The Morse-Smale Complex

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Abstract

The main aim of this paper is to present the construction of the Morse–Smale complex of a compact smooth manifold M with boundary and to establish the connection to the topology of M. This approach to connecting the analysis of an appropriate function $f : M \to \mathbb{R}$ — respectively the dynamical system associated to such a function and a Riemannian metric g — with the topology of the manifold was introduced by Thom (see [21]) and Smale (see [17], [18] and [19]). In the more traditional approach developed by Morse (see [10] for an presentation of these ideas) the function f is used to construct a CW-space of the same homotopy-type as the manifold M. The approach presented here uses the unstable manifold of the negative gradient vector field of f with respect to g to construct a decomposition of M that enables one to extract topological information from it. We remark that the approach developed by Thom and Smale is often more suitable for studying infinite-dimensional manifolds such as loop-spaces than the more traditional approach. However, the techniques used to obtain similar results in the infinite-dimensional setting differ substantially from the techniques used in this paper. For an exposition of Morse Theory as a toy-model of infinite-dimensional issues, see [16], for instance.

In the first chapter we start to introduce the basic terminology concerning Morse Theory and present a prove of the important Morse Lemma.
Then we distinguish special pairs (f, g) - we call them Morse–Smale pairs — of functions $f : M \to \mathbb{R}$ and Riemannian metrices g. The main motivation is to gain control of the behaviour of the negative gradient vector field $-\operatorname{grad}_g(f)$ near critical points. We also introduce some conditions that control the behaviour of f and g on the boundary of M. The boundary conditions considered here are not the most common ones. The choice of these boundary conditions is motivated by the idea that the boundary should fit with the decomposition of M by the unstable manifolds. Consequently the Morse–Smale complex associated to the critical points on the boundary forms a sub-complex of the Morse–Smale complex of the whole manifold. Stable and unstable manifolds are introduced next and the Lyapunov-property is established. We prove that the stable and unstable manifolds are sub-manifolds diffeomorphic to Euclidean spaces and state the Smale condition.
Then we investigate if the conditions we imposed on the Morse–Smale pairs are generic. In order to do this we define jets and cite some facts concerning openness and density of certain subsets of smooth functions. The main result consists of two parts: First it is shown that the set of Morse functions is $\mathcal{C}^\infty$-dense and $\mathcal{C}^2$-open in the set of all extensions of a given Morse-function

on the boundary (given a fixed collar of ∂M). And then we show that if we fix an appropriate f, the set of Riemannian metrices such that (f, g) is a Morse–Smale pair and such that g coincides with the pull-back of the Euclidean metric by a Morse-chart is $\mathcal{C}^\infty$-dense and $\mathcal{C}^1$-residual in the set of all Riemannian metrices that coincide with the pull-back of the Euclidean metric by a Morse-chart.

In the next chapter we continue the investigation of the dynamical system associated to a Morse–Smale pair. We present a way to deal with the boundary and show that the stable respectively unstable manifold really form a decomposition of M.
Then we define the space of trajectories from one critical point to another and equip it with a topology and then with a smooth structure. Next we introduce the space of unparametrised trajectories from one critical point to another critical point. Different ways to interpret this space are presented: as the space of orbits of an $\mathbb{R}$-action on the space of parametrised trajectories, as a subset of the continuous functions from a compact interval to M and as the intersection of the space of parametrised trajectories with a level-hypersurface of f. We prove that the topologies obtained from this different interpretations coincide and then we equip this space with a smooth structure.
The space of broken trajectories from one critical point to another one is defined as the disjoint union of products of certain spaces of unparametrised trajectories. We show that one can interpret this space as a subspace of continuous functions from a compact interval to M and equip it with the subspace topology. We show that this topological space that contains the space of unparametrised trajectories as a subspace is compact.
Next we introduce the notation of a smooth manifold with corners. In Theorem (2.27.) we prove that the space of broken trajectories can be canonically equipped with the structure of a smooth manifold with corners such that the k-boundary can be identified with the k-times broken trajectories. So the space of unparametrised trajectories from p to q possesses a canonical compactifaction which carries the structure of a smooth manifold with corners. To prove this we follow the treatment presented in the expositions [3], [4] and [5].
In the next section we show that the unstable manifolds also possess a canonical compactifaction that can be canonically equipped with the structure of a smooth manifold with corners. The way we proceed is similar to the one used to prove Theorem (2.27.): First we interpret the unstable manifolds as subspaces of continuous maps from a compact interval to M then we define $\hat{W}^-(p)$ and show that these spaces can be interpreted as subspaces of continuous mappings too and that these spaces are compact. In Theorem (2.33.) we state that $\hat{W}^-(p)$ can be canonically equipped with the structure of a smooth manifold with corners.

The end of the second chapter deals with orientations of the unstable manifolds and how these induce orientations on the spaces of unparametrised trajectories and their one-boundaries.

In the last chapter we use the information about the analysis of the negative gradient flow to make contact with topology. We introduce the Morse-Smale complex (over $\mathbb{Z}$) and show that it is a differential complex. The homology of this complex is called Morse homology.
We introduce spectral sequences of filtered complexes, explain convergence and state two important results about spectral sequences.
In the third section a prove is presented that shows that Morse homology is isomorphic to singular homology. The idea is to show that the decomposition of M by the unstable satisfies the most important properties that the relative homology groups of a CW-decomposition would satisfy. Then we show that the Morse-Smale complex can be interpreted as some kind of "cellular" complex of the decomposition of M by unstable manifolds. From this the isomorphism follows and implies the Morse inequalities.
Next it is shown that the Morse cohomology is isomorphic to the deRham cohomology. We define a map Int^* from the real valued differential forms into the dual of the Morse–Smale complex (over $\mathbb{R}$) and prove that it is a chain map. Then M is filtered with the help of the Morse function f and show that Int^* preserves the induced filtrations on $\Omega^*(M)$ and on $C^k(f;\mathbb{R})$. Hence, Int^* induces a map between the spectral sequences associated to the induced filtrations of the differential complexes $\Omega^*(M)$ and $C^k(f;\mathbb{R})$. Int^* induces an isomorphism between the E^1-terms of the spectral sequences and consequently it induces an isomorphism between the cohomology groups of the two differential complexes.

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Contents

Chapter 1

Basic Concepts

We start to investigate the basic notations necessary to define the Morse–Smale complex. If not otherwise stated, M denotes a smooth compact manifold, possibly with boundary, of dimension n.

1.1 Foundations

We introduce the main terminology concerning Morse functions and their local behaviour and prove some basic facts. Most of this material is covered by introductions to Morse Theory, see the classic [10] or the chapter about "Morse Theory" in [8], for instance.
Additionally, an adaption of these concepts to manifolds with boundaries is presented.

1.1. Definition *Critical points, Hessians*
Let $f : M \to \mathbb{R}$ be a smooth real-valued function on M.
A point $x \in M$ is called critical *if the one-form $df \in \Omega^1(M) := \Gamma^\infty(T^*M)$ is zero at x, where $df_p : T_pM \to \mathbb{R}$ is defined by $df_p(X_p) := X_p(f)$ with $X_p \in T_pM$ arbitrary. If a point is not critical it is called* regular.
Critical values *are points in the image of critical points and* regular values *are points in $\mathbb{R}$ with no critical point in the pre-image under f.*
The Hessian H_f *of f at a critical point p is the bilinear map $T_pM \times T_pM \to \mathbb{R}$ given by $(X, Y) \mapsto \tilde{X}(\tilde{Y}(f))_p$ where $\tilde{X}$ and $\tilde{Y}$ are smooth vector fields such that $\tilde{X}_p = X$ and $\tilde{Y}_p = Y$.*

<u>Remark</u>: the Hessian
As can be seen easily, the Hessian of a smooth function at a critical point is bilinear. Its failure to be symmetric can be measured by the difference

$$H_f(X,Y)(p) - H_f(Y,X)(p) = \tilde{X}_p(\tilde{Y}(f)) - \tilde{Y}_p(\tilde{X}(f)) = [\tilde{X},\tilde{Y}]_p(f) = df_p([X,Y]).$$

Consequently, the Hessian is a symmetric bilinear form on T_pM because df_p maps every vector in T_pM to zero, in particular $df_p([X,Y]) = 0$, and so the difference between $H_f(X,Y)(p)$ and $H_f(Y,X)(p)$ vanishes. Furthermore, $H_f(X,Y)(p)$ is independent from the particular choice of extensions of X and Y: on the one hand we know that $\tilde{X}_p(\tilde{Y}(f)) = X(\tilde{Y}(f))$ and hence it is independent from the extension $\tilde{X}$ of X and on the other hand $\tilde{Y}_p(\tilde{X}(f)) = Y(\tilde{X}(f))$ and so it is independent from the extension $\tilde{Y}$ of Y.
So, given a critical point p of f, we have a natural symmetric bilinear form associated to it: $H_f(X,Y)(p) : T_pM \times T_pM \to \mathbb{R}$.

1.2. Definition *Morse functions*
A critical point p is called non-degenerate *if the Hessian of f is non-degenerate at this point, i.e. the induced map*

$$T_pM \to T_p^*M, X_x \longmapsto H_f(X_p, \cdot)(p)$$

is an isomorphism of vector spaces. If all the critical points of a function are non-degenerate, the function is a Morse function*.*

Remark: index of a symmetric bilinear form
In general, given a bilinear form H on a vector space V, we define the *index* of H to be the dimension of a maximal linear subspace of V on which H is negative definite. This subspace is not canonically given what can be seen already in the easiest case: $V := \mathbb{R}^2$ and $H((v_1,v_2),(w_1,w_2)) := v_1 \cdot w_1 - v_2 \cdot w_2$. However, these maximal subspaces on which the non-degenerate bilinear form is negative definite are all of the same dimension, and consequently the index is well-defined nevertheless. This is a direct consequence of the Theorem of Sylvester.

1.3. Definition *index of a critical point, $Cr_k(f)$, $Cr(f)$*
Let f be a Morse function on M and assume p is a critical point of f. The index *of p is defined to be the index of the Hessian of f at p and we write* $\operatorname{ind}(p)$ *for it.*
The set of all critical points of f will be denoted by

$$Cr(f) := \{x \in M : df(x) = 0\}$$

and the set of all critical points of a fixed index k by

$$Cr_k(f) := \{x \in Cr(f) : \operatorname{ind}(x) = k\}.$$

Remark: the situation in charts
Let p be a critical point of a Morse function f and (U, u) a chart centred at p, i.e. $u(p) = 0$, where $U \subset \mathbb{R}^n$. To express the condition $df_p = 0$ in the chart, we pull-back f to a function from $u(U)$ to $\mathbb{R}$. In this chart we calculate $d(f \circ u^{-1})_{u(p)}$. To simplify the notation, we denote $f \circ u^{-1}$ by $\tilde{f}$ and $u(p)$ by $x := (x^1, \ldots, x^n)$. We obtain $d\tilde{f}(\frac{\partial}{\partial x^i}) = (\frac{\partial}{\partial x^i})(\tilde{f}) = \frac{\partial \tilde{f}}{\partial x^i}$ and hence

$$df_x = d\tilde{f}_{(x^1,\ldots,x^n)} = \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial x^i} dx^i \tag{1.1}$$

where dx^i, $i = 1, \ldots, n$ denote the one-forms defined on the chart (U, ψ) by $dx^i(y) = y^i$ with y a vector in T_pU and y^i its i' th component. These one-forms constitute a basis of T_p^*U.
So, our condition $df_x = 0$ is equivalent to the vanishing of all partial derivatives of the pull-back of f, and of course, this condition for "extremal points" is well-known from analysis.
The local expression for the Hessian is well-known too: in a chart containing a critical point, the Hessian computes to:

$$\begin{aligned} H_{\tilde{f}}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(x) &= \left(\frac{\partial}{\partial x^i}(\frac{\partial \tilde{f}}{\partial x^j}) \right)_x \\ &= (\frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j})_x \end{aligned}$$

Consequently the Hessian is the invariant formulation of what is known as the Hessian matrix in analysis and the non-degeneracy of H_f is equivalent to the matrix $(\frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j})_{i,j}$ being non-degenerate — or stated another way — $\det(\frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j})_{i,j} \neq 0$.
It is also known that the Morse function near a critical point is well-behaved: for any critical point p of a Morse function f there is a chart (U, u) centred at this point such that the local expression of f in this chart, i.e. $f \circ u^{-1}$, has the form

$$f(p) - (x^1)^2 - \ldots - (x^k)^2 + (x^{k+1})^2 + \ldots + (x^n)^2$$

where $n = \dim M$, $k = \text{ind}(p)$ and $(x^1, \ldots, x^n) = u(y)$ for $y \in U$. Charts with this properties are called *Morse charts*. Before proving the existence of Morse charts, we follow closely the exposition in [10] and prove:

1.4. Lemma
Let f be a smooth function on a convex neighbourhood V of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then we can find n smooth functions denoted by $g_i, i = 1, \ldots, n$,

on V *with* $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ *such that*

$$f(x^1, \dots, x^n) = \sum_{i=1}^{n} x^i g_i(x^1, \dots, x^n)$$

holds on V*.*

Proof:
By convexity of V we can write

$$\begin{aligned} f(x^1, \dots, x^n) = \int_0^1 \frac{df(tx^1, \dots, tx^n)}{dt} dt &= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(tx^1, \dots, tx^n) \cdot x^i dt \\ &= \sum_{i=1}^{n} x^i \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \dots, tx^n) dt \end{aligned}$$

and consequently $g_i(x^1, \dots, x^n) := \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \dots, tx^n)dt$, $i = 1, \dots, n$, possess the desired properties.

$\square$

1.5. Theorem *Lemma of Morse*
Let p *be a non-degenerate critical point of* f*. Then there is a chart* (U, u) *centred at* p*, such that* f *has the following form in this chart*

$$f(p) - (x^1)^2 - \dots - (x^k)^2 + (x^{k+1})^2 + \dots + (x^n)^2 \tag{1.2}$$

where $(x^1, \dots, x^n)$ *are the local coordinates of this chart and* k *is the index of* f *at* p*.*

Proof:
If we had found such a chart it is clear that k must be the index of p because we can compute the Hessian in this chart and obviously it has index k.
To find such a chart choose an arbitrary chart centred at p, furthermore we can assume that $f(p) = 0$ because we can apply a shift of $-f(p)$ to the whole function. Write $\tilde{f}$ for the local representation of f in the chosen chart. As the chart is centred at p and $f(p) = 0$ we have $\tilde{f}(0) = 0$ and so we can apply Lemma (1.4.) and write $\tilde{f}$ as

$$\tilde{f}(x^1, \dots, x^n) = \sum_{j=1}^{n} x^j g_j(x^1, \dots, x^n)$$

in some neighbourhood of 0 with appropriate smooth functions g_j, $j = 1, \dots, n$. One knows that $g_j(0) = \frac{\partial \tilde{f}}{\partial x^j}(0) = 0$ because $0 \in \mathbb{R}^n$ is a critical point of $\tilde{f}$ and so we can apply Lemma (1.4.) again and obtain

$$g_j(x^1, \dots, x^n) = \sum_{i=1}^{n} x^i h_{ij}(x^1, \dots, x^n)$$

on some small neighbourhood of 0 with smooth functions h_{ij} and consequently

$$\tilde{f}(x^1, \dots, x^n) = \sum_{i,j=1}^{n} x^i x^j h_{ij}(x^1, \dots, x^n).$$

We can assume that $h_{ij} = h_{ji}$, because otherwise we could replace h_{ij} by $1/2(h_{ij} + h_{ji})$ in the formula. Next we calculate

$$\begin{aligned}
\frac{\partial \tilde{f}}{\partial x^m} &= \sum_{j=1}^{n} x^j h_{mj}(x^1, \dots, x^n) + \sum_{i=1} x^i h_{im}(x^1, \dots, x^n) + \sum_{i,j=1}^{n} x^i x^j \frac{\partial h_{ij}}{\partial x^m}(x^1, \dots, x^n) \\
&= 2\sum_{i=1}^{n} x^i h_{im}(x^1, \dots, x^n) + \sum_{i,j=1}^{n} x^i x^j \frac{\partial h_{ij}}{\partial x^m}(x^1, \dots, x^n)
\end{aligned}$$

and so

$$\begin{gathered}
\frac{\partial^2 \tilde{f}}{\partial x^m \partial x^l}(0) = 2h_{ml}(0) \Longrightarrow \\
h_{ij}(0) = \frac{1}{2} \frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j}(0)
\end{gathered}$$

and by non-degeneracy of the critical point 0 the matrix $(h_{ij}(0))_{ij}$ is non-degenerate too.
To construct a chart such that f is of the desired form, we use a sequence of coordinate transformations near 0 in the domain of the chart $u(U)$. Assume that there exist coordinates $y^1, \dots, y^n$ in an open neighbourhood U_1 of 0 such that

$$\tilde{f}(y^1, \dots, y^n) = \pm(y^1)^2 \pm \dots \pm (y^{r-1})^2 + \sum_{i,j \geq r}^{n} y^i y^j H_{ij}(y^1, \dots, y^n)$$

holds on $U_2 \subset U_1$, an open neighbourhood of 0 and assume $H_{ij} = H_{ji}$. Furthermore we can assume that $H_{rr}(0) \neq 0$ as we can always perform linear coordinate transformations in the last $n - r + 1$ coordinates such that this holds. On an open neighbourhood of 0 $g(y^1, \dots, y^n) := \sqrt{|H_{rr}(y^1, \dots, y^n)|}$ is a smooth, non-zero function. We perform the coordinate transformation

$$\begin{aligned}
z^i &= y^i \quad \text{for} \quad i \neq r \quad \text{and} \\
z^r(y^1, \dots, y^n) &= g(y^1, \dots, y^n)\left(y^r + \sum_{i>r} y^i \frac{H_{ir}(y^1, \dots, y^n)}{H_{rr}(y^1, \dots, y^n)}\right).
\end{aligned}$$

To show that this is a valid coordinate transformation on a small neighbourhood of 0 we make use of the inverse function theorem and the fact that the Jacobian of the transformation is non-degenerate. To prove this, it suffices

to observe that $\frac{\partial z^r}{\partial y^r}(0) = g(0,\ldots,0) \neq 0$.
In the new coordinates $\tilde{f}$ is of the form

$$\tilde{f}(z^1,\ldots,z^n) = \pm(z^1)^2 \pm \ldots \pm (z^r)^2 + \sum_{i,j\geq r+1}^{n} z^i z^j H'_{ij}(z^1,\ldots,z^n)$$

with appropriate symmetric H'_{ij}. This is demonstrated by the following calculation:

$$\begin{aligned}
(z^r)^2 &= \pm|H_{rr}(y^1,\ldots,y^n)|\left(y^r + \sum_{i>r} y^i \frac{H_{ir}(y^1,\ldots,y^n)}{H_{rr}(y^1,\ldots,y^n)}\right)^2 \\
&= \pm|H_{rr}(y^1,\ldots,y^n)|(y^r)^2 + 2\sum_{i>r} y^r y^i H_{ir}(y^1,\ldots,y^n) \\
&\pm \sum_{i,j>r} y^i y^r \frac{H_{jr}(y^1,\ldots,y^n)H_{ir}(y^1,\ldots,y^r)}{H_{rr}(y^1,\ldots,y^n)}
\end{aligned}$$

and that coordinates with the desired properties exist follows by induction over r.

□

<u>Remark</u>: the local behaviour of a Morse function near a critical point
By the Lemma of Morse, the local situation near critical points only depends on the index of the critical point. Furthermore, take an arbitrary critical point x and choose a Morse chart (U,u) for it. Clearly, p is the only critical point in U because only there $\frac{\partial \tilde{f}}{\partial x^i} = 0$, for $i = 1,\ldots,n$. So we have found an open neighbourhood U of p in which p is the only critical point. Hence:

1.6. Corollary
The critical points of a Morse function are isolated and because M is assumed to be compact, there are only finitely many of them.

1.2 Morse pairs and stable respectively unstable Manifolds

<u>Remark</u>: collars
Given a smooth manifold M with boundary ∂M. An *open collar* of ∂M is an embedding

$$\varphi : \partial M \times [0,\varepsilon[\hookrightarrow M \quad \text{such that} \quad \varphi(\cdot,0) = id_{\partial M} \quad (\varepsilon > 0).$$

A *closed collar* is an embedding $\varphi : \partial M \times [0, \varepsilon] \hookrightarrow M$ such that $\varphi(\cdot, 0) = id_{\partial M}$ as before. For an arbitrary manifold M with boundary the following theorem holds:

1.7. Theorem *Collaring Theorem*
∂M has a collar.

Proof:
A proof of this fact can be found in [8].

□

Remark: the gradient of a Morse function
If M is equipped with a Riemannian metric, i.e a smooth section of the bundle of fibrewise positive definite inner products on $T_x M$, we have a natural isomorphism $\flat : TM \to T^*M$ and this induces a bijection between $\Gamma^\infty(M; TM)$ and $\Omega^1(M) = \Gamma^\infty(M; T^*M)$ given by $Z \longmapsto g(Z, \cdot)$. The inverse isomorphism $T^*M \to TM$ is denoted by $\sharp$.
We define the *gradient* of a smooth function on M to be $\sharp(df)$ and denote it by $\mathrm{grad}_g(f)$, i.e. $g(\mathrm{grad}_g(f), Y) = df(Y)$. By definition, critical points of f are exactly the zeros of the gradient vector field $\mathrm{grad}_g(f)$.

1.8. Definition *Morse pairs*
Let M be a compact manifold with boundary ∂M, $\iota : \partial M \to M$ denotes the embedding of ∂M into M. For $f \in \mathcal{C}^\infty(M; \mathbb{R})$, f_0 denotes the restriction of f to ∂M, i.e. $f_0 := f \circ \iota$. Let g denote a Riemannian metric on M. $\iota^* g$ is the pull-back of g under $\iota : \partial M \to M$.
We call (f, g) a Morse pair *if the following conditions are satisfied:*
1.) f is a Morse function on M.
2.) There is a collar $\varphi : \partial M \times [0, \varepsilon[\to M$ of ∂M such that $f \circ \varphi$ can be written in the following way:

$$f \circ \varphi(x, t) = h(x) + t^2 \quad \text{for } x \in \partial M, \quad 0 \le t < \varepsilon \tag{1.3}$$

and $h \in \mathcal{C}^\infty(\partial M; \mathbb{R})$. Furthermore the pull-back of g under φ has the form:

$$\varphi^* g = \pi^* g_{\partial M} + dt \otimes dt \tag{1.4}$$

with $g_{\partial M}$ a Riemannian metric on ∂M and $\pi : \partial M \times [0, \varepsilon[\to \partial M$ the projection on the first factor.
3.) For critical points on $M \setminus \partial M$ there is a Morse chart (U, u) such that $g = u^* g_E$ on U where g_E denotes the Euclidean metric on $u(U) \subset \mathbb{R}^n$.
4.) Critical points of f lying on ∂M are critical points of f_0 too and with respect to this Morse function on the manifold ∂M there exists a Morse chart such that the pull-back of the Euclidean metric on $\mathbb{R}^{n-1}$ under this chart coincides with $\iota^* g$.

Remark: condition 2.)
Condition 2.) implies that the critical points of f_0 are exactly the critical points of f on the boundary. With the help of the collar one obtains $df = dh + 2tdt$ for points on the collar and consequently $df = dh$ on the boundary. Furthermore one sees that there are no critical points on the collar except the ones lying on the boundary ∂M.
Furthermore, by condition 2.) $H_f(p)$ is given by

$$\begin{pmatrix} H_{f_0}(p) & 0 \\ 0 & 2 \end{pmatrix}$$

for all critical points on the boundary and that assures that the indices of critical points of f lying on the boundary are equal to the indices when the points are regarded as critical points of f_0.
The gradient vector field of the function f with respect to $\varphi^* g$ coincides with the gradient vector field of $f_0 := f \circ \iota$ on ∂M with respect to $g_{\partial M}$ because $\frac{\partial f}{\partial t} = \frac{\partial t^2}{\partial t} = 0$ on the boundary and $\frac{\partial}{\partial t}$ is orthogonal to ∂M. Especially, the gradient vector field is tangential to the boundary.

