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ON THE STABILITY OF b-INCOMPLETENESS IN THE WHITNEY TOPOLOGY ON THE SPACE OF CONNECTIONS

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Sunto. — Si studia il problema della stabilità del b-bordo di una varietà, nel contesto dello spazio delle connessioni principali sul fibrato dei riferimenti. Alcuni nuovi risultati nella geometria degli spazi di connessioni, che generalizzano risultati ottenuti in un articolo precedente [4], permettono di descrivere una situazione generale in cui la b-incompletezza è conservata nel cambio della connessione data. Questi risultati sembrano adatti per ottenere vari tipi di teoremi di stabilità. In effetti, una conseguenza è che la stabilità, nella topologia C^0 di Whitney, può essere dimostrata in modo piuttosto semplice per punti del cosiddetto "bordo essenziale".

1. - Introduction.

The *b-completion* is usually considered as the main tool for the study of the geometry of space-time singularities. This construction can be done whenever a principal connection on the frame bundle LM of a manifold M is given. In fact, the connection induces a riemannian metric on LM; the metric completion of LM can be quotiented by the group action, and M is dense in this quotient, which is exactly

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the b-completion. We say that M is *b-complete*, with respect to the given connection, if it coincides with its b-completion. This concept is a generalization of that of geodesic completeness, and also of that of metric completeness for riemannian manifolds. If M is b-incomplete, then the b-completion is the union of M with the *b-boundary*, which may be thought of as the set of endpoints of b-bounded curves (i.e. curves with bounded horizontal lift in LM) with no endpoint in M.

Thus, a space-time singularity can be seen as a point in the b-boundary generated by the Levi-Civita connection associated to the Lorentz metric (see [1, 5, 9] for detailed information and discussion). In other terms, the existence of a singularity is related to the b-incompleteness of M. It is then clear that stability problems concerning b-incompleteness and completeness have great interest in relation to the very existence of physical singularities, both from classical and quantistic point of view (for example, see [2, 3, 8, 9]).

The setting of the stability problem requires essentially two things: first, a decision about what a gravitational field is, that is, of which space it is a point (or of which bundle it is a section); second, the assignment of a precise meaning to the idea of "small change" of it. Though the most usual choice for the field is a Lorentz metric, also the connection is an important candidate, as it is suggested by the framework of gauge theories and by that of metric-affine theories of gravitation [6]. The interest for this approach is stimulated also from the theory of the *systems of connections* [10]. This framework seems promising, and in fact we are able to prove a result which can be interpreted as a kind of b-incompleteness stability, since it says that if M is b-incomplete with respect to a connection, then it is also b-incomplete with respect to a new connection, which approaches the former in a certain sense. This result, which is an improvement of a previous version [4], may be not so expressive from an intuitive point of view, but it is rather general and seems a good startpoint for the study of the problem, since it may produce various interesting consequences.

In fact, one such consequence is obtained, in the last section, in a framework which is quite natural for stability problems, i.e. that of the *Whitney topology* on the space of sections of a bundle, in our case the bundle of principal connections of the frame bundle. We shall say that any property depending on the connection is stable if the set of all connections which have the property is open in this topology.

Thus, we are concerned with the study of b-incompleteness stability in this context.

However, only a part of the b-boundary is usally considered as representing the physical singularities, that is the so-called "essential boundary", constituted by limit points of not (partially) trapped curves. Now, the final result of this paper is precisely that stability, in the sense of the Whitney topology, holds at least for these points.

There are still many open problems. For example, note that the choice of a different space of sections (for example Lorentz metrics) as representing the possible gravitational field might give completely different results; in fact the map "riemannian connection" from the space of metrics to that of principal connections is not continuous. It would be important to study which is the largest possible modification of the field which preserves the existence of singularities. This could relate our approach to others in which coarser topologies are used and some kind of instability arises (see [14, 15]). Indeed, many people think that the soundest approach from a physical viewpoint would be the unexplored one based on the space of solutions of a given field equation.

2. - Preliminaries.

This section contains a brief review of basic concepts. For details, see [7, 10, 12].

By $p : E \longrightarrow M$ we shall indicate a fibred manifold of finite dimension. Then, p is a differentiable surjection of maximal rank. T, T^*, V and J are respectively the tangent, cotangent, vertical and first-jet functors. We have the fibered structures $Tp : TE \longrightarrow TM$, $\pi_E : TE \longrightarrow E$, $p_E : JE \longrightarrow E$.

