Growth diagrams and non-symmetric Cauchy identities over near staircases

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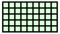
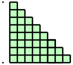
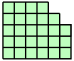
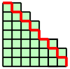
CMUC, University of Coimbra
The 72nd Séminaire Lotharingien de Combinatoire
Lyon

March 25, 2014

Plan

- (Symmetric) Cauchy identity over rectangle shapes 
- Non-symmetric Cauchy identities
    - on staircases 
    - on truncated staircases 
    - on near staircases 

Symmetric Cauchy identity

(Symmetric) Cauchy identity

$$\prod_{(i,j)\in[k]\times[m]} (1 - x_i y_j)^{-1} = \prod_{i=1}^{k}\prod_{j=1}^{m}(1 - x_i y_j)^{-1}$$
$$= \sum_{\nu^+} s_{\nu^+}(x_1, \ldots, x_k) s_{\nu^+}(y_1, \ldots, y_m)$$

over all partitions ν^+ of length $\leq \min\{k, m\}$.

Left hand side is symmetric in the variables x_i and y_j separately.

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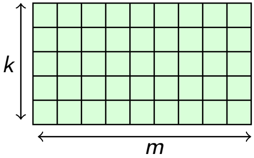

$= (m^k)$

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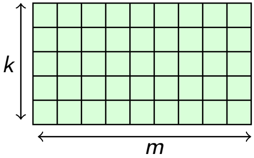
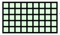


$= (m^k)$

Bijective proof: *D. E. Knuth. Pacific J. Math, 1970.*

RSK: Robinson-Schensted-Knuth correspondence

- RSK correspondence

{multisets of cells of }

$$\rightarrow \quad \biguplus_{\nu^+ \in \mathbb{N}^k} SSYT(\nu^+, k) \times SSYT(\nu^+, m)$$

$$\begin{pmatrix} b_1 & \cdots & b_r \\ a_1 & \cdots & a_r \end{pmatrix} \quad \rightarrow \quad (F, G)$$

- The multivariate generating function for the multisets of cells in (m^k)

$$\prod_{(i,j) \in (m^k)} (1 - x_i y_j)^{-1} = \sum_{\nu^+ \mathbb{N}^k} \sum_{(F,G) \in SSTY(\nu^+,k) \times SSTY(\nu^+,m)} x^F y^G$$

$$= \sum_{\nu^+ \mathbb{N}^k} s_{\nu^+}(x_1, \ldots, x_k) s_{\nu^+}(y_1, \ldots, y_m)$$

- RSK correspondence gives an expansion of the Cauchy kernel in the basis of Schur polynomials.

Schur polynomial

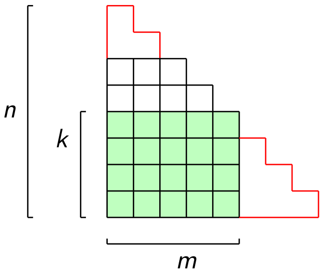
$$s_{\nu^+} = \sum_{T \in SSYT_n(\nu^+)} x^T$$

Non-symmetric Cauchy identity over staircases

Non-symmetric Cauchy identity over staircases   A. Lascoux (2003)

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

The left hand side is no more symmetric on the variables x_i and y_j.



A. Lascoux (2003) RSK for bicrystals in type A.

A. M. Fu, A. Lascoux (2009) algebraic proof

Bases for $\mathbb{Z}[x_1, \ldots, x_n]$

- Linear bases for the ring of integer polynomials $\mathbb{Z}[x_1, \ldots, x_n]$
  - Key polynomials $\{\kappa_\nu : \nu \in \mathbb{N}^n\}$ lift the Schur polynomials s_{ν^+}

$$\kappa_{(\nu_n,\ldots,\nu_1)} = s_{\nu^+}, \quad \nu_n \leq \ldots \leq \nu_1$$