Remark: condition 3.)
The special forms of the Morse function and of the Riemannian metric together in one chart — we will call such a chart *convenient* — imply that the gradient vector field of f is explicitly known near critical points. In a convenient chart one has:

$$\mathrm{grad}_g(f) = \sharp(df) = \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial x^i} \frac{\partial}{\partial x^i}$$

because the isomorphism $\flat : TM \to T^*M$ induced by the Euclidean metric just maps dx^i to $\frac{\partial}{\partial x^i}$. Inserting the explicit expression of a Morse function in a Morse chart one obtains

$$\mathrm{grad}_g(f) = -2 \sum_{i=1}^{k} x^i \frac{\partial}{\partial x^i} + 2 \sum_{i=k+1}^{n} x^i \frac{\partial}{\partial x^i}. \tag{1.5}$$

Remark: condition 4.)
Together with the collar, the special chart that is assumed to exist in condition 4.) can be used to construct charts that are especially suitable. First use a Morse chart (U, u) for $f_0 : \partial M \to \mathbb{R}$ for which the pull-back of the Euclidean metric under u coincides with g. Next, define

$$V := U \times [0, \varepsilon[, \quad v : U \times [0, \varepsilon] \to u(U) \times [0, \varepsilon[, \quad v := u \times id$$

and hence we get a chart centred at p with the following properties:
In this chart f is of the form

$$f(p) - (x^1)^2 - \ldots - (x^k)^2 + (x^{k+1})^2 + \ldots + (x^{n-1})^2 + t^2 \tag{1.6}$$

and the Riemannian metric is of the form

$$g(v,w) := \sum_{i=1}^{n-1} v^i \cdot w^i + v^t \cdot w^t \tag{1.7}$$

where v^t and w^t correspond to the components in the direction $\frac{\partial}{\partial t}$.
The boundary of M corresponds to $\{t = 0\}$ in such a chart. We will call these *convenient charts* for critical points on the boundary.

Remark: the boundary conditions
The boundary conditions we impose on Morse pairs are not the most common ones. The motivation of this choice is the following: The unstable manifolds will be shown to build a decomposition of the manifold M and we want the boundary ∂M to be compatible with that decomposition. An early presentation of the idea to decompose a manifold with the help of an appropriate function and to get topological informations from this decomposition can be found in [21]. We remark that a pair (f, g) — with f a Morse function on a manifold M and g a Riemannian metric — is also called a *generalised triangulation* if for very critical point of f there is a Morse chart such that the pull-back of the Euclidean metric in this chart coincides with g and if (f, g) satisfies the Morse–Smale condition (Definition (1.15)), see [3] for instance.
Another possibility to deal with the boundary is to look at Morse functions f such that the gradient vector field is transversal to the boundary. More about this type of boundary condition can be found in [8] or [18] for instance.

1.9. Definition *the negative gradient flow of a Morse pair*
Given a Morse pair (f, g) on a compact manifold M we can investigate the negative gradient flow, i.e. the solutions of

$$\gamma_x'(t) = -\operatorname{grad}_g(f)(\gamma_x(t)), \quad \gamma_x(0) = x \tag{1.8}$$

called the negative gradient flow *of f.*

Remark: the negative gradient flow of a Morse function
The local existence and uniqueness of this flow for points in the interior of the manifold, i.e. in $M \backslash \partial M$, is implied by the Theorem of Picard-Lindelöf for ODEs: in charts, the flow equation is just a first-order ODE. For points on the boundary, the condition 2.) — especially that $\operatorname{grad}_g(f)$ is tangential

to the boundary — implies local existence and uniqueness: one can look at the flow equation restricted to ∂M and there the flow exists and then this flow also satisfied the flow equation on the whole manifold. Because M is compact, the flow of the negative gradient vector field is even defined globally, i.e. on all of $\mathbb{R}$.
Observe that critical points of f are exactly the stationary solutions of (1.8).

1.10. Definition *stable and unstable manifolds*
Given a Morse pair (f, g) assume p is a critical point of f. We define

$$W^-(p) := \{y \in M : \lim_{t\to-\infty} \gamma_y(t) = p\}$$
$$W^+(p) := \{y \in M : \lim_{t\to+\infty} \gamma_y(t) = p\}$$

where $\gamma_y(\cdot)$ denotes the negative gradient flow of (f, g), and one calls $W^-(p)$ the unstable *and $W^+(p)$ the* stable manifold of p.
In the next chapter the existence of the limits in the definition of the stable and unstable manifolds is shown for every point in M, see Lemma (2.2.).

Remark: stable and unstable manifolds
$W^-(p)$ respectively $W^+(p)$ are all points in M that are transported asymptotically to p under the negative gradient flow (for $t \to -\infty$ respectively $t \to +\infty$).
To justify our terminology it remains to proof that $W^-(p)$ and $W^+(p)$ are really manifolds (with boundary). Obviously, we can define stable and unstable sets for any dynamical system, however they need not form sub-manifold anymore, see [17] and [18] for expositions concerning more general dynamical systems.
Next, we show that the flow of the negative gradient vector field possesses the *Lyapunov-property*, i.e. there is a smooth function that strictly decreases along non–degenerate flow-lines.

1.11. Lemma
Given a Morse pair (f, g) on M. Then f decreases along flow lines of the negative gradient flow. Furthermore, assume that x and y are two points lying on a flow line of the negative gradient flow with $f(x) = b$, $f(y) = a$ and $a < b$. Then this flow line intersects all level hyper surfaces $f^{-1}(c)$ where $a < c < b$, with c regular, and it does so transversally. Additionally the flow line intersects such a hyper surface exactly once.

Proof:
That f decreases along the flow line follows from:

$$\begin{aligned}\frac{d}{dt} f(\gamma_x(t)) &= df(\gamma_x(t))(\gamma_x'(t)) = g(\mathrm{grad}_g(f)(\gamma_x(t)), \gamma_x'(t)) \\ &= -g(\gamma_x'(t), \gamma_x'(t)) = -||\gamma_x'(t)||^2 \le 0.\end{aligned}$$

and strictly smaller 0 when x is not a critical point of f. The flow line gives us a smooth path from x to y and so f must take every value between $f(x) = b$ and $f(y) = a$. As the function strictly decreases away from critical points, every such value is taken exactly once.
That the intersection is transversal can be checked directly: for any $t \in \mathbb{R}$ the level hyper surface is given by $H_x(t) = \{y \in M : f(y) = f(\gamma_x(t))\}$ where $\gamma_x(t)$ is the solution of (1.8.) with $\gamma_x(0) = x$. $H_x(t)$ is a sub-manifold of codimension 1.
So we have a smooth embedding $i : H_x(t) \hookrightarrow M$ and the induced map $Ti : T_{\gamma_x(t)}H_x(t) \hookrightarrow T_{\gamma_x(t)}M$. Choose an arbitrary vector Z tangential to $H_x(t)$, i.e. $Z \in Ti(T_{\gamma_x(t)}H_x(t))$. Now we have:

$$g(\gamma_x'(t), Z) = -g(\mathrm{grad}_g(f)(\gamma_x(t)), Z), = -df(Z)_{\gamma_x(t)} = 0$$

where the equalities follow from the definition of $\gamma_x(t)$, the definition of the gradient vector field and the fact that f restricted to $H_x(t)$ is constant and hence $df = 0$ for vectors in $T_{\gamma_x(t)}H_x(t)$.

$\square$

1.12. Proposition
There are no non-constant flow lines with

$$\lim_{t\to-\infty}(\gamma_x(t)) = \lim_{t\to+\infty}(\gamma_x(t)).$$

Proof:
This is a direct consequence of Lemma (1.11.): If the two limits exist and are equal, f would have the same value along the whole flow line $\gamma_x(t)$, so by the equality

$$\frac{d}{dt}f(\gamma_x(t)) = -||\gamma_x'(t)||^2$$

$\gamma_x'(t) = 0$ and hence $\gamma_x(t)$ would be a stationary point of the flow.

$\square$

Remark: stable and unstable manifolds near critical points
Given a Morse pair (f, g) we can give an explicit description of the stable and unstable manifolds near critical points. Assume that p is a critical point lying on the boundary ∂M. By condition 4) in the definition of Morse–Smale pairs there is a chart centred at p that satisfies (1.6) and (1.7). We remark that if a flow line starts at a point in a local chart for p and then leaves this chart after some time, this flow line can never return into the domain of this chart. This is an immediate consequence of the Lyapunov-property. In the chosen chart we can do a splitting into coordinates having a minus

in front of them and the ones that do not (including the t-coordinate). We define $y := (x^1, \ldots, x^k) \in \mathbb{R}^k$ and $x := (t, x^{k+1}, \ldots, x^{n-1}) \in [0, \varepsilon[\times\mathbb{R}^{n-k-1}$. By (1.5) we obtain

$$- \operatorname{grad}_g(f) = 2\left(\sum_{i=1}^{k} y^i \frac{\partial}{\partial y^i}\right) - 2\left(\sum_{i=1}^{n-k-1} x^i \frac{\partial}{\partial x^i}\right) - 2t\frac{\partial}{\partial t}$$

In our notation equation (1.8) reads:

$$t'(s) = -2t(s), \quad x'(s) = -2x(s), \quad y'(s) = 2y(s) \tag{1.9}$$

and hence

$$t(s) = t_0 e^{-2s}, \quad x(s) = x_0 e^{-2s}, \quad y(s) = y_0 e^{2s}. \tag{1.10}$$

With the help of the Lyapunov-property one sees that points on the stable manifold of p are points z such that $\lim_{s\to+\infty} \gamma_z(s) = 0$. From the explicit description of the negative gradient flow one easily deduces that $W^+(p) \cap U$ is given by

$$([0, \varepsilon[\times\mathbb{R}^{n-k-1} \times 0) \cap u(U) \tag{1.11}$$

where (U, u) denotes the convenient chart. In the same manner one sees that $W^-(p) \cap U$ is given by

$$(0 \times \mathbb{R}^k) \cap u(U) \tag{1.12}$$

and observe that unstable manifolds of a Morse pair on the boundary are "trapped" in the boundary. If a unstable manifold $W^-(p)$ on the boundary would be contained in a unstable manifold of the whole manifold that is larger than $W^-(p)$, a shift in the index would occur at p, seen as a critical point of the boundary on the one hand, and as a critical point of the whole manifold on the other hand. But this is forbidden by condition 4) for Morse pairs. Additionally, the stable manifolds intersect the boundary transversal: We use the collar and observe that $\frac{\partial}{\partial t}$ is always transversal to $T(\partial M)$ in TM restricted to ∂M.
The same calculations can be made for points in the interior of the manifold with the obvious small adaptions.

<u>Remark</u>: stable and unstable spheres
Consider a convenient chart (U, u) centred at a critical point p of index k. The Morse function has the form (1.2) respectively (1.6) in such a chart and we have a splitting into the stable and the unstable part, see (1.11) and (1.12). Without loss of generality we can assume that $f(p) = 0$ and if f is restricted to $(0 \times \mathbb{R}^k) \cap u(U)$ it has the form

$$-(y^1)^2 - \ldots - (y^k)^2;$$

$$(x^1)^2 + \ldots + (x^{n-k})^2$$

if f is restricted to the stable part with $(x^1, \ldots, x^{n-k}, 0, \ldots, 0) \in ([0, \varepsilon[\times \mathbb{R}^{n-k-1} \times 0) \cap u(U)$ or to $(x^1, \ldots, x^{n-k}, 0, \ldots, 0) \in (\mathbb{R}^{n-k} \times 0) \cap u(U)$ respectively. We define

$$S^-_{-d}(p) := \tilde{f}^{-1}(-d) \cap (0 \times \mathbb{R}^k) = 0 \times S^k_{\sqrt{d}} \tag{1.13}$$

$$S^+_d(p) := \tilde{f}^{-1}(d) \cap (\mathbb{R}^{n-k} \times 0) = S^{n-k}_{\sqrt{d}} \times 0 \tag{1.14}$$

for $d > 0$ and call $S^-_d(p)$ the *unstable sphere* of p with radius d and $S^+_d(p)$ the *stable sphere* of p with radius d. We can always find $d > 0$ sufficiently small such that the stable and unstable spheres are contained in $u(U)$.
Let $z \in W^-(p)$ with $0 < f(p) < -d$. By Lemma (1.11.) it is clear that the trajectory of the negative gradient flow that starts at z must intersect $S^-_d(p)$ in exactly one point. If $0 > f(p) > -d$ this is also true because of the explicit form of the trajectories in the convenient chart, see (1.10). Analogous arguments hold for $S^+_d(p)$ and points in $W^+(p)$.

1.13. Theorem *stable and unstable manifolds*
Given a Morse pair (f, g), the stable and unstable manifolds of any critical point are sub-manifolds of M. The dimension of the unstable manifold equals the index of the critical point and the dimension of the stable one is equal to $\dim M$ *minus the index of the critical point.*

Proof:
First assume that p is a critical point of (f, g). Then we can find a convenient chart for p. We computed how the stable respectively unstable manifolds look like in this chart, see (1.11) and (1.12) and obviously the convenient chart is a sub-manifold chart for the stable and unstable manifolds at p.
That the stable and unstable manifolds are sub-manifolds of M can be seen as follows: near the critical point this is obvious. For an arbitrary point x on the stable / unstable manifold fix a time T such that $\gamma_x(\pm T)$ is contained in an appropriate chart. Choose an open neighbourhood U of $\gamma_x(\pm T)$ that is totally contained in the appropriate chart. Now $\gamma_U(\mp T)$ is an open neighbourhood of x that is diffeomorphic to U. Hence it can be used as a sub-manifold chart of the stable / unstable manifold near x. Observe that this works because the diffeomorphisms provided by the negative gradient flow preserve the stable and unstable manifolds and the boundary of M.
The dimensions can be read off the explicit description in a convenient chart.

□

1.14. Proposition
Given a manifold M of dimension n and a Morse pair (f, g) on M. Let p be a critical point of (f, g). If p lies in the interior of M, $W^+(p)$ is diffeomorphic

to $\mathbb{R}^{n-\mathrm{ind}(p)}$ and $W^-(p)$ is diffeomorphic to $\mathbb{R}^{\mathrm{ind}(p)}$. If p lies on the boundary of M, $W^+(p)$ is diffeomorphic to the half space $\{(x_1, \ldots, x_{n-\mathrm{ind}(p)}) \in \mathbb{R}^{n-\mathrm{ind}(p)} : x_1 \geq 0\}$ and $W^-(p)$ is diffeomorphic to $\mathbb{R}^{\mathrm{ind}(p)}$.

Proof:
For $W^+(p) \setminus \{p\}$ with p in the interior of M we can use the following parametrisation:

$$S_d^+(p) \times \mathbb{R} \to W^-(p) \setminus \{p\}, \quad (\theta, s) \mapsto \gamma_\theta(\frac{1}{2}\ln(s)). \tag{1.15}$$

We have seen that this mapping is bijective and smoothness of this mapping is a general result concerning solutions of flow equations, see [13] or [9]. The same argument works for $W^-(p)$ as well.
Now assume $p \in \partial M$. The unstable manifolds are contained in ∂M. And we can use an adaption of the map (1.15.) to parametrise $W^-(p)$ and

$$(S_d^+(p) \cap (s \geq 0)) \times \mathbb{R} \to W^+(p) \setminus \{p\}, \quad (\theta, s) \mapsto \gamma_\theta(\frac{1}{2}\ln(s)) \tag{1.16}$$

to parametrise $W^+(p) \setminus \{p\}$. In a convenient chart

$$S_d^-(p) \times]0, \infty[\to W^-(p), \quad (\theta, s) \mapsto \gamma_\theta(\frac{1}{2}\ln(s))$$

is given by $(\theta, s) \mapsto \theta e^{\ln(s)} = \theta s$ — see (1.10) — and hence is just the parametrisation by polar coordinates. Consequently, it can be extended to a smooth diffeomorphism $\mathbb{R}^{\mathrm{ind}(x)} \cong W^-(p)$.
Analogous arguments work for the stable manifolds. For critical points on the boundary the parametrisation (1.16) implies that the stable sphere is diffeomorphic to the half space $\{(x_1, \ldots, x_{n-\mathrm{ind}(p)} \in \mathbb{R}^{n-\mathrm{ind}(p)} : x_1 \geq 0\}$.

$\square$

1.15. Definition *Morse–Smale condition and Morse–Smale pairs*
We call a pair (f, g), with $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and g a Riemannian metric Morse–Smale*, if*
1.) (f, g) is a Morse pair and
2.) for every pair of critical points p and q, the stable manifold $W^-(p)$ is transversal to the unstable one $W^+(q)$, written $W^-(p) \pitchfork W^+(q)$, i.e. for every $x \in W^-(p) \cap W^+(q)$ we have

$$T_x M = T_x W^-(p) + T_x W^+(q). \tag{1.17}$$

Remark: the Morse–Smale condition
The Morse–Smale condition will guarantee that $W^+(p) \cap W^-(q)$ are manifolds for every $p, q \in Cr(f)$.

1.3 Questions concerning Genericy

So far we have introduced the basic terminology and have established some facts, but we have not clarified the question whether there are Morse–Smale pairs with the desired properties on any manifold with boundary, if there are "plenty" of them, etc. This questions will be dealt with in this section. In [18] similar questions are investigated.

Remark: jet bundles
We outline the most important properties of jets, see [1] for information about jets of sections, [12] for an exposition of the theory of jets on manifolds with corners and [8] for a general introduction.
Let $p : E \to X$ be a smooth vector bundle with finite dimensional fibre, $x_1, x_2 \in X$, (U, u, u_0) a vector bundle chart of $p : E \to X$, i.e. u denotes the trivialization of the vector-bundle over U and u_0 denotes the chart for U induced by u, such that $x_1, x_2 \in U$, and $s_1, s_2 \in \Gamma^\infty(U; E|_U)$, i.e. local sections of the vector bundle over U (by partition of unity we can extend them to sections on the whole manifold, and in the reverse direction, we can "chop" every global section into local ones). One has

$$\begin{aligned} u : p^{-1}(U) &\overset{\cong}{\to} u_0(U) \times F_u \\ s_1^u, s_2^u : u_0(U) &\to F_u \end{aligned}$$

where s_i^u denotes the local representation of the sections s_i in the chart u and F_u is the typical fibre of E.
We can introduce an equivalence relation on pairs $(x, s) \in X \times \Gamma^\infty(X; E)$ by setting $(s_1, x_1) \equiv^k (s_2, x_2) :\Longleftrightarrow$

$$x_1 = x_2 \quad \text{and} \quad (s_1^u(u(x_1)), \ldots, D^k s_1^u(u(x_1))) = (s_2^u(u(x_2)), \ldots, D^k s_2^u(u(x_2)))$$

for one (and then any) vector bundle chart (U, u, u_0). $D^l s^u$ denotes the l'th derivative of the local representative s^u of the section s. Obviously this is an equivalence relation and equivalence classes are denoted by $[s(x)]_k =: j^k s(x)$, the *k-jet* of s at x. Set

$$J^k(X; E) := \{[s(x)]_k : s \in \Gamma^\infty(X; E), x \in X\},$$

this is called the *k-jet bundle* of $p : E \to X$. There are two natural maps

$$j^k s : X \to J^k(X; E), \quad x \mapsto j^k s(x)$$

called the *k-jet extension map* and

$$p^k : J^k(X; E) \to X, \quad j^k(x) \mapsto x$$

the *k-jet projection*. The k-jet projection makes $J^k(X; E)$ into a smooth vector bundle with finite dimensional fibre over X, see [1], and hence its name is justified. Given a vector bundle chart (U, u, u_0) of $p : E \to X$ we obtain a canonical vector bundle chart for $J^k(X; E)$:

$$u^k(j^k s(x)) = (u_0(x), Ds^u(u_0(x)), \ldots, D^k s^u(u_0(x)))$$

and one can see the structure of the fibre from this chart. Furthermore there are maps

$$\pi_l^k : J^k(X; E) \to J^l(X; E), \quad j^k s(x) \mapsto j^l s(x)$$

for $k \geq l$. These maps satisfy

$$\begin{aligned} \pi_k^k &= id_{J^k(X;E)}, \quad \text{and} \\ \pi_m^l \circ \pi_l^k &= \pi_m^k \quad \text{for all} \quad k \geq l \geq m \end{aligned}$$

hence we can build the inverse limit of the system

$$J^0(E; X) \xleftarrow{\pi_0^1} J^1(E; X) \xleftarrow{\pi_1^2} \ldots$$

in the category of Hausdorff topological spaces and obtain $J^\infty(X; E)$, and elements of this space are denoted by $j^\infty s(x)$. There are again mappings

$$\begin{aligned} j^\infty s : X &\to J^\infty(X; E), \\ p^\infty : J^\infty(X; E) &\to X, \\ \pi_k^\infty : J^\infty(X; E) &\to J^k(X; E) \end{aligned}$$

$J^\infty(X; E)$ is a complete metric space, as a countable inverse limit of complete metrical spaces. Next we want to introduce topologies on the space of smooth sections $\Gamma^\infty(X; E)$. We follow the approach used in [1].

1.16. Theorem *topology on the space of continuous sections*
Let $p : E \to X$ be a vector bundle with finite dimensional fibre and with compact base space X. Let φ be a fibre metric on $p : E \to X$. Then:
1.) $||\cdot||_{\sup}$ is a norm on $\Gamma^0(X; E)$
2.) The topology on $\Gamma^0(X; E)$ determined by $||\cdot||_{\sup}$ is independent from the choice of φ.
3.) In this topology $\Gamma^0(X; E)$ is a separable Banach-space.

Proof:
A proof can be found in [1].

□

Remark: the $\mathcal{C}^r$-topology
The $\mathcal{C}^r$*-topology* on $\Gamma^r(X; E)$ is the topology induced by the injection $j^r : \Gamma^r(X; E) \mapsto \Gamma^0(X; J^r(X; E))$, i.e. the coarsest topology on $\Gamma^r(X; E)$ so that

j^r is continuous.
In the case of functions one can also define jets and the $\mathcal{C}^r$-topology in the same manner, see [12] for more details. In particular we remark that $\mathcal{C}^r(X;Y)$ equipped with the $\mathcal{C}^r$-topology is a complete topological space, provided that X is compact.
$\Gamma^\infty(X;E)$ is contained in $\Gamma^r(X;E)$ for $r \in \mathbb{N}$ and we define the *$\mathcal{C}^r$-topology* on $\Gamma^\infty(X;E)$ as the subspace topology $\Gamma^\infty(X;E)$ inherits from $\Gamma^r(X;E)$ equipped with the $\mathcal{C}^r$-topology.
Another description of these topologies on the space of sections uses families of fibre-metrics. $J^\infty(X;E)$ becomes a Fréchet-space in this description.

1.17. Lemma
The map $j^r : \Gamma^r(X;E) \to \Gamma^0(X;J^r(X;E))$, $s \mapsto j^r(s)$, $r = 0, 1, \ldots, \infty$ is a linear continuous injection with closed image.

Proof:
Again, we refer to [1] and [12] for the proof.

$\square$

Next we cite two results from [12] respectively [1] and their proofs can also be found there. We remark that Theorem (1.18.) makes use of the Theorem of Sard, see [13] for instance.

1.18. Theorem *Density of transversal Intersections*
Let X, Y, Z, W be manifolds with corners (in particular, this type of spaces includes manifolds with boundary), let $f : Z \to Y$ be a smooth mapping. Let $\varphi : W \to \mathcal{C}^\infty(X, Y)$ be a mapping. Consider $\Phi : W \times X \to Y$, given by $\Phi(w, x) := \varphi(w)(x)$, and assume that Φ is smooth.
Assume $\Phi \pitchfork f$. Then the set $\{w \in W : \varphi(w) \pitchfork f\}$ is dense in W (in fact: its complement in W has Lebesgue measure 0).

