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VANISHING GEODESIC DISTANCE FOR THE RIEMANNIAN METRIC WITH GEODESIC EQUATION THE KDV-EQUATION

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Abstract. The Virasoro-Bott group endowed with the right-invariant L^2-metric (which is a weak Riemannian metric) has the KdV-equation as geodesic equation. We prove that this metric space has vanishing geodesic distance.

1. Introduction

It was found in [11] that a curve in the Virasoro-Bott group is a geodesic for the right invariant L^2-metric if and only if its right logarithmic derivative is a solution of the Korteweg-de Vries equation, see 2.3. Vanishing geodesic distance for weak Riemannian metrics on infinite dimensional manifolds was first noticed on shape space $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ for the L^2-metric in [7, 3.10]. In [8] this result was shown to hold for the general shape space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ for any compact manifold M and Riemannian manifold N, and also for the right invariant L^2-metric on each full diffeomorphism group with compact support $\operatorname{Diff}_c(N)$. In particular, Burgers' equation is related to the geodesic equation of the right invariant L^2-metric on $\operatorname{Diff}(S^1)$ or $\operatorname{Diff}_c(\mathbb{R})$ and it thus also has vanishing geodesic distance. We even have

Result. [8] *The weak Riemannian L^2-metric on each connected component of the total space $\operatorname{Imm}(M,N)$ for a compact manifold M and a Riemannian manifold (N,g) has vanishing geodesic distance.*

This result is not spelled out in [8] but it follows from there: Given two immersions f_0, f_1 in the same connected component, we first connect their shapes $f_0(M)$ and $f_1(M)$ by a curve of length $< \varepsilon$ in the shape space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ and take the horizontal lift to get a curve of length $< \varepsilon$ from f_0 to an immersion $f_1\circ\varphi$ in the connected component of the orbit through f_1. Now we use the induced metric f_1^*g on M and the right invariant L^2-metric induced on $\operatorname{Diff}(M)_0$ to get a curve in $\operatorname{Diff}(M)$ of length $< \varepsilon$ connecting φ with Id_M. Evaluating at f_1 we get curve in $\operatorname{Imm}(M,N)$ of length $< \varepsilon$ connecting $f_1\circ\varphi$ with f_1.

In this article we show that the right invariant L^2-metric on the Virasoro-Bott groups (see 2.1) has vanishing geodesic distance. This might be related to the fact that the Riemannian exponential mapping is not a diffeomorphism near 0, see [2] for $\operatorname{Diff}(S^1)$ and [3] for the Virasoro group over S^1. See [10] for information on conjugate points along geodesics.

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2. The Virasoro-Bott groups

2.1. **The Virasoro-Bott groups.** Let $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ be the group of diffeomorphisms of $\mathbb{R}$ which rapidly fall to the identity. This is a regular Lie group, see [6, 6.4]. The mapping

$$c : \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}) \times \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}) \to \mathbb{R}$$
$$c(\varphi, \psi) := \frac{1}{2}\int \log(\varphi\circ\psi)' d\log\psi' = \frac{1}{2}\int \log(\varphi'\circ\psi) d\log\psi'$$

satisfies $c(\varphi,\varphi^{-1}) = 0$, $c(\mathrm{Id},\psi) = 0$, $c(\varphi,\mathrm{Id}) = 0$ and is a smooth group cocycle, called the Bott cocycle:

$$c(\varphi_2,\varphi_3) - c(\varphi_1\circ\varphi_2,\varphi_3) + c(\varphi_1,\varphi_2\circ\varphi_3) - c(\varphi_1,\varphi_2) = 0.$$

The corresponding central extension group $\mathrm{Vir} := \mathbb{R}\times_c \operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$, called the Virasoro-Bott group, is a trivial $\mathbb{R}$-bundle $\mathbb{R}\times\operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ that becomes a regular Lie group relative to the operations

$$\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}\begin{pmatrix}\psi\\ \beta\end{pmatrix} = \begin{pmatrix}\varphi\circ\psi\\ \alpha+\beta+c(\varphi,\psi)\end{pmatrix},\quad \begin{pmatrix}\varphi\\ \alpha\end{pmatrix}^{-1} = \begin{pmatrix}\varphi^{-1}\\ -\alpha\end{pmatrix}\quad \varphi,\psi\in\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}),\ \alpha,\beta\in\mathbb{R}.$$

Other versions of the Virasoro-Bott group are the following: $\mathbb{R}\times_c\operatorname{Diff}_c(\mathbb{R})$ where $\operatorname{Diff}_c(\mathbb{R})$ is the group of all diffeomorphisms with compact support, or the periodic case $\mathbb{R}\times_c\operatorname{Diff}^+(S^1)$. One can also apply the homomorphism $\exp(i\alpha)$ to the center and replace it by S^1. To be specific, we shall treat the most difficult case $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ in this article. All other cases require only obvious minor changes in the proofs.