  - Demazure atoms $\{\widehat{\kappa}_\nu : \nu \in \mathbb{N}^n\}$

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  - Demazure atoms $\{\widehat{\kappa}_\nu : \nu \in \mathbb{N}^n\}$

$$\kappa_\nu = \sum_{\beta \leq \nu} \widehat{\kappa}_\beta \qquad s_{\nu^+} = \sum_{\nu \in \mathfrak{S}_n \nu^+} \widehat{\kappa}_\nu$$

The Bruhat ordering on $\mathfrak{S}_n\nu$ is defined to be the transitive closure of the relations

$(\nu_1, \ldots, \nu_i, \ldots, \nu_j, \ldots, \nu_n) < (\nu_1, \ldots, \nu_j, \ldots, \nu_i, \ldots, \nu_n)$, if $\nu_j < \nu_i$.

Combinatorial structure of key polynomials

- Combinatorial rules for monomial expansions of the linear bases $\{\kappa_\alpha : \alpha \in \mathbb{N}^n\}$ and $\{\widehat{\kappa}_\alpha : \alpha \in \mathbb{N}^n\}$
  - Lascoux-Schützenberger (late 80's)

$$SSYT_n(\lambda) = \biguplus_{\alpha \in \mathfrak{S}_n \lambda} \{T \in SSYT_n : K_+(T) = key(\alpha)\}$$

$$key(1,0,4,0,2) = \begin{matrix} 5 & & & \\ 3 & 5 & & \\ 1 & 3 & 3 & 3 \end{matrix}$$

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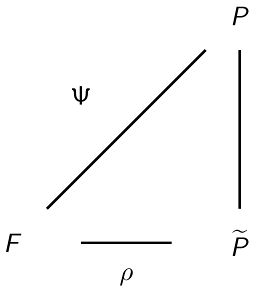
  - Kashiwara crystal bases (early 90's); Haglund, Haiman, Loehr (2005); Mason (2009)

$$\hat{\kappa}_\alpha(x) = \sum_{T \in \widehat{\mathfrak{B}}_\alpha} x^T = \sum_{K_+(T) = key(\alpha)} x^T = \sum_{sh(F)=\alpha} x^F,$$

$$\kappa_\alpha(x) = \sum_{T \in \mathfrak{B}_\alpha} x^T = \sum_{K_+(T) \leq key(\alpha)} x^T = \sum_{sh(F)\leq\alpha} x^F.$$

SSAFs encode SSYTs with the right keys (Mason,2008)

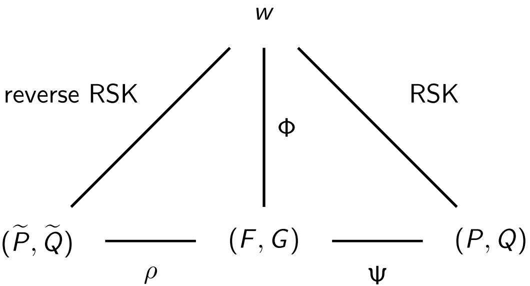


η reverse Schensted row insertion

$$K_+(P) = key(sh(F)) = \begin{array}{lll} 7 & & \\ 5 & & \\ 4 & 4 & \\ 3 & 3 & \\ 1 & 1 & 3 \end{array}$$

$$P = \begin{array}{lll} 7 & & \\ 5 & & \\ 3 & 4 & \\ 2 & 2 & \\ 1 & 1 & 3 \end{array} \qquad \widetilde{P} = \begin{array}{lll} 1 & & \\ 3 & & \\ 4 & 1 & \\ 5 & 2 & \\ 7 & 3 & 2 \end{array} \qquad F = \begin{array}{ccccccc} & & 2 & & & & \\ 1 & & 3 & 2 & & & \\ 1 & & 3 & 4 & 5 & & 7 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

A triangle of Robinson-Schensted-Knuth correspondences (Mason)



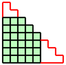
$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q) = sh(\widetilde{P}) = sh(\widetilde{Q})$$

$$key(sh(F)) = K_+(P), \; key(sh(G)) = K_+(Q)$$

RSK analogue restricted to truncated staircases

- RSK analogue for staircases

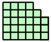
$$\{\text{multisets of cells of } \text{[diagram]} \} \rightarrow \biguplus_{\nu \in \mathbb{N}^n} \{ (F,G) : sh(F)=\nu,\ sh(G) \leq \omega\nu \}$$



$$\begin{pmatrix} b_1 & \cdots & b_r \\ a_1 & \cdots & a_r \end{pmatrix} \rightarrow (F, G)$$