Remark:
Two maps $f : A \to X$ and $g : B \to X$ are called *transversal*, $f \pitchfork g$, if $T_xX = (Tf)(T_aA) + (Tg)(T_bB)$ holds for any $a \in A$ and $b \in B$ with $f(a) = g(b) = x$.

1.19. Lemma
Let X, Y, Z be manifolds with corners with compact X, let $f : Z \to Y$ be a proper smooth mapping, i.e. the pre-images of all compact sets are compact. Then the set $\{g \in \mathcal{C}^\infty(X, Y) : g \pitchfork f\}$ is $\mathcal{C}^1$-open in $\mathcal{C}^\infty(X, Y)$.

With this preparation we can start to construct suitable functions that will be the functions of Morse–Smale pairs. We need the following basic fact:

1.20. Theorem *Morse functions*
Let M be a compact smooth manifold without boundary. Then the set of Morse functions is $\mathcal{C}^\infty$-dense in $\mathcal{C}^\infty(M;\mathbb{R})$ and open with respect to the $\mathcal{C}^2$-topology.

Proof:
Let $f : M \to \mathbb{R}$ be a fixed smooth function on M.
Set $X := M$, $Y := T^*M$, $0_M : M \hookrightarrow T^*M$ the embedding of M into T^*M as the zero-section and $Z := 0_M(M)$. A function has non-degenerate critical points if and only if $df \in \Omega^1(M)$ is transversal to Z. We intend to apply Theorem (1.18.) and Lemma (1.19.).
For any $q \in M$ choose a chart (U,u) and set $h^i_U := x^i$, $i = 1,\dots,n$, in this chart, with $u(y) = (x^1,\dots,x^n)$ the coordinate functions of this chart. Consequently $dh^i_U = dx^i$ in this chart. Outside an open neighbourhood V of q that is contained in U. We extend the local functions h^i_U to smooth functions h^i_V on M. We can find such a V for every $q \in M$ and this family of subsets constitute an open cover of M. By compactness there is a finite sub-cover denoted by $\mathcal{V}$.
Now we set $W := \mathbb{R}^{|\mathcal{V}|n}$ and define

$$\varphi : \mathbb{R}^{|\mathcal{V}|n} \to \mathcal{C}^\infty(M;T^*M), \quad \lambda_{V,j} \mapsto df + \left(\sum_{V\in\mathcal{V}, j=1,\dots,n} \lambda_{V,j} dh^j_V \right)$$

and so

$$\Phi : \mathbb{R}^{|\mathcal{V}|n} \times M \to T^*M, \quad (\lambda_{V,j}, x) \mapsto df(x) + \left(\sum_{V\in\mathcal{V}, j=1,\dots,n} \lambda_{V,j} dh^j_V(x) \right)$$

hence Φ is a smooth mapping.
Next we claim that Φ is transversal to the zero section. So, let $y \in M$ arbitrary, than there is a $W \in \mathcal{V}$ with $y \in W$. On W h^j_W coincides with h^j_U for all $j = 1,\dots,n$ and consequently we obtain

$$d\tilde{f} + \left(\sum_{j=1,\dots,n} \lambda_{W,j} dx^j \right)$$

in the chart if we set $(\lambda)_{V,j} = 0$ for $V \neq W \in \mathcal{V}$ and j arbitrary. Obviously we can span $T^*_y M$ by varying the components of $(\lambda)_{W,j}$.
Now we can apply Theorem (1.18.) and so the set of points $(\lambda)_{V,j}$ in $\mathbb{R}^{|\mathcal{V}|n}$ such that $\varphi((\Lambda)_{V,j})$ is transversal to the zero-section is dense. So we can always find $(\Lambda)_{V,j}$ arbitrary small such that

$$f + \left(\sum_{V\in\mathcal{V}, j=1,\dots,n} \lambda_{V,j} h^j_V \right)$$

has only no-degenerate critical points and consequently the Morse functions are $\mathcal{C}^\infty$-dense in $\mathcal{C}^\infty(M;\mathbb{R})$.
Furthermore, the map $0_M : M \to T^*M$ is proper by compactness of M and so the set $\{g \in \mathcal{C}^\infty(M;\mathbb{R}) : dg \pitchfork 0_M\}$ is $\mathcal{C}^2$-open in $\mathcal{C}^\infty(M;\mathbb{R})$ by Lemma (1.19.). The shift from $\mathcal{C}^1$ to $\mathcal{C}^2$ occurs because a function h must by two times differentiable if dh should be one time differentiable as can be seen in charts - see the remark about the situation in charts in the last section.

□

Fix $\phi : \partial M \times [0,\varepsilon] \to M$ a closed collar of ∂M. Regard ∂M as a compact manifold without boundary and let f_0 be a Morse function on ∂M. Extend f_0 to the collar by setting

$$f(x,t) := f_0(x) + t^2$$

with $x \in \partial M$ and $t \in [0,\varepsilon[$. We can extend f to a smooth function on the whole manifold, for instance, with the help of a bump function. However, extensions of f need not be Morse any more. But we can prove:

1.21. Theorem *extending Morse functions from the collar*
Let f_0 be a Morse function on a fixed closed collar of ∂M of the form $f(x,t) = f(x) + t^2$ on this collar. Consider the set

$$\mathcal{M} := \{h \in \mathcal{C}^\infty(M;\mathbb{R}) : h \textit{ extends } f \textit{ and } h \textit{ is a Morse function on } M\}.$$

$\mathcal{M}$ is $\mathcal{C}^\infty$-dense and $\mathcal{C}^2$-open in the set of all smooth functions that extend f.

Proof:
Let g be an arbitrary smooth extension of f.
Let $A := \phi(\partial M \times [0,\varepsilon])$ be the embedded collar. Then g is a Morse function on A because it has the special form $f_0(x) + t^2$ there. $\mathcal{U}$ should be a finite family of charts of M such that the chart-neighbourhoods $U \in \mathcal{U}$ constitute a finite open cover of M. Such a finite family of charts alway exists as M is compact. For $U \in \mathcal{U}$ set $V := U \setminus A$. This is an finite open cover of $M \setminus A$ denoted by $\mathcal{V}$ and (V, v) are charts of M again where v is the restriction of u to $V \subset U \in \mathcal{U}$. Construct local functions $h_V^j := x^j$ $j = 1,\dots,n$ for all $V \in \mathcal{V}$ and extend these local functions to functions on M with the help of bump functions in a special way:
For $V \in \mathcal{V}$ with $\overline{V} \cap A = \emptyset$ we can extend the functions h_V^j, $j = 1,\dots,n$ so that $supp(h_V^j) \cap A = \emptyset$. For $V \in \mathcal{V}$ with $\overline{V} \cap A \neq \emptyset$ we use the following construction: Since f is Morse there as can bee seen by the special form of f on the collar, there is an open neighbourhood W_y for every $y \in A$ such that

g is Morse on this neighbourhood because $dg \neq 0$ is an open property. The union of these sets W_y is open and denoted by B. We know that g is Morse on B. On $V \cap B$ we can change h_V^j in such a way that $supp(h_V^j) \cap A = \emptyset$ and even find an extension h_V^j such that $supp(h_V^j) \cap A = \emptyset$.
But now we can use arguments similar to the ones used in the proof of Theorem (1.20.). We set $X := M$, $Y := T^*M$, $M \hookrightarrow T^*M$ and $W := \mathbb{R}^{|\mathcal{V}|n}$. We consider the mappings

$$\varphi : \mathbb{R}^{|\mathcal{V}|n} \to \mathcal{C}^\infty(M; T^*M), \quad (\lambda)_{V,j} \mapsto dg + \left(\sum_{V \in \mathcal{V}, j=1,\ldots,n} \lambda_{V,j} dh_V^j \right)$$

and

$$\Phi : \mathbb{R}^{|\mathcal{V}|n} \times M \to T^*M, \quad (\lambda_{V,j}, x) \mapsto dg(x) + \left(\sum_{V \in \mathcal{V}, j=1,\ldots,n} \lambda_{V,j} dh_V^j(x) \right)$$

and Φ is smooth.
Observe that φ leaves dg unchanged on A as all the h_V^j vanish there identically. For points in B the transversality of Φ is clear because g is Morse by construction. For points in the complement of B we can find $\lambda_{V,j}$'s to prove that Φ is transversal to the zero-section as was done the proof of Theorem (1.20.).
Now we can apply Theorem (1.18.) to show that the set $\mathcal{M}$ is $\mathcal{C}^\infty$-dense in the set of all smooth extensions of f and Lemma (1.19.) to argue that $\mathcal{M}$ is $\mathcal{C}^2$- open too: $\{g \in \mathcal{C}^\infty(M; \mathbb{R}) : dg \pitchfork 0_M\}$ is $\mathcal{C}^2$-open in $\mathcal{C}^\infty(M; \mathbb{R})$ and consequently

$$\begin{gathered}\{g \in \mathcal{C}^\infty(M; \mathbb{R}) : dg \pitchfork 0_M \text{ and } g = f \text{ on } A\} = \\ \{g \in \mathcal{C}^\infty(M; \mathbb{R}) : dg \pitchfork 0_M\} \cap \{g \in \mathcal{C}^\infty(M; \mathbb{R}) : g = f \text{ on } A\}\end{gathered}$$

is $\mathcal{C}^2$-open in the space of smooth extensions of f, i.e. in $\{g \in \mathcal{C}^\infty(M; \mathbb{R}) : g = f \text{ on } A\}$.

□

Next we intend to construct Riemannian metrics that serve as the metrics of the Morse–Smale pairs we are interested in:

<u>Remark</u>: construction of convenient Riemannian metrics on the collar
Be aware that Riemannian metrics are not arbitrary sections of a vector bundle. They form a subspace in the space of all sections of the vector bundle of fibrewise symmetric bilinear forms on T^*M, $Symm(T^*M \otimes T^*M) \to M$, so we can equip the space of smooth sections of this bundle with the $\mathcal{C}^r$-topologies and the space of Riemannian metrics inherits the corresponding

subspace topologies.
Let f be a Morse function with the properties described so far. ∂M is a compact smooth manifold without boundary and for all critical points on ∂M we can find a chart (U, u) such that f has the form (1.6). The charts can be chosen in such a way that they have disjoint topological closure. Define local metrics on the chart-neighbourhoods with help of the chart by pulling back the Euclidean metric from the chart domain contained in $\mathbb{R}^{n-1} \times [0, \infty[$. Now we can extend the local metrics near the critical points to an Riemannian metric g_0 on ∂M with the help of bump functions. Then we extend g_0 to a Riemannian metric on the collar by setting $g := g_0 + dt \otimes dt$. Consequently, the charts (U, u) become convenient charts for the critical points on the boundary.
Now we can consider critical points in the interior of the manifold. Again we choose Morse charts for each critical point and pull-back the Euclidean metric on $\mathbb{R}^n$ to these chart-neighbourhoods. Then we find Riemannian metrics on M such that they coincide with the local metrics in the Morse charts and with the fixed Riemannian metric on the collar. So these Morse charts become convenient charts too.
The next task is to deal with the Morse–Smale condition. Before we start with this investigation we cite two facts we will use. The first one is stated in [1] and the second one is an adaption of a result in the same exposition.

1.22. Lemma
Let X be a compact smooth manifold, ξ^0,η vector fields on X, ψ^0 the flow of ξ^0, and ψ^λ (with $\lambda \in \mathbb{R}$) the flow of the vector field $\xi^\lambda := \xi^0 + \lambda\eta$. Then for $x \in X$ and $t \in \mathbb{R}$:

$$\frac{d}{d\lambda}\left(\psi_t^\lambda(x)\right)_{\lambda=0} = \int_0^t (T\psi_s^0) \circ \eta \circ \psi_{-s+t}^0(x) ds.$$

<u>Proof</u>:
A proof of this "Perturbation Theorem" can be found in [1].

□

1.23. Lemma
Let X be a compact smooth manifold with Riemannian metric g , f a Morse function and x a fixed regular point on a trajectory of the negative gradient flow $-\operatorname{grad}_g(f)$. We denote the flow of $\xi^\lambda = -\operatorname{grad}_g(f) + \lambda\eta$ ($\lambda \in \mathbb{R}$) by ψ^λ.
For given $y \neq x$ with $\psi_t^0(y) = x$ for some $t \in \mathbb{R}$ and given $v \in T_xX$ there exists a smooth vector field η supported away from critical points such that

$$\frac{d}{d\lambda}\left(\psi_t^\lambda(x)\right)_{\lambda=0} = v.$$

Proof:
Choose a smooth function $h : \mathbb{R} \to \mathbb{R}$, supported on $[0, t]$ with $\int_0^t h(s)ds = 1$. Define

$$\eta(z) := h(s) \cdot (T\psi^0_{-s})v$$

for $z = \psi^0_{-s+t}(y)$, $0 \leq s \leq t$, on the flow line that contains x and y.
η can be extended to a smooth vector field on X, supported away from critical points, this can be done with the help of the so-called Straightening-out Theorem for instance, which can be found in [1]. Now we calculate

$$\int_0^t T\psi^0_s \circ \eta \circ \psi^0_{-s+t} ds = \int_0^t h(s)v ds = v$$

and the claim follows with the help of the Perturbation Theorem.

□

1.24. Theorem *Morse–Smale pairs*
Given a Morse function f on a compact smooth manifold M and fixed local Riemannian metrics on open neighbourhoods of critical points of f that are totally contained in Morse charts, where the metrices are obtained by pulling back the Euclidean metric with help of the Morse chart.
$\mathcal{G}$ denotes the set of all Riemannian metrics such that (f, g) satisfies the Morse–Smale condition and such that these metrices coincides with the local metrics on the fixed neighbourhoods of the critical points.
The set $\mathcal{G}$ is $\mathcal{C}^\infty$-dense and $\mathcal{C}^1$-residual in the set of all Riemannian metrics that coincide with the local metrics on the fixed neighbourhoods of the critical points.

Proof:
We follow the idea outlined in [2], see [18] and [15] for a similar treatment. Let $\overline{g}$ be an arbitrary Riemannian metric that coincides with the local Riemannian metrics defined on the domains of the Morse charts (U, u). If we modify the Riemannian metric, the critical points remain unchanged and we will change $\overline{g}$ only outside an open neighbourhood of the critical points, so the splitting of $u(U)$ into the stable and unstable part remains the same. Hence we can still use the parametrisation (1.15) of the stable respectively unstable manifolds. Observe that if we change the Riemannian metric in a smooth way, the negative gradient vector field also changes in a smooth way, as can seen in charts and the flow also changes in a smooth manner by the Perturbation Theorem.
First observe that one can reduce the problem of transversality of all stable and unstable manifolds to the question how the stable and unstable spheres intersect: Let p and q be two critical points such that $S^-_e(p)$ intersects

$S_e^+(q)$ transversally in $f^{-1}(e)$. Let $y \in W^-(p) \cap W^+(q)$ arbitrary. But then the stable and unstable spheres of q respectively p intersect transversally if transported to the hyper surface $f^{-1}(f(y))$ because the flow induces diffeomorphisms. The stable and unstable manifolds are flow-invariant in the sense that if a point is contained in them so is the whole trajectory of the negative gradient flow through that point and hence the vector $-\operatorname{grad}_g(f)$ is always contained in the tangential space of stable and unstable manifolds. So $T_yW^-(p)$ and $T_yW^+(q)$ span T_yM for $y \in W^-(p) \cap W^+(q)$. In the other direction, one observes that the dimension of the tangential space of the intersection of a stable and an unstable sphere is the same as the dimension of the tangential space of the stable and unstable manifolds minus one and if $W^-(p)$ and $W^+(q)$ intersect transversally this implies that $S_e^-(p)$ and $S_e^+(q)$ intersect transversally in the submanifold $f^{-1}(e)$.
We denote the parametrisation of the unstable manifold of a critical point p of index k by

$$u : \mathcal{R} \times S^-_{f(p)-d}(p) \times \mathbb{R} \to M, \quad (g, \theta, t) \mapsto u_g(\theta, t)$$

where $u_g(\theta, \cdot)$ denotes the flow line of the negative gradient flow with respect to g starting at $\theta \in S^-_{f(p)-d}(p)$ and $\mathcal{R}$ is the set of Riemannian metrics that coincide with the fixed ones near critical points. We consider

$$u : \mathcal{R} \times \bigcup_{p \in Cr(f)} (S^-_{f(p)-d}(p) \times \mathbb{R}) \to M,$$

$$(g, \theta, t) \mapsto u_g(\theta, t) \tag{1.18}$$

and observe that the Morse–Smale condition is satisfied for $g \in \mathcal{R}$ iff u is transversal to $\bigcup_{q \in Cr(f)} S^+_{f(q)+d}(q)$.
We intend to apply Theorem (1.18.) and hence set

$$X := \bigcup_{p \in Cr(f)} (S^-_{f(p)-d}(p) \times \mathbb{R}),$$

$$Z := M,$$

$$Y := \bigcup_{q \in Cr(f)} S^+_{f(q)+d}(q)$$

but it remains to find a subspace of $\mathcal{R}$ that serves as W. We have

$$\varphi : \mathcal{R} \to \mathcal{C}^\infty(X; Z), \quad g \mapsto u_g$$
$$\Phi : \mathcal{R} \times X \to Z, \quad (g, \theta, t) \mapsto u_g(\theta, t).$$

We have seen that it suffices to find a finite-dimensional family of Riemannian metrics such that u becomes transversal for all combinations of stable and unstable spheres for different critical points. This families can be constructed step by step by starting with a critical point of highest critical value, denoted by p. We consider an unstable sphere of p, $S_d^-(p)$. For every point y on the unstable sphere we can find n vector fields supported in a small neighbourhood of y such that

$$\frac{d}{d\lambda}\left(\psi_{1,t}^{\lambda}(x)\right)_{\lambda=0}, \dots, \frac{d}{d\lambda}\left(\psi_{n,t}^{\lambda}(x)\right)_{\lambda=0}$$

span T_yM. But then these vector fields also span T_zM for z in an open neighbourhood V_y of y. These open neighbourhoods build an open covering of the unstable sphere and by compactness there is a finite sub-cover. Consequently we have found finitely many vector fields such that the map $\lambda \mapsto \psi_t^{\lambda}(x)$ is submersive at $\psi_t^0(x)$, where ψ^{λ} denotes the the flow of the perturbed vector field $-\operatorname{grad}_g(f) + \lambda X$. But if it is submersive it is transversal to any sub-manifold, especially to any stable sphere. The same arguments work for all the other unstable spheres and so we can find finitely many vector fields for which u is transversal to $\bigcup_{q\in Cr(f)} S^+_{f(q)+d}(q)$ step by step. Because there are only finitely many critical points we also obtain finitely many vector fields supported away from the critical points that must be varied such that u is transversal to $\bigcup_{q\in Cr(f)} S^+_{f(q)+d}(q)$. If we denote these vector fields by $X_1, \dots, X_m$ we can set $W := \mathbb{R}^m$ and then

$$\begin{aligned} u : \mathbb{R}^m \times \bigcup_{p\in Cr(f)} (S^-_{f(p)-d}(p) \times \mathbb{R}) &\to M \times M, \\ (\Lambda, \theta, t) &\mapsto u_\Lambda(\theta, t) \end{aligned}$$

where $\Lambda \in \mathbb{R}^m$ and u_Λ is the flows of the vector field

$$-\operatorname{grad}_{\overline{g}}(f) + \sum_{i=1,\dots,m} \Lambda_i X_i.$$

This is a smooth mapping and it is transversal to $\bigcup_{q\in Cr(f)} S^+_{f(q)+d}(q)$. In order to apply Theorem (1.18.), it remains to show that there are Riemannian metrics that coincide with the fixed Riemannian metrics g on the domains of Morse charts near critical points and such that

$$-\operatorname{grad}_{\overline{g}}(f) + \sum_{i=1,\dots,m} \Lambda_i X_i = -\operatorname{grad}_g(f)$$

at least for $(\lambda_1, \dots, \lambda_m)$ sufficiently small. Then the map $g \mapsto \operatorname{grad}_g(f)$ would be submersive and so we would have found a finite-dimensional family of Riemannian metrices such that (1.19.) is transversal to $\bigcup_{q\in Cr(f)} S^+_{f(q)+d}(q)$. So, given $X := \operatorname{grad}_{\overline{g}}(f)$ and Y a vector field that is equal to X outside an open subset U that contains no critical point of f. Y shall be sufficiently close to X in the sense that $df(Y) > 0$ everywhere (this holds for X by definition). Then there is a splitting of T_yM into $< Y_y > \oplus(\ker df)_y$ if y is not a critical point of f. With the help of the splitting we can define a Riemannian metric g as the bilinear form represented by the following matrix:

$$\begin{pmatrix} df(Y) & 0 \\ 0 & \overline{g}_{\ker df} \end{pmatrix}$$

where $\overline{g}_{\ker df}$ is the restriction of $\overline{g}$ to the sub-bundle $\ker df$ in T^*M. For $Y = X$ this coincides with $\overline{g}$ because then $df(X) = g(X, X)$ and the splitting $< X > \oplus \ker df$ is $\overline{g}$-orthogonal. For Y sufficiently small g defines a smooth Riemannian metric and outside U g coincides with $\overline{g}$. Furthermore let Z be an arbitrary vector field with $\mu Y + \tilde{Z}$ the decomposition of Y with respect to the splitting $< Y > \oplus \ker df$. Then

$$df(Z) = df(\mu Y + \tilde{Z}) = \mu df(Y) = g(\mu Y, Y) = g(\mu Y + \tilde{Z}, Y) = g(Z, Y)$$

holds and so the constructed metric satisfies $\operatorname{grad}_g(f) = Y$.
No we can apply Theorem (1.18.) and obtain the result that the set $\mathcal{G}$ is $\mathcal{C}^\infty$-dense in the set of all Riemannian metrics that coincide with the fixed local metrics on chart neighbourhoods of the critical points.
That $\mathcal{G}$ is C^1-residual can be seen as follows: Apply Lemma (1.19.) to

$$X_n := \bigcup_{p \in Cr(f)} (S^-_{f(p)-d}(p) \times [-n, n]),$$

$$Y := \bigcup_{q \in Cr(f)} S^+_{f(q)+d}(q) \quad \text{and} \quad Z := M$$

with $\iota : Y \hookrightarrow Z$ the smooth embedding of the stable spheres into M. We have seen that it suffices to check transversality of the unstable manifolds with the stable spheres to obtain that the Morse–Smale condition is satisfied. By Lemma (1.19.) the set $\{h \in \mathcal{C}^\infty(X_n, Z) : h \pitchfork \iota\}$ is $\mathcal{C}^1$-open in $\mathcal{C}^\infty(X_n, Z)$. Consider the set $\mathcal{B} \subset \mathcal{C}^\infty(X_n, Z)$ consisting of all $h \in \mathcal{C}^\infty(X_n, Z)$ such that $h(\theta, t) = \Phi^g_\theta(t)$, where Φ^g is the flow induced by $-\operatorname{grad}_g(f)$ for some Riemannian metric g that coincides with the pull-back of the Euclidean metric in convenient charts for all the critical points. Then $\mathcal{B} \cap \{\Phi^g \pitchfork \iota\}$ is C^1-open in $\mathcal{B}$ for all $n \in \mathbb{N}$. That this set is $\mathcal{C}^1$-dense too is shown similar to the $\mathcal{C}^\infty$-denseness of Riemannian metrices as before. If we take the intersection over the set of all Riemannian metrices such that $\Phi^g \pitchfork \iota$ for X_n, $n \in \mathbb{N}$, we obtain that the set of Riemannian metrices such that $\Phi^g \pitchfork \iota$ for $\bigcup_{p \in Cr(f)} (S^-_{f(p)-d}(p) \times \mathbb{R})$ is the intersection of countable many $\mathcal{C}^1$-open and dense sets and hence $\mathcal{C}^1$-residual.

