An analogue of Schensted's bumping algorithm in affine type A

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Fock spaces representations of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$

Let $e \in \mathbb{Z}_{>1}$, q an indeterminate.

$\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ = q-deformation of the Lie algebra of affine type A.

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Theorem (Jimbo, Misra, Miwa, Okado 1991)

$\mathcal{F}_{\mathbf{s}}$ is an (integrable) $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module.

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- Symbol of $\boldsymbol{\lambda}$ of shape $\mathbf{s}$:

$$\begin{pmatrix} \dots & -4 & -3 & -2 & & & & & \\ \dots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \dots & -4 & -3 & -2 & -1 & 0 & & & \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 1 & & & & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & & & \end{pmatrix}$$

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Definition

$\mathcal{S}(\boldsymbol{\lambda}, \mathbf{s})$ is *semistandard* if $s_c \leq s_{c+1}$ for all $c = 1, \dots, l$ and if the columns (resp. rows) of $\mathcal{S}(\boldsymbol{\lambda}, \mathbf{s})$ are non-decreasing (resp. increasing).

Crystal graphs

According to Kashiwara, $\mathcal{F}_{\mathbf{s}}$ has a *crystal graph* $B(\mathcal{F}_{\mathbf{s}})$.

- Vertices = all l-partitions.
- Edges of B = directed arrows colored by $i \in \{0, \dots e-1\}$.

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Properties

- *$B(\mathcal{F}_{\mathbf{s}}) = \bigsqcup B$ (connected components).*
- *Each B has a unique vertex $\overset{\bullet}{\boldsymbol{\lambda}}$ with zero-indegree : the source vertex. We denote $B = B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$.*
- *Each $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ writes $\overset{\bullet}{\boldsymbol{\lambda}} \xrightarrow{i_1} \dots \xrightarrow{i_p} \boldsymbol{\lambda}$*
- *$\boldsymbol{\emptyset}$ is always a source vertex.*
- *There is a natural graph isomorphism between $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ and $B(\boldsymbol{\emptyset}, \mathbf{r})$ for some $\mathbf{r} \in \mathscr{S}_e$ where $\mathscr{S}_e = \{\mathbf{t} \in \mathbb{Z}^l \mid 0 \leq t_d - t_c < e$ for $c < d\}$.*

An example

Take $e = 3$,
$\mathbf{s} = (1, 3)$
and $\boldsymbol{\lambda} = (1, 1^2)$.
One can
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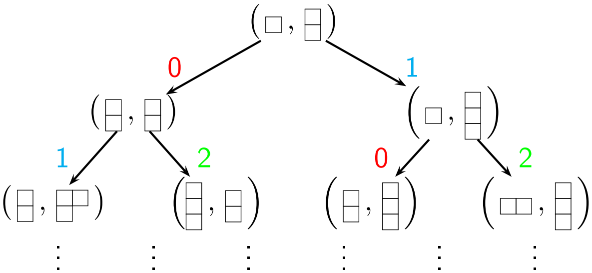
$B(\boldsymbol{\lambda}, \mathbf{s})$:



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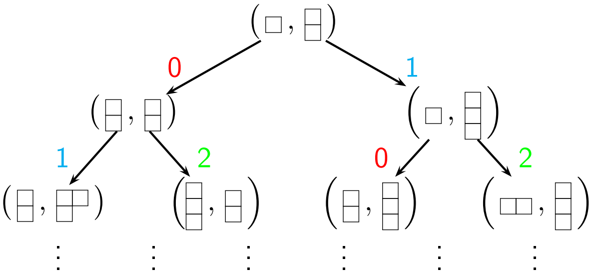
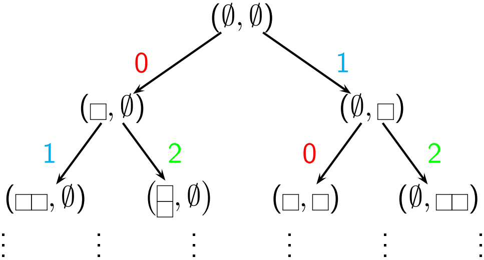


This crystal graph is isomorphic to $B(\emptyset, \mathbf{r})$ with $\mathbf{r} = (0, 1)$:



Notation

We write $|\boldsymbol{\lambda}, \mathbf{s}\rangle \sim |\boldsymbol{\nu}, \mathbf{t}\rangle$ if $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}) \simeq B(\overset{\bullet}{\boldsymbol{\nu}}, \mathbf{t})$ and if $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ appear at the same place in their respective crystal.