2.2. **The Virasoro Lie algebra.** The Lie algebra of the Virasoro-Bott group $\mathbb{R}\times_c\operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ is $\mathbb{R}\times\mathfrak{X}_{\mathcal{S}}(\mathbb{R})$ (where $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}) = \mathcal{S}(\mathbb{R})\partial_x$) with the Lie bracket

$$\left[\begin{pmatrix}X\\ a\end{pmatrix},\begin{pmatrix}Y\\ b\end{pmatrix}\right] = \begin{pmatrix}-[X,Y]\\ \omega(X,Y)\end{pmatrix} = \begin{pmatrix}X'Y - XY'\\ \omega(X,Y)\end{pmatrix}$$

where

$$\omega(X,Y) = \omega(X)Y = \int X'dY' = \int X'Y''dx = \tfrac{1}{2}\int \det\begin{pmatrix}X' & Y'\\ X'' & Y''\end{pmatrix}dx,$$

is the *Gelfand-Fuks Lie algebra cocycle* $\omega : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, which is a bounded skew-symmetric bilinear mapping satisfying the cocycle condition

$$\omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0.$$

It is a generator of the 1-dimensional bounded Chevalley cohomology $H^2(\mathfrak{g},\mathbb{R})$ for any of the Lie algebras $\mathfrak{g} = \mathfrak{X}(\mathbb{R})$, $\mathfrak{X}_c(\mathbb{R})$, or $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}) = \mathcal{S}(\mathbb{R})\partial_x$. The Lie algebra of the Virasoro-Bott Lie group is thus the central extension $\mathbb{R}\times_\omega\mathfrak{X}_{\mathcal{S}}(\mathbb{R})$ induced by this cocycle. We have $H^2(\mathfrak{X}_c(M),\mathbb{R}) = 0$ for each finite dimensional manifold of dimension ≥ 2 (see [4]), which blocks the way to find a higher dimensional analog of the Korteweg-de Vries equation in a way similar to that sketched below.

To complete the description, we add the adjoint action:

$$\mathrm{Ad}\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}\begin{pmatrix}Y\\ b\end{pmatrix} = \begin{pmatrix}\mathrm{Ad}(\varphi)Y = \varphi_*Y = T\varphi\circ Y\circ\varphi^{-1}\\ b + \int S(\varphi)Y\,dx\end{pmatrix}$$

where the *Schwartzian derivative* S is given by

$$S(\varphi) = \Big(\frac{\varphi''}{\varphi'}\Big)' - \frac{1}{2}\Big(\frac{\varphi''}{\varphi'}\Big)^2 = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\Big(\frac{\varphi''}{\varphi'}\Big)^2 = \log(\varphi')'' - \frac{1}{2}(\log(\varphi')')^2$$

which measures the deviation of φ from being a Möbius transformation:

$$S(\varphi)=0 \iff \varphi(x)=\frac{ax+b}{cx+d} \text{ for } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}).$$

The Schwartzian derivative of a composition and an inverse follow from the action property:

$$S(\varphi\circ\psi)=(S(\varphi)\circ\psi)(\psi')^2+S(\psi),\quad S(\varphi^{-1})=-\frac{S(\varphi)}{(\varphi')^2}\circ\varphi^{-1}$$

2.3. **The right invariant L^2-metric and the KdV-equation.** We shall use the L^2-inner product on $\mathbb{R}\times_\omega \mathfrak{X}_{\mathcal{S}}(\mathbb{R})$:

$$\left\langle \begin{pmatrix} X\\ a\end{pmatrix}, \begin{pmatrix} Y\\ b\end{pmatrix}\right\rangle := \int XY\,dx+ab.$$

We use the induced right invariant weak Riemannian metric on the Virasoro group.

According to [1], see [9] for a proof in the notation and setup used here, a curve $t\mapsto \binom{\varphi(t,\ \ )}{\alpha(t)}$ in the Virasoro-Bott group is a geodesic if and only if

$$\begin{aligned}
&\begin{pmatrix} u_t\\ a_t\end{pmatrix} = -\operatorname{ad}\begin{pmatrix} u\\ a\end{pmatrix}^{\top}\begin{pmatrix} u\\ a\end{pmatrix} = \begin{pmatrix} -3u_xu-au_{xxx}\\ 0\end{pmatrix} \qquad \text{where}\\
&\begin{pmatrix} u(t)\\ a(t)\end{pmatrix} = \partial_s\begin{pmatrix} \varphi(s)\\ \alpha(s)\end{pmatrix}.\begin{pmatrix} \varphi(t)^{-1}\\ -\alpha(t)\end{pmatrix}\Big|_{s=t} = \partial_s\begin{pmatrix} \varphi(s)\circ\varphi(t)^{-1}\\ \alpha(s)-\alpha(t)+c(\varphi(s),\varphi(t)^{-1})\end{pmatrix}\Big|_{s=t},\\
&\begin{pmatrix} u\\ a\end{pmatrix} = \begin{pmatrix} \varphi_t\circ\varphi^{-1}\\ \alpha_t-\int\frac{\varphi_{tx}\varphi_{xx}}{2\varphi_x^2}dx\end{pmatrix},
\end{aligned}$$

since we have

$$\begin{aligned}
2\partial_s c(\varphi(s),\varphi(t)^{-1})|_{s=t} &= \partial_s\int \log(\varphi(s)'\circ\varphi(t)^{-1})\,d\log((\varphi(t)^{-1})')|_{s=t}\\
&=\int \frac{\varphi_t(t)'\circ\varphi(t)^{-1}}{\varphi(t)'\circ\varphi(t)^{-1}}\left(-\frac{\varphi(t)''\circ\varphi(t)^{-1}}{(\varphi(t)'\circ\varphi(t)^{-1})^2}\right)dx\\
&=-\int\left(\frac{\varphi_t'\varphi''}{(\varphi')^2}\right)(t)dy=-\int\left(\frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\right)(t)dx.
\end{aligned}$$