$\square$

Remark:
Given a function that satisfies all conditions imposed on functions of a Morse pair we can apply this result as follows: Consider critical points on the boundary, find Morse charts for them and extend these charts with the help of the collar. Near critical points fix Riemannian metrices by pulling back the Euclidean metric defined on the domain of the charts. By the previous Lemma, the Riemannian metrics that coincide with these local metrics and that satisfy the Morse–Smale condition are $\mathcal{C}^\infty$-dense. Then extend such a Riemannian metric to the collar and fix local Riemannian metrics on Morse charts for critical points in the interior of M. We can extend these local metrices and the Riemannian metric on the collar to a Riemannian metric on M. We have seen that we can always find "many" perturbations of this extension such that the the perturbed Riemannian metrices satisfy the Morse–Smale condition and the perturbed Riemannian metrics differ from the one we started with only on small neighbourhoods of the unstable spheres.

Chapter 2

The Space of Trajectories

2.1 Properties of the negative gradient Flow

In the previous chapter we have introduced some notations concerning the dynamical system given by the negative gradient flow of a Morse(–Smale) pair (f, g). Now we continue this investigation and explain some of the properties of the negative gradient flow. We follow the presentation in [9].

Remark: situation at the boundary
We use the following result:

2.1. Lemma
Each manifold with boundary is a sub-manifold of a manifold without boundary of the same dimension.

Proof:
A slightly more general fact concerning smooth manifolds with corners is proved in [12].
With the help of Theorem (1.7.) we can outline a proof: Construct a vector field X on a collar $\phi : \partial M \times [0, \varepsilon[\to M$, $(x, t) \mapsto \phi(x, t)$ that coincides with $\frac{\partial}{\partial t}$ on $\partial M \times [0, \varepsilon/4]$ and vanishes on $\partial M \times [\varepsilon/2, \varepsilon]$. X can be extended to M by setting it 0 outside the collar. Then we consider the flow Φ generated by this vector field and observe that M is diffeomorphically mapped into the interior of M and so we obtain a manifold without boundary $M \setminus \partial M$ that contains $\Phi(M)$ as a sub-manifold and because $M \cong \Phi(M)$ the claim follows.

$\square$

We call a smooth manifold $\tilde{M}$ without boundary that contains the original manifold M with boundary ∂M as a sub-manifold a smooth extension of M. Given a Morse pair (f, g) on M we can extend f to a smooth function

$\tilde{f}$ of $\tilde{M}$ because M is a closed subset of $\tilde{M}$. Then there is a smooth extension of M and a smooth extension of f such that no critical points lie in $\tilde{M} \setminus M$. Indeed, let $\overline{M}$ be a smooth extension on M and $\overline{f}$ a smooth extension of f to $\overline{M}$. Observe that $\overline{f}$ is Morse on M. First assume that $x \in \partial M$ is no critical point. But then we have $d\overline{f}_x = df_x \neq 0$ and so there is an open neighbourhood U_x of x in $\overline{M}$ such that $d\overline{f} \neq 0$ on U_x and without loss of generality U_x can be assumed to be a chart neighbourhood of x. If $x \in \partial M$ is a critical point we can find a Morse chart of $\overline{f}$ in $\overline{M}$ and then x is the only critical point in the chart neighbourhood U_x. Now we can define $\tilde{M} := M \cup \left(\bigcup_{x\in\partial M} U_x\right) \subset \overline{M}$ and observe that this is an smooth extension of M. Furthermore we define $\tilde{f}$ to be the restriction of $\overline{f}$ to $\tilde{M}$. It is obvious that $\tilde{f}$ is a smooth extension of f such that all critical points of $\tilde{f}$ are contained in M.
Another consequence is that flow lines with points in M are totally contained in M: we have seen that flow lines with points on the boundary stay in the boundary because the gradient vector field is tangential to the boundary. Additionally, unstable manifolds with points on the boundary are trapped in the boundary and unstable manifolds containing points in in the interior of M are totally contained in the interior of M.
All in all we can use the existence of such a smooth extension $\tilde{M}$ to deal with the dynamical system associated to (f, g) as if M would be a manifold without boundary. So in the following we can assume without loss of generality that M is a compact manifold without boundary.

2.2. Lemma
Let $\gamma_x(\cdot)$ be a flow line of the negative gradient flow associated to a Morse pair (f, g). Then

$$\gamma_x(+\infty) := \lim_{t\to+\infty} \gamma_x(t)$$
$$\gamma_x(-\infty) := \lim_{t\to-\infty} \gamma_x(t)$$

exist and these limits are critical points of f.

Proof:
First we show that $grad_g(f)$ converges to 0 along every flow line $\gamma_x(t)$ for $t \to +\infty$. Define

$$f_{+\infty} := \lim_{t\to+\infty} f(\gamma_x(t)) > -\infty$$

because M is compact, so f has a minimum and is bounded from below by it. Furthermore, as f decreases along flow lines as proved in Lemma (1.11.), we have

$$f(x) = f(\gamma_x(0)) \geq f(\gamma_x(t)) \geq f_{+\infty}$$

and $\frac{d}{dt}f(\gamma_x(t)) = -||\gamma_x'(t)||^2$ — see the proof of Lemma (1.11.) — implies

$$\int_0^\infty ||\gamma_x'(t)||^2 dt = \int_0^\infty -\frac{d}{dt}f(\gamma_x(t))dt = f(x) - f_{+\infty} < \infty.$$

Since $\gamma_x'(t) = -grad_g(f(\gamma_x(t)))$ we obtain

$$\int_0^\infty ||grad_g(f(\gamma_x(t)))||^2 dt < \infty$$

and consequently

$$\lim_{t\to+\infty} grad_g(f(\gamma_x(t))) = 0$$

and similar the claim is proved for $t \to -\infty$.
Compactness of M implies that we can find $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}$ with $\lim_{n\to\infty} t_n = +\infty$ such that $\gamma_x(t_n)$ converges to some critical point p of f. It remains to show that $\lim_{t\to+\infty} x(t)$ exists and that

$$\lim_{t\to+\infty} \gamma_x(t) = p$$

holds. This follows directly from the local form of the stable manifolds provided that (f, g) is a Morse pair: The explicit behaviour of the negative gradient vector field around p, see (1.10), implies that there is a neighbourhood U of the critical point p such that any flow line in that neighbourhood containing p as an accumulation point of some sequence $\gamma_x(t_n)$, $t_n \to +\infty$, is contained in the stable manifold of p.
The same argument works if we consider the case with $t \to -\infty$.

$\square$

Remark: decomposition of the manifold by the stable/unstable manifolds
The last lemma ensures that the stable respectively unstable manifolds really decompose the manifold M, i.e. for every point x the limits

$$\lim_{t\to+\infty} \gamma_x(t) =: q \quad \text{respectively} \quad \lim_{t\to-\infty} \gamma_x(t) =: p$$

exist and consequently $x \in W^+(q)$ and $x \in W^-(p)$.
Furthermore it is clear that different unstable manifolds must not intersect because if they would, the points in their intersection would have two different critical points as the limit of the flow line for $t \to -\infty$. Obviously, the same argument holds for stable manifolds. Next we introduce the space of trajectories between two critical points:

2.3. Definition *Space of Trajectories*
Given a Morse pair (f, g) and let p and q be two of its critical points. For $p \neq q$ we set

$$\mathcal{M}(p, q) := W^-(p) \cap W^+(q)$$

$\mathcal{M}(p, q)$ *is the* space of trajectories from p to q.

Remark: uniform convergence
As M is a compact Riemannian manifold, we have a metric on M coming from the geodesic distance between points and this metric also induces the topology of the manifold. Because M is metrisable, it makes sense to speak of "uniform convergence" of mappings from a metrical space into M, in particular uniform convergence of a family of trajectories is defined.

Remark: convenient topologies on the space of trajectories
By arguments used before, every point lies in a space of trajectories between two critical points $\mathcal{M}(p, q)$. Clearly, when a point x lies in $\mathcal{M}(p, q)$, so does the whole trajectory $\gamma_x(\cdot)$ as both $W^-(p)$ and $W^+(q)$ are flow-invariant and hence their intersection is too.
The most obvious way to equip $\mathcal{M}(p, q)$ with a topology is to consider the subspace topology it inherits from M. Another possibility is to embed $\mathcal{M}(p, q)$ into $\mathcal{C}(\mathbb{R}, M)$ — the space of continuous functions from $\mathbb{R}$ to M — via

$$\Gamma : \mathcal{M}(p, q) \to \mathcal{C}(\mathbb{R}, M), \quad x \mapsto \gamma_x(\cdot).$$

This map is injective: given two maps in $\Gamma(\mathcal{C}(\mathbb{R}, M))$ we can apply

$$ev_0 : \Gamma(\mathcal{C}(\mathbb{R}, M)) \to \mathcal{M}(p, q), \quad \gamma_x(\cdot) \mapsto \gamma_x(0) = x$$

and so we see that $\gamma_x(\cdot) = \gamma_y(\cdot)$ implies $x = y$. We equip $\mathcal{C}(\mathbb{R}, M)$ with the topology of uniform convergence and denote this topological space by $\mathcal{C}^0(\mathbb{R}, M)$. $\mathcal{M}(p, q)$ obtains the subspace topology of this embedding. To show that these two topologies coincide, we prove:

2.4. Lemma
Let $(x_n)_{n\in\mathbb{N}}$ a convergent sequence of points in $\mathcal{M}(p, q)$ with limit $z \in \mathcal{M}(p, q)$. Then $\gamma_{x_n}(\cdot)$ converges uniformly to $\gamma_z(\cdot)$.

Proof:
As solutions of the flow equation depend smoothly on initial values, $\gamma_x(t)$ depends smoothly on x. Hence $\gamma_{x_n}(t)$ and $\gamma'_{x_n}(t)$ are convergent for every $t \in \mathbb{R}$.
Being solutions of a flow equation, they are locally uniformly convergent.

There we can perform the following calculation:

$$\begin{aligned}(\lim_{n\to\infty}\gamma_{x_n}(t))' &= \lim_{n\to\infty}\gamma'_{x_n}(t) = \lim_{n\to\infty} -grad_g(f)\circ\gamma_{x_n}(t)\\ &= -grad_g(f)\circ\lim_{n\to\infty}\gamma_{x_n}(t)\end{aligned}$$

and because of

$$\lim_{n\to\infty}\gamma_{x_n}(0) = \lim_{n\to\infty} x_n = z = \gamma_z(0)$$

the uniqueness of solutions of ODEs with given initial values implies

$$\lim_{n\to\infty}\gamma_{x_n}(t) = \gamma_z(t)$$

and this convergence is uniform on compact subsets of $\mathbb{R}$.
To get uniform convergence everywhere on $\mathbb{R}$, we must look at the situation near the critical points p and q. We look at the situation near p, the situation near q is totally analogous. As convergence of points implies the convergence of the associated points on the level-hyper surfaces, for instance on $S^-_{f(p)-d}(p)$ in a convenient chart (U,u). To show uniform convergence there, we use the explicit form of the trajectories near critical points, given by (1.10): Denote the series of convergent points on $S^-_{f(p)-d}(p)$ by $(x_n)_{n\in\mathbb{N}}$ and its limit by x. The trajectories through this points are given by

$$\gamma_{x_n}(t) = x_n e^{2t} \quad \text{and} \quad \gamma_x(t) = xe^{2t}$$

and we are only interested in the domain where $t<0$. There we get

$$||\gamma_{x_n}(t)-\gamma_x(t)|| = ||x_n - x||e^{2t} \leq ||x_n - x||$$

and hence uniform convergence.

□

2.5. Proposition
The two topologies on $\mathcal{M}(p,q)$ mentioned above are equivalent.

Proof:
First we consider $\Gamma : \mathcal{M}(p,q) \to \mathcal{C}^0(\mathbb{R},M)$, $x \mapsto \gamma_x(\cdot)$. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathcal{M}(p,q)$ with $x_n \to x \in \mathcal{M}(p,q)$. By Lemma (2.4.) $\gamma_{x_n}(\cdot)\to\gamma_x(\cdot)$ uniformly in $\mathcal{C}^0(\mathbb{R},M)$. Hence the map we considered is continuous.
In the other direction we can look at $ev_0 : \Gamma(\mathcal{M}(p,q)) \to \mathcal{M}(p,q)$, $\gamma_x(\cdot)\mapsto \gamma_x(0) = x$. Assume $\gamma_{x_n}(\cdot)\to\gamma_x(\cdot)$ in $\mathcal{C}^0(\mathbb{R},M)$. But uniform convergence implies pointwise convergent and hence $\gamma_{x_n}(0)\to\gamma_x(0)$ and consequently $x_n \to x$ in $\mathcal{M}(p,q)$, so ev_0 is continuous too.
Furthermore ev_0 and Γ are inverse to each other as maps between $\mathcal{M}(p,q)$ and $\Gamma(\mathcal{M}(p,q))$.

□

Next we prove two lemmas we will use to show that $\mathcal{M}(p, q)$ is a sub-manifold of M, provided that (f, g) is a Morse–Smale pair.

2.6. Lemma
Given $X_2 \subset X_1 \subset X$ with X a smooth manifold of dimension n, X_1 a sub-manifold of X of dimension n_1 and X_2 a sub-manifold of X_1 of dimension n_2.
Then X_2 is a sub-manifold of X of dimension n_2.

Proof:
For arbitrary $x \in X_2 \subset X_1$ there is a sub-manifold chart (U, u) for X_1 in X centred at x, i.e.

$$u : U \longrightarrow u(U) \subset \mathbb{R}^n, \text{ such that } u(X_1 \cap U) = u(U) \cap (\mathbb{R}^{n_1} \times 0)$$

and because u is a local diffeomorphism on U, $u(X_2 \cap U)$ is a sub-manifold of $u(X_1 \cap U)$ and hence there is a sub-manifold chart (V, v) of $u(X_2 \cap U)$ in $u(X_1 \cap U)$ centred at $u(x)$:

$$v : V \longrightarrow v(V) \subset \mathbb{R}^{n_1}, \text{ such that } v(u(X_2 \cap U) \cap V) = v(V) \cap (\mathbb{R}^{n_2} \times 0)$$

There is an open neighbourhood W of $u(x)$ in $u(U) \cap (\mathbb{R}^{n_1} \times 0)$ and an $\varepsilon > 0$ such that $W \times]-\varepsilon, \varepsilon[\subset u(U)$ with $W \subset v^{-1}(V)$ and

$$\psi : W \longrightarrow \psi(W), \quad \psi := v \times id$$

and ψ is a diffeomorphism on W. Consider

$$\psi \circ u : u^{-1}(W \times]-\varepsilon, \varepsilon[) \longrightarrow \psi(W \times]-\varepsilon, \varepsilon[)$$

and this map satisfies

$$\begin{aligned} \psi \circ u(X_2 \cap u^{-1}(W \times]-\varepsilon, \varepsilon[)) &= \psi \circ u((W \times]-\varepsilon, \varepsilon[) \cap (\mathbb{R}^{n_2} \times 0)) \\ &= \psi(W \times]-\varepsilon, \varepsilon[) \cap (\mathbb{R}^{n_2} \times 0) \end{aligned}$$

and hence $(u^{-1}(W \times]-\varepsilon, \varepsilon[), \psi \circ u)$ is a sub-manifold chart of X_2 in X centred at x.

□

2.7. Lemma
Given $i : M \longrightarrow N$ an embedding of a smooth manifold into another smooth manifold. Let $f : P \longrightarrow N$ be a smooth map from a smooth manifold P to N that is transversal to i.
Then $f^{-1}(i(M))$ is a sub-manifold of P.

Proof:
A poof of this Lemma can be found in [13], for instance.
We outline the idea of a proof: $i(M)$ is a sub-manifold of N, hence, locally it can be written as the pre-image of 0 of a smooth vector-valued function g where 0 can be assumed to be a regular value of 0. But then, locally we have $f^{-1}(i(M)) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$. Because of our the transversality-assumption we can apply the implicit function theorem and hence $f^{-1}(i(M))$ describes a sub-manifold of P.

□

2.8. Corollary
Given two sub-manifolds X_1, X_2 of a smooth manifold X that intersect transversal. Then their intersection $X_1 \cap X_2$ is a sub-manifold of X of dimension $dimX_1 + dimX_2 - dimX$.

Proof:
Consider the maps $i_1 : X_1 \hookrightarrow X$ and $i_2 : X_2 \hookrightarrow X$ and these maps are transversal because X_1 and X_2 are transversal. Now we can apply Lemma (2.6.) and so $X_1 \cap X_2 = i_1^{-1}(i_2(X))$ is a sub-manifold of X_1 and by Lemma (2.7.) it is also a sub-manifold of X.

□

2.9. Proposition
Let $p \neq q$ be two critical points of a Morse–Smale pair (f, g). If $\mathcal{M}(p, q)$ is non-empty, it is a manifold of dimension $\mathrm{ind}(p) - \mathrm{ind}(q)$. *$\mathcal{M}(p, q)$ does not contain any critical points.*

Proof:
This follows from Corollary (2.8.) because the Morse–Smale condition assures the transversality of $W^-(p)$ and $W^+(q)$.
Because $p \neq q$ there cannot be critical points in $\mathcal{M}(p, q)$, otherwise assume that x is a critical point in $\mathcal{M}(p, q)$. But then $p = \lim_{t \longrightarrow -\infty} \gamma_x(t) = x = \lim_{t \longrightarrow +\infty} \gamma_x(t) = q$ and hence we would have $p = q$.

□

Remark: canonical parametrisation
Often, a parametrisation of the flow lines different from the one obtained by solving the flow equation (1.8), is more useful. A canonical way to parametrise the flow lines can be constructed with the help of the Morse function f. Usually we will denote the flow lines that come along with the parametrisation from solving the flow equation by γ and the ones that are canonically

parametrised by σ.
So, let $\sigma(t)$ be a reparametrisation of $\gamma_x(t)$ with $x \in \mathcal{M}(p,q)$ and set

$$f(p) =: b > a := f(q)$$

the inequality follows from the fact that f decreases along the flow lines. We require that

$$f(\sigma(t)) = a + b - t \quad \text{for} \quad a < t < b.$$

To prove existence of such a parametrisation, we show that $\sigma(t)$ satisfies a flow-equation at non-critical points by performing the following calculations:

$$\begin{aligned} -1 = \frac{d}{dt}(a+b-t) &= \frac{d}{dt}(f(\sigma(t))) = df_{\sigma(t)}(\sigma'(t)) \\ &= g(grad_g(f)(\sigma(t)), \sigma'(t)) \end{aligned}$$

and we know that $\sigma'(t)$ points in the direction of $\gamma'(t)$ at the same points, as the former is assumed to be a reparametrisation of the latter, hence $\sigma'(t) = \lambda grad_g(f)$ at every point (λ is a smooth function $\mathcal{C}^\infty(M;\mathbb{R})$). If we plug this into the previous equality we obtain

$$\lambda = -\frac{1}{||grad_g(f)||^2}$$

and consequently

$$\sigma'(t) = -\frac{grad_g(f)}{||grad_g(f)||^2} \circ \sigma(t) \tag{2.1}$$

and so existence is obvious. Observe that Lemma (1.11.) remains true and that the Morse function f still decreases along $\sigma(\cdot)$.

Remark: canonical parametrisation near critical points
Given a point z_0 in the unstable manifold near the critical point p, see (1.11) and (1.12) for the explicit description of this situation. The trajectory starting at z_0 is given by (1.10). We now want to calculate the reparametrisation of γ_{x_0} that is solution of

$$f(\gamma_{z_0}(\phi(t))) = -t \quad \text{for} \quad t > 0$$

hence, a shifted version of the canonical parametrisation. f has the form $-||y||^2 - ||x||^2$ in the convenient chart and so one obtains

$$f(\gamma_{z_0}(\phi(t))) = f(z_0 e^{2\phi(t)}) = -||z_0||^2 e^{4\phi(t)} = -t \quad \Longrightarrow$$

$$\phi(t) = \frac{1}{4}\ln(\frac{t}{||z_0||^2}) \quad \Longrightarrow$$

$$\sigma(t) = \gamma_{z_0}(\phi(t)) = \frac{z_0}{||z_0||}\sqrt{t}. \tag{2.2}$$

Observe that one can extend $\sigma(\cdot)$ continuously from $]a, b[$ with $a := f(q)$ and $b := f(p)$ to $[a, b]$ and that this unique continuation is not smooth. Unlike in the usual parametrisation, the associated critical points of the flow line are arrived at in finite time.

Remark: the space of parametrised trajectories
So far we have considered spaces of trajectories where we have distinguished between points lying on the same trajectory, or - what is equivalent - we have distinguished between trajectories even if the are just reparametrisation of one another. In the next section we investigate the space of unparametrised trajectories.

2.2 The Space of unparametrised Trajectories

Remark: the $\mathbb{R}$-action on $\mathcal{M}(p, q)$
There is a natural $\mathbb{R}$-action on the space $\mathcal{M}(p, q)$ for arbitrary critical points $p \neq q$, given by:

$$\mathbb{R} \times \mathcal{M}(p, q) \longrightarrow \mathcal{M}(p, q), \quad (t, x) \mapsto \gamma_x(t). \tag{2.3}$$

If one identifies $\mathcal{M}(p, q)$ with the subspace $\Gamma(\mathcal{M}(p, q))$ of $\mathcal{C}(\mathbb{R}, M)$ this action takes the form $(t, \gamma_x(\cdot)) \mapsto \gamma_x(\cdot + t)$, hence a reparametrisation via a shift in the argument. Clearly this is an action and we can consider the quotient space:

2.10. Definition *the Space of unparametrised Trajectories*
We denote the space of orbits of this action by

$$\mathcal{T}(p, q) := \mathcal{M}(p, q)/\mathbb{R}$$

and call it the space of all unparametrised trajectories *from p to q.*

Next we intend to equip $\mathcal{T}(p, q)$ with a topology. Beforehand we prove:

2.11. Proposition
Let (f, g) be a Morse–Smale pair and assume p and q are critical points of f. For arbitrary $f(q) < c < f(p)$, $\mathcal{M}(p, q) \cap f^{-1}(c)$ is a sub-manifold of M. If d is another value with $f(q) < d < f(p)$ then $\mathcal{M}(p, q) \cap f^{-1}(d)$ is diffeomorphic to $\mathcal{M}(p, q) \cap f^{-1}(c)$.

Proof:
Because there are only finitely many critical points, there are only finitely many critical values and hence we can find a regular value e with $f(q) < e < f(p)$. By regularity of e, $f^{-1}(e)$ is a sub-manifold of M and it intersects $\mathcal{M}(p,q)$ transversally, so $\mathcal{M}(p,q) \cap f^{-1}(e)$ is a sub-manifold of M by Corollary (2.8.).
Now let d be an arbitrary value $a := f(q) < d < f(q) =: b$. We show that $\mathcal{M}(p,q) \cap f^{-1}(d)$ is diffeomorphic to $\mathcal{M}(p,q) \cap f^{-1}(e)$: Define a map $\tau_{e,d} : \mathcal{M}(p,q) \cap f^{-1}(e) \longrightarrow \mathcal{M}(p,q) \cap f^{-1}(d)$ by $x \mapsto \Phi_{e-d}(x)$, where Φ denotes the flow of the vector field $\frac{-grad_g(f)}{||grad_g(f)||^2}$. This map is smooth with smooth inverse $x \mapsto \Phi_{d-e}(x)$, so $\tau_{e,d}$ is a diffeomorphism. Consequently $\mathcal{M}(p,q) \cap f^{-1}(d)$ is a sub-manifold for arbitrary $f(q) < d < f(p)$ and all such sub-manifolds are diffeomorphic as they are all diffeomorphic to $\mathcal{M}(p,q) \cap f^{-1}(e)$.

