Bijection between Tamari intervals and flows on rooted trees

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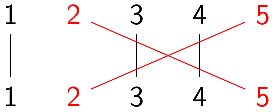
Introduction
- Order on permutations
- Order on trees
- Link between these orders
- Flows of rooted trees

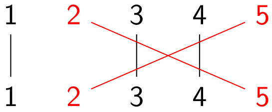
Bijection between Tamari intervals and flows
- Main result
- Interval-posets
- Example of bijection

Transpositions



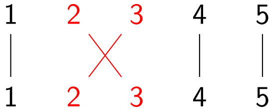
$(2,5)$

Transpositions



$(2, 5)$

Simples transpositions



$(2, 3) = s_2$

Right weak order

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

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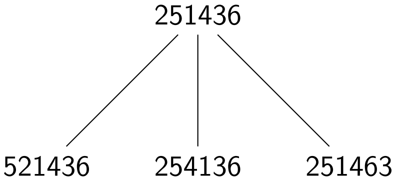

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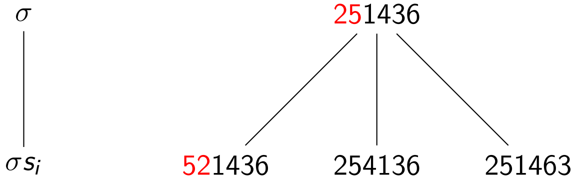

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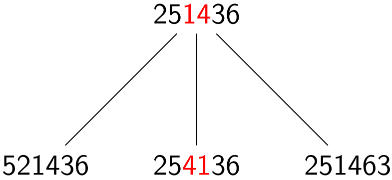

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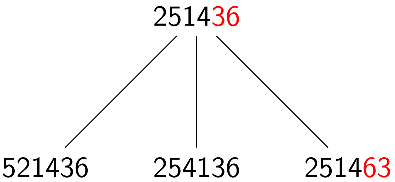

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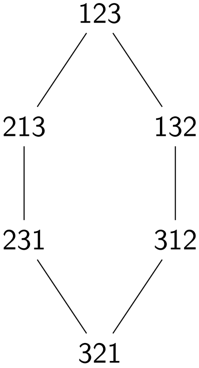

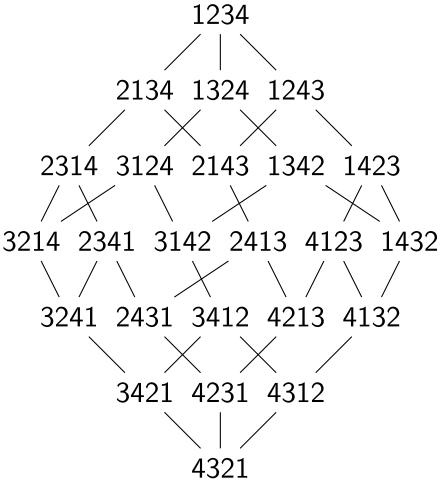

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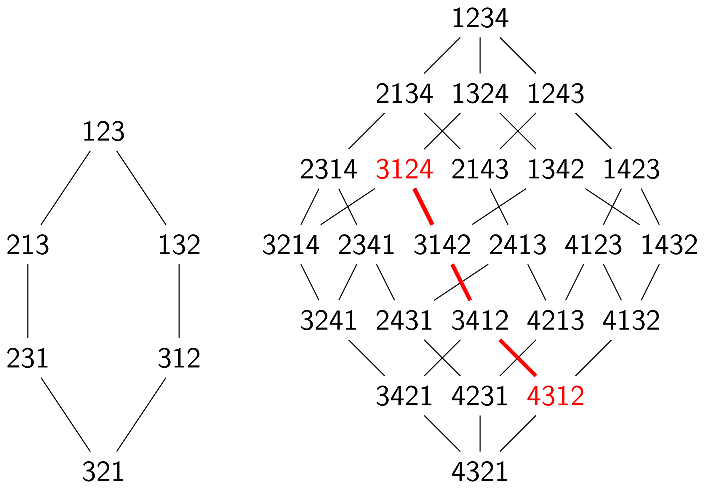

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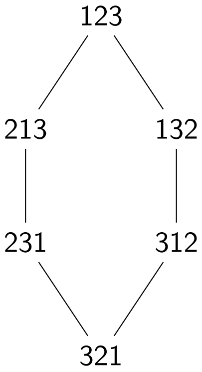

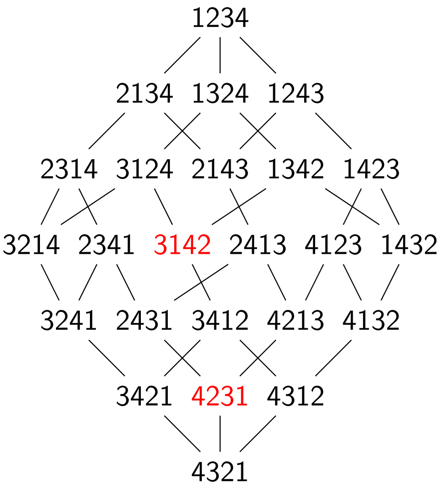

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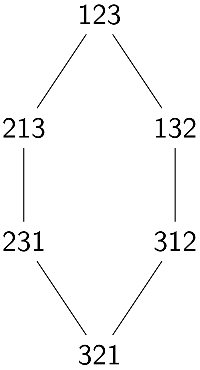