Thus a is a constant in time and the geodesic equation is hence the *Korteweg-- de Vries equation*

$$u_t+3u_xu+au_{xxx}=0$$

with its natural companions

$$\varphi_t=u\circ\varphi,\qquad \alpha_t=a+\int\frac{\varphi_{tx}\varphi_{xx}}{2\varphi_x^2}dx.$$

To be complete, we add the invariant momentum mapping J with values in the Virasoro algebra (via the weak Riemannian metric). We need the transpose of the adjoint action:

$$\begin{aligned}
\left\langle \operatorname{Ad}\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}^{\top}\begin{pmatrix}Y\\ b\end{pmatrix},\begin{pmatrix}Z\\ c\end{pmatrix}\right\rangle &= \left\langle\begin{pmatrix}Y\\ b\end{pmatrix},\operatorname{Ad}\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}\begin{pmatrix}Z\\ c\end{pmatrix}\right\rangle\\
&=\left\langle\begin{pmatrix}Y\\ b\end{pmatrix},\begin{pmatrix}\varphi_*Z\\ c+\int S(\varphi)Z\,dx\end{pmatrix}\right\rangle
\end{aligned}$$

$$= \int Y((\varphi' \circ \varphi^{-1})(Z \circ \varphi^{-1})\, dx + bc + \int bS(\varphi) Z\, dx$$
$$= \int ((Y \circ \varphi)(\varphi')^2 + bS(\varphi)) Z\, dx + bc$$

Thus, the invariant momentum mapping is given by

$$J\left(\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}, \begin{pmatrix}Y\\ b\end{pmatrix}\right) = \mathrm{Ad}\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}^{\top}\begin{pmatrix}Y\\ b\end{pmatrix} = \begin{pmatrix}(Y \circ \varphi)(\varphi')^2 + bS(\varphi)\\ b\end{pmatrix}.$$

Along a geodesic $t \mapsto g(t, \quad) = \binom{\varphi(t, \quad)}{\alpha(t)}$, the momentum

$$J\left(\begin{pmatrix}\varphi\\ \alpha\end{pmatrix}, \begin{pmatrix}u = \varphi_t \circ \varphi^{-1}\\ a\end{pmatrix}\right) = \begin{pmatrix}(u \circ \varphi)\varphi_x^2 + aS(\varphi)\\ a\end{pmatrix} = \begin{pmatrix}\varphi_t \varphi_x^2 + aS(\varphi)\\ a\end{pmatrix}$$

is constant in t.

2.4. **Lifting curves to the Virasoro-Bott group.** We consider the extension

$$\mathbb{R} \xrightarrow{i} \mathbb{R} \times_c \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}) \xrightarrow{p} \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}).$$

Then p is a Riemannian submersion for the right invariant L^2-metric on $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$, i.e., Tp is an isometry on the orthogonal complements of the fibers. These complements are not integrable; in fact, the curvature of the corresponding principal connection is given by the Gelfand-Fuks cocycle. For any curve $\varphi(t)$ in $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ its horizontal lift is given by

$$\begin{pmatrix}\varphi(t)\\ a(t) = a(0) - \int_0^t \int \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\, dx\, dt\end{pmatrix}$$

since the right translation to $(\mathrm{Id}, 0)$ of its velocity should have zero vertical component, see 2.3. The horizontal lift has the same length and energy as φ.

3. Vanishing of the geodesic distance

3.1. **Theorem.** *On all Virasoro-Bott groups mentioned in 2.1 geodesic distance for the right invariant L^2-metric vanishes.*

The rest of this section is devoted to the proof of theorem 3.1 for the most difficult case $\mathbb{R} \times_c \mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$.

3.2. **Proposition.** *Any two diffeomorphisms in $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ can be connected by a path with arbitrarily short length for the right invariant L^2-metric.*

In [8] for $\mathrm{Diff}_c(\mathbb{R})$ it was first shown that there exists one non-trivial diffeomorphism which can be connected to Id with arbitrarily small length. Then, it was shown that the diffeomorphisms with this property form a normal subgroup. Since $\mathrm{Diff}_c(\mathbb{R})$ is a simple group this concluded the proof. But $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ is not a simple group since $\mathrm{Diff}_c(\mathbb{R})$ is a normal subgroup. So, we have to elaborate on the proof of [8] as follows.

Proof. We show that any rapidly decreasing diffeomorphism can be connected to the identity by an arbitrarily short path. We will write this diffeomorphism as $\mathrm{Id} + g$, where $g \in \mathcal{S}(\mathbb{R})$ is a rapidly decreasing function with $g' > -1$. For $\lambda = 1 - \varepsilon < 1$ we define

$$\varphi(t, x) = x + \max(0, \min(t - \lambda x, g(x))) - \max(0, \min(t + \lambda x, -g(x))).$$

This is a (non-smooth) path defined for $t \in (-\infty, \infty)$ connecting the identity in $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ with the diffeomorphism $(\mathrm{Id} + g)$. We define $\psi(t,x) = \varphi(\tan(t), x) \star G_\varepsilon(t,x)$, where $G_\varepsilon(t,x) = \frac{1}{\varepsilon^2} G_1(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ is a smoothing kernel with $\mathrm{supp}(G_\varepsilon) \subseteq B_\varepsilon(0)$ and $\iint G_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = 1$. Thus ψ is a smooth path defined on the finite interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$ connecting the identity in $\mathrm{Diff}_S(\mathbb{R})$ with a diffeomorphism arbitrarily close to $(\mathrm{Id} + g)$ for ε small. (Compare figure 1 for an illustration.)