We shall deal with the fibered product ot two fibered manifolds $p : E \longrightarrow M$ and $q : F \longrightarrow M$. This has three different fibered structures:

$$\pi_1 : E \times_M F \longrightarrow E, \quad \pi_2 : E \times_M F \longrightarrow F, \quad p \circ \pi_1 = q \circ \pi_2 : E \times_M F \longrightarrow M .$$

The reciprocal image of a section $s : M \longrightarrow E$ is

$$q^*s \equiv (s \circ q, id_F) : F \longrightarrow E \times_M F .$$

A connection on $E \longrightarrow M$ is a section $\Gamma : E \longrightarrow JE$; the assignment of Γ is equivalent to that of a 1-form on TE with values in VE, $\omega_\Gamma : E \longrightarrow T^*E \otimes_E VE$, called connection form, which is an affine morphism $TE \longrightarrow VE$ over $\pi_1 : E \times_M TM \longrightarrow E$ and, restricted to VE, is the identity.

The space of all sections $M \longrightarrow E$ will be denoted by $\mathcal{S}ec\,(E \longrightarrow M)$. Given a topology on this space, a property depending on the section is called *stable* if the set of all sections which have the property is open. We shall use the W^0 topology (or *Whitney* or *wholly open* C^0 topology): a basis of this topology is constituted by the family of subsets:

$$\mathcal{E}\,(U) \equiv \{s \in \mathcal{S}ec\,(E \longrightarrow M) : s\,(M) \subset U\} ,$$

where U is an open submanifold of E.

A *system of connections* in constituted by a fibered manifold $\pi : C \longrightarrow M$, together with a fibered morphism $\gamma : C \times_M E \longrightarrow JE$. Any section $\tilde{\Gamma} : M \longrightarrow C$ determines a connection $\Gamma \equiv \gamma \circ (p^*\tilde{\Gamma})$. We remark that we cannot obtain all possible connections of $E \longrightarrow M$ with this procedure, unless we take C to be infinite-dimensional. However, finite-dimensional systems of connections arise in important situations, like in the case of principal connections. By *space of connections* we mean the (infinite-dimensional) space $\mathcal{S}ec\,(C \longrightarrow M)$.

Given a system of connections, there is a canonical connection Λ on the fibered manifold $K \equiv C \times_M E \longrightarrow C$; its connection form is characterized by the relation

$$\langle \omega_\Lambda , X \rangle = (\pi_1\,(X), \langle \omega_\Gamma, T\pi_2\,(X) \rangle) \in C \times_M VE \subset C \times_M TE = VK ,$$

where Γ is any connection such that $\tilde{\Gamma}\,(\pi \circ \pi_1\,(X)) = \pi_1\,(X)$.

Given a principal bundle $(P, p, M; G)$, the action of the group G on P can be naturally extended to an action on JP, and a principal connection is exactly a section $P \longrightarrow JP$ which is invariant with respect to this action. Thus, principal connections arise from a finite-dimensional system, where $C \equiv JP/G$. This bundle can be shown to be affine, with associated vector bundle $T^*M \otimes_M VP/G$. Moreover, if P admits a global section or G is abelian, then VP/G is isomorphic, though in general not canonically, to $M \times \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of G.

3. - The frame bundle and related structures.

This section is a summary of some further preliminary results; some of these were established in [4], and the others can be found in [1, 5, 9]. Henceforth $K \equiv (JP/G) \times_M P \equiv C \times_M P$ and $P \equiv LM$, the frame bundle with projection p onto M (then $K \simeq JP$). We recall that any principal connection Γ determines a riemannian metric on $TLM \longrightarrow LM$ (independently discovered by Schmidt [13] and Marathe [10]), and that the b-completeness of M is equivalent to the metric completeness of LM. This metric is defined by:

$$g_\Gamma : TLM \times_{LM} TLM \longrightarrow \boldsymbol{R} : (X, Y) \longmapsto \theta(X) \cdot \theta(Y) + \omega_\Gamma(X) \cdot \omega_\Gamma(Y)$$

where:

$\theta : TLM \longrightarrow \boldsymbol{R}^m : X \longmapsto \langle (\pi_{LM}(X))^*, Tp(X) \rangle$ is the canonical 1-form, $(\pi_{LM}(X))^*$ is the dual basis of $\pi_{LM}(X)$;

$\omega_\Gamma : TLM \longrightarrow \mathcal{G} = \boldsymbol{R}^{m^2}$ is the connection form, when we take into account the natural isomorphism $VLM \simeq LM \times \mathcal{G}$;

$\cdot$ is the standard inner product in $\boldsymbol{R}^m$ and $\boldsymbol{R}^{m^2}$.