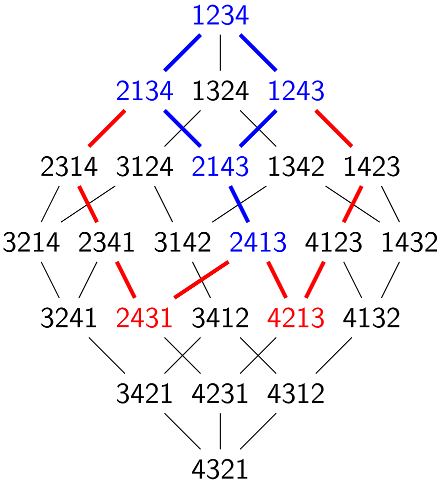

Binary trees

Recursive definition:

- empty tree or
- a root with a left and a right subtree

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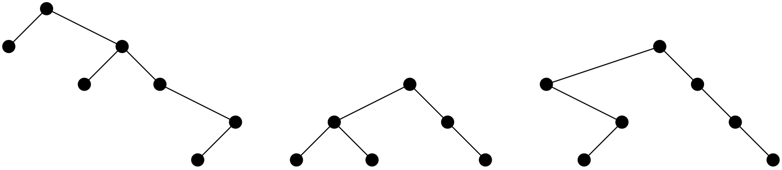
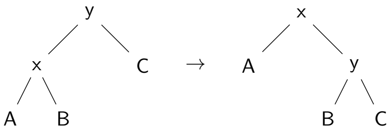
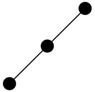
Examples



Right rotation



With x and y being nodes and A, B and C being subtrees.



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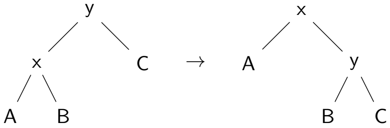

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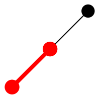

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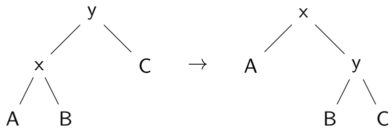

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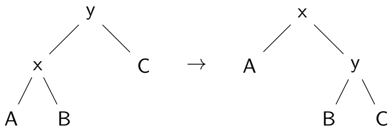

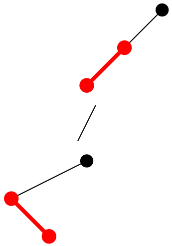

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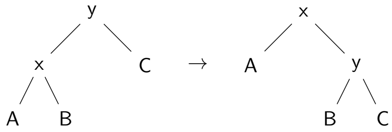

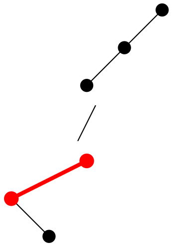

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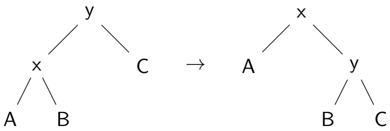

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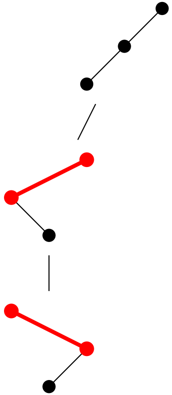

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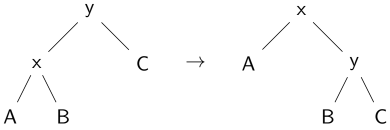

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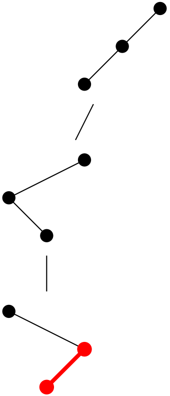

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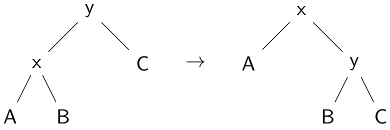

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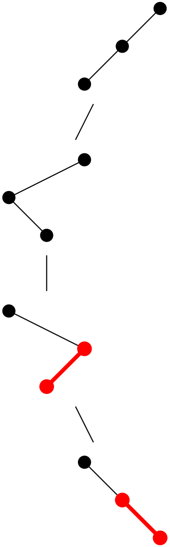

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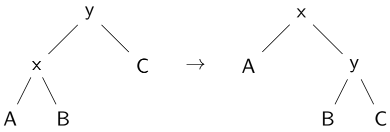

