Bell polynomials in combinatorial Hopf algebras

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Séminaire Lotharingien de Combinatoire (72nd)

March 25, 2014

Presentation

✓ Introduction

- The commutative partial multivariate Bell polynomials have been defined by E.T. Bell in 1934.
- given by :

$$B_{n,k}(a_1, a_2, \dots) = \sum \frac{n!}{k_1!k_2!\dots k_n!} \left(\frac{a_1}{1!}\right)^{k_1} \left(\frac{a_2}{2!}\right)^{k_2} \dots \left(\frac{a_n}{n!}\right)^{k_n}$$

where $k_1 + k_2 + \cdots k_n = k$ and $k_1 + 2k_2 + 3k_3 + \cdots nk_n = n$

- Applications :
  - Combinatorics : set partitions
  - Analysis, Algebra : *Lagrange* inversion theorem, *Faà di Bruno's* formula
  - Probabilities : *Gibbs* distributions.

Presentation

- Some of the simplest formulæ are related to the enumeration of combinatorial objects
- *Stirling* numbers of the first kind $s_{n,k} = \left[{n \atop k}\right]$ (A008275)
- count the number of permutations according to their number of cycles.

$$\left[{n \atop k}\right] = B_{n,k}(0!, 1!, 2!, \dots)$$

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Example

- $s(4,2) = 11$: the symmetric group on 4 objects has
  - 3 permutations of the form $(**)(**)$: 2 orbits, each of size 2
  - 8 permutations of the form $(***)(*)$: 1 orbit of size 3 and 1 orbit of size 3.

Présentation

- *Stirling* numbers of the second kind $S_{n,k} = \left\{ {n \atop k} \right\}$ (A106800)
- count the number of ways to partition a set of n objects into k non-empty subsets.

$$\left\{ {n \atop k} \right\} = B_{n,k}(1, 1, \ldots, 1)$$

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Example

- $S(4, 2) = 7$

- *Lah* numbers, : $L(n, k) = \binom{n-1}{k-1} \frac{n!}{k!}$ (Sloane: A008297)
- count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets.

$$L(n, k) = B_{n,k}(1!, 2!, \dots, (n-k+1)!)$$

Motivation

- Find the main identities from symmetric functions
- Give analogues of these formulæ in some Hopf algebras :
  - The algebra of symmetric functions *Sym*
    ([33211] is a partition of the integer 10)
  - The algebra of word symmetric functions **WSym**
    ($\{\{1,3\},\{4\}\,\{2,5\}\}$ is a set partition of $\{1,2,3,4,5\}$)
  - The bi-indexed word algebra **BWSym**
    whose bases are indexed by set partitions into lists which can be constructed from a set partition by ordering each block.
    $\{[3,1],[2]\} \sim \binom{321}{\{\{1,3\},\{2\}\}}$ set partitions into lists of $\{1,2,3\}$

Presentation

- The PhD thesis of *M. Mihoubi* present some applications of these polynomials and several examples
- *Dominique Manchon et al.* (Noncommutative Bell polynomials, quasideterminants and incidence Hopf algebras - 2014)
  - various descriptions, commutative and noncommutative Bell polynomials
  - construct commutative and noncommutative Bell polynomials and explain how they give rise to Faà di Bruno's Hopf algebras.

Outline

1. Combinatorial Hopf algebras
2. Bell polynomials
3. Bell polynomials in combinatorial Hopf algebras
4. Conclusion

Combinatorial Hopf algebras

✓ combinatorial objects :

- words : $\mathbb{C} < \mathbb{A} >$
- permutations : **FQSym**
- integer partitions : *Sym*
- compositions : *QSym*
- binary trees : **PBT**
- set compositions : **WQSym**
- set partitions : **WSym**
- set partitions in lists : **BWSym**

How do we define a combinatorial Hopf algebra ?

Minimum requirements

- bases indexed by a combinatorial object
- has a product and a coproduct
- graded
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Additional conditions

- can be realized as subalgebras of a polynomial algebra with an infinite number of variables
- has distinguished basis which has positive product and coproduct structure coefficients
- related to representation theory

The algebra of *symmetric functions* : *Sym*

The algebra of *symmetric functions*

- The algebra of *symmetric functions*, $Sym(\mathbb{X})$, is the space of the polynomials that are invariant under permutations of the variables
- bases indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$.

