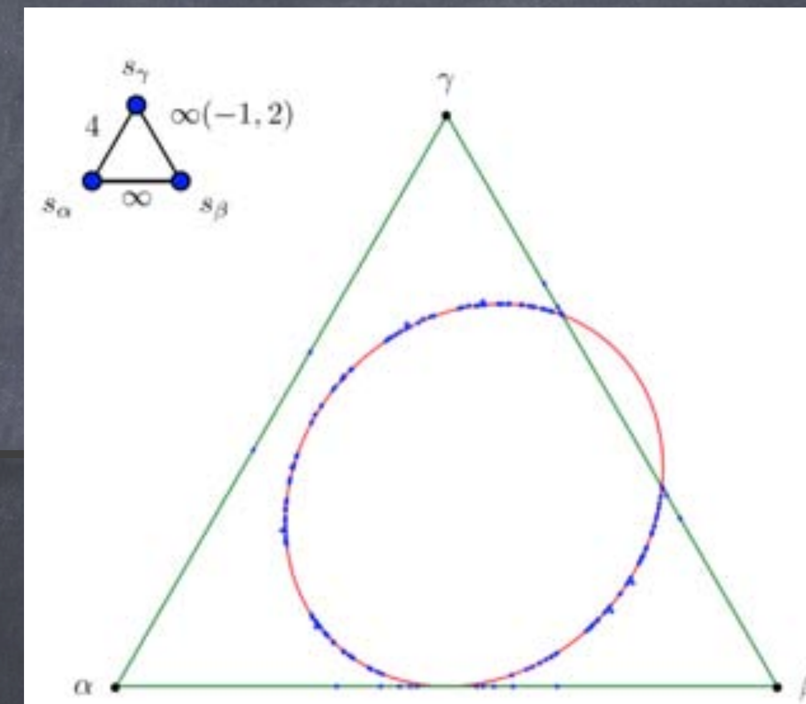

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iii) A is finite biclosed iff finite separable iff $A = N(w), \ w \in W$

Theorem (CH & JP Labbé). In rank 3, Biconvex sets with inclusion is a lattice: $\hat{A} \vee \hat{B} = \mathrm{conv}(\hat{A} \cup \hat{B}) \cap \hat{\Phi}$

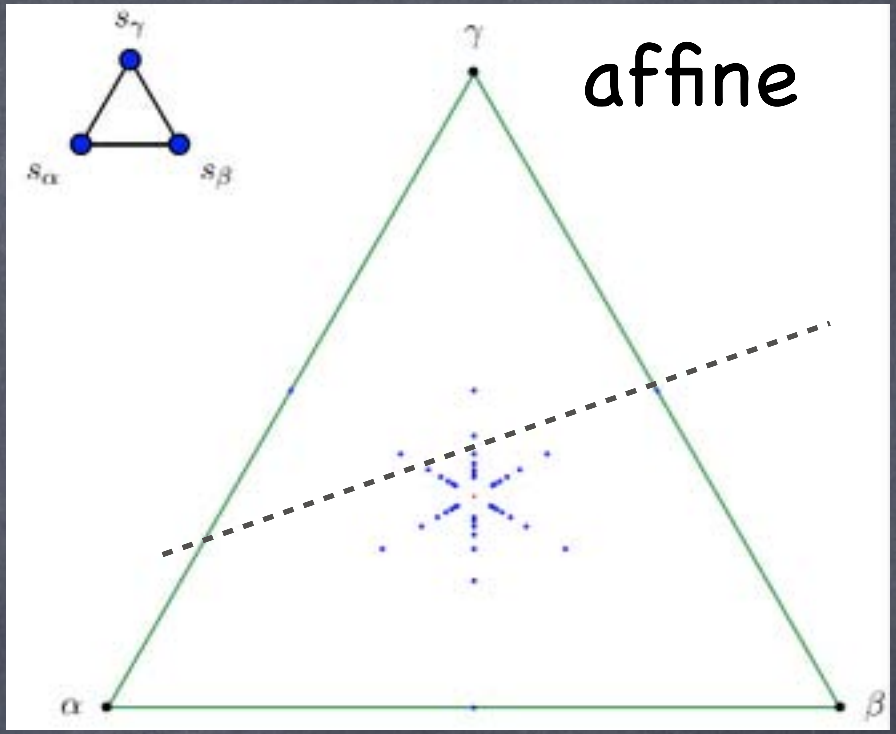
Remarks.