$\square$

Remark: convenient topologies on $\mathcal{T}(p,q)$
There are different ways to topologise $\mathcal{T}(p,q)$ but they will be seen to be equivalent. First, we observe that the natural $\mathbb{R}$-action on $\mathcal{M}(p,q)$ is continuous. Consequently, we can equip $\mathcal{T}(p,q)$ with the quotient topology induced by

$$P : \mathcal{M}(p,q) \longrightarrow \mathcal{T}(p,q) = \mathcal{M}(p,q)/\mathbb{R}, \quad x \mapsto [x]$$

where $[x]$ denotes the equivalence class of x. Hence, open sets in $\mathcal{T}(p,q)$ are those sets that have an open pre-image in $\mathcal{M}(p,q)$ under P. An immediate consequence is that maps from $\mathcal{T}(p,q)$ into any other topological space are continuous if and only if the map obtained by composition with P is a continuous map from $\mathcal{M}(p,q)$ into this space.
Next, we can use the canonical parametrisation of trajectories to obtain a topology on $\mathcal{T}(p,q)$. If $\gamma_x(\cdot)$ and $\gamma_y(\cdot)$ are two trajectories of the negative gradient flow such that their images in M coincide — hence they are just reparametrised versions of one another — the corresponding canonical parametrised trajectories coincide. Hence, $\sigma(\cdot)$ does not depend on x but only on $[x]$. For points in different equivalence classes, the corresponding canonically parametrised trajectories are different, and so there is a bijection between all canonically parametrised trajectories from p to q and $\mathcal{T}(p,q)$. Let $a := f(q)$ and $b := f(p)$ and by construction canonically parametrised trajectories are elements of $\mathcal{C}([a,b],M)$. So, by equipping $\mathcal{C}([a,b],M)$ with a topology, $\mathcal{T}(p,q)$ inherits the subspace-topology. We have observed that M has a metric and so we can equip $\mathcal{C}([a,b],M)$ with the topology coming from uniform convergence and this topological space will be denoted by $\mathcal{C}^0([a,b],M)$.

The third way to topologise $\mathcal{T}(p,q)$ uses the bijection between $\mathcal{T}(p,q)$ and $\mathcal{M}(p,q) \cap f^{-1}(c)$, $a < c < b$. Every trajectory from p to q intersects $\mathcal{M}(p,q) \cap f^{-1}(c)$ exactly once as was proved in Lemma (1.11.). So we use the topology on $\mathcal{M}(p,q) \cap f^{-1}(c)$ to equip $\mathcal{T}(p,q)$ with a topology. Observe that this topology does not depend on the particular value $a < c < b$ one chooses as all level-hyper surfaces in $\mathcal{M}(p,q)$ are diffeomorphic, see Proposition (2.11.). So, one can even use the bijection to obtain a differentiable structure on $\mathcal{T}(p,q)$.
Next we will proof that:

2.12. Proposition
All the topologies on $\mathcal{T}(p,q)$ described before are equivalent.

Proof:
That the topologies coming from $\mathcal{C}^0([a,b],M)$ and from $\mathcal{M}(p,q) \cap f^{-1}(c)$ are equivalent is similar to the proof of Lemma (2.4.): If $\sigma_n(\cdot)$ converge uniformly to $\sigma(\cdot)$, they do so pointwise. Hence $\sigma_n(a+b-c)$ converges to $\sigma(a+b-c)$ and by definition $\sigma_n(a+b-c)$ and $\sigma(a+b-c)$ are lying in $\mathcal{M}(p,q) \cap f^{-1}(c)$. In the other direction, if points converge in $\mathcal{M}(p,q) \cap f^{-1}(c)$, the canonical parametrised trajectories through them converge pointwise. Like in the proof of Lemma (2.4.) outside convenient charts of the critical points, the canonical parametrised trajectories satisfy a flow equation and uniform convergence follows from this. Near critical points we use the explicit behaviour of the canonical parametrised trajectories — see (2.2) — to demonstrate uniform convergence.
To prove the equivalence of the topology coming from $\mathcal{M}(p,q) \cap f^{-1}(c)$ and the one induced by $P : \mathcal{M}(p,q) \longrightarrow \mathcal{M}(p,q)/\mathbb{R} =: \mathcal{T}(p,q)$ we construct two continuous maps between these spaces that are inverse to each other:
First consider

$$\mathcal{M}(p,q) \cap f^{-1}(c) \overset{\iota}{\hookrightarrow} \mathcal{M}(p,q) \xrightarrow{P} \mathcal{T}(p,q).$$

This is a composition of continuous maps and hence continuous. On the other hand one considers

$$\mathcal{M}(p,q) \longrightarrow \mathcal{M}(p,q) \cap f^{-1}(c), \quad x \mapsto \Phi_{f(x)-c}(x)$$

with Φ the flow induced by $\frac{-grad_g(f)}{||grad_g(f)||^2}$, hence it is smooth and in particular continuous. Furthermore this map is invariant under the $\mathbb{R}$-action. So it induces a continuous map

$$\Lambda : \mathcal{T}(p,q) \to \mathcal{M}(p,q) \cap f^{-1}(c),$$

that is inverse to $P \circ \iota : \mathcal{M}(p,q) \cap f^{-1}(c) \to \mathcal{T}(p,q)$.

□

Remark: more about the $\mathbb{R}$-action on $\mathcal{M}(p,q)$
One way to obtain a differentiable structure on $\mathcal{T}(p,q)$ works via the bijection between $\mathcal{T}(p,q)$ and $\mathcal{M}(p,q) \cap f^{-1}(c)$ for $f(q) < c < f(p)$.
Another way to equip $\mathcal{T}(p,q)$ with a differentiable structure is by analysing the properties of the $\mathbb{R}$-action on $\mathcal{M}(p,q)$ and apply general results about the space of orbits of actions of Lie-groups on manifolds. Observe that no problems with the boundary can arise, as we can apply the procedure explained before to make the situation take place in the setting of manifolds without boundary, and by considering only the part that is of interest (which is invariant under the $\mathbb{R}$-action), so all assumptions can be taken for granted. The action is smooth, as $\mathbb{R} \times \mathcal{M}(p,q) \longrightarrow \mathcal{M}(p,q), \quad (t,x) \mapsto \gamma_x(t)$ is smooth. Moreover, this action is free, as no critical points are contained in $\mathcal{M}(p,q)$. Another important property of a smooth action on topological spaces is defined next:

2.13. Definition *proper actions*
A smooth action $G \times M \longrightarrow M$ is called proper *if the map*

$$G \times M \longrightarrow M \times M, \quad (g,x) \mapsto (g \cdot x, x)$$

is proper.

2.14. Lemma
A smooth action $G \times M \longrightarrow M$ is proper if and only if for all $(x_n)_{n\in\mathbb{N}} \subset M$, $(g_n)_{n\in\mathbb{N}} \subset G$ for which $x_n \to x$ and $g_n \cdot x_n \to y$, there is a convergent subsequent of $(g_n)_{n\in\mathbb{N}}$.

Proof:
The proof is straight-forward and can be found in the lecture notes [14].

$\square$

2.15. Proposition
Given a smooth proper free action of a Lie-Group on a smooth manifold the space of orbits admits a unique smooth structure such that the projection $P : M \longrightarrow M/G$ is a surjective submersion.

Remark: on the proof
In [14] it is shown that in the presence of a single orbit type one can construct charts from normal slices for proper actions. If the action under consideration is free, there is just one orbit type given by e where $e \in G$ is the neutral element.

2.16. Proposition
The $\mathbb{R}$-action on $\mathcal{M}(p,q)$ given by

$$\mathbb{R} \times \mathcal{M}(p,q) \longrightarrow \mathcal{M}(p,q), \quad (t,x) \mapsto \gamma_x(t)$$

is proper.

Proof:
We use Lemma (2.14) and assume $(x_n)_{n\in\mathbb{N}} \subset \mathcal{M}(p,q)$, $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ for which $x_n \to x \in \mathcal{M}(p,q)$ and $\gamma_{x_n}(t_n) \to y \in \mathcal{M}(p,q)$.
Assume that the sequence $(t_n)_{n\in\mathbb{N}}$ is not contained in a compact subset of $\mathbb{R}$. Then it must be unbounded and after selecting a subsequence we can assume without loss of generality that $t_n \to \infty$. By Lemma (2.6) we know that $\gamma_{x_n}(\cdot)$ converges uniformly to $\gamma_x(\cdot)$ and hence for arbitrary ε

$$d(\gamma_{x_n}(t), \gamma_x(t)) < \varepsilon$$

holds for all t and n sufficiently large. But $\gamma_x(t_n) \to q$, and $\gamma_{x_n}(t_n) \to y$, hence $y = q$ This is a contradiction to our assumption that y is not a critical point.

$\square$

2.17. Corollary
The orbit space $\mathcal{T}(p,q)$ admits a unique smooth structure such that the projection $P : \mathcal{M}(p,q) \to \mathcal{T}(p,q)$ is a surjective submersion.

Proof:
The action has been proved to be proper and free, so one can apply Proposition (2.16.) to the $\mathbb{R}$-action on $\mathcal{M}(p,q)$.

$\square$

Remark: the differentiable structure on $\mathcal{T}(p,q)$
One can easily check that the two ways to equip $\mathcal{T}(p,q)$ with a differentiable structure are equivalent. On one hand, we consider the map given by

$$\mathcal{M}(p,q) \cap f^{-1}(c) \overset{\iota}{\hookrightarrow} \mathcal{M}(p,q) \xrightarrow{P} \mathcal{T}(p,q)$$

again and as ι and P are smooth, this map is smooth too. On the other hand consider the map

$$\Lambda : \mathcal{T}(p,q) \longrightarrow \mathcal{C}^0([a,b], M) \longrightarrow \mathcal{M}(p,q) \cap f^{-1}(c)$$

can be composed with P and we obtain

$$\mathcal{M}(p,q) \xrightarrow{P} \mathcal{T}(p,q) \xrightarrow{\Lambda} \mathcal{M}(p,q) \cap f^{-1}(c)$$

and in the next lemma we show that the fact that P is a surjective submersion implies that $\Lambda \circ P$ is smooth iff Λ is smooth and smoothness of $\Lambda \circ P$ follows from the fact that this map is equal to $x \mapsto \Phi_{f(x)-c}(x)$, where Φ is again the flow generated by $\frac{-grad_g(f)}{||grad_g(f)||^2}$. Moreover Λ and $P \circ \iota$ are inverse to each other and so they are even diffeomorphisms.

2.18. Lemma
Given a surjective submersion $p : M \longrightarrow M'$. Then a map $\phi : M' \longrightarrow N$ is smooth if and only if $\phi \circ p : M \longrightarrow N$ is smooth.

Proof:
That the composition of smooth maps is smooth is clear, so we only need to prove the other direction of this equivalence. On the other hand one obtains smooth local sections $\sigma : U \to M$, U an open subset in M' with the help of the implicit function theorem. Hence $f = f \circ (p \circ \sigma) = (f \circ p) \circ \sigma$ in an open neighbourhood of an arbitrary point of M' and consequently smoothness of $f \circ p$ implies smoothness of f.

□

Remark: dimension of $\mathcal{T}(p, q)$
In the last section we have deduced that

$$dim(\mathcal{M}(p, q)) = \mathrm{ind}(p) - \mathrm{ind}(q)$$

and consequently we obtain

$$dim(\mathcal{T}(p, q)) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1$$

Consider the special case where $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$. Then the space of unparametrised trajectories from p to q is zero-dimensional (assumed that it is non-empty). Later we will prove that in this case $\mathcal{T}(p, q)$ is compact and hence it is a finite collection of points, so there are only a finite number of trajectories between these points. Next we will introduce the space of (unparametrised) broken trajectories:

2.19. Definition *the Space of (unparametrised) broken Trajectories*
Given a Morse–Smale pair (f, g) on M, let p and q be two critical points of f. Define the space of k-times broken trajectories *from p to q as*

$$\hat{\mathcal{T}}_k(p, q) := \bigsqcup_{p=:y_0, y_1, \ldots, y_k, y_{k+1}:=q} \mathcal{T}(y_0, y_1) \times \mathcal{T}(y_1, y_2) \times \ldots \times \mathcal{T}(y_k, y_{k+1}).$$

The space of broken trajectories *from p to q is*

$$\hat{\mathcal{T}}(p, q) := \bigsqcup_{k \geq 0} \hat{\mathcal{T}}_k(p, q).$$

Remark: a topology on the space of broken trajectories
The way of interpreting $\mathcal{T}(p,q)$ as a subspace of $\mathcal{C}^0([a,b],M)$ with $f(p) =: b > a := f(q)$ can be applied to $\hat{\mathcal{T}}(p,q)$ as well: An element of $\hat{\mathcal{T}}(p,q)$ is a collection of canonically parametrised trajectories such that successive trajectories fit together at the critical point where the one trajectory ends and the new one starts. Because the canonical parametrised trajectories are parametrised with the function f the can be fit together at the critical values and one obtains a continuous function from $\tau : [a,b] \to M$ that satisfies:
1.) $\tau(a) = p$, $\tau(b) = q$.
2.) $f(\tau(t)) = a + b - t$
3.) for $t \in]a,b[$ such that $\tau(t) \notin Cr(f)$ the derivative $\tau'(t)$ exists and

$$\tau'(t) = -\frac{grad_g(f)}{||grad_g(f)||^2} \circ \tau(t)$$

holds.
One the other hand, each $\tau(\cdot) \in \mathcal{C}([a,b],M)$ that satisfies 1.), 2.) and 3.) can be interpreted as a broken trajectories from p to q: By 1.) $\tau(\cdot)$ starts at p and ends at q. By 2.) only finitely many critical points lie in the image of $\tau(\cdot)$ and we order these decreasing with f, $p =: y_0, y_1, \ldots, y_k, y_{k+1} := q$. On $]f(y_i), f(y_{i-1})[$, $i = 1, \ldots, k+1$, $\tau(\cdot)$ satisfies the flow equation and hence is an unbroken canonical parametrised trajectory from y_{i-1} to y_i. By continuity of $\tau(\cdot)$ the unbroken canonical parametrised trajectory from y_{i-1} to y_i and the one from y_i to y_{i+1} fit together at y_i and by 2.) $\tau(t) = y_i$ is only satisfied for $t = a + b - f(y_i)$.
So the broken trajectories from p to q can be identified with the subspace of $\mathcal{C}([a,b],M)$ described by the conditions 1.), 2.) and 3.). Again we consider the topology of uniform convergence on $\mathcal{C}([a,b],M)$ and equip $\hat{\mathcal{T}}(p,q)$ with the subspace topology it inherits from $\mathcal{C}^0([a,b],M)$. Observe that the subspace-topology on $\hat{\mathcal{T}}_k(p,q) \subset \hat{\mathcal{T}}(p,q)$ coincides with the topology that $\hat{\mathcal{T}}_k(p,q)$ inherits from using the definition as a product of (unbroken) canonical parametrised trajectories: Uniform convergence of the different unbroken trajectories implies uniform convergence of the whole broken trajectory and vice versa. In particular the map

$$\mathcal{T}(p,q) \hookrightarrow \hat{\mathcal{T}}(p,q)$$

induces a homeomorphism

$$\mathcal{T}(p,q) \stackrel{\cong}{\to} \hat{\mathcal{T}}_0(p,q).$$

Hence, the unbroken trajectories build a subspace of the broken ones and we will prove that $\hat{\mathcal{T}}(p,q)$ equipped with the subspace-topology coming from $\mathcal{C}^0([a,b],M)$ is compact. Hence, the topological closure of $\mathcal{T}(p,q)$ lies in $\hat{\mathcal{T}}(p,q)$ and in the next section it will follow that it is exactly the closure.

We state the Theorem of Arzela-Ascoli as we will use it to prove the compactness of $\hat{\mathcal{T}}(p,q)$:

2.20 Theorem *Arzela-Ascoli*
Let S be a compact metrical topological space and T a metrical space. A subset A of $\mathcal{C}(S,T)$ is compact with respect to uniform convergence if and only if it is bounded, closed and equicontinuous, i.e. for every $s_0 \in S$ and every $\varepsilon > 0$ there is an $\delta > 0$ such that $d(f(s), f(s_0)) < \varepsilon$ for all $s \in S$ such that $d(s, s_0) < \delta$ for every $f \in A$.

Proof:
The proof of this theorem can be found in most text books about functional analysis.

□

2.21. Theorem *Compactness of the Space of broken Trajectories*
Given a Morse–Smale pair on M and two critical points p and q. Then $\hat{\mathcal{T}}(p,q)$ is compact.

Proof:
We verify the three conditions in order to apply the Theorem of Arzela-Ascoli to $\hat{\mathcal{T}}(p,q)$:

claim a) $\hat{\mathcal{T}}(p,q)$ is bounded
M is compact and so the metric on M being induced by the Riemannian metric on M is bounded by the Theorem of Hopf–Rinov. Hence $\mathcal{C}^0([a,b],M)$ is bounded and so is $\hat{\mathcal{T}}(p,q)$.

claim b) $\hat{\mathcal{T}}(p,q)$ is closed
Given a sequence of broken canonical parametrised trajectories $(\sigma_n(\cdot))_{n\in\mathbb{N}} \subset \hat{\mathcal{T}}(p,q)$ that converges in $\mathcal{C}^0([a,b],M)$. We denote the limit by $\sigma(\cdot)$ and show that it must lie in $\hat{\mathcal{T}}(p,q)$. It is clear that $\sigma(\cdot)$ starts at p and ends at q because uniform convergence implies pointwise convergence and hence

$$\sigma(a) = \lim_{n\to\infty} \sigma_n(a) = \lim_{n\to\infty} p = p$$

and similar one shows that $\sigma(b) = q$. Furthermore $\sigma(\cdot)$ is continuous as the limit of continuous maps under uniform convergence. As $f(\sigma_n(t)) = a+b-t$ holds for all $n \in \mathbb{N}$ one obtains

$$f(\sigma(t)) = f(\lim_{n\to\infty} \sigma_n(t)) = \lim_{n\to\infty} f(\sigma_n(t)) = a + b - t$$

and so $\sigma(\cdot)$ is again parametrised by f and there can be only a finite number of critical points on $\sigma(\cdot)$. If $\sigma(s)$ is not a critical point of f then we can assume without loss of generality that all $\sigma_n(s)$ are regular points too. Near $\sigma(s)$ one has $\sigma_n(\cdot) \to \sigma(\cdot)$ uniformly. Furthermore

$$\begin{aligned}
\lim_{n\to\infty} \sigma_n'(s) &= \lim_{n\to\infty} \frac{-grad_g(f)}{||grad_g(f)||^2} \circ \sigma_n(s) \\
&= \frac{-grad_g(f)}{||grad_g(f)||^2} \circ \lim_{n\to\infty} \sigma_n(s) \\
&= \frac{-grad_g(f)}{||grad_g(f)||^2} \circ \sigma(s)
\end{aligned}$$

but as $\sigma_n(\cdot) \to \sigma(\cdot)$ local uniformly and $\sigma_n'(s)$ converges for s in an neighbourhood of t one obtains that $\sigma'(t)$ exists and that it is equal to

$$\lim_{n\to\infty} \sigma_n'(t) = -grad_g(f)/||grad_g(f)||^2 \circ \sigma(t).$$

Hence

$$\sigma'(t) = \frac{-grad_g(f)}{||grad_g(f)||^2} \circ \sigma(t)$$

for all t such that $\sigma(t)$ is not a critical point of f. So the claim follows.

claim c) $\hat{\mathcal{T}}(p,q)$ is equicontinuous:
First we prove the statement for points that are not critical. There we can make the following estimate:

$$d(\sigma(s), \sigma(s_0)) \leq \int_{s_0}^{s} ||\sigma'(t)||dt \leq \int_{s_0}^{s} ||\frac{grad_g(f)}{||grad_g(f)||^2} \circ \sigma(t)||dt \leq C \cdot |s - s_0|$$

where C is the maximum of $\frac{1}{||grad_g(f)||}$ in a closed neighbourhood of $\sigma(s_0)$ not containing a critical point.
At critical points we make use of the explicit form of the trajectories we have computed in (2.2): all broken trajectories that arrive at the critical point and go through it have the following representation in a convenient chart:

$$\sigma(t) = \frac{x_-}{||x_-||}\sqrt{t} \quad \text{for} \quad t \in [0, d]$$

where $x_- \in S^-_{f(p)-d}(p)$ and similar for elements on $S^+_{f(p)+d}(p)$.
Furthermore we must look at trajectories "near" broken trajectories in a neighbourhood of a critical point. We use a convenient chart and compute the canonically parametrised (possibly broken) trajectories to be given by

$$\sigma(t) = \left(\frac{x}{||x||} \sqrt{\frac{\sqrt{t^2 + 4||x||^2||y||^2} - t^2}{2}}, \frac{y}{||y||} \sqrt{\frac{\sqrt{t^2 + 4||x||^2||y||^2} + t^2}{2}} \right)$$

where the trajectory passes through $(x, y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. Broken trajectories are given by

$$\sigma(t) = \begin{cases} (\frac{x}{||x||}\sqrt{-t}, 0) & \text{for } t < 0 \\ (0, \frac{y}{||y||}\sqrt{t}) & \text{for } t > 0 \end{cases}$$

and from this equicontinuous continuity follows directly.

□

2.3 Compactification of the Space of unparametrised Trajectories

We follow the treatment presented in [3], [4] and [5]. Beforehand we need to introduce manifolds with corners. The main source is [12] where much more material about this kind of manifolds is presented.

2.22. Definition *the positive quadrant, corners*
The positive quadrant of dimension n*,* $Q_n \subset \mathbb{R}^n$*, is the subspace*

$$Q_n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0\}.$$

If $x = (x_1, \dots, x_n) \in Q_n$ *satisfies* $x_i = 0$ *for exactly* m *i's with* $0 \leq i \leq n$ *then we call* x *a* corner of index m*.*

2.23. Definition *smooth manifolds with corners and basic terminology*
A smooth manifold with corners X *of dimension* n *is a topological (second countable and Hausdorff) space together with an atlas of charts* $\phi_\alpha : U_\alpha \longrightarrow Q_n$*,* $\alpha \in A$ *where the family* $(U_\alpha)_{\alpha \in A}$ *forms an open cover of* X*,* Q_n *denotes the positive quadrant in* $\mathbb{R}^n$ *and* ϕ_α *is a homeomorphism onto its image for every* $\alpha \in A$ *(using the initial topology given by* $\iota : Q_n \hookrightarrow \mathbb{R}^n$*). Furthermore the transition functions* $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$ *are smooth maps, i.e. they can be extended to smooth maps between open neighbourhoods of* $\phi_\alpha(U_\alpha \cap U_\beta)$ *and* $\phi_\beta(U_\alpha \cap U_\beta)$ *(in the standard topology of* $\mathbb{R}^n$*).*
A point $x \in X$ *is called a* corner of index k *if there is chart for* x *such that the image of* x *is a corner of index* k*. As diffeomorphisms map corners of given index into corners of the same index, corners and their indices are well-defined.*
The set of all corners of a fixed index k *is called the* k-boundary*,* $\partial_k X$*, and the* boundary *is the union of all k-boundaries for* $0 \leq k \leq n$*. By* inner points *we denote points not lying on the boundary of* X*.*

Remark: the definition of manifolds with corners
We observe that boundaries come along with a natural grading, given by the boundaries of a fixed index. Furthermore, every k-boundary lies in the topological closure of a l-boundary for $0 \leq l \leq k$, what can be easily seen from

the definition. We define $\partial_{\geq k}X := \bigcup_{j\geq k}\partial_j X$ and we have $\partial X = \partial_{\geq 1}X$. The boundary of a manifold is closed and inherits the structure of a topological manifold. The pair $(X, \partial X)$ is a topological manifold with boundary.
We will make use of the next lemma for recognising manifolds with corners later on:

2.24. Lemma
Let $\mathcal{P}$ be a manifold with corners, and $\mathcal{S}$ respectively $\mathcal{O}$ smooth manifolds. Suppose that $p : \mathcal{P} \longrightarrow \mathcal{O}$ is a smooth mapping and $s : \mathcal{S} \longrightarrow \mathcal{O}$ and embedding that are transversal *to each other. (This means that p restricted to every k-boundary of $\mathcal{P}$ is transversal to s.)*
Then $p^{-1}(s(\mathcal{S}))$ is a smooth manifold with corners and it admits a unique smooth structure such that the inclusion $i : p^{-1}(s(\mathcal{S})) \hookrightarrow \mathcal{P}$ is an immersion.