Now, the canonical 1-form can be extended to a 1-form on $TK \longrightarrow K$, denoted by the same symbol, by putting $\theta(W) \equiv \theta \circ T\pi_2(W)$; thus we can define a symmetric bilinear form:

$$f : TK \times_k TK \longrightarrow \boldsymbol{R} : (W, Z) \longmapsto \theta(W) \cdot \theta(Z) + \omega_\Lambda(W) \cdot \omega_\Lambda(Z) .$$

This form vanishes on π_2-vertical vectors, and is then degenerate. It can be proved that the assignment of a connection on $C \longrightarrow M$, $\Delta : C \longrightarrow JC$, induces naturally a riemannian metric g_Δ on $TK \longrightarrow K$, whose restriction to π_1-vertical vectors coincides with f.

If $\tilde{\Gamma}: M \longrightarrow C \equiv JLM/G$ is a section, then $p^*\tilde{\Gamma}: LM \longrightarrow K$ is a section, and the submanifold $S_\Gamma \equiv p^*\tilde{\Gamma}(LM) \subset K$ is diffeomorphic to LM. Moreover, the restriction f_Γ of f to S_Γ is a riemannian metric, and $p^*\tilde{\Gamma}$ is an isometry between (LM, g_Γ) and (S_Γ, f_Γ). Thus, metric completeness of (LM, g_Γ), and consequently b-completeness of M with respect to Γ, can be studied on S_Γ.

We shall indicate by $\overline{LM}_\Gamma$ the metric completion of LM with respect to g_Γ, and we shall put $\partial_\Gamma LM \equiv \overline{LM}_\Gamma - LM$; then, $\overline{M}_\Gamma \equiv \overline{LM}_\Gamma/G$ and $\partial_\Gamma M \equiv \overline{M}_\Gamma - M = \partial_\Gamma LM/G$ are the b-completion and the b-boundary of M respectively.

Thus, $\partial_\Gamma LM$ can be identified with ∂S_Γ, the f_Γ-boundary of S_Γ. If there exists a connection $\Delta: C \longrightarrow JC$ such that the immersion $(S_\Gamma, f_\Gamma) \hookrightarrow (K, g_\Delta)$ is uniformly continuous, then ∂S_Γ can be also identified with a subspace of $\partial_\Delta K$, the g_Δ-boundary of K; for example, this is the case when $\tilde{\Gamma}$ is Δ-horizontal.

Next, we recall that $\overline{M}_\Gamma$ is not, in general, a Hausdorff space, and that this fact is related to the existence of (partially) trapped curves $c: [0, 1) \longrightarrow M$ with endpoint in $\partial_\Gamma M$. In other terms, if $\hat{c}: [0, 1) \longrightarrow LM$ is a curve with endpoint in $\partial_\Gamma LM$, its projection $c \equiv p \circ \hat{c}: [0, 1) \longrightarrow M$ may have the property that there exists a compact set $H \subset M$ such that, for all $\bar{\lambda} \in [0, 1)$, there is $\lambda \in (\bar{\lambda}, 1)$ such that $c(\lambda) \in H$. As a consequence, c is not a proper map.

Then we define the essential boundary of M as the portion of $\partial_\Gamma M$ constituted by those points which are endpoints of at least one not trapped curve. Such curve is then proper (however, note that a point of the essential boundary can be also the endpoint of a trapped curve). We shall call *essential completeness* (*incompleteness*) the property of a manifold of having empty (not empty) essential boundary.

3. - f-bounded open submanifolds of K.

The concept of f-bounded open submanifold of K was used in [4]. In this section we give a more precise definition, and prove the existence of such sets directly by showing one possible construction. This is important for the interpretation of the preliminar result of § 4 as a kind of generalized stability, and also for the proof of the last proposition about Whitney stability.