The L^2-energy of ψ is

$$E(\psi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} (\psi_t \circ \psi^{-1})^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 \psi_x \,\mathrm{d}x\,\mathrm{d}t$$

where $\psi^{-1}(t,x)$ stands for $\psi(t, \quad)^{-1}(x)$. We have

$$\partial_a \max(0, \min(a,b)) = \mathbb{1}_{0 \le a \le b}, \qquad \partial_b \max(0, \min(a,b)) = \mathbb{1}_{0 \le b \le a},$$

and therefore

$$\begin{aligned}
\psi_x(t,x) &= \varphi_x(\tan(t), x) \star G_\varepsilon \\
&= \big(1 - \lambda \mathbb{1}_{0 \le \tan(t) - \lambda x \le g(x)} \quad + g'(x) \mathbb{1}_{0 \le g(x) \le \tan(t) - \lambda x} \\
&\qquad - \lambda \mathbb{1}_{0 \le \tan(t) + \lambda x \le -g(x)} + g'(x) \mathbb{1}_{0 \le -g(x) \le \tan(t) + \lambda x}\big) \star G_\varepsilon, \\
\psi_t(t,x) &= \big((1 + \tan(t)^2) \varphi_t(\tan(t), x)\big) \star G_\varepsilon \\
&= \big((1 + \tan(t)^2)(\mathbb{1}_{0 \le \tan(t) - \lambda x \le g(x)} - \mathbb{1}_{0 \le \tan(t) + \lambda x \le -g(x)})\big) \star G_\varepsilon.
\end{aligned}$$

Note that these functions have disjoint support when $\varepsilon = 0, \lambda = 1 - \varepsilon = 1$.

Claim. The mappings $\varepsilon \mapsto \psi_t$ and $\varepsilon \mapsto (\psi_x - 1)$ are continuous into each L^p with p even. (The proofs are simpler when p is even because there are no absolute values to be taken care of.) To prove the claim, we calculate

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \big((1 + \tan(t)^2) \varphi_t(\tan(t), x)\big)^p \,\mathrm{d}x\,\mathrm{d}t &= \iint_{\mathbb{R}^2} \varphi_t(t,x)^p (1 + t^2)^{p-1} \,\mathrm{d}x\,\mathrm{d}t \\
= \iint_{\mathbb{R}^2} (\mathbb{1}_{0 \le t - \lambda x \le g(x)} + \mathbb{1}_{0 \le t + \lambda x \le -g(x)}) (1 + t^2)^{p-1} \,\mathrm{d}x\,\mathrm{d}t \\
= \int_{g(x) \ge 0} \int_{\lambda x}^{\lambda x + g(x)} (1 + t^2)^{p-1} \,\mathrm{d}t\,\mathrm{d}x + \int_{g(x) < 0} \int_{\lambda x + g(x)}^{\lambda x} (1 + t^2)^{p-1} \,\mathrm{d}t\,\mathrm{d}x \\
= \int_{\mathbb{R}} \Big| F(t) \Big|_{t = \lambda x}^{t = \lambda x + g(x)} \Big| \,\mathrm{d}x = \int_{\mathbb{R}} |F(\lambda x + g(x)) - F(\lambda x)| \,\mathrm{d}x,
\end{aligned}$$

where $F(\lambda x + g(x)) - F(\lambda x)$ is a polynomial without constant term in $g(x)$ with coefficients also powers of λx. Integrals of the form $\int_{\mathbb{R}} |(\lambda x)^{k_1} g(x)^{k_2}| \,\mathrm{d}x$ with $k_1 \ge 0, k_2 > 0$ are finite and continuous in $\lambda = 1 - \varepsilon$ since g is rapidly decreasing. This shows that $\|(1 + \tan(t)^2) \varphi_t(\tan(t), x)\|_p$ depends continuously on ε . Furthermore the sequence $(1 + \tan(t)^2) \varphi_t(\tan(t), x)$ converges almost everywhere for $\varepsilon \to 0$, thus it also converges in measure. By the theorem of Vitali, this implies convergence in L^p, see for example [12, theorem 16.6]. Convolution with G_ε acts as approximate unit in each L^p, which proves the claim for ψ_t. For $\psi_x - 1$ it follows similarly.

The above claim implies that

$$E(\psi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 \psi_x \,\mathrm{d}x\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 (\psi_x - 1) \,\mathrm{d}x\,\mathrm{d}t + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}} \psi_t^2 \,\mathrm{d}x\,\mathrm{d}t$$

viewed as a mapping on $L^4 \times L^4 \times L^2$ (first summand) and on $L^2 \times L^2$ (second summand) is continuous in ε. It also vanishes at $\varepsilon = 0$ since then ψ_x and ψ_t have disjoint support. The Cauchy-Schwarz inequality $L(\psi)^2 < \pi E(\psi)$ implies that $L(\psi)$ goes to zero as well. Ultimately, $\psi(\frac{\pi}{2}) = (\mathrm{Id} + g) \star G_\varepsilon$ is arbitrarily close to $\mathrm{Id} + g$. $\square$

3.3. **Lemma.** *For any $a \in \mathbb{R}$ there exists an arbitrarily short path connecting $\binom{\mathrm{Id}}{0}$ and $\binom{\mathrm{Id}}{a}$, i.e.,* $\operatorname{dist}^{L^2}_{\mathrm{Vir}}\left(\binom{\mathrm{Id}}{0}, \binom{\mathrm{Id}}{a}\right) = 0$.