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- *Sym* is generated by the monomials as a vector space
- *Sym* is generated as an algebra by :

1. The *power sum symmetric functions* ; $p_n(\mathbb{X})$ is defined by :

$$p_n(\mathbb{X}) = \sum_{i \geqslant 1} x_i^n$$

2. The nth *complete symmetric functions* ; $h_n(\mathbb{X})$ the sum of all the monomials of degree n

The algebra of *symmetric functions* : *Sym*

Example

- for an alphabet $\mathbb{X} = \{x_1, x_2, x_3\}$

$$m_{21} = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$$

$$\begin{aligned} h_3 &= m_3 + m_{21} + m_{111} \\ &= x_1^3 + x_2^3 + x_3^3 + m_{21} + x_1x_2x_3. \end{aligned}$$

$$\begin{aligned} p_{21} &= p_2p_1 \\ &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) \\ &= m_3 + m_{21}. \end{aligned}$$

Newton formula

- The generating function of the $h_n(\mathbb{X})$ is given by the *Cauchy* function :

$$\sigma_t(\mathbb{X}) = \sum_{n\geqslant 0} h_n(\mathbb{X}) t^n = \prod_{i\geqslant 1} (1 - x_i t)^{-1}$$

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Newton formula

These two free families of generators of *Sym* are linked by the *Newton formula* :

$$\sigma_t(\mathbb{X}) = \exp\{\sum_{n\geqslant 1} p_n(\mathbb{X})\frac{t^n}{n}\}$$

where $\mathbb{X} = \{x_1, x_2, \ldots\}$ is an infinite set of commuting variables

Transformations of alphabets

- let $\mathbb{X}$, $\mathbb{Y}$ be two alphabets and $\alpha \in \mathbb{C}$
- the sum of two alphabets $\mathbb{X} + \mathbb{Y}$ is defined by :

$$p_n(\mathbb{X} + \mathbb{Y}) = p_n(\mathbb{X}) + p_n(\mathbb{Y})$$

or equivalently

$$\sigma_t(\mathbb{X} + \mathbb{Y}) = \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y})$$

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or equivalently

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- the product of two alphabets :

$$p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})$$

and

$$\sigma_t(\alpha\mathbb{X}) = [\sigma_t(\mathbb{X})]^{\alpha}$$

eq

$$p_n(\alpha\mathbb{X}) = \alpha p_n(\mathbb{X})$$

The algebra of word symmetric functions

Definition of *WSym*

Let $\mathbb{A}$ be an alphabet.

✓ $\mathbb{C} < \mathbb{A} >$ ={ linear combinations of words with the concatenation product}

✓ The algebra of word symmetric functions is a way to construct a noncommutative analogue of *Sym*.

The algebra of word symmetric functions

Definition of *WSym*

Let $\mathbb{A}$ be an alphabet.

✓ $\mathbb{C} < \mathbb{A} >$ ={ linear combinations of words with the concatenation product}

✓ The algebra of word symmetric functions is a way to construct a noncommutative analogue of *Sym*.

- its bases are indexed by set partitions
  - *power sum symmetric functions* : $\Phi := \{\Phi^\pi\}_\pi$: $\Phi^\pi(\mathbb{A}) = \sum_w a_1 a_2 \ldots a_n$ where $i, j \in \pi_k \Rightarrow a_i = a_j$
  - word monomial functions defined by $\Phi^\pi = \sum_{\pi \leq \pi'} M_{\pi'}$

Example

$$\Phi^{\{1,3\}\{2\}}\Phi^{\{1,4\}\{2,5,6\}\{3,7\}\{8\}} = \Phi^{\{1,3\}\{2\}\{4,7\}\{5,8,9\}\{6,10\}\{11\}}.$$

$$\Phi^{\{1,4\}\{2,5,6\}\{3,7\}} = M_{\{1,4\}\{2,5,6\}\{3,7\}} + M_{\{1,2,4,5,6\}\{3,7\}} + M_{\{1,3,4,7\}\{2,5,6\}} + M_{\{1,4\}\{2,3,5,6,7\}} + M_{\{1,2,3,4,5,6,7\}}.$$

Outline

Notations and background

✓ The *Bell polynomials*

- The (complete) *Bell polynomials* are usually defined on an infinite set of commuting variables $\{a_1, a_2, \ldots\}$ by the following generating function

$$\sum_{n\geqslant 0} A_n(a_1, a_2, \ldots, a_p, \ldots)\frac{t^n}{n!} = \exp\left(\sum_{m\geqslant 1} a_m \frac{t^m}{m!}\right)$$

where A_n is the number of partitions of a set of size n.