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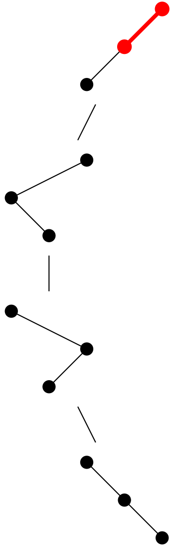

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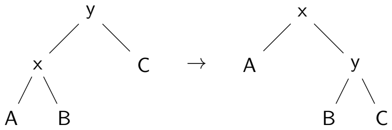

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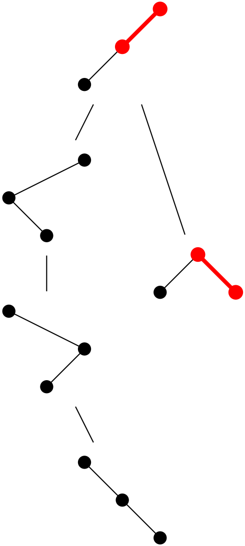

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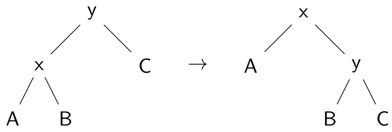

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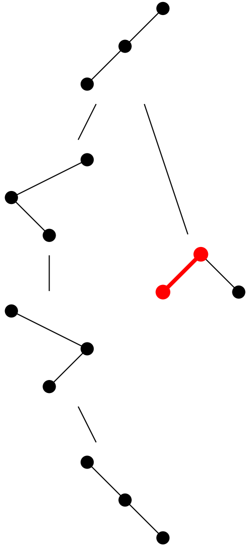

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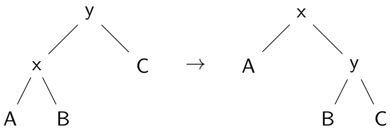

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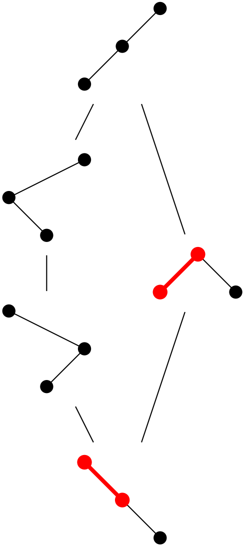

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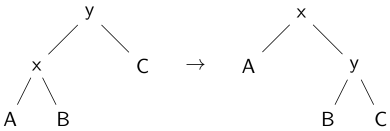

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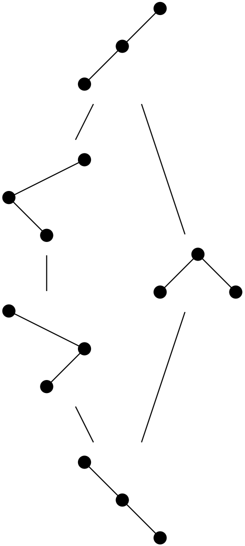

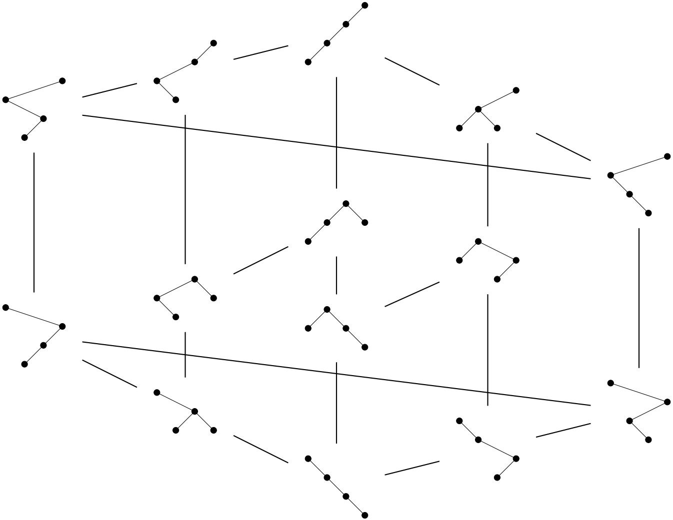

Some results on the Tamari order:

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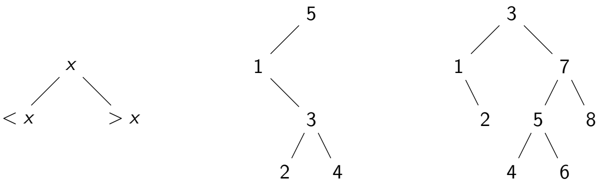
- 2013, Pournin: flip distance in the associahedra

Link with the weak order

Canonical labelling

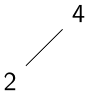


Binary search tree insertion

15324 $\longrightarrow$ 4

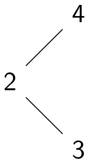
Binary search tree insertion

15324 →



Binary search tree insertion

15324 $\rightarrow$



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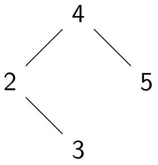