Proof. The aim of the following argument is to construct a family of paths in the diffeomorphism group, parametrized by ε, with the following properties: all paths in the family start and end at the identity and their length in the diffeomorphism group with respect to the L^2 metric tends to 0 as $\varepsilon \to 0$. By letting ε be time-dependent, we are able to control the endpoint $a(T)$ of the horizontal lift for certain diffeomorphisms.

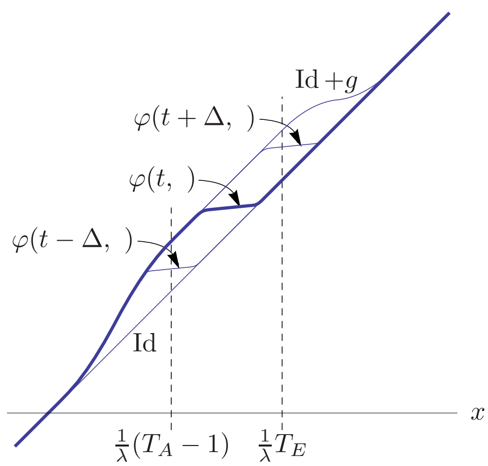

FIGURE 1. The path $\varphi(t, \ )$ defined in 3.3 connecting Id to Id $+g$, plotted at $t - \Delta < t < t + \Delta$. Between the dashed lines, $g \equiv 1$ is constant.

We consider the function

$$
\begin{aligned}
f(z, a, \varepsilon) &= \max(0, \min(z, a)) \star G_\varepsilon(z) G_\varepsilon(a) \\
&= \iint \max(0, \min(z - \overline{z}, a - \overline{a})) G_\varepsilon(\overline{z}) G_\varepsilon(\overline{a}) \,\mathrm{d}\overline{z} \,\mathrm{d}\overline{a} \\
&= \iint \max(0, \min(z - \varepsilon\overline{z}, a - \varepsilon\overline{a})) G_1(\overline{z}) G_1(\overline{a}) \,\mathrm{d}\overline{z} \,\mathrm{d}\overline{a} \qquad (1) \\
&= \varepsilon f(\frac{z}{\varepsilon}, \frac{a}{\varepsilon}, 1)
\end{aligned}
$$

where $G_\varepsilon(z) = \frac{1}{\varepsilon} G_1(\frac{z}{\varepsilon})$ is a function with $\operatorname{supp}(G_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$ and $\int G_\varepsilon \,\mathrm{d}x = 1$. Furthermore, let $g : \mathbb{R} \to [0, 1]$ be a function with compact support contained in $\mathbb{R}_{>0}$ and $g' > -1$, so that $x + g(x)$ is a diffeomorphism. For $0 < \lambda < 1$ and $t \in [0, T]$ let

$$\varphi(t, x) = x + f(t - \lambda x, g(x), \varepsilon(t))$$

be the path going away from the identity (since $\operatorname{supp}(g) \subset \mathbb{R}_{>0}$, see also figure 1). For given $\varepsilon_0 > 0$, let

$$\psi(t,x) = x + f(T - t - \lambda x, g(x), \varepsilon_0)$$

the path leading back again. The only difference to [8] is that the parameter ε may vary along the path.

We shall need some derivatives of φ and f:

$$\begin{aligned}
\varphi_t(t,x) &= f_z(t-\lambda x, g(x), \varepsilon(t)) + \dot\varepsilon(t) f_\varepsilon(t-\lambda x, g(x), \varepsilon(t))\\
\varphi_x(t,x) &= 1 - \lambda f_z(t-\lambda x, g(x), \varepsilon(t)) + f_a(t-\lambda x, g(x), \varepsilon(t)) g'(x)\\
f_z(z,a,\varepsilon) &= \int_{-\infty}^{z}\int_{-\infty}^{a-z} G_\varepsilon(w) G_\varepsilon(w+b)\,\mathrm{d}b\,\mathrm{d}w\\
f_a(z,a,\varepsilon) &= \int_{-\infty}^{a}\int_{-\infty}^{z-a} G_\varepsilon(w) G_\varepsilon(w+b)\,\mathrm{d}b\,\mathrm{d}w\\
f_\varepsilon(z,a,\varepsilon) &= \frac{1}{\varepsilon}\Big(f(z,a,\varepsilon) - z f_z(z,a,\varepsilon) - a f_a(z,a,\varepsilon)\Big)\\
f_{zz}(z,a,\varepsilon) &= G_\varepsilon(z)\int_{-\infty}^{a} G_\varepsilon(b)\,\mathrm{d}b - \int_{-\infty}^{z} G_\varepsilon(w) G_\varepsilon(w-(z-a))\,\mathrm{d}w
\end{aligned}$$

Claim 1. The path φ followed by ψ still has arbitrarily small length for the L^2-metric.