Proof:
The proof that $p^{-1}(s(\mathcal{S}))$ is a smooth submanifold of $\mathcal{P}$ is similar to the one of Lemma (2.7.). Locally, $p^{-1}(s(\mathcal{S}))$ can be written as $g^{-}(0)$ where g is a smooth vector-valued function with regular value 0. But then $(p \circ g)^{-1}(0)$ is a submanifold of $\mathcal{P}$ transversal to every k-boundary.
That $p^{-1}(s(\mathcal{S}))$ admits an unique smooth structure such that the inclusion is an immersion follows from the universal property that for every smooth manifold with corners Z a mapping $f : Z \to p^{-1}(s(\mathcal{S}))$ is smooth if and only if $i \circ f : Z \to \mathcal{P}$ is smooth. A detailed proof of this statement can be found in [13], for instance.

□

Remark: some useful notations
Given a Morse–Smale pair (f, g), there is also only a finite number of critical values of f, as there are only finitely many critical points. We order them by increasing value:

$$c_1 < c_2 < \ldots < c_s.$$

Obviously, we can find an $\varepsilon > 0$ such that $c_i + \varepsilon < c_{i+1} - \varepsilon$ for all $i = 1, \ldots, s-1$. We set

$$Cr(i) := Cr(f) \cap f^{-1}(c_i), \quad c_i^+ := c_i + \varepsilon \quad \text{and} \quad c_i^- := c_i - \varepsilon$$

and

$$M_i := f^{-1}(c_i), \quad M_i^+ := f^{-1}(c_i^+), \quad M_i^- := f^{-1}(c_i^-).$$

Observe that M_i^+ and M_i^- are smooth sub-manifolds of co-dimension 1 because c_i^+ and c_i^+ are regular values of f. M_i is not a manifold, but

$$\dot{M}_i := M_i \backslash Cr(i)$$

is a smooth sub-manifold of co-dimension 1: we have just removed the critical points and away from these there are always open neighbourhoods that are diffeomorphic to open neighbourhoods on M_i^+ where the diffeomorphism is induced by $\frac{-grad_g(f)}{||grad_g(f)||^2}$. Furthermore, let $x \in Cr(i)$ and define

$$S^+(x) := W^+(x) \cap M_i^+, \quad S^-(x) := W^-(x) \cap M_i^-$$

which are just special stable respectively unstable spheres of x, furthermore set

$$\mathcal{S}_x := S^+(x) \times S^-(x)$$

and it will be convenient to write

$$S_i^+ := \bigcup_{x \in Cr(i)} S^+(x), \quad S_i^- := \bigcup_{x \in Cr(i)} S^-(x), \quad \mathcal{S}_i := \bigcup_{x \in Cr(i)} \mathcal{S}_x.$$

$$\dot{M}_i^+ := M_i^+ \backslash S_i^+ \quad \text{and} \quad \dot{M}_i^- := M_i^- \backslash S_i^-$$

are manifolds too: Near a critical point consider the situation in a convenient chart where the stable respectively unstable manifolds are of the form (1.13) and (1.14). But now we see that S_i^+ is just a closed subset, and hence $\dot{M}_i^+ := M_i^+ \backslash S_i^+$ is an open subset in M_i^+ which is a sub-manifold as remarked before. Consequently $\dot{M}_i^+$ is itself a sub-manifold. The same argument holds for $\dot{M}_i^-$ with the obvious adaptations.

Next we define diffeomorphisms:

$$\psi_i : M_i^- \longrightarrow M_{i-1}^+, \quad x \mapsto \Phi_{c_i - c_{i-1} - 2\varepsilon}(x)$$

where $\Phi_{c_i - c_{i-1} - 2\varepsilon}(\cdot)$ is the diffeomorphism generated by the flow of $\frac{-grad_g(f)}{||grad_g(f)||^2}$ at the time when the flow lines intersect M_{i-1}^+. This is a diffeomorphism, as no critical values lie in $[c_{i-1} + \varepsilon, c_i - \varepsilon]$.

Additionally, we have

$$\begin{aligned} \varphi_i^+ &: \dot{M}_i^+ \longrightarrow \dot{M}_i, \quad x \mapsto \Phi_{+\varepsilon}(x) \\ \varphi_i^- &: \dot{M}_i^- \longrightarrow \dot{M}_i, \quad x \mapsto \Phi_{-\varepsilon}(x) \end{aligned}$$

and Φ is again the flow generated by $\frac{-grad_g(f)}{||grad_g(f)||^2}$ but this time problems would occur on points lying on S_i^+ as points on S_i^+ would be mapped to the critical point under the flow and the flow equation would break down. But away from this points no problems occur. The same holds for φ_i^-.

One can extend φ_i^+ to M_i^+ by mapping points on S_i^+ to the associated critical points on whose stable manifold the point is lying. This extension is the unique continuous extension of φ_i^+. That this works can be checked in one of the convenient charts for the critical point. Consider a critical point $z \in Cr(i)$ and assume $\varepsilon > 0$ to be small enough such that $S^+(z) = ((x,0) \in \mathbb{R}^{n-k} \times 0 : ||x|| = \varepsilon)$ is totally contained in this chart, it will do no harm to assume $\varepsilon = 1$. Next we have to work out the expression for φ_i^+ in this chart: let $x = (x^+, y^+)$ be a point in $\dot{M}_i^+$ on M_i^+ in the convenient chart, y^+ lies in the unstable part and x^+ in the stable part, see (1.11) and (1.12). Without loss of generality we can assume that $f(z) = 0$. We have already computed that the solution of the negative gradient flow is given by

$$x^+(t) = x^+ e^{-2t}, \quad y^+(t) = y^+ e^{2t}$$

see (1.10). And we search the point where this trajectory intersects $M_i := \{-||y||^2 + ||x||^2\} = f(z) = 0$. By substituting $x^+(t)$ and $y^+(t)$ into the local expression for f we can conclude that $t = \frac{1}{4}\ln(\frac{||x^+||}{||y^+||})$ and hence:

$$\varphi_i^+ : (x^+, y^+) \mapsto (x^+\sqrt{\frac{||y^+||}{||x^+||}}, y^+\sqrt{\frac{||x^+||}{||y^+||}}) \tag{2.4}$$

in the convenient chart centred at z. We remark that no problems with the denominators can occur, as $y^+ \neq 0$ because otherwise $(x^+, y^+) \in S^+(z)$, and by $||x^+||^2 = 1 + ||y^+||^2$, x^+ is also non-vanishing.
But from this local expression it is evident that φ_i^+ can be extended to the whole of M_i^+ by setting φ_i^+ equal to $0 \in \mathbb{R}^n$ on $S^+(z)$ in all convenient charts centred at critical points in $Cr(i)$ (assumed that the index of the critical points is not 0). By letting y^+ approach 0 we see that $\varphi_i^+(x^+, y^+)$ approaches $(0,0) = 0 \in \mathbb{R}^n$ too and so this extension is continuous. Uniqueness follows from the observation that all points in $S^+(x)$ can be approached by a sequence of points in $\dot{M}_i^+$ by choosing a sequence of points in $\mathbb{R}^{n-k}$, $(x_n^+)_{n\in\mathbb{N}}$, converging to x^+, with $||x_n^+|| > 1$ and finding $(y_n^+)_{n\in\mathbb{N}}$ near 0 such that $-||y_n^+||^2 + ||x_n^+||^2 = 1$ is satisfied for all $n \in \mathbb{N}$. By regularity of the condition $-||y_n^+||^2 + ||x_n^+||^2 = 1$ this is alway possible.
Up to obvious changes, this works for φ_i^- as well, as can be seen easily (assumed that the index of the critical point is not $n = \dim(M)$).

2.25. Definition *model space for the space of broken trajectories*
With the help of the terminology developed in the last remark we define

$$P_i := \{(u,v) \in M_i^+ \times M_i^- : \varphi_i^+(u) = \varphi_i^-(v)\}. \tag{2.5}$$

Remark:
This set represents all unparametrised trajectories, possibly broken, at a critical point with critical value c_i, from M_i^+ to M_i^-.

There are two obvious functions on P_i, the projections on the first respectively second factor:

$$p_i^+ : P_i \longrightarrow M_i^+, \quad p_i^- : P_i \longrightarrow M_i^-$$

A priori, it is only clear that P_i is a closed subset of $M_i^- \times M_i^+$. The next proposition clarifies the structure of P_i:

2.26. Proposition

$P_i \subset M_i^+ \times M_i^-$ is a smooth sub-manifold with boundary. Especially the following holds:
1.) $p_i^+ : P_i \backslash \partial P_i \longrightarrow \dot{M}^+$ respectively $p_i^- : P_i \backslash \partial P_i \longrightarrow \dot{M}^-$ are diffeomorphisms.
2.) ∂P_i is diffeomorphic to $\mathcal{S}_i$ under the restriction of $p_i^- \times p_i^+$ on ∂P_i, p_i^+ respectively p_i^+ restricted to ∂P_i are the projections on S_i^- respectively on S_i^+.

Proof:
Consider

$$\dot{M}_i^+ \times \dot{M}_i^- \to \dot{M}_i \times \dot{M}_i, \quad (u, v) \mapsto (\varphi_i^+(u), \varphi_i^-(v)).$$

This map is submersive because away from the stable and unstable spheres φ_i^+ and φ_i^- are diffeomorphisms. The pre-image of $\triangle_{\dot{M}_i} := \{(u, u) \in \dot{M}_i \times \dot{M}_i : u \in \dot{M}_i\}$ under $\varphi_i^+ \times \varphi_i^-$ is $P_i \setminus (S_i^+ \times S_i^-)$. Hence $P_i \setminus (S_i^+ \times S_i^-)$ is a sub-manifold of $M_i^+ \times M_i^-$.
Now we assume v lies on $S^-(z)$. Again, we choose $\varepsilon > 0$ so small, that $S^+(z)$ is totally contained in a convenient chart centred at z, and then we assume without loss of generality that this ε is 1. v is mapped to 0 in the chart, and so we search for $u \in M_i^+$ such that $\varphi_i^+(u) = 0$ and one sees that this are exactly the points in $S^+(z)$. The argument works the other way round too, so, if one point is contained in the stable manifold, the other one must be contained in the unstable manifold and vice versa.
Now consider the smooth mapping

$$\begin{aligned} \zeta : S^+(z) \times S^-(z) \times [0, \varepsilon[&\longrightarrow M_i^+ \times M_i^- \\ (\theta^+, \theta^-, t) &\mapsto (\sqrt{1+t^2}\theta^+, t\theta^-, t\theta^+, \sqrt{1+t^2}\theta^-) \end{aligned}$$

we check that ζ maps into P_i, i.e. $\varphi_i^+ \circ pr_{1,2} \circ \zeta = \varphi_i^- \circ pr_{3,4} \circ \zeta$ where $pr_{1,2}$ denote the projection of the first two components and $pr_{3,4}$ the projection of the third and the fourth. We first check equality for points in $P_i \setminus (S^+(z) \times$

$S^-(z)$):

$$\begin{aligned}
\varphi_i^+ \circ pr_{1,2}(\zeta(\theta^+,\theta^-,t)) &= \varphi_i^+(\sqrt{1+t^2}\theta^+, t\theta^-) \\
&= \left(\sqrt{1+t^2}\theta^+ \frac{\sqrt{t||\theta^-||}}{\sqrt{\sqrt{1+t^2}||\theta^+||}}, t\theta^- \frac{\sqrt{\sqrt{1+t^2}||\theta^+||}}{\sqrt{t||\theta^-||}}\right) \\
&= \left((\sqrt{t\sqrt{1+t^2}})\theta^+, (\sqrt{t\sqrt{1+t^2}})\theta^-\right)
\end{aligned}$$

$$\begin{aligned}
\varphi_i^- \circ pr_{3,4}(\zeta(\theta^+,\theta^-,t)) &= \varphi_i^-(t\theta^+, \sqrt{1+t^2}\theta^-) \\
&= \left(t\theta^+ \frac{\sqrt{\sqrt{1+t^2}||\theta^-||}}{\sqrt{t||\theta^+||}}, \sqrt{1+t^2}\theta^- \frac{\sqrt{t||\theta^+||}}{\sqrt{\sqrt{1+t^2}||\theta^-||}}\right) \\
&= \left((\sqrt{t\sqrt{1+t^2}})\theta^+, (\sqrt{t\sqrt{1+t^2}})\theta^-\right)
\end{aligned}$$

and by continuous extension of this formulas equality follows for all points of P_i. Moreover ζ can be smoothly extended to a map $S^+(z) \times S^-(z) \times]-\delta, \varepsilon[\to M_i^+ \times M_i^-$ with $\delta > 0$ such that the extension is injective: given (θ^+,θ^-,t) and (ω^+,ω^-,s) in $S^+(z) \times S^-(z) \times]-\delta, \varepsilon[$ such that $\zeta(\theta^+,\theta^-,t) = \zeta(\omega^+,\omega^-,s)$. By projection to first respectively forth component one obtains $\theta^+ = \omega^+$ and $\theta^- = \omega^-$. By projecting to the second component $t = s$ follows. Furthermore the tangential mapping of ζ is given by the matrix

$$\begin{pmatrix} \sqrt{1+t^2} & 0 & \frac{t}{\sqrt{1+t^2}}\theta^+ \\ 0 & t & \theta^- \\ t & 0 & \theta^+ \\ 0 & \sqrt{1+t^2} & \frac{t}{\sqrt{1+t^2}}\theta^- \end{pmatrix}$$

and so the extension of ζ is immersive.
Consequently ζ must be the parametrisation of a collar of a part of the boundary and hence $\partial P_i = S^+(z) \times S^-(z)$ near the critical point z.

Now the two claims follow directly by pasting together the results we have obtained for one neighbourhood of a critical point for all critical points with the same critical value.

□

2.27. Theorem *the smooth structure of $\hat{\mathcal{T}}(p,q)$*
Given a Morse–Smale pair (f,g) and critical points p and q, $\hat{\mathcal{T}}(p,q)$ admits a canonical smooth structure of a manifold with corners such that the map

$$\hat{\mathcal{T}}_k(p,q) \hookrightarrow \hat{\mathcal{T}}(p,q)$$

induces a diffeomorphism

$$\hat{\mathcal{T}}_k(p,q) \stackrel{\cong}{\to} \partial_k \hat{\mathcal{T}}(p,q). \tag{2.6}$$

<u>Proof</u>:
We set $f(p) = c_{r+1}$ and $f(q) = c_{r-k-1}$ and remark that $\hat{\mathcal{T}}(p,q)$ is only non-empty provided $k \geq 0$, because f decreases along the flow lines. Define

$$\mathcal{P} := \mathcal{P}_{r,r-k} := P_r \times P_{(r-1)} \times \ldots \times P_{r-k}$$

and as a product of manifolds with boundaries, this is a smooth manifold with corners. $\mathcal{P}$ can be interpreted as the space that represents pieces of broken trajectories that need not fit together. Additionally we have:

$$\mathcal{O} := \prod_{r}^{r-k} (M_i^+ \times M_i^-)$$

$$\mathcal{S} := S_p^- \times M_r^- \times \ldots \times M_{r-k+1}^- \times S_q^+$$

that are smooth manifolds. Next we define smooth maps

$$\omega_i : M_i^- \longrightarrow M_i^- \times M_{i-1}^+, \quad x \mapsto (x, \psi_i(x))$$

$$\tilde{p}_i : P_i \longrightarrow M_i^+ \times M_i^-, \quad y \mapsto (p_i^+(y), p_i^-(y))$$

and

$$\alpha : S_p^- \longrightarrow M_k^+$$

denotes the restriction of $\psi_{r+1} : M_{r+1}^- \longrightarrow M_r^+$ to S_p^- and

$$\beta : S_q^+ \longrightarrow M_{r-k}^-$$

is the restriction of $\psi_{r-k}^{-1} : M_{r-k-1}^+ \longrightarrow M_{r-k}^-$ to S_q^+.
We can put these maps together:

$$s := \alpha \times \omega_r \times \ldots \times \omega_{r-k-1} \times \beta : \mathcal{S} \longrightarrow \mathcal{O}$$

$$p := \tilde{p}_r \times \ldots \times \tilde{p}_{r-k} : \mathcal{P} \longrightarrow \mathcal{O}$$

and observe that there is a bijection between $p^{-1}(s((S))$ and $\hat{\mathcal{T}}(p,q)$:

$$\hat{\mathcal{T}}(p,q) \to p^{-1}(s(\mathcal{S}))$$

which is given by evaluating broken trajectories at the levels M_r^+, M_r^-,..., M_{r-k}^+, M_{r-k}^-. This map is continuous because convergence of broken trajectories in $\hat{\mathcal{T}}(p,q)$ means uniform convergence and this implies pointwise

convergence in the level-hypersurfaces. That this map is bijective follows easily from the definition of $p^{-1}(s(\mathcal{S}))$. Assume a point in $\mathcal{O}$ that lies in the image of s, then each pair of successive components of this point are associated to one canonical parametrised trajectory. That a point lies in the pre-image of p implies that the different parts of the broken trajectory fit together. Hence one can find exactly one broken trajectory from p to q that is mapped to a given point in $p^{-1}(s(\mathcal{S}))$. So, the map is bijective and continuous and we know that $\hat{\mathcal{T}}(p,q)$ is compact, consequently this map is a homeomorphism.
$s : \mathcal{S} \longrightarrow \mathcal{O}$ is an embedding and next we will see that p and s are transversal. Consider the diagram (diag 1):

$$\begin{tikzcd}
S_p^- \arrow[d, "\alpha"] & & & M_r^- \arrow[dl, "id"'] \arrow[dr, "\psi_r"] & & & & \cdots \arrow[dl, "id"'] & \\
M_r^+ & & M_r^- & & M_{r-1}^+ & & M_{r-1}^- & & \cdots \\
& P_r \arrow[ul, "p_r^+"] \arrow[ur, "p_r^-"'] & & & & P_{r-1} \arrow[ul, "p_{r-1}^+"] \arrow[ur, "p_{r-1}^-"'] & & & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & M_{r-k+1}^- \arrow[dl, "id"'] \arrow[dr, "\psi_{r-k+1}"] & & & S_q^+ \arrow[d, "\beta"] \\
\cdots & & M_{r-k}^+ & & M_{r-k}^- \\
\cdots & & & P_{r-k} \arrow[ul, "p_{r-k}^+"] \arrow[ur, "p_{r-k}^-"'] &
\end{tikzcd}$$

Choose a point in $\partial_l \mathcal{P}$ and consider P_i for $i = r, \ldots, r-k$ arbitrary. Then l of the components of the chosen point lie in $\mathcal{S}_i$ and the other $k-l$ lie in $P_i \setminus \mathcal{S}_i$. Assume the case where the component lies in $\mathcal{S}_{\rangle}$. The map $p_i^+ \times p_i^- : P_i \rightarrow M_i^+ \times M_i^-$ restricted to $\mathcal{S}_{\rangle}$ equals the product map of the identities on S_i^+ and on S_i^-, respectively. But then we can change the diagram by replacing the part

$$\begin{tikzcd}
& M_{i+1}^- \arrow[dl, "id"'] \arrow[dr, "\psi_{i+1}"] & & & & M_i^- \arrow[dl, "id"'] \arrow[dr, "\psi_i^+"] & \\
M_{i+1}^- & & M_i^+ & & M_i^- & & M_{i-1}^+ \\
& & & P_i \arrow[ul, "p_i^+"] \arrow[ur, "p_i^-"'] & & &
\end{tikzcd}$$

by

$$
\begin{tikzcd}
 & & S_i^+ \arrow[d] & S_i^- \arrow[d] & & \\
M_{i+1}^+ & & M_{i+1}^- & M_{i-1}^+ & & M_{i-1}^- \\
 & P_{i+1} \arrow[ul, "p_{i+1}^+"'] \arrow[ur, "p_{i+1}^-"] & & & P_{i-1} \arrow[ul, "p_{i-1}^+"'] \arrow[ur, "p_{i-1}^-"] &
\end{tikzcd}
$$

because transversality of the two small diagrams would imply transversality of the diagram we started with. Consequently, in this case we can "split" the diagram in two smaller ones. In the other case one observes that we can replace the map $p_i^+ \times p_i^- : P_i \setminus \mathcal{S}_i \to \dot{M}_i^+ \times \dot{M}_i^-$ by the map $\Phi_{2\varepsilon} : \dot{M}_i^+ \to \dot{M}_i^-$ where Φ is the flow of the vector field $-\frac{grad_g(f)}{||grad_g(f)||^2}$. Because we restrict this map to $\dot{M}_i^+$ no problems occur. Consequently we can replace

$$
\begin{tikzcd}
M_{i+1}^- \arrow[dr, "\psi_{i+1}"] & & & & M_i^- \arrow[dl, "id"'] \arrow[dr, "\psi_i"] & \\
 & M_i^+ & & M_i^- & & M_{i-1}^- \\
 & & P_i \arrow[ul, "p_i^+"'] \arrow[ur, "p_i^-"] & & &
\end{tikzcd}
$$

with

$$
\begin{tikzcd}
M_{i+1}^- \arrow[dr] & \\
 & M_{i-1}^+
\end{tikzcd}
$$

where the map $M_{i+1}^- \to M_{i-1}^+$ is given by $\Phi_{c_{i+1}-c_{i-1}-2\varepsilon}$. At $i = r$ and $i = r - k$ small adaptions are made but nothing essential changes.
All in all we observe that the diagram can be reduced to smaller diagrams step-by-step and in the end we get diagrams of the form $S_i^+ \to S_j^-$ or $S_i^- \to S_j^+$ where the maps are given by the flow of $-\frac{grad_g(f)}{||grad_g(f)||^2}$. Consequently, it suffices that all unstable spheres are transversal to the stable ones and in the proof of Theorem (1.23.) this has been shown to be equivalent to the Morse–Smale condition.
Consequently we can apply Lemma (2.24.) and obtain that $\hat{\mathcal{T}}(p,q)$ possesses a canonical structure of a smooth manifold with corners. The k-boundary $\partial_k\hat{\mathcal{T}}(p,q)$ is given by $\partial_k\mathcal{P} \cap p^{-1}(s(\mathcal{S}))$ and this corresponds exactly to the

k-times broken trajectories. We have identified $\hat{\mathcal{T}}(p,q)$ and $p^{-1}(s(\mathcal{S}))$ as topological spaces and in the last chapter we have seen that the topology of $\hat{\mathcal{T}}_k(p,q)$ as a subspace of $\hat{\mathcal{T}}(p,q)$ coincides with the product topology if we regard $\hat{\mathcal{T}}_k(p,q)$ as the product

$$\bigsqcup_{p=:y_0,y_1,\ldots,y_k,y_{k+1}:=q} \mathcal{T}(y_0,y_1)\times\mathcal{T}(y_1,y_2)\times\ldots\times\mathcal{T}(y_k,y_{k+1}).$$

We have to show that the smooth structure on $\hat{\mathcal{T}}_k(p,q)$ obtained from this two description also coincide. So, consider

$$T := \mathcal{T}(p=:y_0,y_1)\times\ldots\times\mathcal{T}(y_k,y_{k+1}:=q)\subset\hat{\mathcal{T}}_k(p,q)\subset\hat{\mathcal{T}}(p,q).$$

From the way (diag 1) splits up and collapses one sees that the induced smooth structure of T is the one obtained by identifying $\mathcal{T}(y_i,y_{i+1})$ with the space of trajectories from y_i to y_{i+1} intersected with a level-hypersurface, and then equipping T with the corresponding product structure.

□

2.4 Compactification of the unstable Manifolds

In this section we intend to compactify the unstable manifolds of critical points. In many respects the treatment will resemble the treatment in the previous section. Again we follow [3], [4] respectively [5].