- The *partial Bell polynomials* are defined by

$$\sum_{n\geqslant 0} B_{n,k}(a_1, a_2, \ldots, a_p, \ldots)\frac{t^n}{n!} = \frac{1}{k!}(\sum_{m\geqslant 0} a_m \frac{t^m}{m!})^k$$

where $B_{n,k}$ counts the number of partitions of a n-set into k blocks.

examples

Example

- *Stirling* number of :

$$\textit{the first kind}: \quad B_{n,k}(0!, 1!, 2!, \dots) = \begin{bmatrix} n \\ k \end{bmatrix} (A008275)$$

$$\textit{the second kind}: \quad B_{n,k}(1, 1, \dots) = \begin{Bmatrix} n \\ k \end{Bmatrix} (A106800)$$

$$B_{6,2}(x_1, x_2, x_3, x_4, x_5) = 6x_5x_1 + 15x_4x_2 + 10x_3^2$$

- 6 set partitions of 6 elements of the form $5 + 1$
- 15 set partitions of 6 elements of the form $4 + 2$
- 10 set partitions of 6 elements of the form $3 + 3$

Notations and background

remark

$$A_n(a_1, a_2, \dots, a_{n-k}, a_{n-k+1}) = \sum_{k=1}^{n} B_{n,k}(a_1, a_2, \dots, a_{n-k}, a_{n-k+1})$$

is called the *n*th complete *Bell* polynomial

- Without loss of generality, we can assume $a_1 = 1$
  - if $a_1 \neq 0$,

$$B_{n,k}(a_1, a_2, \dots, a_p, \dots) = a_1^k B_{n,k}(1, \frac{a_2}{a_1}, \dots, \frac{a_p}{a_1}, \dots)$$

  - if $a_1 = 0$ and $k \leqslant n$,

$$B_{n,k}(0, a_2, \dots, a_p, \dots) = \frac{n!}{(n-k)!} B_{n,k}(a_2, \dots, a_p, \dots)$$

  - if $a_1 = 0$ and $n < k$, $B_{n,k}(0, a_2, \dots, a_p, \dots) = 0$

Observation

- These polynomials are related to several combinatorial sequences which involve set partitions.

Observation

it seems natural to investigate analogous formulæ on Bell polynomials which involve combinatorial objects :

- partitions
- permutations
- set partitions in lists *etc*

in some combinatorial Hopf algebra with bases indexed by these objects.

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1 Combinatorial Hopf algebras

2 Bell polynomials

3 Bell polynomials in combinatorial Hopf algebras
- Bell polynomials in *Sym* (sum and product)
- Bell polynomials in the Faà di Bruno algebras
- Bell polynomials in WSym algebras

4 Conclusion

Cauchy function

$\sigma_t(\mathbb{X})$ is the generating function of the $h_n(\mathbb{X})$

$$\sigma_t(\mathbb{X}) = \sum_{n\geqslant 0} h_n(\mathbb{X}) t^n$$

remark

- several equalities on Bell polynomials can be proved by manipulating generating functions.
- they are easily proved using symmetric functions and virtual alphabets.

Bell polynomials and *Cauchy* function

- Consider $\mathbb{X}$ a virtual alphabet satisfying $a_i = i!h_{i-1}(\mathbb{X})$ for any $i \geq 1$ and for simplicity, let $\widetilde{h}_n(\mathbb{X}) := n!h_n(\mathbb{X})$.
- One has :

$$\sum_{n\geqslant 0} B_{n,k}(a_1, a_2, \dots)\frac{t^n}{n!} = \frac{t^k}{k!}\left(\sum_{i\geqslant 1}\frac{a_i}{i!}t^{i-1}\right)^k$$

$$= \frac{t^k}{k!}\left(\sum_{i\geqslant 0} h_i(\mathbb{X})t^i\right)^k$$

$$= \frac{t^k}{k!}\left(\sigma_t(\mathbb{X})\right)^k$$

$$= \frac{t^k}{k!}\sigma_t(k\mathbb{X}).$$

Bell polynomials in terms of *Cauchy* function

for each i, $a_i = i!h_{i-1}(\mathbb{X})$

$$B_{n,k}(1, 2!h_1, \ldots, (m+1)!h_m(\mathbb{X}), \ldots) = \frac{n!}{k!}h_{n-k}(k\mathbb{X})$$
$$= \binom{n}{k}\widetilde{h}_{n-k}(k\mathbb{X})$$