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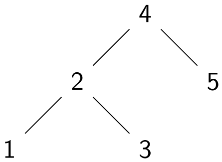

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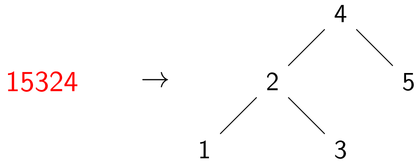
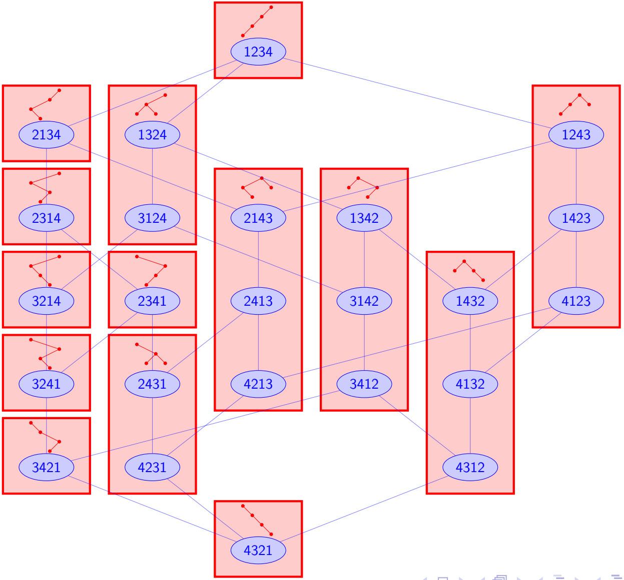


Characterization: the permutations which give the same tree are its linear extensions.
15324, 31254, 35124, 51324, . . .


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

Rooted tree

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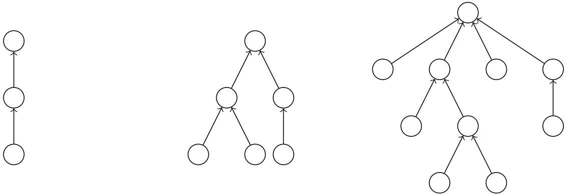

Forest of rooted trees

An ordered list of rooted trees.

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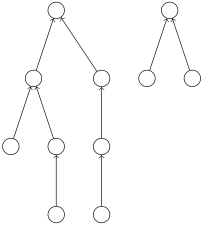


Flow on a forest

Forest of rooted trees with weight on nodes such that:

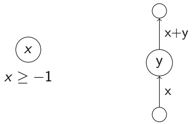
Flow on a forest

Forest of rooted trees with weight on nodes such that:

$x \geq -1$

Flow on a forest

Forest of rooted trees with weight on nodes such that:



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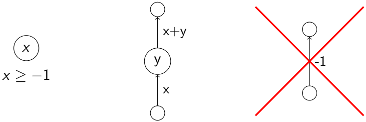

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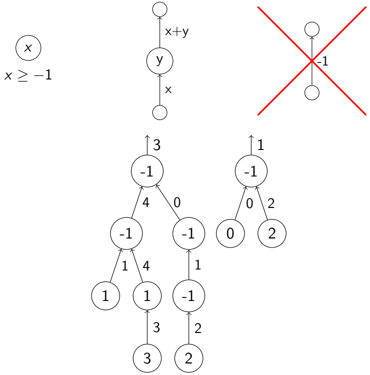

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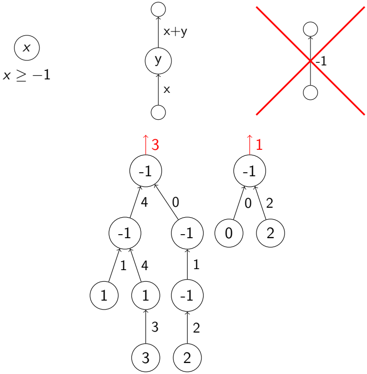

The *exit rate* of a forest is the sum of the exit rates of the trees.

Flow on a forest

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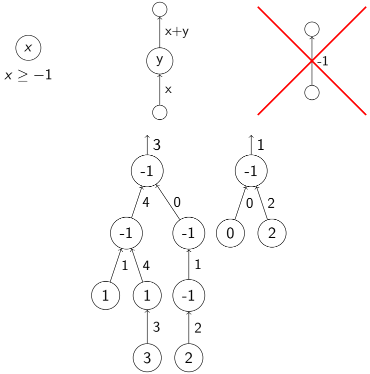

The *exit rate* of a forest is the sum of the exit rates of the trees.

A *closed flow* is a forest with exit rate 0.

Theorem (Chapoton, C., Pons)

The number of closed flows of a given forest F is the number of elements smaller than or equal to a certain binary tree $T(F)$ in the Tamari order.