We are working with a fixed time interval $[0, 2T]$. Thus arbitrarily small length is equivalent to arbitrarily small energy. The energy is given by

$$\iint \varphi_t^2 \varphi_x \,\mathrm{d}x\,\mathrm{d}t = \iint (f_z + \dot\varepsilon f_\varepsilon)^2 (1 - \lambda f_z + f_a g')\,\mathrm{d}x\,\mathrm{d}t \tag{2}$$

Looking at the formula for f_ε we see that $\varepsilon f_\varepsilon$ is bounded on a domain with bounded a. Thus $\|\dot\varepsilon f_\varepsilon\|_\infty \to 0$ can be achieved by choosing ε, such that $|\dot\varepsilon| \le C\varepsilon^{3/2}$. We will see later that this is possible. Inspecting $\varphi_t(t,x)$ and looking at the formulas for f_z and f we see that for $t - \lambda x < -\varepsilon(t)$ and for $t - \lambda x - g(x) > 2\varepsilon(t)$ we have $\varphi_t(t,x) = 0$. Thus the domain of integration is contained in the compact set

$$[0,T] \times [-\frac{T + \|g\|_\infty + 2\|\varepsilon\|_\infty}{\lambda}, \frac{T + \|\varepsilon\|_\infty}{\lambda}].$$

Therefore, it is enough to show that the L^∞-norm of the integrand in (2) goes to zero as $\|\varepsilon\|_\infty$ goes to zero. For all terms involving $\dot\varepsilon f_\varepsilon$ this is true by the above assumption since $(1 - \lambda f_z + f_a g')$ and $\varepsilon f_\varepsilon$ are bounded. For the remaining parts $f_z^2(1-\lambda f_z)$ and $f_z^2 f_a g'$ we follow the argumentation of [8]. For t fixed and λ close to 1, the function $1 - \lambda f_z$, when restricted to the support of f_z, is bigger than $\varepsilon(t)$ only on an interval of length $O(\varepsilon(t))$. Hence we have

$$\int_0^T \int_{\mathbb{R}} f_z^2 (1 - \lambda f_z)\,\mathrm{d}x\,\mathrm{d}t \le \|f_z\|_\infty^2 \int_0^T \int_{\mathbb{R}} (1 - \lambda f_z)\,\mathrm{d}x\,\mathrm{d}t = O(\|\varepsilon\|_\infty).$$

For the last part, we note that the support of $f_z^2 f_a$ is contained in the set $|g(x) - (t - \lambda x)| \le 2\varepsilon$. Now we define $x_0 < x_1$ by $g(x_0) + \lambda x_0 = T - 2\|\varepsilon\|_\infty$ and $g(x_1) + \lambda x_1 = T + 2\|\varepsilon\|_\infty$. Then

$$\int_0^T \int_{\mathbb{R}} f_z^2 f_a g'\,\mathrm{d}x\,\mathrm{d}t \le T\|f_z\|_\infty^2 \|f_a\|_\infty \int_{\operatorname{supp}(f_z^2\, f_a)} g'\,\mathrm{d}x$$

$$= T(g(x_1) - g(x_0)) \le 4T\|\varepsilon\|_\infty.$$

The estimate for ψ is similar and easier. This proves claim 1.

Claim 2. For every $a \in \mathbb{R}$ and $\delta > 0$ we may choose $\varepsilon(t)$ with $\|\varepsilon\|_\infty < \delta$ such that

$$\int_0^T \int_{\mathbb{R}} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} \frac{\psi_{tx}\psi_{xx}}{\psi_x^2}\,\mathrm{d}x\,\mathrm{d}t = a.$$

We will subject ε and g to several assumptions. First, we partition the interval $[0,T]$ equidistantly into $0 < T_A < T_E < T$ and the (t,x)-domain into two parts, namely $A_1 = ([0,T_A] \cup [T_E,T]) \times \mathbb{R}$ and $A_2 = [T_A,T_E] \times \mathbb{R}$. We want $g(x) \equiv 1$ on a neighborhood of the interval $[\frac{1}{\lambda}(T_A - 1), \frac{1}{\lambda}T_E]$. We choose $\varepsilon(t)$ to be constant $\varepsilon(t) \equiv \varepsilon_0$ on $[0,T_A] \cup [T_E,T]$ and to be symmetric in the sense, that $\varepsilon(t) = \varepsilon(T-t)$. In addition, we want $\varepsilon(t)$ small enough, such that $g(x) \equiv 1$ on $[\frac{1}{\lambda}(T_A - 1 - 2\varepsilon(t)), \frac{1}{\lambda}(T_E + \varepsilon(t))]$.