where $\widetilde{h}_{n-k}(k\mathbb{X}) := (n-k)!h_{n-k}(\mathbb{X})$

Bell polynomials in terms of *Cauchy* function

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remark

In the sequel for any alphabet $\mathbb{X}$, we will denote by $B_{n,k}$ the symmetric function defined by :

$$B_{n,k}(\mathbb{X}) := \binom{n}{k} \widetilde{h}_{n-k}(k\mathbb{X}).$$

Examples

- *Lah* numbers (number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets) :
  - Specialization $a_i = i!, \forall i$
  - It implies $h_i(\mathbb{X}) = 1, \forall i$
  - The generating function is given by :

$$\sigma_t(k\mathbb{X}) = (\sum_{n\geqslant 0} h_n(\mathbb{X})t^n)^k$$
$$= (\sum_{n\geqslant 0} t^n)^k = (\frac{1}{1-x})^k.$$

- with this specialization ($a_i = i!$),

$$B_{n,k}(1!, 2!, \ldots, m!, \ldots) = \binom{n-1}{k-1}\frac{n!}{k!} = L_{n,k}$$

Sums of alphabets

As a consequence of

$$h_n(\mathbb{X}+\mathbb{Y}) = \sum_{i=0}^{n} h_i(\mathbb{X}) h_{n-i}(\mathbb{Y})$$

we have

$$\widetilde{h}_n((k_1+k_2)\mathbb{X}) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{h}_i(k_1\mathbb{X}) \widetilde{h}_{n-i}(k_2\mathbb{X})$$

So that

$$B_{n,k_1+k_2}(\mathbb{X}) = \binom{n}{k_1+k_2} \tilde{h}_{n-k_1-k_2}((k_1+k_2)\mathbb{X}) = \sum_{i=0}^{n} \tilde{h}_{i-k_1}(k_1\mathbb{X}) \tilde{h}_{n-k_2-i}(k_2\mathbb{X}).$$

Hence

$$\binom{k_1+k_2}{k_1} B_{n,k_1+k_2} = \sum_{i=0}^{n} \binom{n}{i} B_{i,k_1} B_{n-i,k_2}.$$

Sums of alphabets

- for two alphabets $\mathbb{X}$ and $\mathbb{Y}$, we deduce that

$$\begin{aligned} B_{n-k,k}(\mathbb{X}+\mathbb{Y}) &= \frac{(n-k)!}{k!} h_{n-2k}(k(\mathbb{X}+\mathbb{Y})) \\ &= \frac{(n-k)!}{k!} \sum_{i=0}^{n-2k} h_i(k\mathbb{X}) h_{n-i-2k}(k\mathbb{Y}) \\ &= \frac{(n-k)!}{k!} \sum_{i_1+i_2=n} h_{i_1-k}(k\mathbb{X}) h_{i_2-k}(k\mathbb{Y}). \end{aligned}$$

Observation

$$B_{n-k,k}(\mathbb{X}+\mathbb{Y}) = \binom{n}{k}^{-1} \sum_{i_1+i_2=n} \binom{n}{i_1} B_{i_1,k}(\mathbb{X}) B_{i_2,k}(\mathbb{Y})$$

Bell polynomials and binomial functions

- The partial binomial polynomials are known to be involved in interesting identities on binomial functions.
- In this section we want to prove the equality :

Bell polynomials and binomial polynomials

$$B_{n,k}(1,\ldots,if_{i-1}(a),\ldots)=\binom{n}{k}f_{n-k}(ka)$$

$\forall n\leqslant k\leqslant 1$, where $(f_n)_{n\in\mathbb{N}}$ is a binomial function satisfying

$$\begin{cases} f_0(x)=1 \\ f_n(a+b)=\sum_{k=0}^{n}\binom{n}{k}f_k(a)f_{n-k}(b)\end{cases}$$

- This last identity is nothing but the sum of two alphabets stated in terms of modified complete functions $\widetilde{h}_n$.