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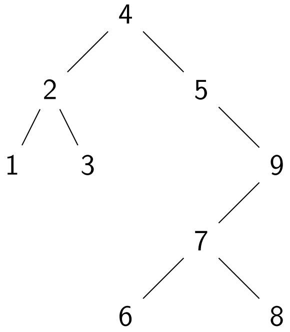
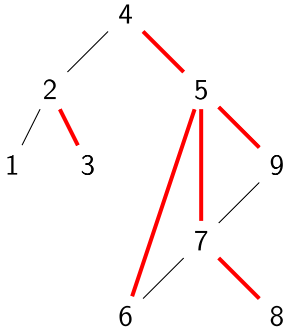
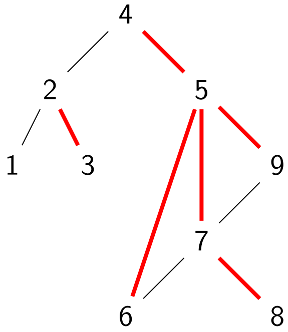
The number of closed flows of a given forest F is the number of elements smaller than or equal to a certain binary tree $T(F)$ in the Tamari order.

The proof is a bijection between all the closed flows on a rooted forest and Tamari intervals having the same maximal element.


4
2
5
1
3
9
7
6
8

Final forest $F_{\geq}(T)$





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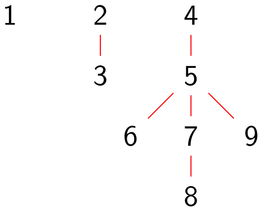

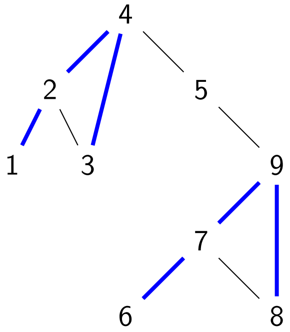

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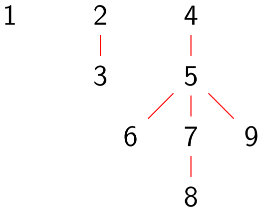
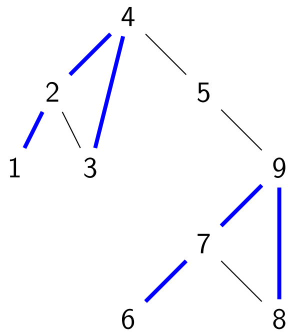
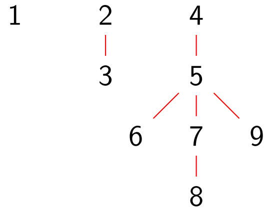