- RSK analogue for truncated staircases

$$\{\text{multisets of cells of } \text{[diagram]} \} \rightarrow \biguplus_{\nu \in \mathbb{N}^k} \{ (F,G) : sh(F)=\nu,\ (sh(G), 0^{n-m}) \leq (0^{n-k}, \omega\nu) \}$$



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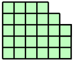
Bijective proof

$$\prod_{(i,j)\in}$$



$$(1 - x_i y_j)^{-1} = \sum_{\nu \in \mathbb{N}^k} \sum_{\substack{F,G\, \in SSAF_n \\ sh(G)=\beta\in\mathbb{N}^m,\ sh(F)=\nu \\ (\beta,0^{n-m})\leq(0^{n-k},\omega\nu)}} x^F y^G$$

Bijective proof

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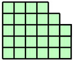


$$= \sum_{\nu\in\mathbb{N}^k} \widehat{\kappa}_\nu(x) \sum_{\substack{Q\in\mathfrak{B}_{(0^{n-k},\omega\nu)}\\ entries\leq m}} y^Q$$

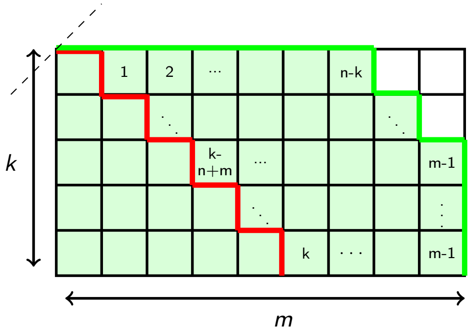
$$= \sum_{\nu\in\mathbb{N}^k} \widehat{\kappa}_\nu(x)\pi_{\sigma(\lambda,SE)}\kappa_{(\omega\nu,0^{n-k})}(y)\ (Lacoux, 2003)$$

$$= \sum_{\nu\in\mathbb{N}^k} \widehat{\kappa}_\nu(x)\kappa_{(0^{m-k},\alpha)}(y)$$

The action of Demazure operators π_i on key polynomials κ_ν can be realised via bubble sorting operators acting on ν, swapping entries i and $i+1$ in the weak composition ν, if $\nu_i > \nu_{i+1}$, and doing nothing, otherwise,

$$\pi_i\kappa_\alpha = \begin{cases} \kappa_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ \kappa_\alpha & \text{if } \alpha_i \leq \alpha_{i+1} \end{cases}.$$

$$\sum_{\nu \in \mathbb{N}^k} \widehat{\kappa}_\nu(x) \pi_{\sigma(\lambda, SE)} \kappa_{(\omega\nu, 0^{n-k})}(y) \; (Lascoux, 2003)$$



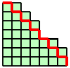
$1 \le k \le m \le n$, and $n - k \le m - 1$,

$$\sigma(\lambda, SE) = (s_{n-k} \ldots s_1) \cdots (s_{m-2} \cdots s_{k-(n+m)-1})(s_{m-1} \ldots s_{k-n+m}) \cdots (s_{m-1} \cdots s_k)$$

Non-symmetric Cauchy identity over near staircases

We want to give a bijective proof for the identity:

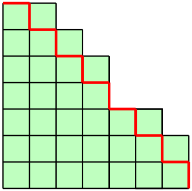
$$\prod_{(i,j)\in} (1 - x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \pi_{r_1}\ldots\pi_{r_p}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

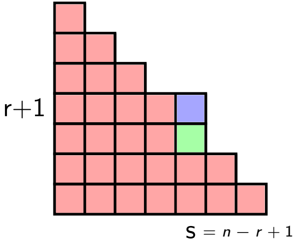


$$= \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\pi_{n-r_p}\ldots\pi_{n-r_1}\kappa_{\omega\nu}(y)$$