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Problem: Give a combinatorial description of the relation $\sim$.

In other terms, starting from arbitrary $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, determine $\mathbf{r} \in \mathscr{S}_e$ and $\boldsymbol{\mu} \in B(\emptyset, \mathbf{r})$ such that $|\boldsymbol{\lambda}, \mathbf{s}\rangle \sim |\boldsymbol{\mu}, \mathbf{r}\rangle$.

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$\rightsquigarrow$ Interesting because only $B(\emptyset, \mathbf{r})$ has an explicit combinatorial description (Foda-Leclerc-Okado-Thibon-Welsh).

$\rightsquigarrow$ Natural interpretations in terms of representations of the complex reflection group $G(l, 1, n)$ (Lascoux-Leclerc-Thibon, Ariki, Shan, Losev).

1^{st} tool: the cyclage

For $\mathbf{s} = (s_1, \dots, s_l)$ and $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$, we define

$$\xi(\mathbf{s}) = (s_l - e, s_1, \dots, s_{l-1}) \quad \text{and} \quad \xi(\boldsymbol{\lambda}) = (\lambda^l, \lambda^1, \dots, \lambda^{l-1}).$$

Example: Take $e = 3$, $\mathbf{s} = (2, 0, 1)$ and $\boldsymbol{\lambda} = (3.2, 1, 4^2)$. Then

$$\xi(\mathbf{s}) = (-2, 2, 0) \quad \text{and} \quad \xi(\boldsymbol{\lambda}) = (4^2, 3.2, 1).$$

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Proposition

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle \sim |\xi(\boldsymbol{\lambda}), \xi(\mathbf{s})\rangle.$$

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Aim: construct a semistandard symbol, starting from an arbitrary symbol.

Example: Let $\mathbf{s} = (2, 0)$ and $\boldsymbol{\lambda} = (3.2, 3^2)$

$$\mathcal{S}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} 0 & 4 & 5 & & \\ 0 & 1 & 2 & 5 & 7 \end{pmatrix}.$$

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Reading of $\boldsymbol{\lambda}$: $w = 54075210$.

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Theorem (Kashiwara 90's)

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle \sim |\boldsymbol{\nu}, \mathbf{t}\rangle.$$

Denote **RS** : $|\boldsymbol{\lambda}, \mathbf{s}\rangle \longmapsto |\boldsymbol{\nu}, \mathbf{t}\rangle$.

There exists $M \in \mathbb{N}$ such that $|\boldsymbol{\nu}, \mathbf{t}\rangle = (\mathbf{RS} \circ \xi)^M(|\boldsymbol{\lambda}, \mathbf{s}\rangle)$ verifies:

1. $\mathcal{S}(\boldsymbol{\nu}, \mathbf{t})$ is semistandard, and
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Theorem (Foda-Leclerc-Okado-Thibon-Welsh 1999)

$|\boldsymbol{\nu}, \mathbf{t}\rangle \in B(\emptyset, \mathbf{t})$ *except if* $\boldsymbol{\nu}$ *contains* e *parts of the same size* $\alpha > 0$ *such that*

$$\{\alpha - a_i + s_{c_i} \mod e \; ; \; i = 1, \dots, e\} = \{0, \dots, e-1\},$$

where a_i= *row fo the* i*-th part, and* c_i= *component of the* i*-th part.*

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$\rightsquigarrow$ How to get rid of the "bad" parts?

3^{rd} tool : The reduction isomorphism

Let $e = 4$, $l = 3$, $\mathbf{t} = (5, 6, 8)$, and $\boldsymbol{\nu} = (4^2.2.1^2, 3.2^2.1^2, 4.2^3.1)$.

$$|\boldsymbol{\nu}, \mathbf{t}\rangle = \left(\begin{array}{|c|c|c|c|} \hline 5 & 6 & 7 & 8 \\ \hline 4 & 5 & 6 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 6 & 7 & 8 \\ \hline 5 & 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|} \hline 8 & 9 & 10 & 11 \\ \hline 7 & 8 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \right)$$

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5	6	7	8
4	5	6	7
3	4		
2			
1			

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5	6	
4	5	
3		
2		

8	9	10	11
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4			

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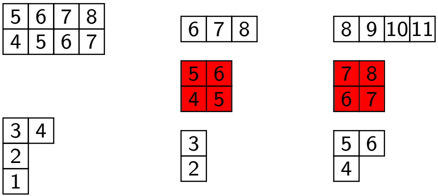

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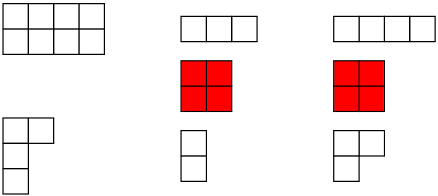
Let $e = 4$, $l = 3$, $\mathbf{t} = (5, 6, 8)$, and $\boldsymbol{\nu} = (4^2.2.1^2, 3.2^2.1^2, 4.2^3.1)$.