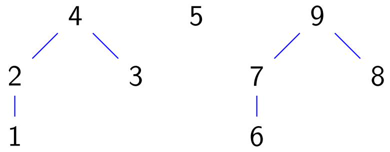
Initial forest $F_{\leq}(T)$

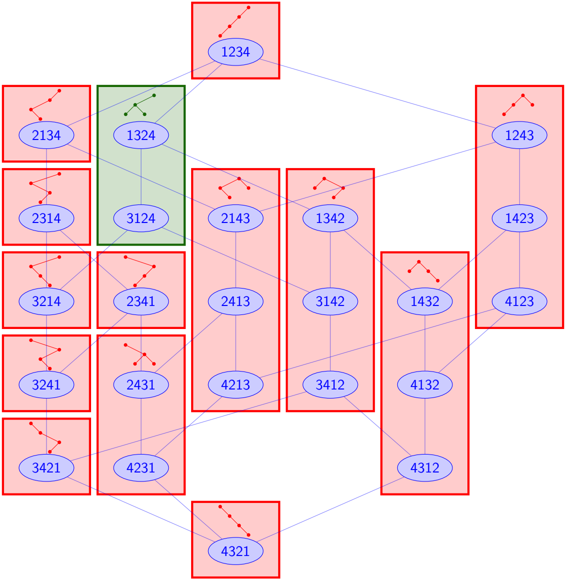


Final forest $F_{\geq}(T)$



Initial forest $F_{\leq}(T)$

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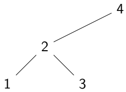
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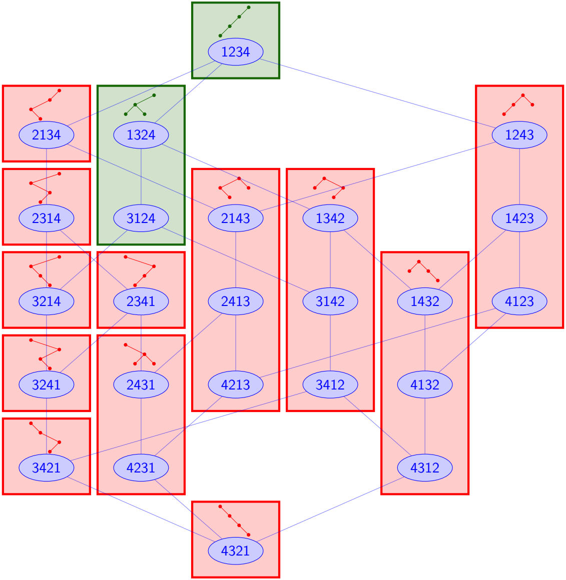
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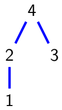
$F_{\leq}(T)$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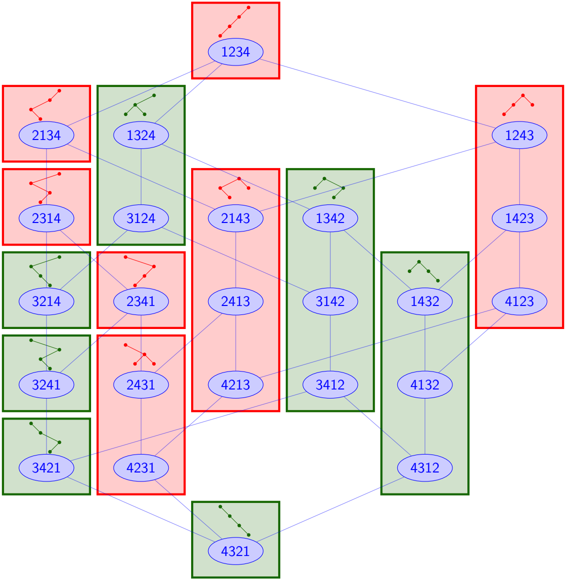
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$F_{\geq}(T)$

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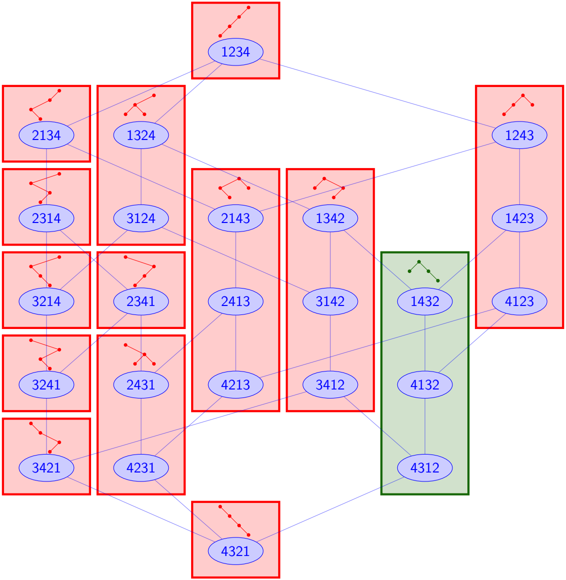
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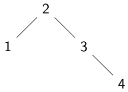
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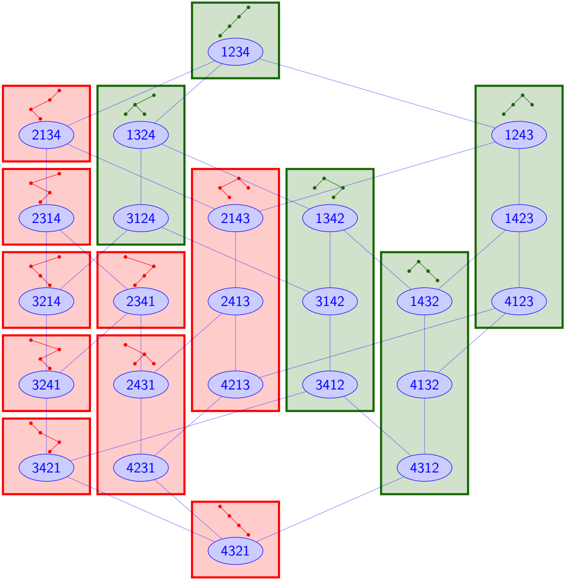
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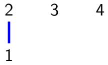
$F_{\leq}(T')$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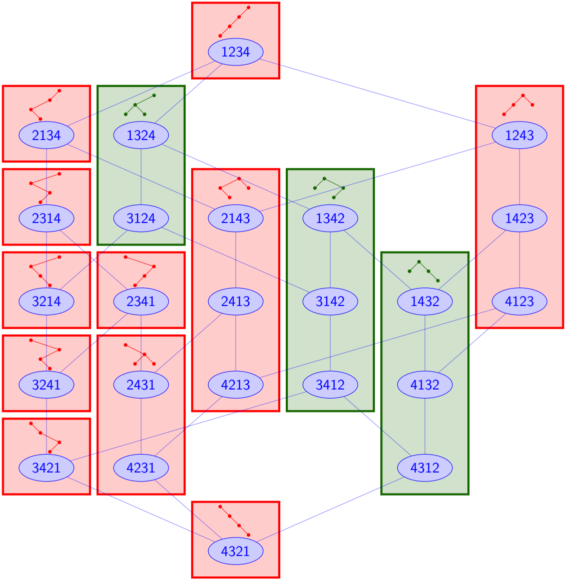
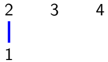
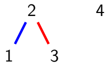