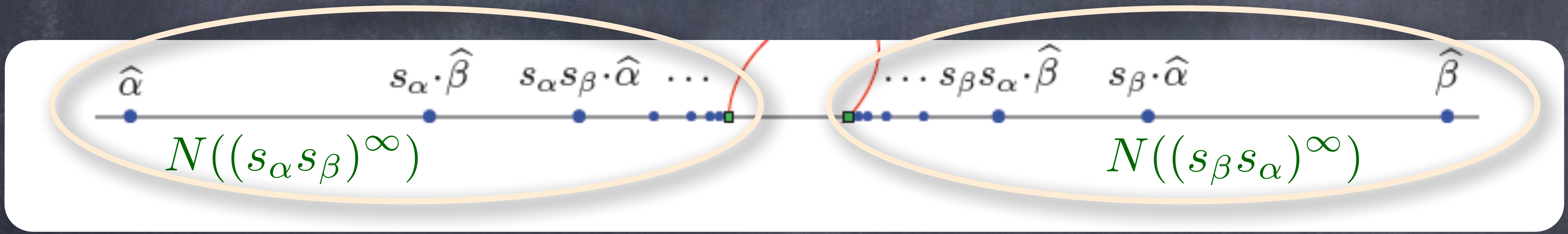
- The theorem and the converse of Prop (ii) is false in general, counterexample in rank 4 not affine nor finite;
- In rank 3 or affine type, does biconvex = biclosed?

Inversion sets of infinite words

Infinite reduced words on S. For an infinite word $w = s_1 s_2 s_3 \ldots,\ s_i \in S$, write:

- $w_i = s_1 s_2 s_3 \cdots s_i;$
- $\beta_0 = \alpha_{s_1}$ and $\beta_i = w_i(\alpha_{s_{i+1}}) \in \Phi^+.$
- w is reduced if the w_i's are.
- Inversion set: $N(w) = \{\beta_i \mid i \in \mathbb{N}\}.$





Remark. P. Cellini & P. Papi, K. Ito studied biclosed sets for Kac-Moody root systems (imaginaty root). They form a subclass: A or A^c verify $\mathrm{conv}(\hat{A}) \cap Q = \emptyset$.

Inversion sets of infinite words

Theorem (Cellini & Papi, 1998). Let the root system be affine, i.e., Q is a singleton. Let $A \subseteq \Phi^+$ s.t. $\mathrm{conv}(\hat{A}) \cap Q = \emptyset$. Then:

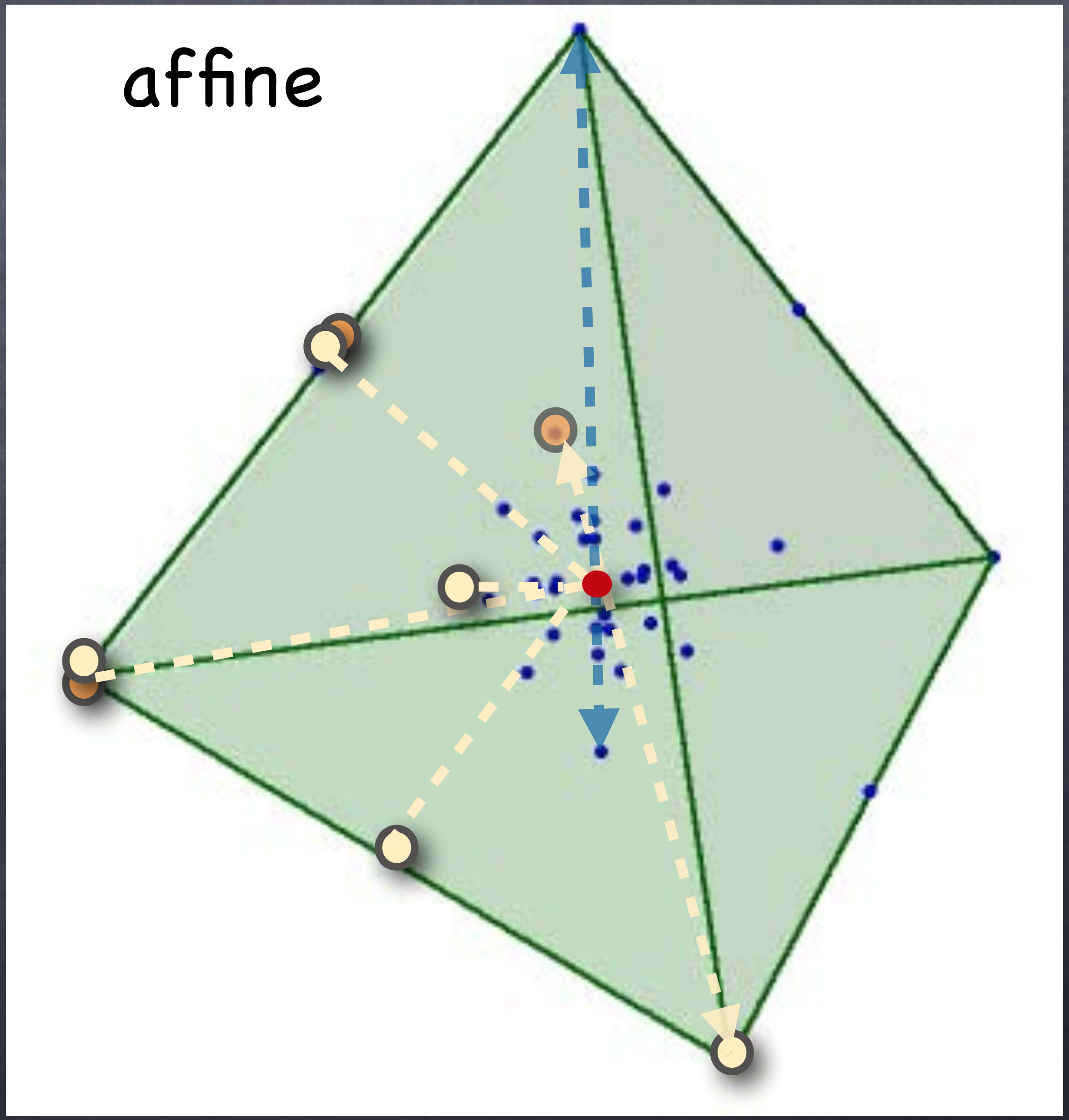
A biclosed iff A separable iff $A = N(w)$, w finite or infinite.

Remark. The class of $A \subseteq \Phi^+$ s.t. A or A^c verify $\mathrm{conv}(\hat{A}) \cap Q = \emptyset$ is not satisfying (negative answer to a question asked by Lam & Pylyavskyy; Baumann, Kamnitzer & Tingley)

$$\hat{N}(21321) \vee \hat{N}(214) = \bullet \vee \bullet$$

$$= \mathrm{conv}(\bullet \cup \bullet) \cap \hat{\Phi}$$

does not arise as an inversion set of a word (finite or infinite)

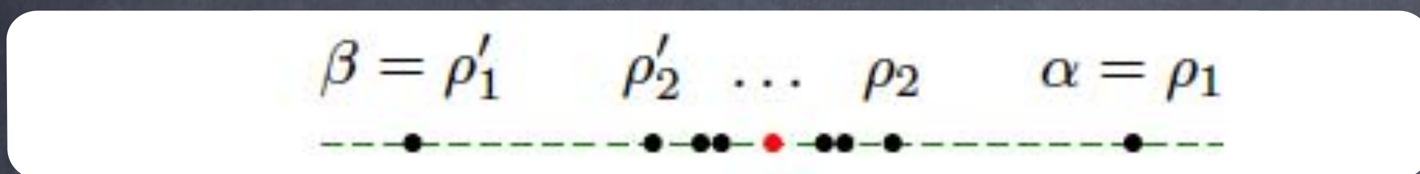


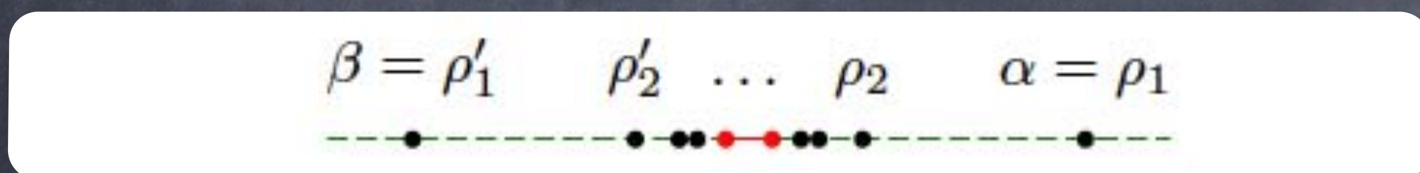
Christophe Hohlweg, 2014

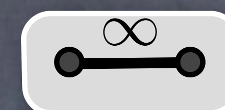
Inversion sets of infinite words (CH & JP Labbé)