(Lascoux 2003)

- Look at the biggest staircase contained inside the near staircases





$\lambda_1 =$ red shape $\cup$ green box $\cup$ blue box
$\lambda_2 =$ red shape $\cup$ green box
$\lambda_3 =$ red shape

$$F_{\lambda_3} = \prod_{(i,j)\in\lambda_3} (1 - x_i y_j)^{-1}, \quad F_{\lambda_2} = (1 - x_r y_s)^{-1} F_{\lambda_3} = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

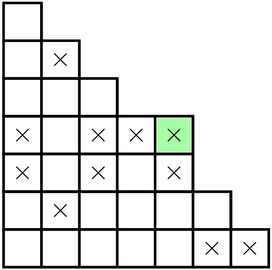
$$\pi_r F_{\lambda_2} = (\pi_r(1 - x_r y_s)^{-1}) F_{\lambda_3} = F_{\lambda_2}(1 - x_{r+1} y_s)^{-1} = F_{\lambda_1}$$

$$F_{\lambda_1} = \sum_{\nu\in\mathbb{N}^n} \pi_r \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\pi_{n-r}\kappa_{\omega\nu}(y).$$

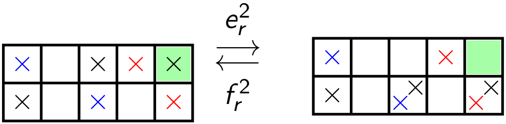
The operator π_r reproduce cells.

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 3 & 4 & 1 & 1 \end{pmatrix}.$$

The biword w in the Ferrers shape $\lambda = (7, 6, 5, 5, 3, 2, 1)$ is represented by putting a cross $\times$ in the cell (i, j) of λ if $\begin{pmatrix} \mathrm{j} \\ \mathrm{i} \end{pmatrix}$ is a biletter of w.

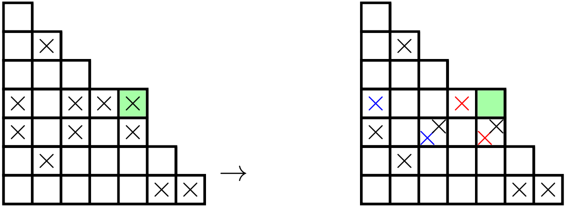


$\lambda = (7, 6, 5, 5, 3, 2, 1)$



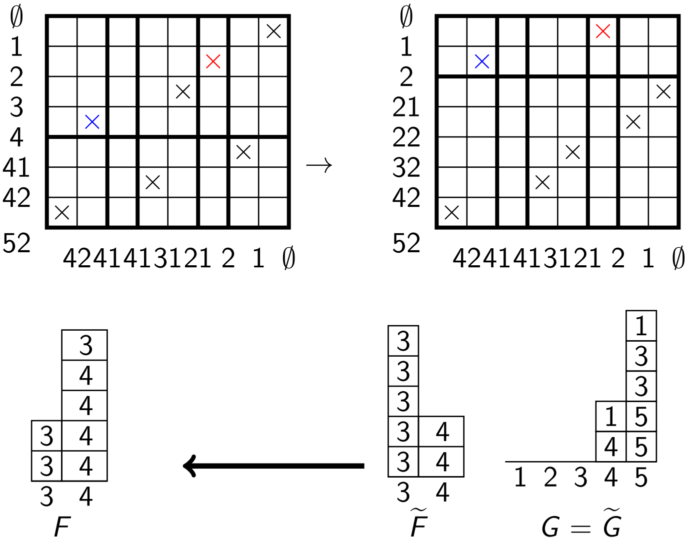
$$\begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & 4 & 3 & 3 & 4 & 3 & 3 \end{pmatrix}$$

Apply the crystal operator e_r as long as it is possible to the second row of the biword w.