$F_{\geq}(T)$

$F_{\leq}(T')$



Interval-poset
$[T, T']$



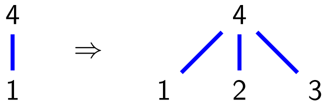
Theorem (C., Pons)

The Tamari order intervals are in bijection with the posets with labels in $1, \ldots, n$ *of size* n *such that:*

Theorem (C., Pons)

The Tamari order intervals are in bijection with the posets with labels in $1, \ldots, n$ of size n such that:

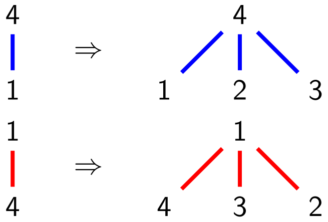
- *If $a < c$ and $a \lhd c$ then $b \lhd c$ for all $a < b < c$.*

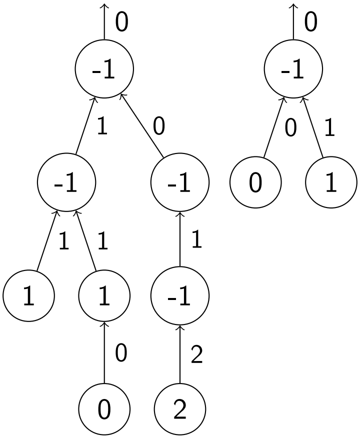


Theorem (C., Pons)

The Tamari order intervals are in bijection with the posets with labels in $1, \ldots, n$ *of size* n *such that:*

- *If* $a < c$ *and* $a \lhd c$ *then* $b \lhd c$ *for all* $a < b < c$.
- *If* $a < c$ *and* $c \lhd a$ *then* $b \lhd a$ *for all* $a < b < c$.

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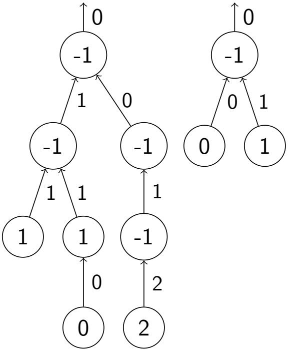
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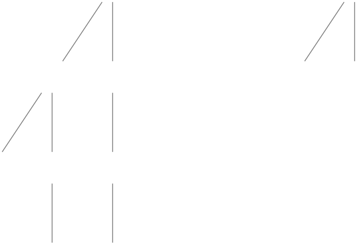