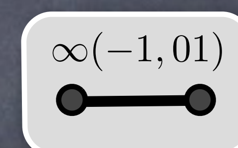
Let $A \subseteq \Phi^+$, we say that:

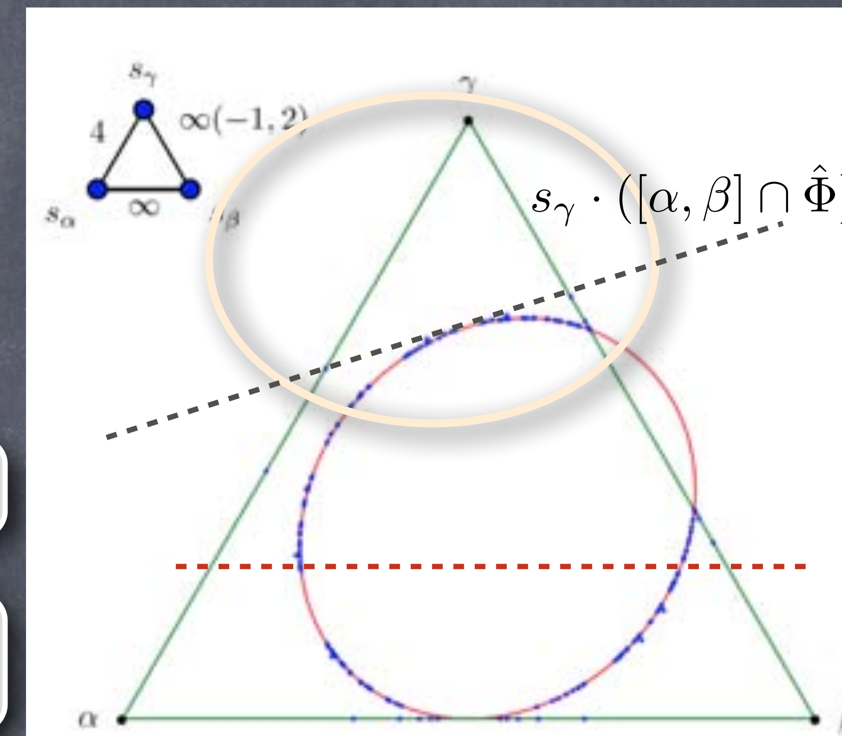
- A **avoids** E if $[\hat{\alpha}, \hat{\beta}] \cap Q = \emptyset, \ \forall \alpha, \beta \in A$
- A **strictly avoids** E if $\mathrm{conv}(\hat{A}) \cap E = \emptyset$.











strictly avoids $\Longrightarrow$ avoids

Proposition. Let $A \subseteq \Phi^+$ be finite.

i) if A is closed then A avoids E ;

ii) if A is convex then A strictly avoids E.

