Bijective proofs of character evaluations using trace forest of the jeu de taquin

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Characters of the symmetric group

Irreducible characters of S_n are very useful in combinatorics.

- Combinatorial maps
- Limit form of partitions
- etc...

There is also a beautiful combinatorial theory.

- **Standard Young tableaux**, semi-standard tableaux
- Robinson-Schensted-Knuth correspondance
- **Jeu de taquin**
- Jucys-Murphy elements, **contents**
- etc...

-3	-2			
-2	-1	0		
-1	0	1		
0	1	2	3	4

$\lambda = (5, 3, 3, 2)$

A dual vision, expressed in contents

For each partition $\lambda \vdash n$, we have a character χ^λ of S_n. When evaluated on conjugacy classes indexed by $\mu \vdash n$, it is noted as χ^λ_μ. We denote $f^\lambda = \chi^\lambda_{[1^n]}$ its dimension.
We fix $\mu \vdash k$, and for $\lambda \vdash n$, we want to express the map:

$$\lambda \mapsto \chi^\lambda_{[\mu, 1^{n-k}]}.$$

They can be expressed as power sum of contents. ($\lambda \vdash n$)

$$n(n-1)\chi^\lambda_{2,1^{n-2}} = 2f^\lambda \left(\sum_{w\in\lambda} c(w)\right)$$
$$n(n-1)(n-2)\chi^\lambda_{3,1^{n-3}} = 3f^\lambda \left(\sum_{w\in\lambda} (c(w))^2 + n(n-1)/2\right)$$
$$n(n-1)(n-2)(n-3)\chi^\lambda_{4,1^{n-4}} = 4f^\lambda \left(\sum_{w\in\lambda} (c(w))^3 + (2n-3)\sum_{w\in\lambda} c(w)\right)$$

Previous work

$$n(n-1)\chi^{\lambda}_{2,1^{n-2}} = 2f^{\lambda}\left(\sum_{w\in\lambda} c(w)\right)$$
$$n(n-1)(n-2)\chi^{\lambda}_{3,1^{n-3}} = 3f^{\lambda}\left(\sum_{w\in\lambda} (c(w))^2 + n(n-1)/2\right)$$
$$n(n-1)(n-2)(n-3)\chi^{\lambda}_{4,1^{n-4}} = 4f^{\lambda}\left(\sum_{w\in\lambda} (c(w))^3 + (2n-3)\sum_{w\in\lambda} c(w)\right)$$

Much effort was devoted into such expressions.

- Frobenius in 1900 the first, then Ingram and others
- Diaconis and Greene for several cases (Jucys-Murphy elements)
- Kerov and Olshanski gave expression in shifted symmetric functions
- Corteel, Goupil and Schaeffer proved them always content sums
- Lassalle gave explicit expression (symmetric functions)

All algebraic. Can we do it **combinatorially**?

Standard Young tableaux

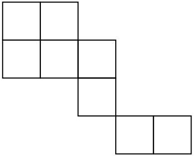
For $\lambda \vdash n$, a *standard Young tableau* (or SYT) is a row-and-column-increasing filling from 1 to n of its Young diagram.

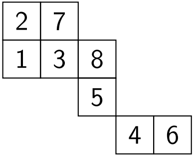
6	12			
4	8	13		
3	7	10		
1	2	5	9	11

We denote $f^{\lambda} = \#SYT$ of shape λ.

Skew tableaux

We can define SYT for *skew shapes*, i.e. a pair of partitions λ/ν with λ covering ν.

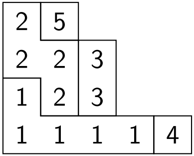




Here is an example for $(5, 3, 3, 2)/(3, 2)$.
We denote $f^{\lambda/\nu} = \#SYT$ of shape λ/ν.

Murnaghan-Nakayama rule

The Murnaghan-Nakayama rull says that characters χ^{λ}_{μ} can be expressed in *ribbon tableaux* of shape λ and ribbon sizes μ.