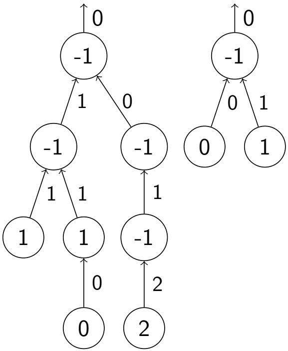
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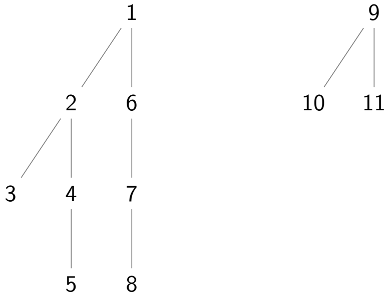
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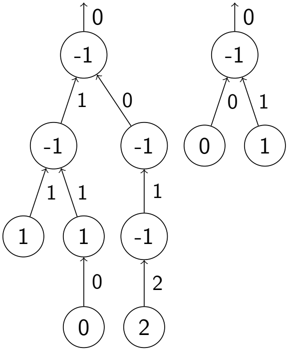
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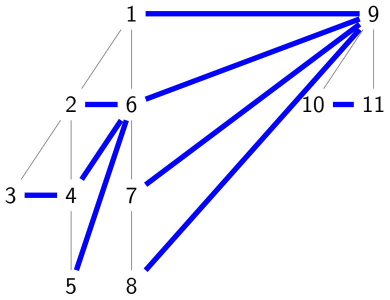
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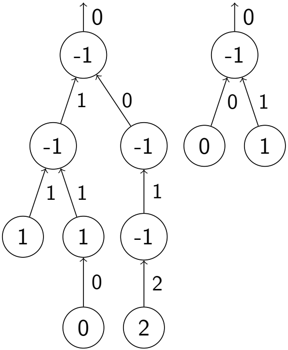
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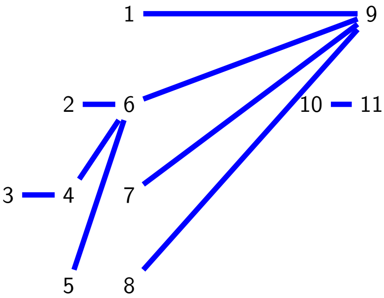
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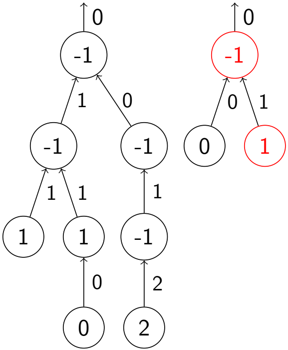
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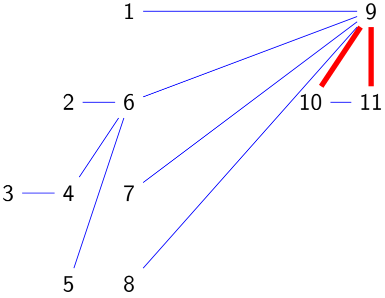
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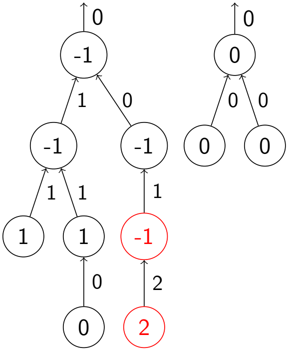
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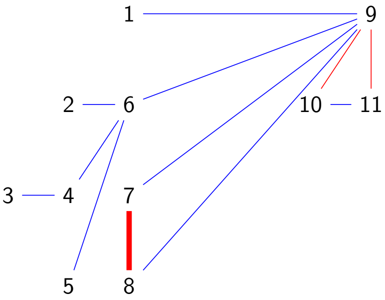
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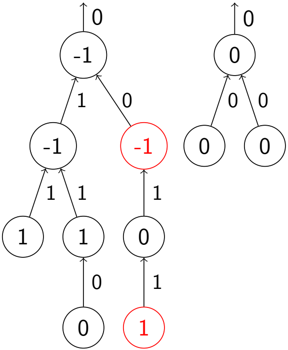
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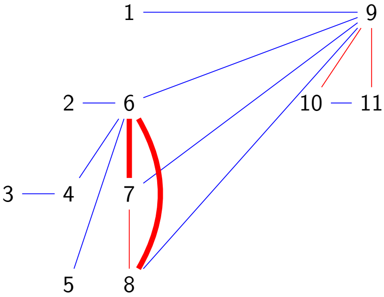
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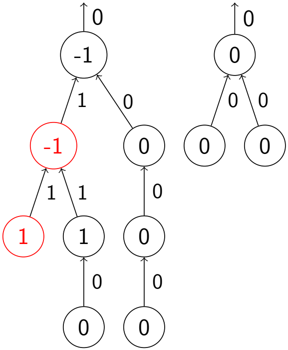
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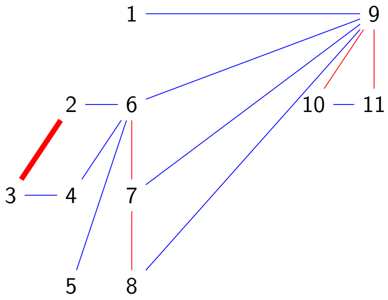
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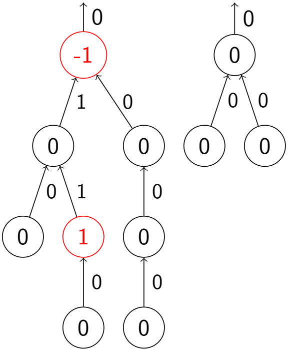
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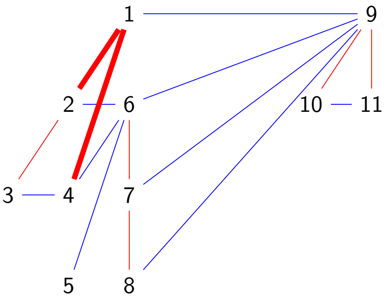
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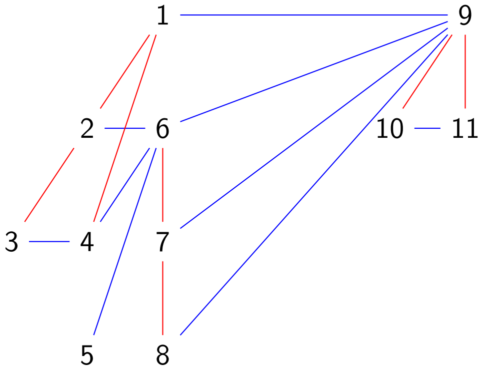
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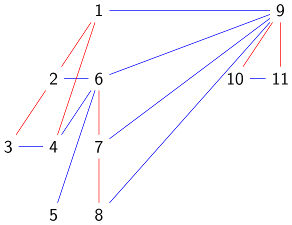
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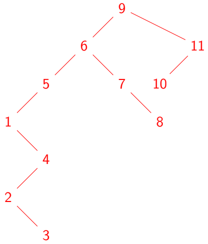
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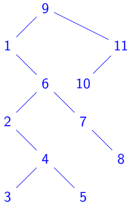
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Thank you for your attention !