DEFINITION. - An open submanifold $V \subset K$ is *f-bounded* if $V \cap S_\Gamma$ is f_Γ-bounded for all $\tilde{\Gamma} : M \longrightarrow C$ such that $\pi_2 (V \cap S_\Gamma) = \pi_2 (V)$.

LEMMA. - *A connected open submanifold $V \subset K$ is f-bounded if the following properties hold:*

a) *There exists $R \in \boldsymbol{R}^+$ such that $f(u, u) / f(v, v) < R$ for all $u, v \in TV$ such that $T\pi_2 (u) = T\pi_2 (v)$.*

b) *There exists one $\tilde{\Gamma} : M \longrightarrow C$ such that $\pi_2 (S_\Gamma \cap V) = \pi_2 (V)$, and $\pi_2 (V)$ is g_Γ-bounded.*

PROOF. - Since $\pi_2 (V)$ is g_Γ-bounded, there exists $\varrho \in \boldsymbol{R}^+$ such that, for any two points $a, b \in \pi_2 (V)$, there is a curve $c : [0, 1] \longrightarrow \pi_2 (V)$ of length not greater than ϱ and such that $c(0) = a$, $c(1) = b$.

Let $\tilde{B} : M \longrightarrow C$, $a, b \in \pi_2 (V \cap S_B) = \pi_2 (V)$.

Then $f_B (dc, dc) < R f_\Gamma (dc, dc)$, and then the f_B-length of c is lesser then $R\varrho$. $\square$

LEMMA. - *Let $\tilde{\Gamma} : M \longrightarrow C$ be a connection and $A \subset LM$ a g_Γ-bounded open submanifold.*

Then, there exists an f-bounded open submanifold $V \subset K$ such that $p^* \tilde{\Gamma} (A) \subset V$.

PROOF. - We shall construct one such V.

For each $a \in LM$, let $\mathcal{S}(a) \equiv \{u \in T_a LM : g_\Gamma (u, u) = 1\}$ be the unit g_Γ-sphere at a, and for any $\hat{a} \in (\pi_2)^{-1}(a)$ put $U(\hat{a}) \equiv T_{\hat{a}} K \cap (T\pi_2)^{-1}(U(a))$.

For all $\hat{u} \in U(\hat{a})$ we put $q(\hat{u}) \equiv f(\hat{u}, \hat{u}) / g_\Gamma (u, u)$, where $u \equiv T\pi_2 (\hat{u})$.

Since $\mathcal{S}$ is compact, it is clear that for all $\hat{a} \in (\pi_2)^{-1}(a)$ there exists $Q(\hat{a}) \equiv \max \{q(\hat{u}) : \hat{u} \in U(\hat{a})\}$; moreover $Q : \hat{a} \longmapsto Q(\hat{a})$ is a continuous function on $(\pi_2)^{-1}(LM)$, and $Q(S_\Gamma) = \{1\}$.

Now, let $1 < R \in \boldsymbol{R}^+$; we put $V \equiv \{\hat{a} \in (\pi_2)^{-1}(A) : 1/R < Q(\hat{a}) < R\}$. We must show that V is open and f-bounded.

First, let $\hat{a} \in V$. Since Q is continuous and $Q(\hat{a}) < R$, there is an open neighbourhood $W_{\hat{a}}$ of $\hat{a}$ in K such that $Q(W_{\hat{a}}) \subset (R, 1/R)$. $W_{\hat{a}} \cap ((\pi_2)^{-1}(A))$ is then an open neighbourhood of $\hat{a}$ contained in V. Thus V is open.

Now, let $w, w' \in TV$ be such that $T\pi_2 (w) = T\pi_2 (w')$, $\hat{a} \equiv \pi_K (w)$, $\hat{a}' \equiv \pi_K (w')$. Then, $w = \lambda \hat{u}$, $w' = \lambda \hat{u}'$, where $\lambda \in \boldsymbol{R}$, $\hat{u} \in U(\hat{a})$, $\hat{u}' \in U(\hat{a}')$.