Corollary

For $\lambda \vdash n$ and $\mu \vdash k$, $\chi^{\lambda}_{\mu 1^{n-k}}$ is a linear combination of $f^{\lambda/\nu}$ for partitions $\nu \vdash k$.

Computing $\chi^{\lambda}_{\mu 1^{n-k}}$ with fixed μ $\Leftrightarrow$ Computing $f^{\lambda/\nu}$ with fixed ν

First attempt

We now try to prove the following combinatorially.

$$n(n-1)\chi^{\lambda}_{2,1^{n-2}} = 2f^{\lambda}\left(\sum_{w\in\lambda} c(w)\right)$$

According to Murnaghan-Nakayama rule, we have

$$\chi^{\lambda}_{2,1^{n-2}} = f^{\lambda/(2)} - f^{\lambda/(1,1)}$$

Because it is nearly standard, with two ways for the ribbon of size 2. Now we need to compute the number of SYT in skew shape.

Jeu de taquin

6	12			
4	8	13		
3	7	10		
1	2	5	9	11

Jeu de taquin

6	**12**			
4	8	13		
3	7	10		
1	2	5	9	11

Jeu de taquin

6	*			
4	8	13		
3	7	10		
1	2	5	9	11

(12)

Jeu de taquin

6	**8**				(12)
4	*	13			
3	7	10			
1	2	5	9	11	

Jeu de taquin

6	8			
4	*	13		
3	7	10		
1	2	5	9	11

(12)

Jeu de taquin

6	8			
4	**7**	13		
3	*	10		
1	2	5	9	11

(12)

Jeu de taquin

6	8				(12)
4	7	13			
3	*	10			
1	2	5	9	11	

Jeu de taquin

6	8				(12)
4	7	13			
*	3	10			
1	2	5	9	11	

Jeu de taquin

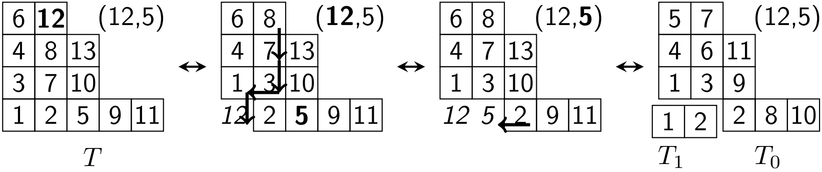
6	8		(12)	
4	7	13		
1	3	10		
*	2	5	9	11

Jeu de taquin

6	8			
4	7	12		
1	3	10		
	2	5	9	11

(12)

Skew-tableaux via jeu de taquin



The jeu de taquin gives a bijection between:

- (T, a, b), with T STY of shape λ, and $1 \leq a, b \leq n$, $a \neq b$,
- (T_0, T_1, a, b), with T_0 a skew tableau of shape λ/μ, T_1 a SYT of shape μ of entries $1, 2$, and $1 \leq a, b \leq n$, $a \neq b$. μ can be (2) or $(1, 1)$.

Just do two consecutive jeu de taquin on a then on b. This extends natually on more entries.

We have nearly finished!

What we want to prove:

$$n(n-1)\chi^{\lambda}_{2,1^{n-2}} = 2f^{\lambda}\left(\sum_{w\in\lambda} c(w)\right).$$

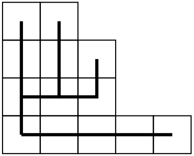
What we have in bijection:

- (T, a, b): f^{λ} SYT T
- (T_0, T_1, a, b): 1 for $T_1 = \square\square$, -1 for $T_1 = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$ $\Rightarrow$
  $n(n-1)(f^{\lambda/(2)-f^{\lambda/(1,1)}}) = n(n-1)\chi^{\lambda}_{2,1^{n-2}}$

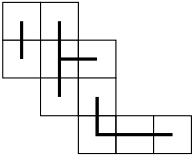
We only need to count how many (a, b) give $T_1 = \square\square$ or $T_1 = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$.