On A_1 we have $\varepsilon(t) \equiv \varepsilon_0$. This implies $\psi_{tx}(t,x) = -\varphi_{tx}(T-t,x)$, $\psi_x(t,x) = \varphi_x(T-t,x)$ and $\psi_{xx}(t,x) = \varphi_{xx}(T-t,x)$. Hence

$$\iint_{A_1} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t + \iint_{A_1} \frac{\psi_{tx}\psi_{xx}}{\psi_x^2}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Let $A_2 = [T_A,T_E] \times \mathbb{R}$ be the region, where $\varepsilon(t)$ is not constant. In the interior, where

$$\begin{aligned} -\varepsilon(t) &< t - \lambda x < g(x) + 2\varepsilon(t) \\ t - g(x) - 2\varepsilon(t) &< \lambda x < t + \varepsilon(t) \end{aligned}$$

we have by assumption $g(x) \equiv 1$. Therefore, one has in this region:

$$\begin{aligned} \varphi_x(t,x) &= -\lambda f_z(t - \lambda x, 1, \varepsilon(t)) \\ \varphi_{xx}(t,x) &= \lambda^2 f_{zz}(t - \lambda x, 1, \varepsilon(t)) \\ \varphi_{tx}(t,x) &= -\lambda f_{zz}(t - \lambda x, 1, \varepsilon(t)) - \lambda f_{\varepsilon z}(t - \lambda x, 1, \varepsilon(t))\dot\varepsilon(t) \end{aligned}$$

We divide the integral over A_2 into two symmetric parts

$$\int_{T_A}^{T/2} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t + \int_{T/2}^{T_E} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t$$

and apply the following variable substitution to the second integral

$$\widetilde{t} = T - t, \quad \widetilde{x} = x + \frac{1}{\lambda}(\widetilde{t} - t).$$

Thus $\widetilde{t} - \lambda\widetilde{x} = t - \lambda x$. Together with $\varepsilon(t) = \varepsilon(\widetilde{t})$ this implies

$$\varphi_x(t,x) = \varphi_x(\widetilde{t},\widetilde{x}), \quad \varphi_{xx}(t,x) = \varphi_{xx}(\widetilde{t},\widetilde{x}).$$

Since $\dot\varepsilon(t) = -\dot\varepsilon(\widetilde{t})$ changes sign, the term containing $\dot\varepsilon(t)$ cancels out and leaves only

$$\varphi_{tx}(t,x) + \varphi_{tx}(\widetilde{t},\widetilde{x}) = -2\lambda f_{zz}(t - \lambda x, 1, \varepsilon(t)).$$

A simple calculation shows that the integration limits transform

$$\int_{T/2}^{T_E} \int_{\frac{1}{\lambda}(t-1-2\varepsilon)}^{\frac{1}{\lambda}(t+\varepsilon)} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t = \int_{T_A}^{T/2} \int_{\frac{1}{\lambda}(\widetilde{t}-1-2\varepsilon)}^{\frac{1}{\lambda}(\widetilde{t}+\varepsilon)} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{t}$$

to those of the first integral. Therefore, the sum of the integrals gives

$$\iint_{A_2} \frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t = -2\lambda^3 \int_{T_A}^{T/2} \int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))} \frac{f_{zz}(t - \lambda x, 1, \varepsilon(t))^2}{\big(1 - \lambda f_z(t - \lambda x, 1, \varepsilon(t))\big)^2}\,\mathrm{d}x\,\mathrm{d}t.$$

From formula (1) we see:

$$f_z(z,a,\varepsilon)=f_z(\frac{z}{\varepsilon},\frac{a}{\varepsilon},1),\qquad f_{zz}(z,a,\varepsilon)=\frac{1}{\varepsilon}f_{zz}(\frac{z}{\varepsilon},\frac{a}{\varepsilon},1).$$

We can use this to rewrite the above integral:

$$\begin{aligned}\iint_{A_2}\frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t&=-2\lambda^3\int_{T_A}^{T/2}\int_{\frac{1}{\lambda}(t-1-2\varepsilon(t))}^{\frac{1}{\lambda}(t+\varepsilon(t))}\frac{f_{zz}(t-\lambda x,1,\varepsilon(t))^2}{\big(1-\lambda f_z(t-\lambda x,1,\varepsilon(t))\big)^2}\,\mathrm{d}x\,\mathrm{d}t\\&=-2\lambda^2\int_{T_A}^{T/2}\int_{-\varepsilon(t)}^{2\varepsilon(t)+1}\frac{f_{zz}(z,1,\varepsilon(t))^2}{\big(1-\lambda f_z(z,1,\varepsilon(t))\big)^2}\,\mathrm{d}z\,\mathrm{d}t\\&=-2\lambda^2\int_{T_A}^{T/2}\int_{-\varepsilon(t)}^{2\varepsilon(t)+1}\frac{1}{\varepsilon(t)^2}\frac{f_{zz}(\frac{z}{\varepsilon(t)},\frac{1}{\varepsilon(t)},1)^2}{\big(1-\lambda f_z(\frac{z}{\varepsilon(t)},\frac{1}{\varepsilon(t)},1)\big)^2}\,\mathrm{d}z\,\mathrm{d}t\\&=-2\lambda^2\int_{T_A}^{T/2}\int_{-1}^{2+\frac{1}{\varepsilon(t)}}\frac{1}{\varepsilon(t)}\frac{f_{zz}(z,\frac{1}{\varepsilon(t)},1)^2}{\big(1-\lambda f_z(z,\frac{1}{\varepsilon(t)},1)\big)^2}\,\mathrm{d}z\,\mathrm{d}t\end{aligned}$$