Bell polynomials and binomial functions

- With the specialization

$$\widetilde{h}_n(\mathbb{A}) := f_n(a) \text{ and } \widetilde{h}_n(\mathbb{B}) := f_n(b)$$

- the last equality is equivalent to the classical

$$\widetilde{h}_n(\mathbb{A}+\mathbb{B}) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{h}_k(\mathbb{A})\widetilde{h}_{n-k}(\mathbb{B})$$

- which is a direct consequence of $\sigma_t(\mathbb{A}+\mathbb{B}) = \sigma_t(\mathbb{A})\sigma_t(\mathbb{B})$

As a direct consequence of

$$B_{n,k}(1, 2!h_1, \dots, (m+1)!h_m(\mathbb{X}), \dots) = \binom{n}{k} \widetilde{h}_{n-k}(k\mathbb{X})$$

we obtain

$$B_{n,k}(1, \dots, if_{i-1}(a), \dots) = B_{n,k}(\mathbb{A}) = \binom{n}{k} \widetilde{h}_{n-k}(k\mathbb{A}) = \binom{n}{k} f_{n-k}(ka).$$

Product of two alphabets

- Let $(a_n)_n$ and $(b_n)_n$ be two sequences of numbers such that $a_1 = b_1 = 1$ and $a_{-n} = b_{-n} = 0$ for each $n \in \mathbb{N}$, $k = k_1 k_2$.
- the following identity seems laborious to prove :

$$B_{n,k}\left(\dots, n! \sum_{\lambda \vdash n-1} \det\left|\frac{a_{\lambda_i - i + j + 1}}{(\lambda_i - i + j + 1)!}\right| \det\left|\frac{b_{\lambda_i - i + j + 1}}{(\lambda_i - i + j + 1)!}\right|, \dots\right) =$$

$$\frac{n!}{k!} \sum_{\lambda \vdash n-k} (k_1! k_2!)^{\ell(\lambda)} \det\left|\frac{B_{\lambda_i - i + j + k_1, k_1}(a_1, \dots)}{(\lambda_i - i + j + k_1)!}\right| \det\left|\frac{B_{\lambda_i - i + j + k_2, k_2}(b_1, \dots)}{(\lambda_i - i + j + k_2)!}\right|.$$

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$$\frac{n!}{k!} \sum_{\lambda \vdash n-k} (k_1! k_2!)^{\ell(\lambda)} \det\left|\frac{B_{\lambda_i - i + j + k_1, k_1}(a_1, \dots)}{(\lambda_i - i + j + k_1)!}\right| \det\left|\frac{B_{\lambda_i - i + j + k_2, k_2}(b_1, \dots)}{(\lambda_i - i + j + k_2)!}\right|.$$

Observation

But it looks rather simpler when we recognize

$$B_{n,k}(\mathbb{X}\mathbb{Y}) = \frac{n!}{k!} h_{n-k}(k\mathbb{X}\mathbb{Y})$$

and apply $h_n(k\mathbb{X}\mathbb{Y}) = \sum_{\lambda \vdash n} s_\lambda(k_1\mathbb{X}) s_\lambda(k_2\mathbb{Y})$, where $s_\lambda = \det\left|h_{\lambda_i - i + j}\right|$.

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specialization with the power sum functions p_n

Bell polynomials in *Sym* again

$$\sigma_t(\mathbb{X}) = \sum_{n\geqslant 0} h_n(\mathbb{X})t^n = \exp\{\sum_{n\geq 1} p_n(\mathbb{X})\frac{t^n}{n}\}$$

$$\sum_{n\geqslant 0} A_n(a_1, a_2, \dots, a_p, \dots)\frac{t^n}{n!} = \exp\left(\sum_{m\geqslant 1} a_m \frac{t^m}{m!}\right)$$

we can consider the complete Bell polynomials A_n as the complete functions $\widetilde{h}_n(\mathbb{X})$. Here we define

$A^p_n(\mathbb{X}) := \widetilde{h}_n(\mathbb{X}) = A_n(0!p_1(\mathbb{X}), 1!p_2(\mathbb{X}), \dots, (n-1)!p_n(\mathbb{X}), \dots)$

$$B^p_{n,k} = B_{n,k}(0!p_1(\mathbb{X}), 1!p_2(\mathbb{X}), \dots, (n-1)!p_n(\mathbb{X}), \dots) = n! \sum_{\substack{\lambda\vdash n \\ \#\lambda=k}} \frac{1}{z_\lambda} p^\lambda(\mathbb{X})$$

where $z_\lambda = \prod_i m_i(\lambda)! i^{m_i(\lambda)}$.