Trace forest

6	12			
4	8	13		
3	7	10		
1	2	5	9	11



2	8			
1	6	7		
	3	4		
		5	9	10



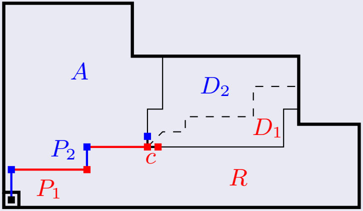
Trace forest: union of all jeu de taquin paths.

Construction: for each cell, an arc pointing to 

Effect of jeu de taquin

Lemma (Reformulation of Krattenthaler(1999))

Let c be a cell in a skew tableau T be a tableau, suppose that a jeu de taquin on the entry in c gives the tableau T_a.



T_a divides into two parts: any jeu de taquin acting on the red (resp. blue) part will give ◻◻ *(resp.* ⊟*).*

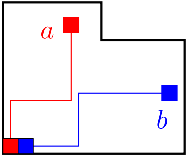
Proof: Case analysis

Thus finished the combinatorial proof

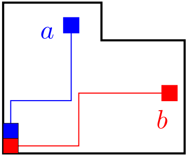
$$n(n-1)\chi^{\lambda}_{2,1^{n-2}} = 2f^{\lambda}\left(\sum_{w\in\lambda} c(w)\right).$$

$(T,a,b) \Leftrightarrow (T_0,T_1,a,b)$, with $+1$ for $T_1 = \square\square$, -1 for $T_1 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. For $a<b$, we look at (T,a,b) and (T,b,a). Two cases:

- a, b not on the same path



$+1$



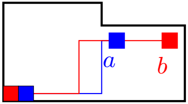
-1

No total contribution.

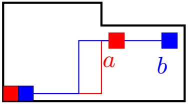
Thus finished the combinatorial proof

$$n(n-1)\chi^{\lambda}_{2,1^{n-2}} = 2f^{\lambda}\left(\sum_{w\in\lambda} c(w)\right).$$

- a, b on the same path. Suppose b on (i, j).
  - The path points to a horizontally. (There are i such a)

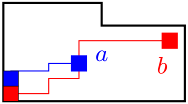


$+1$



$+1$

  - The path points to a vertically. (There are j such a)



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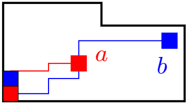


-1

In total, $2i - 2j = 2c(b)$.

Thus finished the combinatorial proof

$$n(n-1)\chi^{\lambda}_{2,1^{n-2}} = 2f^{\lambda}\left(\textstyle\sum_{w\in\lambda} c(w)\right).$$

- a, b not on the same path $\Rightarrow$ Contribution: 0
- a, b on the same path $\Rightarrow$ Contribution: $2c(b)$

Therefore, in the bijection between (T, a, b) and (T_0, T_1, a, b),

- (T, a, b): f^{λ} SYT T, each contributes $2\sum_{b\in T} c(b)$, thus $2f^{\lambda}\left(\sum_{w\in\lambda} c(w)\right)$
- (T_0, T_1, a, b): 1 for $T_1 = \square\square$ and -1 for $T_1 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$ $\Rightarrow$ $n(n-1)(f^{\lambda/(2)-f^{\lambda/(1,1)}}) = n(n-1)\chi^{\lambda}_{2,1^{n-2}}$

Remarks

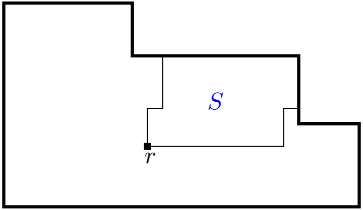
- Purely combinatorial
- Too complicated for other cases
- Computing $\chi_{\mu}^{\lambda} \Leftrightarrow$ Computing $f^{\lambda/\nu}$ for several ν
- Works the same for any T and any trace forest
- Works even for a subtree of the trace forest of T

- *Relative content* c_a *for a cell* a: $c_a(w) = c(w) - c(a)$
- *Content powersum* cp_a^{α}: $cp_a^{(k)}(C) = \sum_{w \in C} c_a^{k-1}(w)$