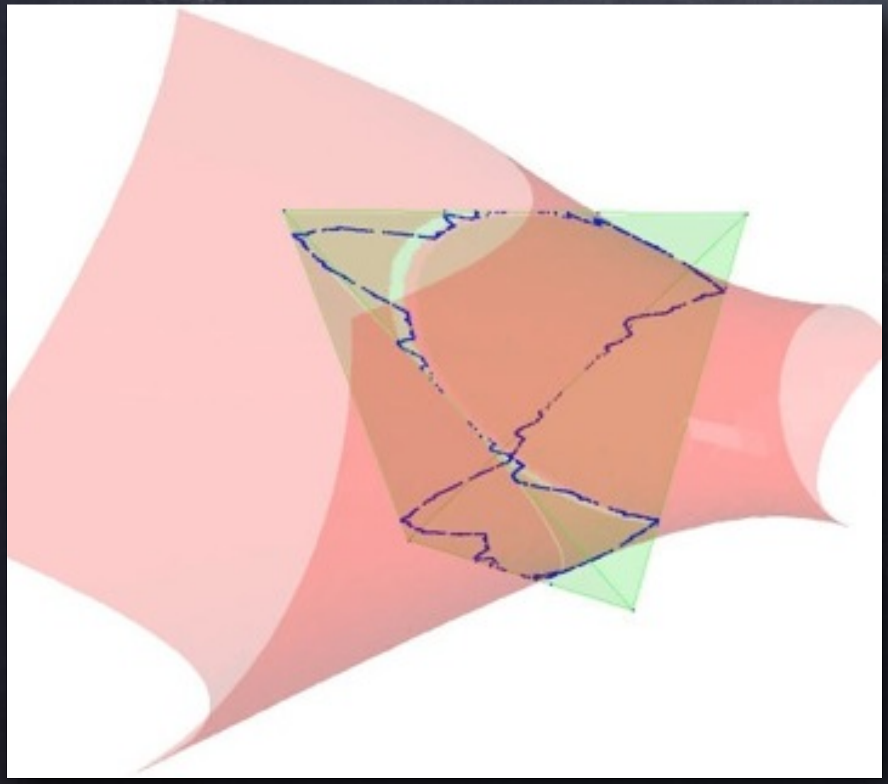
Inversion sets of infinite words (CH & JP Labbé)

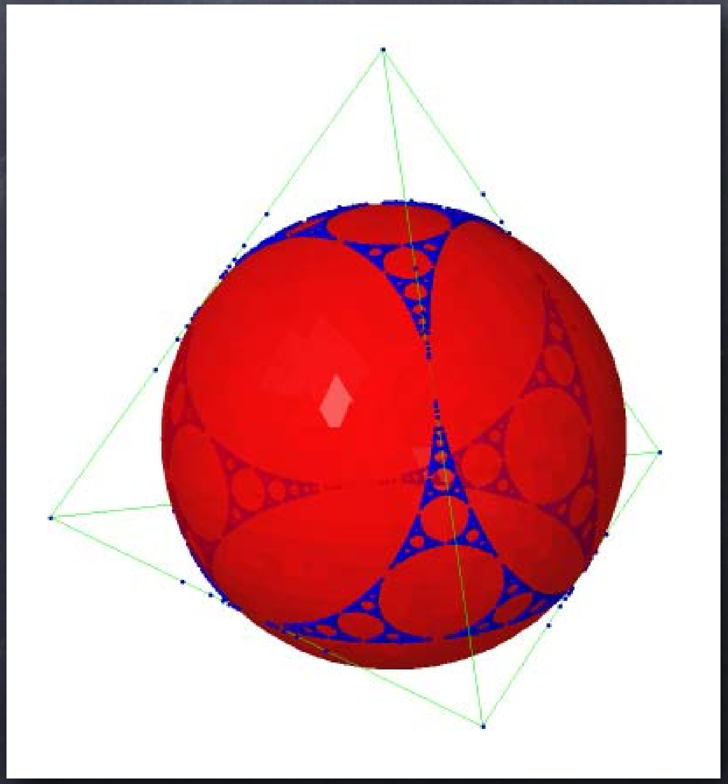
Corollary. If $A = N(w)$ with w reduced infinite or finite word, then A strictly avoids E and is biconvex.

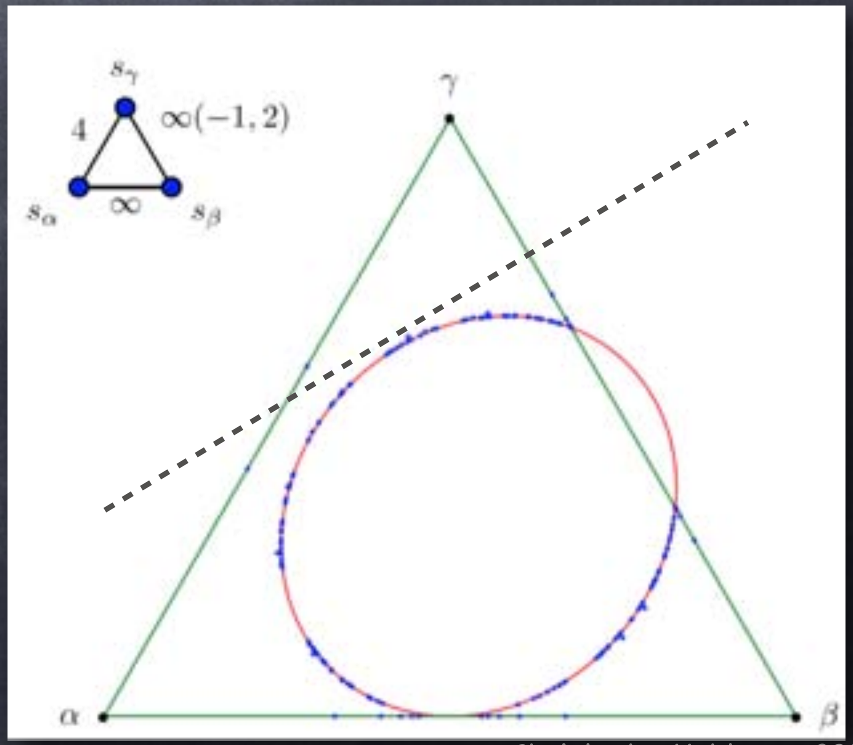
Questions:

i) the converse is true? (true for affine by Cellini & Papi);

ii) $|\mathrm{Acc}(N(w)| \leq 1$?; obviously true for finite and affine; true for weakly hyperbolic (H. Chen & JP Labbé, 2014)







Inversion sets of infinite words (CH & JP Labbé) and $\mathrm{conv}(E)$

Assume the root system to be not finite nor affine

For a reduced $w = s_1 s_2 s_3 \ldots,\ s_i \in S$, recall that:

- $w_i = s_1 s_2 s_3 \cdots s_i$; reduced; $\beta_0 = \alpha_{s_1}$ and $\beta_i = w_i(\alpha_{s_{i+1}}) \in \Phi^+$.
- Inversion set: $N(w) = \{\beta_i \mid i \in \mathbb{N}\}$.

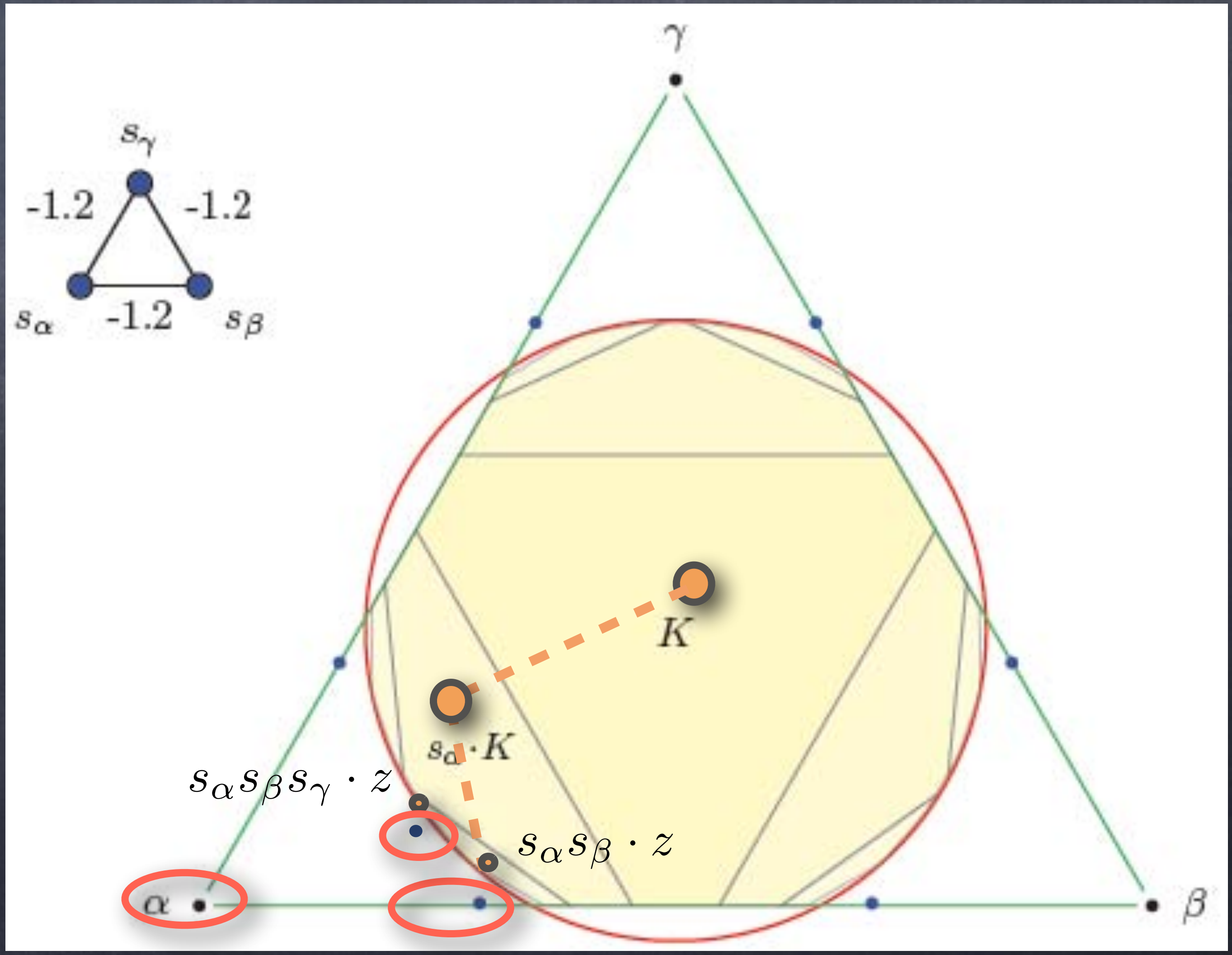
Representation in $\mathrm{conv}(E)$:

$z \in \mathrm{relint}(K)$ and $\{w_i \cdot z,\ i \in \mathbb{N}\}$

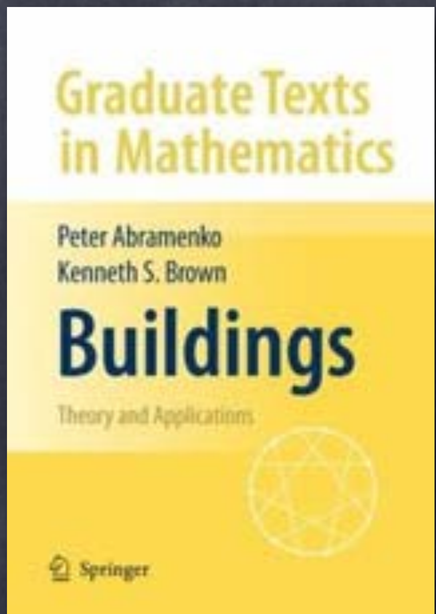
Conjecture.

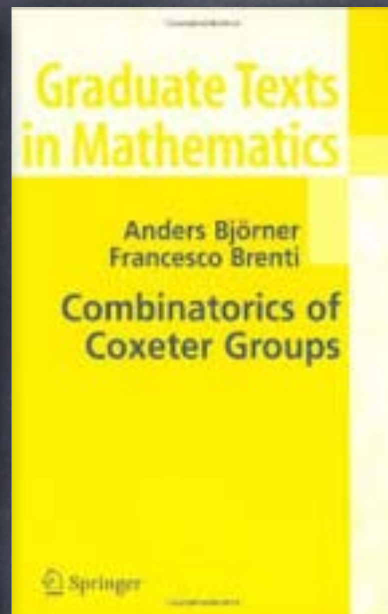
$$\mathrm{Acc}(\hat{N}(w) = \mathrm{Acc}(\{w_i \cdot z,\ i \in \mathbb{N}\})$$

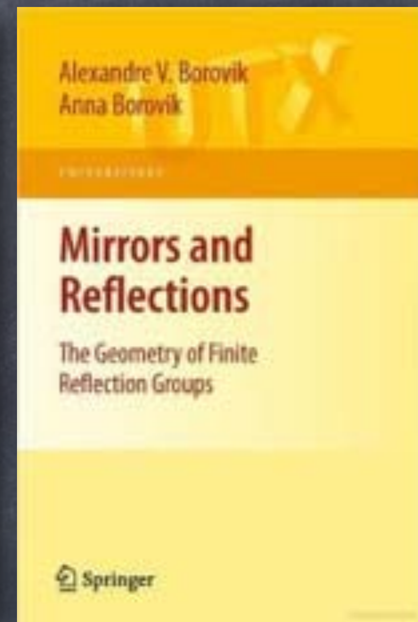
Questions. Link with Lam & Thomas, 2013? Geometric realization of the Davis complex?