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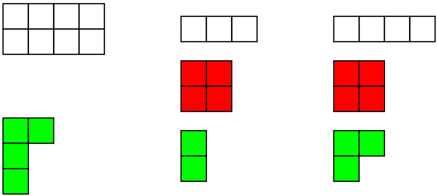

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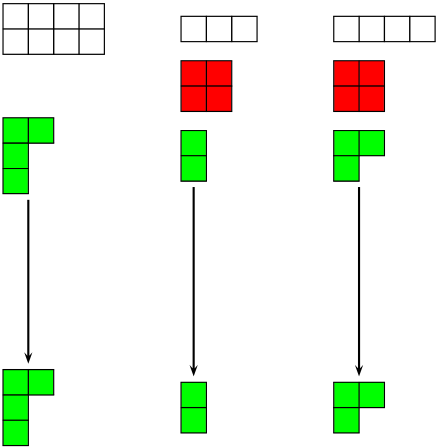

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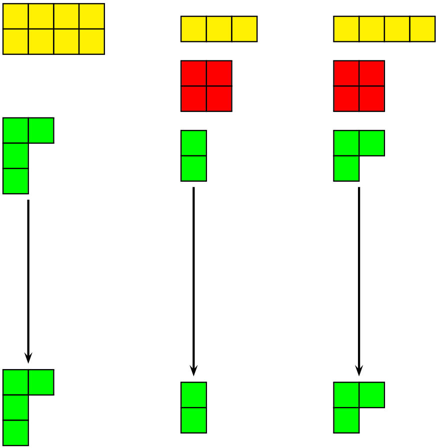

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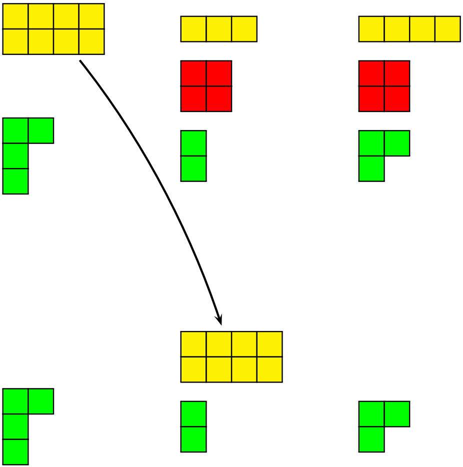

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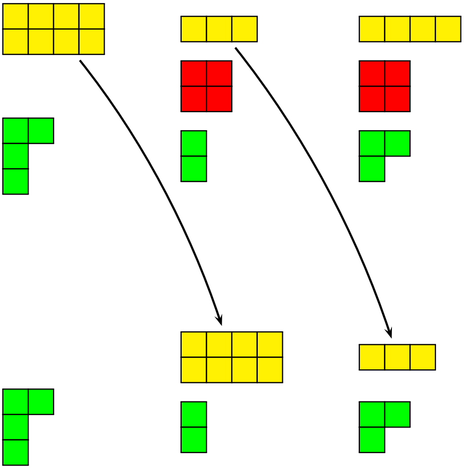

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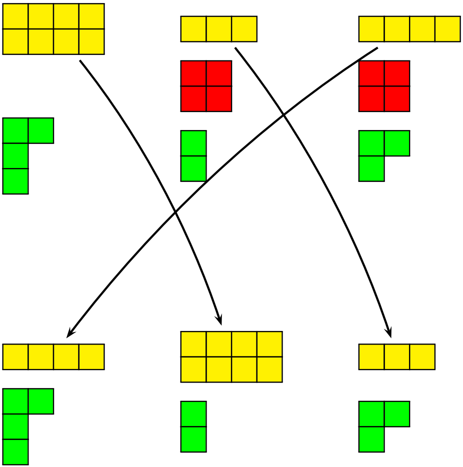