<u>Remark</u>: another interpretation of unstable manifolds
In section 2 of this chapter we interpreted the space of unparametrised trajectories between two critical points as a subspace of the continuous functions from some compact interval to the manifold. We intend to do the the same for the unstable manifolds.
Let (f,g) be a Morse–Smale pair on M and let p be a critical point of f. Set $b := f(p)$ and $e := \min f$. x denotes an arbitrary point in $W^-(p)$. Then we can consider the unique canonical parametrised trajectory that starts at p and goes through x. It shall be parametrised such that $\sigma(t) = b-t$, hence we start at $t=0$. When this trajectory arrives at x at $t = b-f(x)$ it shall become stationary, i.e. $\sigma(t) = x$ for all $t\in[b-f(x), b-e]$. On the other hand, assume we have a map $\tau\in\mathcal{C}([0,b-e],M)$ such that
1.) $\tau(0) = p$
2.) there is exactly an $c\in[0,b-e]$ such that
2.a) $\tau(t)\notin Cr(f)$ for $t\in]0,c]$
2.b) $\tau'(t)$ exists for $t\in]0,c[$ and $\tau'(t) = \frac{-\operatorname{grad}_g(f)}{||\operatorname{grad}_g(f)||^2}\circ\tau(t)$ holds,
2.c) $f(\tau(t)) = b-t$ for $t\in[0,c]$,
2.d) $\tau(t) = \tau(c)$ for $t\in[c,b-e]$

Then this map corresponds to a unique point in $W^-(p)$, namely $\tau(c) = \tau(b-e)$. So there is a one-to-one correspondence between points on $W^-(p)$ and the subspace of $\mathcal{C}([0,b-e],M)$ which is described by the conditions 1.), 2.). We equip $\mathcal{C}([0,b-e],M)$ with the topology induced by uniform convergence and so $W^-(p)$ inherits a topology as a subset of $\mathcal{C}^0([0,b-e],M)$. Next we prove:

2.28. Proposition
The topology that $W^-(p)$ inherits from $\mathcal{C}^0([0,b-e],M)$ coincides with the topology on $W^-(p)$ induced by the embedding $W^-(p) \hookrightarrow M$.

Proof:
Let $(\tau_n(\cdot))_{n\in\mathbb{N}}$ be a subset in $\mathcal{C}^0([0,b-e],M)$ such that every element satisfies conditions 1.), 2.) and that converges uniformly to $\tau(\cdot)$ that satisfies the three conditions too. But uniform convergence implies pointwise convergence, consequently

$$\tau_n(b-e) \to \tau(b-e).$$

Hence uniform convergence of the maps implies pointwise convergence of the points in $W^-(p)$.
On the other hand, let $(x_n)_{n\in\mathbb{N}}$ be a convergent subset in $W^-(p)$ with limit $x \in W^-(p)$. We must show that the corresponding maps converge uniformly. But this is similar to the first part of the proof of Proposition (2.12.). Denote the maps corresponding to x_n by $\tau_n(\cdot)$ and the one corresponding to x by $\tau(\cdot)$. Let $c_n \in [0,b-e]$ denote the value characterised in condition 2.) for the map $\tau_n(\cdot)$, respectively c for the map $\tau(\cdot)$. By condition 2.c) we have $f(\tau_n(c_n)) = b - c_n$ and $f(\tau(c)) = b - c$. But this implies

$$\lim_{n\to\infty} b - c_n = \lim_{n\to\infty} f(\tau_n(c_n)) = f(\lim_{n\to\infty} x_n) = f(x) = b - c$$

so $\lim_{n\to\infty} c_n = c$.
For $t > c$ exists $N \in \mathbb{N}$ such that $c_n < t$ for all $n \geq N$. Hence for n sufficiently large all the maps are constant for such a t and from this uniform convergence follows. For $t < c$ we can find N such that $c_n > t$ for all $n \geq N$ and consequently for n large enough the flow equation stated in 2.b) is satisfied. Now uniform convergence follows as in the proof of Proposition (2.12.). It remains to check locally uniform convergence for $t = c$. Because of $\tau(t) \notin Cr(f)$ there is an open neighbourhood of $\tau(t)$ that does not contain critical points. We can continue the trajectories through x_n $n \in \mathbb{N}$ respectively through x from some time. One uses the fact that the canonically parametrised trajectories intersect the function-hyperlevels perpendicular (Lemma (1.11.)) to estimate the distance between a canonically parametrised trajectory and a trajectory that has become stationary. If follows that $\tau_n(\cdot)$ converges locally uniformly to $\tau(\cdot)$ near $t = c$ too.

□

2.29. Definition $\hat{W}^-(p)$
Given a Morse–Smale pair (f,g) on M. Let p be a critical point of f. Define

$$\hat{W}_k^-(p) := \bigsqcup_{p=:y_0,y_1,\ldots,y_k} \mathcal{T}(y_0,y_1)\times\ldots\times\mathcal{T}(y_{k-1},y_k)\times W^-(y_k)$$

and

$$\hat{W}^-(p) := \bigsqcup_{k\geq 0} \hat{W}_k^-(p).$$

Furthermore we define a map

$$\hat{i}_p : \hat{W}^-(p) \to M$$

by setting $\hat{i}_p := pr : \hat{W}_k^-(p) \to M$ on $\hat{W}_k^-(p)$ with pr being the projection of the last factor.

<u>Remark</u>: topology on $\hat{W}^-(p)$
$\hat{W}^-(p)$ can be interpretated as a subspace of $\mathcal{C}^0([0,b-e],M)$: It corresponds to the subspace of functions $\tau(\cdot)$ that satisfy:
1.) $\tau(0)=p$
2.) there exists exactly one $c\in[0,b-e]$ such that
2.a) $\tau'(t)$ exists for $t\in]0,c[$ if $\tau(t)\notin Cr(f)$ and for such t we have

$$\tau'(t) = \frac{-\operatorname{grad}_g(f)}{||\operatorname{grad}_g(f)||^2}\circ\tau(t),$$

2.b) $f(\tau(t)) = b-t$ for $t\in[0,c]$
2.c) $\tau(t)=\tau(c)$ for $t\in[c,b-e]$.
$\tau(c)$ is contained in $W^-(q)$ for some $q\in Cr(f)$ and $\tau(\cdot)$ describes a broken trajectory from p to q. So this subspace of $\mathcal{C}^0([0,b-e],M)$ is in bijection with $\hat{W}^-(p)$ and the topology of $\hat{W}_k^-(p)\subset\hat{W}^-(p)$ coincides with the topology coming from the definition of $\hat{W}_k^-(p)$ as $\bigsqcup_{p=:y_0,y_1,\ldots,y_k}\mathcal{T}(y_0,y_1)\times\ldots\times\mathcal{T}(y_{k-1},y_k)\times W^-(y_k)$: indeed, uniform convergence of all the unbroken pieces implies uniform convergence of the map $\tau\in\mathcal{C}^0([0,b-e],M)$ and vice versa. In particular $W^-(p)\hookrightarrow\hat{W}^-(p)$ induces a homeomorphism

$$W^-(p)\xrightarrow{\cong}\hat{W}_0^-(p).$$

Observe that the function $\hat{i}_p:\hat{W}^-(p)\to M$ is continuous: If we have a sequence of maps satisfying condition 1.), 2.), uniform convergence implies pointwise convergence and $\hat{i}_p$ coincides with the map $\tau(\cdot)\mapsto\tau(b-e)$ if we interpret points in $\hat{W}^-(p)$ as elements of $\mathcal{C}^0([0,b-e],M)$.

2.30. Theorem *$\hat{W}^{-}(p)$ is compact*
Given a Morse–Smale pair (f, g) on M and a critical point p of f. Then $\hat{W}^{-}(p)$ is compact.

Proof:
We proceed similar as we have done in the proof of Theorem (2.21.) and verify all the conditions in order the apply the Theorem of Arzela–Ascoli (Theorem (2.20.).

claim a) $\hat{W}^{-}(p)$ is bounded:
See the proof of Theorem (2.21.).

claim b) $\hat{W}^{-}(p)$ is closed:
Let $(\tau_n(\cdot))_{n\in\mathbb{N}}$ be a sequence of maps in $\mathcal{C}^0([0, b-e], M)$ satisfying conditions 1.), 2.) stated in the last remark. Suppose that these maps converge uniformly to $\tau(\cdot) \in \mathcal{C}^0([0, b-e], M)$. We have to show that $\tau(\cdot)$ satisfies the 3 conditions. The first condition follows easily as uniform convergence implies pointwise convergence and so $\tau(0) = \lim_{n\to\infty} \tau_n(0) = \lim_{n\to\infty} p = p$. Define c_n to be the values in $[0, b-e]$ as described in condition 2.) for the map $\tau_n(\cdot)$. Furthermore set $x_n := \tau_n(c_n)$. Uniform convergence of $(\tau_n(\cdot))_{n\in\mathbb{N}}$ implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence. By compactness of M there is a limit of $(x_n)_{n\in\mathbb{N}}$, called $x \in M$. By compactness of $[0, b-e]$ there is a convergent subsequence of $(c_n)_{n\in\mathbb{N}}$, so without loss of generality we can assume that $(c_n)_{n\in\mathbb{N}}$ is convergent and by uniform convergence

$$\tau(\lim_{n\to\infty} c_n) = \lim_{n\to\infty} \tau(c_n) = \lim_{n\to\infty} \tau_n(c_n) = \lim_{n\to\infty} x_n = x$$

and we set $c := \lim_{n\to\infty} c_n$. Additionally

$$\lim_{n\to\infty} f(\tau_n(c_n)) = f(x)$$

and together with $f(\tau_n(c_n)) = b - c_n$ this implies $f(x) = b - c$.
Assume $t \in [0, c[$. Because of $c_n \to c$ we can find an $N \in \mathbb{N}$ such that $t < c_n$ for all $n \geq N$. But then we can proceed as in the proof of Theorem (2.21.). For $t \in]c, b-e]$ there is an $N \in \mathbb{N}$ such that $t > c_n$ for $n \geq N$ and consequently

$$\tau(t) = \lim_{n\to\infty} \tau_n(t) = \lim_{n\to\infty} (\tau_n(c_n)) = x = \tau(c)$$

. It remains to investigate the case $t = c$. If $\tau(c)$ is not a critical point one can derive locally uniform convergence as was done at the end of the proof of Proposition (2.28.). If $\tau(c)$ is a critical point one uses a convenient chart and the explicit form of the canonically parametrised trajectories to show uniform convergence — see the proof of Theorem (2.21.).

claim c) $\hat{W}^-(p)$ is equicontinuous:
For the interval where the maps represent broken trajectories, this is done as in the proof of Theorem (2.21.). For the other part the maps get stationary and the claim is trivial.

$\square$

Remark: some useful notations
Additionally to the notation that was introduced in the last section, we set

$$M(i) := f^{-1}(]c_{i-1}, c_{i+1}[)$$

and this is an open sub-manifold of M. Furthermore

$$W_i^+(x) := W^+(x) \cap M(i),$$
$$W_i^-(x) := W^-(x) \cap M(i),$$
$$\mathcal{S}W_i(x) := S^+(x) \times W_i^-(x)$$

for $x \in Cr(i)$ and these are smooth sub-manifolds too. We will use

$$W^+(i) := \bigcup_{x \in Cr(i)} W^+(x),$$

$$W^-(i) := \bigcup_{x \in Cr(i)} W^-(x),$$

$$\mathcal{S}W(i) := \bigcup_{x \in Cr(i)} S^+(x) \times W_i^-(x).$$

Next we define maps

$$\varphi(i) : M(i) \setminus \big(W^-(i) \cup W^+(i)\big) \to \dot{M}_i, \quad x \mapsto \Phi_{f(x)-c_i}(x)$$

where Φ denotes the diffeomorphism induced by the flow of $\frac{-\operatorname{grad}_g(f)}{||\operatorname{grad}_g(f)||^2}$ again. $\varphi(i)$ can be extended continuously to $M(i)$ (assuming that the critical point under consideration has index not equal to $\dim(M)$) by setting $\varphi(i)(x) = \lim_{t\to+\infty} \gamma_x(t)$ for $x \in W^+(i)$ and $\varphi(i) = \lim_{t\to-\infty} \gamma_x(t)$ for $x \in W^-(i)$ and this extension is continuous. One can verify that this extension is continuous and that it is unique, similar as was done for φ_i^+ respectively φ_i^- in the last section. Again we consider the situation in a convenient chart near a critical point in $Cr(i)$ and use the explicit form of the flow lines there. One obtains that $\varphi(i)$ is given by

$$\varphi(i) : (x, y) \mapsto (x\sqrt{\frac{||y||}{||x||}}, y\sqrt{\frac{||x||}{||y||}}).$$

2.31. Definition
Using the notations from the last remark we set

$$Q(i) := \{(u,v) \in M_i^+ \times M(i) : \varphi_i^+(u) = \varphi(i)(v) \textit{ and}$$

$$\Phi_{f(v)-c_i^+}(v) = u \textit{ for } v \in M(i) \setminus W^-(i)\}$$

where Φ denotes the flow induced by $\frac{-\operatorname{grad}_g(f)}{||\operatorname{grad}_g(f)||^2}$. We denote the two canonical projections from $Q(i)$ to M_i^+ respectively to $M(i)$ by

$$l_i : Q(i) \to M_i^+ \quad \textit{and} \quad r_i : Q(i) \to M(i).$$

2.32. Proposition
$Q(i) \subset M_i^+ \times M(i)$ is a smooth sub-manifold with boundary $\partial Q(i)$ diffeomorphic to $\mathcal{S}W(i) \subset M_i^+ \times M(i)$. Especially

1. *$r_i : Q(i) \setminus \partial Q(i) \to M(i) \setminus W^-(i)$ is a diffeomorphism and l_i restricted to $Q(i) \setminus (S_i^+ \times M(i))$ respectively to $(Q(i) \setminus \partial Q(i)) \cap (S_i^+ \times M(i))$ is a smooth bundle with fibre an open segment.*
2. *The restriction of $l_i \times r_i$ to $\partial Q(i)$ is a diffeomorphism to $\mathcal{S}W(i)$, i.e. l_i respectively r_i restricted to $\partial Q(i)$ are the projections onto S_i^+ and $W^-(i)$.*

hold.

Proof:
That $Q(i) \setminus \mathcal{S}W(i)$ is a sub-manifold of $M_i^+ \times M(i)$ can be seen as follows: Consider

$$\operatorname{id} \times \Phi_{f(\cdot)-c_i^+} : M_i^+ \times (M(i) \setminus W^-(i)) \to M_i^+ \times M_i^+$$

and observe that this map is submersive and one obtains that

$$Q(i) \setminus \mathcal{S}W(i) = (\operatorname{id} \times \Phi_{f(\cdot)-c_i^+})^{-1}(\triangle_{M_i^+})$$

is a sub-manifold of $M_i^+ \times (M(i) \setminus W^-(i))$ and consequently a sub-manifold of $M_i^+ \times M(i)$, too.
We set $R := S_i^+ \times W^-(i)$.

$$r_i : Q(i) \setminus R \longrightarrow M(i) \setminus W^-(i)$$

is a smooth mapping with smooth inverse

$$M(i) \setminus W^-(i) \longrightarrow Q(i) \setminus R, \quad v \mapsto (\Phi_{f(v)-c_i^+}(v), v)$$

and hence it is a diffeomorphisms.

$$l_i : Q(i) \setminus (S_i^+ \times M(i)) \to \dot{M}_i^+$$

is smooth and the bundle structure is described by

$$U \times]c_i^+ - c_{i+1}, c_i^+ - c_{i-1}[\to Q(i) \setminus (S_i^+ \times M(i)), \quad (u,t) \mapsto (u, \Phi_t(u))$$

where U is a chart neighbourhood of u in $\dot{M}_i^+$. The fibre over u is given by $\Phi_{]c_i^+ - c_{i+1}, c_i^+ - c_{i-1}[}(x)$. Geometrically this is quite obvious: the points in the fibre over a point on $\dot{M}_i^+$ are exactly all points y on the canonical parametrised trajectory going through u such that $c_{i-1} < f(y) < c_{i+1}$.

$$l_i : (Q(i) \setminus R) \cap (S_i^+ \times M(i)) \to S_i^+$$

is a smooth mapping and the bundle structure is described by

$$U \times]c_i^+ - c_{i+1}, c_i^+ - c_i[\to (Q(i) \setminus R) \cap (S_i^+ \times M(i)), \quad (u,t) \mapsto (u, \Phi_t(u))$$

where U is an open neighbourhood of u in S_i^+. The fibre consists of all points y on the canonical parametrised trajectory through u such that $c_i < f(y) < c_{i+1}$.

For $x \in Cr(i)$ consider a convenient chart and the smooth mapping

$$\eta : S^+(x) \times W_i^-(x) \times [0, \varepsilon[\longrightarrow Q(i)$$

$$(\theta, y, t) \mapsto (\sqrt{1 + t^2||y||^2}\theta, ty, t\sqrt{1 + t^2||y||^2}\theta, y)$$

and observe that η really maps into $Q(i)$ because one can easily check that it maps into $M_i^+ \times M(i)$ and $\varphi_i^+ \circ pr_{1,2} \circ \eta = \varphi(i) \circ pr_{3,4} \circ \eta$. For points in $Q(i) \setminus R$ we get

$$\begin{aligned}
\varphi_i^+ \circ pr_{1,2}(\eta(\theta, y, t)) &= \varphi_i^+(\sqrt{1 + t^2||y||^2}\theta, ty) \\
&= \ldots = (\sqrt{t\sqrt{1 + t^2||y||^2}}||y||\theta, \sqrt{t\sqrt{1 + t^2||y||^2}}\frac{y}{||y||})
\end{aligned}$$

where we skipped steps of the calculation that are totally analogous to the ones made in the proof of Proposition (2.26.). Then we calculate

$$\begin{aligned}
\varphi(i) \circ pr_{3,4}(\eta(\theta, y, t)) &= \varphi(i)(t\sqrt{1 + t^2||y||^2}\theta, y) \\
&= \ldots = (\sqrt{t\sqrt{1 + t^2||y||^2}}||y||\theta, \sqrt{t\sqrt{1 + t^2||y||^2}}\frac{y}{||y||})
\end{aligned}$$

and hence equality on $Q(i) \setminus R$ and by continuous extension of this formulas the result follows for all points of $Q(i)$ as before. Furthermore observe that the second condition for points in $Q(i)$ is satisfied: This condition has to be checked only for points $(u, v) \in M_i^+ \times M(i)$ with $v \in W^+(i) \setminus W^-(i)$ (so v is contained in one of the stable manifolds but no critical point). But then $y = 0$ and $t \neq 0$ and $v = (t\theta, 0)$, $u = (\theta, 0)$ and clearly v is mapped to u under $\Phi_{f(\cdot)-c_i^+}$.
We can extend η smoothly to a map $S^+(x) \times W_i^-(x) \times]-\delta, \varepsilon[\to M_i^+ \times M(i)$ for $\delta > 0$ and η restricted to $S^+(x) \times W_i^-(x) \times \{0\}$ maps to $S^+(x) \times W_i^-(x)$. The extended η is injective because if (θ, y, t) and (ω, z, s) are such that $\eta(\theta, y, t) = \eta(\omega, z, s)$ one obtains $y = z$ by projecting the last factor. Then $t = s$ and consequently $\theta = \omega$. Additionally η is immersive: The tangential mapping is given by the matrix

$$\begin{pmatrix} \sqrt{1+t^2||y||^2} & \frac{t^2||y||}{\sqrt{1+t^2||y||^2}}\theta & \frac{t||y||^2}{\sqrt{1+t^2||y||^2}}\theta \\ 0 & t & y \\ t\sqrt{1+t^2||y||^2} & \frac{t^3||y||}{\sqrt{1+t^2||y||^2}}\theta & \frac{1+t^2||y||(1+||y||)}{\sqrt{1+t^2||y||^2}}\theta \\ 0 & 1 & 0 \end{pmatrix}$$

and so the extension is immersive.
Consequently η must be the parametrisation of a collar of part of the boundary and hence $\partial Q(i) = S_i^+(x) \times W^-(x)$ in the convenient chart for x.

Now the two claims follow directly by pasting together the results we have obtained for one neighbourhood of a critical point for all critical points with the same critical value.

□

2.33. Theorem *the smooth structure of* $\hat{W}^-(p)$
Given a Morse–Smale pair (f, g) *on* M *and critical points* p *of* f *the set* $\hat{W}^-(p)$ *admits a canonical smooth structure of a manifold with corners such that the map*

$$\hat{W}_k^-(p) \hookrightarrow \hat{W}^-(p)$$

induces a diffeomorphism

$$\hat{W}_k^-(p) \xrightarrow{\cong} \partial_k \hat{W}^-(p)$$

and such that

$$\hat{i}_p : \hat{W}^-(p) \to M$$

is a smooth extension of the inclusion $W^-(p) \hookrightarrow M$.

Proof:
Assume $p \in Cr_{r+1}(f)$ and define $X(r-k) := (\hat{i}_p)^{-1}(M(r-k))$ with $M(k) := f^{-1}(]c_{k-1}, c_{k+1}[)$ as before.
The first step of the proof is to equip all the $X(r-k)$ with a topology and the structure of a smooth manifold with corners such that $\hat{i}_p : X(r-k) \longrightarrow M(r-k)$ is smooth. We proceed as in the proof of Theorem (2.27.) and set

$$\begin{aligned}
\mathcal{P} &:= P_r \times P_{r-1} \times \ldots \times P_{r-k+1} \times Q(r-k) \\
\mathcal{O} &:= (M_r^+ \times M_r^-) \times \ldots \times (M_{r-k+1}^+ \times M_{r-k+1}^-) \times M_{r-k}^+ \\
\mathcal{S} &:= S_p^- \times M_r^- \times \ldots \times M_{r-k+1}^- \\
p &:= \tilde{p}_r \times \ldots \times \tilde{p}_{r-k+1} \times l_{r-k} \\
s &:= \alpha \times \omega_r \times \ldots \times \omega_{r-k+1}
\end{aligned}$$

where $p : \mathcal{P} \longrightarrow \mathcal{O}$ and $s : \mathcal{S} \longrightarrow \mathcal{O}$. $\mathcal{P}$ is a smooth manifold with corners and $\mathcal{S}$ and $\mathcal{O}$ are smooth manifolds. Again we can consider the corresponding diagram

$$\begin{tikzcd}
S_p^- \arrow[d, "\alpha"] & & & M_r^- \arrow[dl, "id"'] \arrow[dr, "\psi_r"] & & & & \cdots \arrow[dl, "id"'] \\
M_r^+ & & M_r^- & & M_{r-1}^+ & & M_{r-1}^- & \cdots \\
& P_r \arrow[ul, "p_r^+"] \arrow[ur, "p_r^-"'] & & & & P_{r-1} \arrow[ul, "p_{r-1}^+"] \arrow[ur, "p_{r-1}^-"'] & & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & M_{r-k+1}^- \arrow[dl, "id"'] \arrow[dr, "\psi_{r-k+1}"] & & \\
\cdots & & M_{r-k}^+ & \\
\cdots & & & Q(r-k) \arrow[ul, "l_{r-k}"']
\end{tikzcd}$$

and observe that transversality of p and s is equivalent to the Morse–Smale condition, because the diagram can again be reduced to smaller ones — see the proof of Theorem (2.27.). So we can apply Lemma (2.24.) and obtain a structure of a smooth manifolds with corners on $p^{-1}(s(\mathcal{S}))$.
Next we check that $\hat{i}_p : X(r-k) \to M(r-k)$ is smooth: in local coordinates a point in $X(r-k)$ can be represented as

$$(x_r^+, x_r^-, \ldots, x_{r-k+1}^+, x_{r-k+1}^-, x_{r-k}^+, y_{r-k}^-) \mapsto y_{r-k}^-$$

where (x_i^+, x_i^-) are contained in P_i for $i = r, \ldots, r-k+1$ and $(x_{r-k}^+, y_{r-k}^-) \in Q_{r-k}$ and the map $\hat{i}_p$ is simply given by projecting out the last component.