Looking at the formula for f_{zz}

$$f_{zz}(z,\tfrac{1}{\varepsilon},1)=G_1(z)-\int_{-\infty}^{z}G_1(w)G_1(w-(z-\tfrac{1}{\varepsilon}))\,\mathrm{d}w$$

we see that $f_{zz}(z,\frac{1}{\varepsilon},1)$ is non-zero only on the intervals $|z|<1$ and $|z-\frac{1}{\varepsilon}|<2$. For small ε, these are two disjoint regions. Therefore, the above integral equals

$$\begin{aligned}\iint_{A_2}\frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t=&-2\lambda^2\int_{T_A}^{T/2}\frac{1}{\varepsilon(t)}\int_{-1}^{1}\frac{f_{zz}(z,\frac{1}{\varepsilon(t)},1)^2}{\big(1-\lambda f_z(z,\frac{1}{\varepsilon(t)},1)\big)^2}\,\mathrm{d}z\,\mathrm{d}t-\\&-2\lambda^2\int_{T_A}^{T/2}\frac{1}{\varepsilon(t)}\int_{-2}^{2}\frac{f_{zz}(z+\frac{1}{\varepsilon(t)},\frac{1}{\varepsilon(t)},1)^2}{\big(1-\lambda f_z(z+\frac{1}{\varepsilon(t)},\frac{1}{\varepsilon(t)},1)\big)^2}\,\mathrm{d}z\,\mathrm{d}t\end{aligned}$$

For z bounded and sufficiently small $\varepsilon(t)$, the functions under the integral do not depend on $\varepsilon(t)$ any more as can be seen from the definitions of f_z and f_{zz}. Thus

$$I=\lambda^2\int_{-1}^{1}\frac{f_{zz}(z,\frac{1}{\varepsilon(t)},1)^2}{\big(1-\lambda f_z(z,\frac{1}{\varepsilon(t)},1)\big)^2}\,\mathrm{d}z+\lambda^2\int_{-2}^{2}\frac{f_{zz}(z+\frac{1}{\varepsilon(t)},\frac{1}{\varepsilon(t)},1)^2}{\big(1-\lambda f_z(z+\frac{1}{\varepsilon(t)},\frac{1}{\varepsilon(t)},1)\big)^2}\,\mathrm{d}z,$$

is independent of t and we have

$$\iint_{A_2}\frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}\,\mathrm{d}x\,\mathrm{d}t=-I\int_{T_A}^{T_E}\frac{1}{\varepsilon(t)}\,\mathrm{d}t.$$

The same calculations can be repeated for the return path ψ, where $\varepsilon\equiv\varepsilon_0$ is constant in time:

$$\iint_{A_2}\frac{\psi_{tx}\psi_{xx}}{\psi_x^2}\,\mathrm{d}x\,\mathrm{d}t=I\int_{T_A}^{T_E}\frac{1}{\varepsilon_0}\,\mathrm{d}t.$$

Note that the sign is positive now, which comes from the t-derivative. Putting everything together gives us

$$a=\iint\frac{\varphi_{tx}\varphi_{xx}}{\varphi_x^2}+\frac{\psi_{tx}\psi_{xx}}{\psi_x^2}\,\mathrm{d}x\,\mathrm{d}t=I\int_{T_A}^{T_E}\Big(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon(t)}\Big)\,\mathrm{d}t$$

Let $\varepsilon(t) = \varepsilon_0 + \varepsilon_1\varepsilon_0^{3/2}b(t)$ where $b(t)$ is a bump function with height 1 and ε_1 is a small constant. Note that $\varepsilon(t)$ satisfies $|\dot\varepsilon| \le \|\dot b\|_\infty\, \varepsilon_1\varepsilon_0^{3/2}$. Choosing ε_0 and ε_1 small independently we may produce any $a \in \mathbb{R}$. $\square$

Proof of Theorem 3.1. Let $(\varphi, a) \in \mathbb{R} \times_c \operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$. By proposition 3.2 we get a smooth family $\varphi(\delta, t, x)$ for $\delta > 0$ and $t \in [0,1]$ such that $\varphi(\delta, t, \quad) \in \operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$, $\varphi(\delta, 0, \quad) = \mathrm{Id}_{\mathbb{R}}$, $\varphi(\delta, 1, \quad) = \varphi$, and such that the length of $t \mapsto \varphi(\delta, t, \quad)$ is $< \delta$.

Using 2.4 consider the horizontal lift $(\varphi(\delta, t, \quad), a(\delta, t)) \in \operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ of this family which connects $\binom{\mathrm{Id}}{0}$ with $\binom{\varphi}{a(\delta,1)}$ for each $\delta > 0$ and has length $< \delta$. But one can see from the proof of lemma 3.3 that $a(\delta, 1)$ becomes unbounded for $\delta \to 0$.

Using lemma 3.3 we can find a horizontal path $t \mapsto \binom{\psi(\delta,t,\quad)}{b(\delta,t)}$ for $t \in [0,1]$ in the Virasoro group of length $< \delta$ connecting $\binom{\mathrm{Id}}{0}$ with $\binom{\mathrm{Id}}{a-a(\delta,1)}$. Then the curve $t \mapsto \binom{\psi(\delta,t,\quad)}{b(\delta,t)} \cdot \binom{\varphi}{a(\delta,1)} = \binom{\psi(\delta,t)\circ\varphi}{b(\delta,t)+a(\delta,1)+c(\psi(\delta,t),\varphi)}$ connects $\binom{\varphi}{a(\delta,1)} = \binom{\mathrm{Id}}{0} \cdot \binom{\varphi}{a(\delta,1)}$ with $\binom{\varphi}{a} = \binom{\mathrm{Id}}{a-a(\delta,1)} \cdot \binom{\varphi}{a(\delta,1)}$ and it has length $< \delta$. $\square$

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