We then have:

$$f(w, w) = \lambda^2 f(\hat{u}, \hat{u}); \; f(w', w') = \lambda^2 f(\hat{u}', \hat{u}') .$$

$$f(w, w) / f(w', w') = f(\hat{u}, \hat{u}) / f(\hat{u}', \hat{u}') < R .$$

Since $A = \pi_2(V)$ is g_Γ-bounded, V is f-bounded by virtue of the previous lemma. □

COROLLARY. - *Let $\tilde{\Gamma} : M \longrightarrow C$ be a section. Let $\bar{a} \in \partial_\Gamma LM$. Then, there exist a curve $c : [0, 1) \longrightarrow LM$ with endpoint $\bar{a}$, and an f-bounded open submanifold $V \subset K$ such that $(p^*\tilde{\Gamma} \circ c)([0, 1)) \subset V$.*

PROOF. - Since $\overline{LM}$ is Hausdorff, there exist a g_Γ-bounded open submanifold $A \subset LM$ and a curve $c : [0, 1) \longrightarrow A$ with endpoint $\bar{a}$; thus we can construct V as in the proof of the previous lemma. □

4. - Stability of b-boundary points.

PROPOSITION. - Let $\tilde{\Gamma} : M \longrightarrow C$ be a section such that M is b-incomplete with respect to Γ.

Let $c : [0, 1) \longrightarrow LM$ be a curve with endpoint in $\partial_\Gamma LM$.

Let $V \subset K$ be an f-bounded open submanifold, such that $(p^*\tilde{\Gamma}) \circ c[0, 1) \subset V$.

Then, M is b-incomplete with respect to any connection B such that $(p^*\tilde{B}) \circ c[0, 1) \subset V$, since c has finite g_B-lenght.

PROOF. - Essentially the same of the first proposition in § 4 of [4].

We observe that, in a sense, the previous proposition may be thought as establishing a stability property for b-incompleteness, since it says that if M is b-incomplete with respect to Γ, then it is also b-incomplete with respect to a new connection B which approaches Γ sufficiently well in a certain sense. It is also noticeable that the behaviour of B far away from the singularity has no influence (this statement has meaning at least in LM, whose completion in Hausdorff). In fact, the last corollary of the previous section enables us to construct the manifold V over any g_Γ-bounded neighbourhood of the singularity in LM.

This result seems suitable for obtaining various kinds of stability properties. In fact we shall use it, together with those of the previous section, for establishing a particularly expressive stability theorem: the stability of essential incompleteness in the Whitney topology.

PROPOSITION. - Essential incompleteness is a stable property in the C^0 Whitney topology on the space of principal connections of $LM \longrightarrow M$.

PROOF. - *Let* $\bar{a} \in \partial_\Gamma LM$ be such that its projection onto $\partial_\Gamma M$ is in the essential boundary. Then, we have a curve $c : [0, 1) \longrightarrow LM$ with endpoint $\bar{a}$, such that $p \circ c : [0, 1) \longrightarrow M$ is proper. Let $A \subset LM$ be the interior of the ball in $\overline{LM}_\Gamma$ of centre $\bar{a}$ and radius $\varrho \in \mathbf{R}$. Then $c(\lambda) \in A$ for λ sufficiently close to 1, and then, after a possible affine reparametrization, we have $c : [0, 1) \longrightarrow A$.

Let $V \subset K$ be the f-bounded open submanifold constructed as in the second lemma of the previous section.

Since $p \circ c : [0, 1) \longrightarrow M$ is a proper map, the map

$$\mathcal{S}ec\,(C \longrightarrow M) \longrightarrow C^0\,([0, 1), C) : \tilde{B} \longmapsto \tilde{B} \circ p \circ c$$

is W^0-continuous [12]. Thus, also the map

$$\Phi : \mathcal{S}ec\,(C \longrightarrow M) \longrightarrow \mathcal{V} \equiv C^0\,([0, 1), V) : \tilde{B} \longmapsto (c, \tilde{B} \circ p \circ c) \equiv (p^* \tilde{B}) \circ C$$

is W^0-continuous. Thus $\Phi^{-1}(\mathcal{V})$ is W^0-open in $\mathcal{S}ec\,(C \longrightarrow M)$.

If $\tilde{B} \in \Phi^{-1}(\mathcal{V})$, then $\Phi(\tilde{B}) \in \mathcal{V}$, which means that

$$(p^* \tilde{B}) \circ c\,([0, 1)) \subset V\,.$$

Then, by virtue of the previous proposition, c has finite g_B-length and, being not partially trapped, defines a point in the essential boundary of M with respect to B. $\square$

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