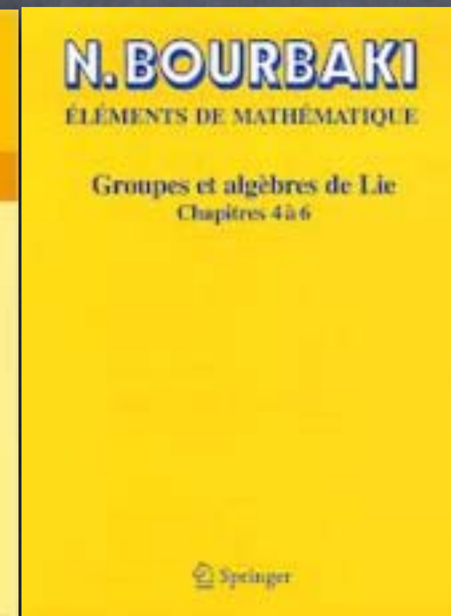


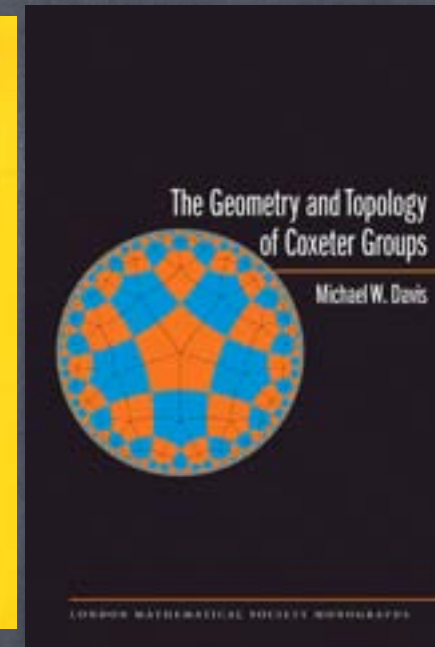
Selected bibliography and other readings

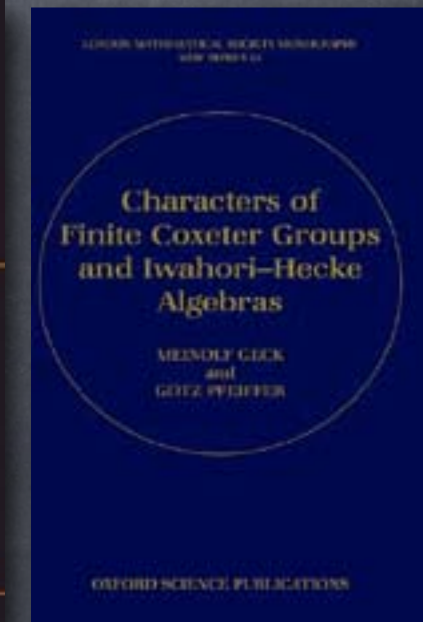


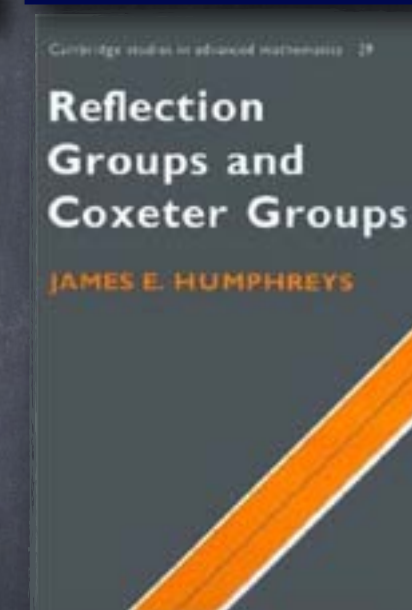












And articles already cited + from

- Brigitte Brink, Bill Casselman, Fokko du Cloux, Bob Howlett, Xiang Fu (regarding automaton and comb.)
- Matthew Dyer (imaginary cones, weak order(s))
- CH & coauthors (Matthew Dyer, Jean-Philippe Labbé, Jean-Philippe Préaux, Vivien Ripoll). A good start for limit of roots and imaginary convex bodies is the survey of the case of Lorentzian spaces (CH, Ripoll, Préaux)
- P Papi and Ken Ito (limit weak order)
- Hao Chen and Jean-Philippe Labbé (Sphere packing)
- …