Arbogast(1800) - Faà di Bruno formula

- Faà di Bruno formula can be expressed in terms of Bell polynomials

$$\frac{d^n}{dt^n}f(g(t)) = \sum_{k\geqslant 0}\sum_{\lambda=(\lambda_1,\dots,\lambda_k)\vdash n}\frac{n!}{z_\lambda}f^{(k)}(g(t))\prod_{j=1}^{k}\frac{g^{(\lambda_j)}(t)}{(\lambda_j-1)!}.$$

- for $\sigma_x(\mathbb{X}) = \exp\{\sum_{n\geq 1}\frac{g^{(n)}(t)}{n!}x^n\}$
- in other words, $p_n(\mathbb{X}) = \frac{g^{(n)}(t)}{(n-1)!}$

  We deduce

$$n!\sum_{\lambda\vdash n}\frac{1}{z_\lambda}\prod_{j=1}^{k}\frac{g^{(\lambda_j)}(t)}{(\lambda_j-1)!} = B^p_{n,k}(\mathbb{X})$$

so that

$$\frac{d^n}{dt^n}f(g(t)) = \sum_{k\geqslant 0}f^{(k)}(g(t))B^p_{n,k}(\mathbb{X}).$$

Operation on alphabets

- set $h_n(\mathbb{X}) = \frac{g^{(n+1)}(t)}{(n+1)!g'(t)}$
- we obtain the equivalent expression

$$\frac{d^n}{dt^n} f(g(t)) = \sum_{k\geqslant 0} (g'(t))^k f^{(k)}(g(t)) B_{n,k}(\mathbb{X})$$

- we define a new operation on alphabets :

$$\sigma_t(\mathbb{X}\Diamond\mathbb{Y}) := (\sigma_t(\mathbb{X}) \circ t\sigma_t(\mathbb{Y})).$$

- assuming that $f(t) = \sigma_t(\mathbb{X})$ and $g(t) = t\sigma_t(\mathbb{Y})$
- we obtain :

$$h_n(\mathbb{X}\Diamond\mathbb{Y}) = \sum_{k=1}^{n} \frac{k!}{n!} h_k(\mathbb{X}) B_{n,k}(\mathbb{Y}).$$

Faà di Bruno's algebra

- the operation $\lozenge$ does not define a coproduct which is compatible with the classical product in *Sym*.
- the relationship with Bell polynomials can be established by observing that, from the Faà di Bruno's composition given by :

$$\sigma_t(\mathbb{X} \circ \mathbb{Y}) = \sigma_t(\mathbb{Y})\sigma_t(\mathbb{X}\lozenge\mathbb{Y})$$

we have

$$h_n(\mathbb{X} \circ \mathbb{Y}) = \sum_{k=0}^{n} \frac{(k+1)!}{(n+1)!} h_k(\mathbb{X}) B_{n+1,k+1}(\mathbb{Y})$$

Faà di Bruno's algebra

- We define for each alphabet $\mathbb{X}$ an alphabet $\mathbb{X}^{\langle -1\rangle}$ satisfying

$$\sigma_t(\mathbb{X}\circ\mathbb{X}^{\langle -1\rangle}) = 1$$

We have

$$h_n(\mathbb{X}^{\langle -1\rangle}) = \frac{h_n(-(n+1)\mathbb{X})}{n+1} = \frac{n!}{(2n+1)!(n+1)} B_{2n+1,n}(-\mathbb{X}).$$

Lagrange-Bürmann's formula

- set $\omega(t)$,$\omega(0)=0$ and $\phi(t)$ such that $\omega(t)=t\phi(t\omega(t))$
- the classical Lagrange-Bürmann formula for any formal power series F :

$$F(\omega(t))=F(0)+\sum_{n\geq 0}\frac{d^{n-1}}{du^{n-1}}\left[F'(u)(\phi(u))^m\right]\Big|_{u=0}\frac{t^n}{n!}.$$

Remark that if we suppose $F(t)=\sigma_t(\mathbb{X})$ and $\omega(t)=t\sigma_t(\mathbb{Y})$:

$$\sigma_t(\mathbb{X}\Diamond\mathbb{Y})=1+\sum_{n\geq 1}\frac{d^{n-1}}{du^{n-1}}[\sigma'_u(\mathbb{X})\sigma_u(-n\mathbb{Y}^{\langle -1\rangle})]_{|u=0}\frac{t^n}{n!}.$$