Lemma

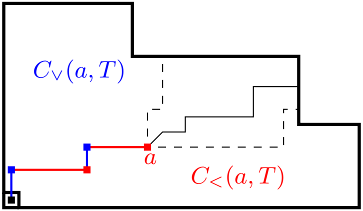
For a subtree S *rooted at* r *of the trace forest of a tableau* T*, the number of pairs* (a, b) *in* S *such that* $(T, a, b) \leftrightarrow (T_0, T_1, a, b)$ *is with* $T_0 = \square\square$ *is*

$$G_{(2)}(S) = \frac{1}{2} cp_r^{(1,1)}(S) + cp_r^{(2)}(S) - \frac{1}{2} cp_r^{(1)}(S) = |S|(|S|-1)/2 + \sum_{w \in S} c_r(w).$$



Bootstrap

For a SYT T and a its entry, we note $C_<(a,T)$ (resp. $C_\vee(a,T)$) the tree on the right (resp. below) of T_a.



Compute $f^{\lambda/(3)} \Leftrightarrow$ Compute $G_{(3)}(T) = \sum_{a \in T} G_{(2)}(C_<(a,T))$.

Inductive method

Direct computation impossible.
The tree structure reminds induction. For a subtree S in trace forest, let $S_<$ and $S_\vee$ be its subtrees on the right and above.
For a function f on a subtree F in the trace forest, its *inductive form* is $(\Delta f)(S) = f(S) - f(S_<) - f(S_\vee)$.

Lemma

For two functions f, g on binary trees with $f(\varnothing) = g(\varnothing) = 0$, $\Delta f = \Delta g \Rightarrow f = g$.

Inductive form

For a subtree S in trace forest rooted at r and a partition α, we define

$$<^{(\alpha)}(S) = cp_r^{\alpha}(S_<), \quad \vee^{(\alpha)}(S) = cp_r^{\alpha}(S_\vee).$$

Lemma

For any partition α, Δcp_r^{α} is a polynomial in some $<^{(\nu)}$ and $\vee^{(\nu)}$.

To compute a function f (formed by cp_r^{α}), we only need to know Δf.

Computing the inductive form

$$\Delta G_{(3)}(S) = \sum_{a \in S} G_{(2)}(C_<(a,S)) - \sum_{a \in S_<} G_{(2)}(C_<(a,S_<)) - \sum_{a \in S_\vee} G_{(2)}(C_<(a,S_\vee))$$

We break the first sum in 3 cases: a is root, $a \in S_<$, $a \in S_\vee$. In each case we know exactly $C_<(a,S)$.
Only nasty part: sums of $cp_r^\alpha(C_<(a,S_<))$ and $cp_r^\alpha(C_<(a,S_\vee))$.

Miracles

Miracle 1: these sums sum up to $<^{(\nu)}(S)$ and $\vee^{(\nu)}(S)$.
Miracle 2: the final result is Δf for some f combination of cp^{α}.

$$G_{(3)} = \frac{1}{6}cp^{(1,1,1)} + cp^{(2,1)} + cp^{(3)} - cp^{(1,1)} - 2cp^{(2)} + \frac{5}{6}cp^{(1)}$$

With some tricks it leads to

$$(n)_3\chi^{\lambda}_{(3,1^{n-3})}/f^{\lambda} = 3cp^{(3)}(\lambda) - \frac{3}{2}cp^{(1,1)}(\lambda) + \frac{3}{2}cp^{(1)}(\lambda) = 3\sum_{w\in\lambda}(c(w))^2 - 3\binom{n}{2}.$$

We can define $G_{(4)}(T) = \sum_{a\in T} G_{(3)}(C_<(a,T))$, and it leads to

$$(n)_4\chi^{\lambda}_{(4,1^{n-4})}/f^{\lambda} = 4\sum_{w\in\lambda}(c(w))^3 + 4(2n-3)\sum_{w\in\lambda}c(w)$$

Nasty computation, but **entirely automatic**.

How to explain?

Totally no clue.
Any idea?

Thank you for your attention!