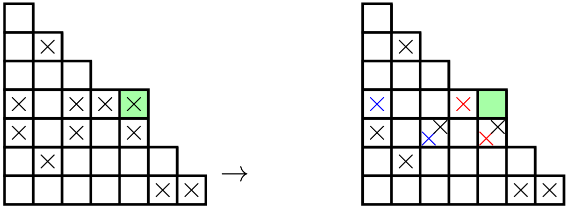


$$\begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 3 & 4 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 3 & 4 & 3 & 3 & 1 & 1 \end{pmatrix}$$

Growth diagram for the analogue of RSK



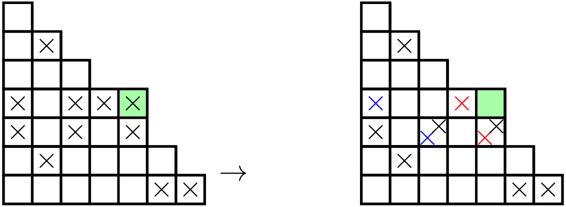
The shape of SSAF changes



$$(F, G) \quad \longleftarrow \quad (\tilde{F}, G)$$

$$sh(F) = s_r sh(\tilde{F}) > sh(\tilde{F})$$

The shape of SSAF changes



$$(F,G) \quad \longleftarrow \quad (\tilde{F},G)$$

$$sh(F) = s_r sh(\tilde{F}) > sh(\tilde{F})$$

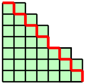
$$sh(G) \le \omega sh(\tilde{F}) = \omega s_r sh(F)$$

$$sh(G) \not\le \omega sh(F)$$

RSK analogue restricted to near staircases

- RSK analogue for near staircases

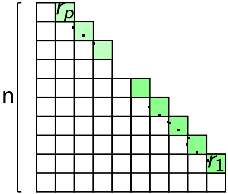
$$\left\{ \begin{array}{c} \text{multisets of cells} \\ \text{with the sited boxes of} \end{array} \right.$$



$$\left. \right\} \to \mathcal{A}_z^{H_z} = \left\{ (F,G) : \begin{array}{c} sh(F)=\nu \\ sh(G) \not\leq \omega s_{r_{i_z}} \cdots \widehat{s}_{r_{i_m}} \cdots s_{r_{i_1}} \nu, \\ m=1,2,\dots,z \\ sh(G) \leq \omega s_{r_{i_z}} \cdots s_{r_{i_1}} \nu \end{array} \right\}$$

$$\begin{pmatrix} b_1 & \cdots & b_r \\ a_1 & \cdots & a_r \end{pmatrix} \to (F, G)$$

Let $0 \leq p < n$ and $0 < r_1 < r_2 < \cdots < r_p \leq n$.



- $$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \prod_{i+j\leq n+1} (1 - x_i y_j)^{-1} \prod_{i=1}^{p} (1 - x_{r_i+1} y_{n-r_i+1})^{-1}$$
$$= \sum_{(F,G)\in\mathcal{A}} x^F y^G + \sum_{z=1}^{p} \sum_{H_z \in \binom{[p]}{z}} \sum_{(F,G)\in\mathcal{A}_z^{H_z}} x^F y^G$$

$$\sum_{\nu\in\mathbb{N}^n}(\pi_{r_1}\dots\pi_{r_p}\widehat{\kappa}_\nu(x))\kappa_{\omega\nu}(y)=\pi_{r_1}\left(\sum_{\nu\in\mathbb{N}^n}\pi_{r_2}\dots\pi_{r_p}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)\right)$$

$$=\pi_{r_1}\left(\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G\right)$$

$$=\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\left(\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G+\sum_{(F,G)\in\mathcal{A}_{z+1}^{H^1_{z+1}}}x^Fy^G\right)$$

$$=\sum_{z=0}^{p}\sum_{H_z\in\binom{[p]}{z}}\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G$$

$$\pi_{r_1}\widehat{\kappa}_\alpha=\begin{cases}\widehat{\kappa}_{s_{r_1}\alpha}+\widehat{\kappa}_\alpha & \text{if } \alpha_r>\alpha_{r+1}\\ \widehat{\kappa}_\alpha & \text{if } \alpha_{r_1}=\alpha_{r_1+1}\\ 0 & \text{if } \alpha_{r_1}<\alpha_{r_1+1}\end{cases}.$$