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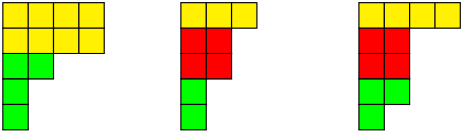

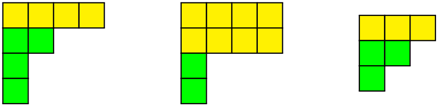

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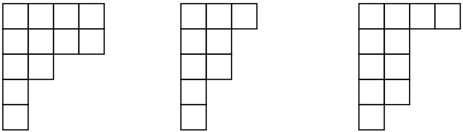

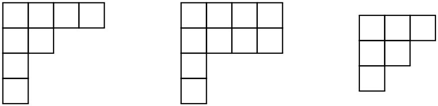

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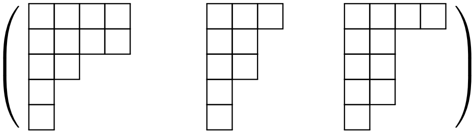
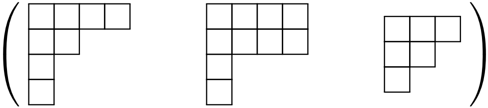
$\boldsymbol{\nu} =$



$\rho(\boldsymbol{\nu}) =$



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$$|\boldsymbol{\nu}, \mathbf{t}\rangle = \left(\begin{array}{llll} 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \\ 3 & 4 & & \\ 2 & & & \\ 1 & & & \end{array} \quad \begin{array}{lll} 6 & 7 & 8 \\ 5 & 6 & \\ 4 & 5 & \\ 3 & & \\ 2 & & \end{array} \quad \begin{array}{llll} 8 & 9 & 10 & 11 \\ 7 & 8 & & \\ 6 & 7 & & \\ 5 & 6 & & \\ 4 & & & \end{array} \right)$$

$$|\rho(\boldsymbol{\nu}), \xi(\mathbf{t})\rangle = \left(\begin{array}{llll} 4 & 5 & 6 & 7 \\ 3 & 4 & & \\ 2 & & & \\ 1 & & & \end{array} \quad \begin{array}{llll} 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \\ 3 & & & \\ 2 & & & \end{array} \quad \begin{array}{lll} 6 & 7 & 8 \\ 5 & 6 & \\ 4 & & \end{array} \right)$$

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Theorem (G. 2013)

$$|\boldsymbol{\nu}, \mathbf{t}\rangle \sim |\rho(\boldsymbol{\nu}), \xi(\mathbf{t})\rangle.$$

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$$|\rho(\boldsymbol{\nu}), \xi(\mathbf{t})\rangle = \left(\begin{array}{llll} 4 & 5 & 6 & 7 \\ 3 & 4 & & \\ 2 & & & \\ 1 & & & \end{array} \quad \begin{array}{llll} 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \\ 3 & & & \\ 2 & & & \end{array} \quad \begin{array}{lll} 6 & 7 & 8 \\ 5 & 6 & \\ 4 & & \end{array} \right)$$

Theorem (G. 2013)

$$|\boldsymbol{\nu}, \mathbf{t}\rangle \sim |\rho(\boldsymbol{\nu}), \xi(\mathbf{t})\rangle.$$

$\rightsquigarrow$ There exists $N \in \mathbb{N}$ such that $|\rho^N(\boldsymbol{\nu}), \xi^N(\mathbf{t})\rangle \in B(\boldsymbol{\emptyset}, \xi^N(\mathbf{t}))$.

Analogy with the non-affine case

	Finite type A	Affine type A
Quantum group	$\mathcal{U}_q(\mathfrak{sl}_e)$	$\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$
Fock space	Basis = Young tableaux	Basis = charged l-partitions/symbols
Interesting connected component of the crystal	Vertices = semistandard tableaux	Vertices = FLOTW symbols
Crystal isomorphisms	**RS** for tableaux	**RS** for symbols
		cyclage ξ
		reduction ρ
Equivalence relation expected	$\mathbf{T}_1 \sim \mathbf{T}_2$ iff $\mathbf{RS}(\mathbf{T}_1) = \mathbf{RS}(\mathbf{T}_2)$	$\boldsymbol{\lambda}_1 \sim \boldsymbol{\lambda}_2$ iff $\Phi(\boldsymbol{\lambda}_1) = \Phi(\boldsymbol{\lambda}_2)$ where $\Phi = \rho^N \circ (\mathbf{RS} \circ \xi)^M$