Lagrange-Bürmann formula

In other words,

$$\begin{aligned} h_n(\mathbb{X}\Diamond\mathbb{Y}) &= \tfrac{1}{n}\textstyle\sum_{i+j=n-1}(i+1)h_{i+1}(\mathbb{X})h_j(-n\mathbb{Y}^{\langle -1\rangle}) \\ &= \tfrac{1}{n}\textstyle\sum_{k=1}^{n} kh_k(\mathbb{X})h_{n-k}(-n\mathbb{Y}^{\langle -1\rangle}) \end{aligned}$$

so that

$$h_{n-k}(-n\mathbb{Y}^{\langle -1\rangle}) = \frac{(k-1)!}{(n-1)!}B_{n,k}(\mathbb{Y}).$$

as a consequence,

$$B_{n,k}(1, h_1(2\mathbb{X}), \ldots, m!h_m((m+1)\mathbb{X}), \ldots) = \frac{(n-1)!}{(k-1)!}h_{n-k}(n\mathbb{X}).$$

Bell polynomials of compositions of alphabets

- from the Cauchy series :

$$\sigma_t(\mathbb{X}\Diamond\mathbb{Y}) := (\sigma_t(\mathbb{X}) \circ t\sigma_t(\mathbb{Y})).$$

- we give formulas involving Bell polynomials and composition of alphabets

1. $$\binom{n}{k}^{-1} B_{n,k}(\mathbb{X}\Diamond\mathbb{Y}) = \sum_{i=1}^{n-k} \binom{i+k}{i}^{-1} B_{i+k,k}(\mathbb{X}) B_{n-k,i}(\mathbb{Y}),$$
2. $$\binom{n+k}{n} B_{n,k}\left(\mathbb{X}\circ\mathbb{Y}\right) = \sum_{i=0}^{n-k} \binom{n+k}{i+k} B_{i+k,k}(\mathbb{X}\Diamond\mathbb{Y}) B_{n-i,k}(\mathbb{Y}).$$

Outline

1. Combinatorial Hopf algebras
2. Bell polynomials
3. Bell polynomials in combinatorial Hopf algebras
   - Bell polynomials in *Sym* (sum and product)
   - Bell polynomials in the Faà di Bruno algebras
   - Bell polynomials in WSym algebras
4. Conclusion

Bell polynomials in other Hopf algebras

- in the algebra of word symmetric functions, we obtain

$$\mathcal{B}_{n,k}(S^{\{\{1\}\}}(\mathbb{A}),\dots,S^{\{\{1,\dots,m\}\}}(\mathbb{A}),\dots) = \sum_{\substack{\#\pi=k\\ \pi\vDash n}} S^{\pi}(\mathbb{A}).$$

- the bi-indexed word algebra BWSym

$$\mathcal{B}_{n,k}\left(\mathcal{S}_1,\mathcal{S}_{12}+\mathcal{S}_{21},\dots,\sum_{\sigma\in\mathfrak{S}_m}\mathcal{S}_\sigma,\dots\right) = \sum_{\substack{\hat{\Pi}\Vdash n\\ \#\hat{\Pi}=k}} \mathcal{S}_{\hat{\Pi}}.$$

- the Hopf algebra $\mathfrak{S}QSym$
- denoting by C_n the set of the cycles of size n
- we obtain

$$\mathcal{B}_{n,k}(M_1,M_{21},M_{231}+M_{312},\dots,\sum_{\sigma\in C_n} M_{\{\{1,2,\dots,m\}\}},\dots) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \#\mathrm{supp}(\sigma)=k}} M_\sigma.$$

Outline

1. Combinatorial Hopf algebras
2. Bell polynomials
3. Bell polynomials in combinatorial Hopf algebras
4. Conclusion

Conclusion

- The algebra *Sym* can be used to encode equalities on *Bell* polynomials
- we inverstigate analogues of Bell polynomials in other combinatorial Hopf algebras
  - **WSym**
  - **BWSym**
  - the Faà di Bruno's algebra
- express the $r-$ Bell polynomials in combinatorial Hopf algebras (*Sym*).
- we use properties of symmetric functions to prove known identities about $r-$Bell polynomials as well as some new ones.
- Link : (1402.2960)