The second step is to show that the topology and the structure of a smooth manifold with corners induced by $X(m)$ and $X(m')$ on $X(m) \cap X(m')$ is the same for various m and m'. Observe that $X(m)$ and $X(m')$ only intersect if $|m - m'| \leq 1$. So it suffices to investigate the case $m = r - l$ and $m' = r - l - 1$.
In local charts in $X(r-k)$ we have

$$(x_r^+, x_r^-, \ldots, x_{r-l+1}^+, x_{r-l+1}^-, x_{r-l}^+, y_{r-l}^-)$$

and as before: $(x_i^+, x_i^-) \in P_i$ for $i = r, \ldots, r-l+1$ and $(x_{r-l}^+, y_{r-l}^-) \in Q_{r-k}$, in particular $y_{r-l}^- \in M(r-l)$. In $X(r-k-1)$ one gets

$$(u_r^+, u_r^-, \ldots, u_{r-l+1}^+, u_{r-l+1}^-, u_{r-l}^+, u_{r-l}^-, u_{r-l-1}^+, v_{r-l-1}^-)$$

with $(u_i^+, u_i^-) \in P_i$ for $i = r, \ldots, r-l$ and $(u_{r-l-1}^+, v_{r-l-1}^-) \in Q_{r-l-1}$, especially $v_{r-l-1} \in M(r-l-1)$. We have $M(r-l) \cap M(r-l-1) = f^{-1}(]c_{r-l-1}, c_{r-l}[)$ and this set contains no critical points. If we apply $\hat{i}_p$ we obtain y_{r-l}^- respectively v_{r-l-1}^- and consequently $y_{r-l}^- = v_{r-l-1}^- \in M(r-l) \cap M(r-l-1)$. Now we present smooth coordinate transformations on $X(r-l) \cap X(r-l-1)$:

$$\begin{aligned} &(x_r^+, x_r^-, \ldots, x_{r-l+1}^+, x_{r-l+1}^-, x_{r-l}^+, y_{r-l}^-) \mapsto \\ &(x_r^+, x_r^-, \ldots, x_{r-l+1}^+, x_{r-l+1}^-, x_{r-l}^+, \Phi_{c_{r-l}-\varepsilon-f(y_{r-l}^-)}(y_{r-l}^-), \Phi_{c_{r-l-1}+\varepsilon-f(y_{r-l}^-)}(y_{r-l}^-), y_{r-l}^-) \end{aligned}$$

where Φ denotes the flow of $\frac{-\operatorname{grad}_g(f)}{||\operatorname{grad}_g(f)||^2}$. In the other direction we have

$$\begin{aligned} &(u_r^+, u_r^-, \ldots, u_{r-l+1}^+, u_{r-l+1}^-, u_{r-l}^+, u_{r-l}^-, u_{r-l-1}^+, v_{r-l-1}^-) \mapsto \\ &\qquad (u_r^+, u_r^-, \ldots, u_{r-l+1}^+, u_{r-l+1}^-, u_{r-l}^+, v_{r-l-1}^-) \end{aligned}$$

and one can easily check that these are inverse to each other.

Consequently, we obtain a topological space X with the structure of a smooth manifold with corners. There is a bijection between $\hat{W}^-(p)$ and X which is given by interpreting $\hat{W}^-(p)$ as a subspace of $\mathcal{C}^0([0, b-e])$ and evaluating $\tau(\cdot) \in \hat{W}-(p)$ at the different level hypersurfaces and at $b-e$. Similar as was done in the proof of Theorem (2.27.), one observes that this gives a continuous bijection between $\hat{W}^-(p)$ and X. By compactness of $\hat{W}^-(p)$ (Theorem 2.30.) this map is even a homeomorphism and hence we can identify $\hat{W}^-(p)$ and X as topological spaces and we equip $\hat{W}^-(p)$ with the structure of a smooth manifold with corners coming from X.
With the help of the right charts one sees that the k-boundary is given by

$$\partial_k \hat{W}^-(p) = \bigsqcup_{p=:y_0, y_1, \ldots, y_k} \mathcal{T}(y_0, y_1) \times \ldots \times \mathcal{T}(y_{k-1}, y_k) \times W^-(y_k).$$

Similar as was done in the proof of Theorem (2.27.) for the broken trajectories, one checks that the smooth structures of $\hat{W}_k^-(p)$ — once as $\bigsqcup_{p=:y_0,y_1,\ldots,y_k} \mathcal{T}(y_0,y_1)\times \ldots \times \mathcal{T}(y_{k-1},y_k) \times W^-(y_k)$ and once as a subset of $\hat{W}^-(p)$ — coincide.

□

2.5 Orientations

In order to develop the Morse–Smale complex with integer coefficients in the next chapter we have to introduce orientations of the unstable manifolds and the space of unparametrised trajectories and need to understand how these orientations fit together.

Remark: conventions
In Proposition (1.14.) it was shown that $W^-(p) \cong \mathbb{R}^{\mathrm{ind}(x)}$ for all $p \in Cr(f)$. Hence the unstable manifolds are orientable and we choose orientations θ_p for all of them. In a convenient chart we can identify $T_pM/T_pW^+(p)$ with $T_pW^-(p)$ with the help of the Riemannian metric g of the Morse–Smale pair (f,g). Because $W^+(p)$ is contractible, the bundle $TM/TW^+(p)$ — where TM is restricted to $W^+(p)$ — can be identified with $W^+(p)\times T_pW^-(p)$. Consequently all the stable manifolds obtain a co-orientation (hence the quotient bundle of TM modulo $TW^+(p)$ is oriented).
Next we want to define orientations on $\mathcal{M}(p,q) := W^-(p)\cap W^+(q)$. We will make use of the following convention: Given a short exact sequence of vector bundles

$$0 \to E \to F \to G \to 0$$

then the orientation of E_x followed by the orientation of G_x yields an orientation of F_x. Hence, if orientations on two of the three vector bundles are given an orientation on the third vector bundle is induced.
We can apply this to the short exact sequence

$$0 \to T_z(W^-(p)\cap W^+(q)) \overset{i}{\hookrightarrow} T_zW^-(p) \overset{p}{\to} T_zM/T_zW^+(q) \to 0$$

with $z \in \mathcal{M}(p,q)$, i the map induced by the inclusion $W^-(p)\cap W^+(q) \to W^-(p)$ and p the composition of the inclusion $T_zW^-(p) \hookrightarrow T_zM$ and the projection $T_zM \to T_zM/T_zW^+(q)$. That p is surjective follow from the transversality of $W^-(p)$ and $W^+(q)$. So we obtain orientations for $\mathcal{M}(p,q)$ for all $p \neq q \in Cr(f)$ (convention 1.)).
Now we can define orientations on all spaces of unparametrised trajectories. One way to equip $\mathcal{T}(p,q)$ with a differentiable structure was to identify it with $\mathcal{M}(p,q)\cap f^{-1}(c)$ where $f(p) =: b > c > a := f(q)$. $f^{-1}(c)$ is co-oriented and $\mathcal{M}(p,q)$ is oriented, so we can orient $\mathcal{T}(p,q)$ such that $-\operatorname{grad}_g(f)$ followed by the orientation $\mathcal{T}(p,q)$ yields the orientation of $\mathcal{M}(p,q)$ (convention

2.)).

Because $W^-(p)$ is the interior of the smooth manifold with corners $\hat{W}^-(p)$, the orientation θ_p induces an orientation of $\partial_1\hat{W}^-(p)$. Here we follow the convention that a outward pointing vector followed by an oriented bases of the tangential space of the boundary should coincide with the orientation on the whole manifold. On the other hand we have seen in Theorem (2.33.) that $\partial_1\hat{W}^-(p) = \bigsqcup_{q\in Cr(f)} \mathcal{T}(p,q)\times W^-(q)$ and so the orientations of $\mathcal{T}(p,q)$ and $W^-(q)$ define an orientation on $\partial_1\hat{W}^-(p)$ too. The orientation on a product is given be an oriented base of the first factor followed by an oriented base of the second factor. The next Proposition states that these two orientations coincide:

2.34. Proposition
The orientation on $\partial_1\hat{W}^-(p)$ induced by the orientation θ_p of $W^-(p)$ is the same as the orientation that $\partial_1\hat{W}^-(p)$ inherits from

$$\partial_1\hat{W}^-(p) = \bigsqcup_{q\in Cr(f)} \mathcal{T}(p,q)\times W^-(q).$$

<u>Proof</u>:
We choose a convenient chart for $q \in Cr_k(f)$ and analyse the local situation. In the convenient chart we have the splitting into a stable and an unstable part, $\mathbb{R}^{n-k}$ respectively $\mathbb{R}^k$. The Morse function f has the form

$$f : \mathbb{R}^{n-k}\times\mathbb{R}^k \to \mathbb{R}, \quad (x,y)\mapsto -||y||^2 + ||x||^2$$

and the action of the negative gradient flow is given by $(x,y)\mapsto s\cdot(x,y) = (e^{-2s}x, e^{2s}y)$, see (1.10). Define

$$Q := S^{n-k-1}\times\mathbb{R}^k\times[0,\infty[$$

and maps

$$l : Q \to f^{-1}(1)\subset\mathbb{R}^{n-k}\times\mathbb{R}^k, \quad (\theta,y,t)\to(\sqrt{1+t^2}\theta, ty),$$
$$r : Q\to\mathbb{R}^{n-k}\times\mathbb{R}^k, \quad (\theta,y,t)\mapsto(t\sqrt{1+t^2||y||^2}\theta, y)$$

which are just local expressions for $Q(i)$ and the maps l_i and r_i we defined in section 4 of this chapter. We parametrise the 1-level by

$$\varphi : S^{n-k-1}\times\mathbb{R}^k\to f^{-1}(1), \quad (\theta,y)\mapsto(\sqrt{1+||y||^2}\theta, y).$$

Furthermore we set

$$\tilde{l} := \varphi^{-1}\circ l : Q\to S^{n-k-1}\times\mathbb{R}^k, (\theta,y,t)\mapsto(\theta,ty).$$

Define $N := W^-(p) \cap f^{-1}(1) \subset S^{n-k-1} \times \mathbb{R}^k \cong f^{-1}(1)$ — this is a submanifold transversal to $S^{n-k-1} \times \{0\}$ of dimension $l-1$. Define

$$\mathcal{T} := N \cap (S^{n-k-1} \times \{0\}),$$

a manifold of dimension $l-k-1$ and

$$\hat{W} := \tilde{l}^{-1}(N) \subset Q,$$

a manifold of dimension $l-k$ and observe that this are just local representations of the spaces $\mathcal{T}(p,q)$ and $\hat{W}^-(p)$. Note that

$$\partial \hat{W} = \hat{W} \cap \partial Q = \mathcal{T} \times \mathbb{R}^k \times \{0\} \subset S^{n-k-1} \times \mathbb{R}^k \times [0,\infty[.$$

We define the following projections:

$$\begin{aligned}
p_1 &: S^{n-k-1} \times \mathbb{R}^k \to S^{n-k-1}, \\
p_2 &: S^{n-k-1} \times \mathbb{R}^k \to \mathbb{R}^k, \\
\pi_1 &: Q = S^{n-k-1} \times \mathbb{R}^k \times [0,\infty[\to S^{n-k-1}, \\
\pi_2 &: Q = S^{n-k-1} \times \mathbb{R}^k \times [0,\infty[\to \mathbb{R}^k, \\
\pi_3 &: Q = S^{n-k-1} \times \mathbb{R}^k \times [0,\infty[\to [0,\infty[.
\end{aligned}$$

Let $\omega_p \in \Omega^l(W^-(p))$ be a volume form that gives rise to the orientation on $W^-(p) \subset \mathbb{R}^{n-k} \times \mathbb{R}^n$ and $\omega_q \in \Omega^k(\mathbb{R}^k)$ shall represent the orientation on $W^-(q) = \mathbb{R}^k$. We can extend ω_p to a l-form $\tilde{\omega}_p$ on $\mathbb{R}^{n-k} \times \mathbb{R}^k$. Furthermore we orient N such that $-\operatorname{grad}_g(f)$ followed by orientation of N yields the orientation of $W^-(p)$. This orientation is represented by $\mu \in \Omega^{l-1}(N)$ which can be extended to a $(l-1)$-form $\tilde{\mu}$ on $S^{n-k-1} \times \mathbb{R}^k$.
Furthermore $\mathcal{T}$ — the local model for $\mathcal{T}(p,q)$ — is oriented such that $-\operatorname{grad}_g(f)$ followed by the orientation of $\mathcal{T}$ yields the orientation of $\mathcal{M}(p,q)$ and this space is oriented by convention 2.). Let $\tau \in \Omega^{l-k-1}(\mathcal{T})$ be a volume form that gives rise to the described orientation on $\mathcal{T}$. We extend it to a $(l-k-1)$-form $\tilde{\tau}$ on S^{n-k}. By convention 1.), 2.) and the definition of $\tilde{\mu}$ it follows that $p_1^*\tilde{\tau} \wedge p_2^*\tilde{\omega}_q$ represents the same orientation as $\tilde{\mu}$ on N. Hence

$$p_1^*\tilde{\tau} \wedge p_2^*\tilde{\omega}_q = \lambda\tilde{\mu} \tag{2.7}$$

on N for a smooth function $\lambda : S^{n-k-1} \times \mathbb{R}^k \to \mathbb{R}$ with $\lambda \equiv 1$ on $\mathcal{T}$.
On $\hat{W} \setminus \partial\hat{W}$

$$\frac{\partial}{\partial s}\pi_3(s \cdot (\theta, y, t)) = \frac{\partial}{\partial s} e^{-2s} t = -2e^{-2s} t < 0$$

holds, so the orientation on $\hat{W} \setminus \partial\hat{W}$ can be described by $-d\pi_3 \wedge \tilde{l}^* p_2^*\tilde{\mu}$. This follows from the definition of $\tilde{\mu}$ and the fact that π_3 decreases along the flow lines. With the help of (2.7.) we obtain that

$$-d\pi_3 \wedge \tilde{l}^* p_1^*\tilde{\tau} \wedge \tilde{l}^* p_2^*\tilde{\omega}_q \tag{2.8}$$

describes the orientation of $\hat{W}$(near $\partial\hat{W}$). On the other hand we see that $-d\pi_3$ represents the outward pointing vector on $\hat{W}$ and by our conventions for the product orientation it follows that

$$-d\pi_3 \wedge \pi_1^*\tilde{\tau} \wedge \pi_2^*\tilde{\omega}_q \tag{2.9}$$

gives the orientation on $\hat{W}^-$ that it inherits from the boundary. So we have to compare the two differential forms (2.8) and (2.9). We compute $(p_1 \circ \tilde{l})(\theta, y, t) = \theta = \pi_1(\theta, y, t)$ and $(p_2 \circ \tilde{l})(\theta, y, t) = ty = t\pi_2(\theta, y, t)$. It follows that

$$-d\pi_3 \wedge \tilde{l}^* p_1^* \tilde{\tau} \wedge \tilde{l}^* p_2^* \tilde{\omega}_q = -d\pi_3 \wedge \pi_1^* \tilde{\tau} \wedge \lambda' \pi_2^* \tilde{\omega}_q$$

where $\lambda' > 0$ on $\hat{W} \setminus \partial\hat{W}$ near $\partial\hat{W}$. So the two orientations on $\hat{W}$ coincide.

□

2.35. Proposition
The orientation of $\partial_1\hat{\mathcal{T}}(p,q)$ induced by $\mathcal{T}(p,q)$ coincides with the orientation of $\partial_1\hat{\mathcal{T}}(p,q)$ that it inherits from

$$\partial_1\hat{\mathcal{T}}(p,q) = \bigsqcup_{z \in Cr(f)} \mathcal{T}(p,z) \times \mathcal{T}(z,q)$$

up to the factor $(-1)^{\mathrm{ind}(p)-\mathrm{ind}(z)}$.

<u>Proof</u>:
Like in the proof of Proposition (2.34.), let $z \in Cr(f)$ of index k and fix a convenient chart for z. Define

$$P := S^{n-k-1} \times S^{k-1} \times [0, \infty[$$

and maps

$$p_+ : P \to f^{-1}(1) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k, \quad (\theta^+, \theta^-, t) \mapsto (\sqrt{1+t^2}\theta^+, t\theta^-),$$

$$p_- : P \to f^{-1}(-1) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k, \quad (\theta^+, \theta^-, t) \mapsto (t\theta^+, \sqrt{1+t^2}\theta^-).$$

Again, P and these maps are local versions of P_i and p_i^+ respectively p_i^-, see section 3 of this chapter. Next we parametrise the 1-level by

$$\varphi_+ : S^{n-k-1} \times \mathbb{R}^k \to f^{-1}(1), \quad (\theta^+, y) \mapsto (\sqrt{1+||y||^2}\theta^+, y)$$

and the -1-level by

$$\varphi_- : \mathbb{R}^{n-k} \times S^{k-1} \to f^{-1}(-1), \quad (x, \theta^-) \mapsto (x, \sqrt{1+||x||^2}\theta^-).$$

Furthermore we define

$$\tilde{\varphi}_+ := \varphi_+^{-1} \circ p_+ : P \to S^{n-k-1} \times \mathbb{R}^k, \quad (\theta^+, \theta^-, t) \mapsto (\theta^+, t\theta^-),$$

$$\tilde{\varphi}_- := \varphi_-^{-1} \circ p_- : P \to \mathbb{R}^{n-k} \times S^{k-1}, \quad (\theta^+, \theta^-, t) \mapsto (t\theta^+, \theta^-),$$

a lot of projections

$$p_1 : S^{n-k-1} \times \mathbb{R}^k \to S^{n-k-1},$$

$$p_2 : S^{n-k-1} \times \mathbb{R}^k \to \mathbb{R}^k,$$

$$\begin{array}{rcl} \pi_1 : P & = & S^{n-k-1} \times S^{k-1} \times [0, \infty[\to S^{n-k-1}, \\ \pi_2 : P & = & S^{n-k-1} \times S^{k-1} \times [0, \infty[\to S^{k-1}, \\ \pi_3 : P & = & S^{n-k-1} \times S^{k-1} \times [0, \infty[\to [0, \infty[, \end{array}$$

and two inclusions

$$i_1 : S^{k-1} \hookrightarrow \mathbb{R}^k,$$

$$i_2 : S^{n-k-1} \times \mathbb{R}^k \hookrightarrow \mathbb{R}^{n-k} \times \mathbb{R}^k.$$

Choose an orientation of $\mathbb{R}^k \cong W^-(z)$ and let $\omega_z \in \Omega^k(\mathbb{R}^k)$ be a volume form that induces this orientation.
We denote a volume form that gives rise to the orientation of $W^-(p)$ by $\omega_p \in \Omega^{l^-}(W^-(p))$ and extend ω_p to a l^--form on $\mathbb{R}^{n-k} \times \mathbb{R}^k$. Set $N^- := W^-(p) \cap f^{-1}(1) \subset S^{n-k-1} \times \mathbb{R}^k$. This is a submanifold transversal to $S^{n-k-1} \times \{0\}$ of dimension $l^- - 1$. We orient N^- as in the proof of Proposition (2.34.), let $\mu^- \in \Omega^{l^- -1}(N^-)$ be a volume form that gives rise to this orientation. Now define $\mathcal{T}^- := N^- \cap (S^{n-k-1} \times \{0\})$, this is a submanifold of dimension $l^- - k - 1$ and inherits an orientation as was described in the proof of Proposition (2.34.). Let $\tau^- \in \Omega^{l^- -k-1}(\mathcal{T})$ be a volume form that induces this orientation, choose an extension of τ^-, denoted by $\tilde{\tau}^- \in \Omega^{l^- -k-1}(S^{n-k-1})$. By definition of the orientation of $\mathcal{T}^-$ the relation

$$p_1^* \tilde{\tau}^- \wedge p_2^* \omega_z = \lambda_1 \tilde{\mu}^- \in \Omega^{l^- -1}(N^-) \tag{2.10}$$

for a smooth functions $\lambda_1 : N^- \to \mathbb{R}$ with $\lambda_1 \equiv 1$ on $\mathcal{T}^-$ holds.
Additionally, let $N^+ := W^+(q) \cap f^{-1}(-1) \subset \mathbb{R}^{n-k} \times S^{k-1}$. This is a submanifold transversal to $\{0\} \times S^{k-1}$ of dimension $l^+ - 1$. Let $\omega_q \in \Omega^{n-l^+}(W^+(q)$ be a form that gives rise to the co-orientation on $W^+(q)$ and extend it to a $(n - l^+)$-form $\tilde{\omega}_q$ on $\mathbb{R}^{n-k} \times \mathbb{R}^k$. We can co-orientate N^+ with the the form $i_2^* \tilde{\omega}_q$. Next we define $\mathcal{T}^+ := N^+ \cap (\{0\} \times S^{k-1})$ and this is a submanifold of dimension $l^+ + k - 2 - n$. $\mathcal{T}^+$ is a local model for $\mathcal{T}(z, q)$ and by convention 1.), 2.) and the definition of the co-orientation of $\tilde{\mu}^+$ it is oriented. Let $\tau^+ \in \Omega^{l^+ -k-n-}(\mathcal{T}^+)$ be a volume form for this orientation and extend it to

a form $\tilde{\tau}^+$ on S^{k-1}.
Next we define

$$\tilde{\mathcal{T}} := \tilde{\varphi}_+^{-1}(N^-) \cap \tilde{\varphi}_-^{-1}(N^+) \subset P$$

and observe that

$$\partial\tilde{\mathcal{T}} = \tilde{\mathcal{T}} \cap \partial P = \mathcal{T}^- \times \mathcal{T}^+ \times \{0\} \subset S^{n-k-1} \times S^{k-1} \times [0, \infty[.$$

$\tilde{\mathcal{T}}$ is an oriented submanifold of dimension $l^+ + l^- - n - 2$, let $\tau \in \Omega^{l^+ + l^- - n - 1}(\tilde{\mathcal{T}})$ be a volume form that gives rise to this orientation. $\tilde{\tau} \in \Omega^{l^+ + l^- - n - 2}(P)$ denotes a smooth extension of τ. The relation that $\tilde{\tau}$ must satisfy — see convention 1.), 2.) and the definitions of $\tilde{\mu}_-$ and $\tilde{\mu}_+$ — is

$$\tilde{\tau} \wedge (\tilde{\varphi}_-)^* \tilde{\mu}^+ = \lambda_2 (\tilde{\varphi}_+)^* \tilde{\mu}^-. \tag{2.11}$$

We calculate

$$\frac{\partial}{\partial s} \pi_3(s \cdot (\theta^+, \theta^-, t)) = \frac{\partial}{\partial s} e^{-2s} t = -2e^{-2s} t < 0$$

and consequently the orientation of $\tilde{\mathcal{T}}$ it inherits from the orientation of the boundary is represented by $-d\pi_3 \wedge \pi_1^* \tilde{\tau}^- \wedge \pi_2^* \tilde{\tau}^+$. We want to understand how these two orientations fit together and hence, we have to calculate

$$-d\pi_3 \wedge \pi_1^* \tilde{\tau}^- \wedge \pi_2^* \tilde{\tau}^+ \wedge (\tilde{\varphi}_-)^* \tilde{\mu}^+. \tag{2.12}$$

Convention 1.) and 2.) and the definition of $\tilde{\mu}^+$ imply that

$$d\pi_3 \wedge \pi_2^* \tilde{\tau}^+ \wedge (\tilde{\varphi}_-)^* \tilde{\mu}^+$$

induces the same orientation as $\pi_2^* i_1^* \omega_z$ and so

$$-d\pi_3 \wedge \pi_1^* \tilde{\tau}^- \wedge \pi_2^* \tilde{\tau}^+ \wedge (\tilde{\varphi}_-)^* \tilde{\mu}^+$$

describes the same orientation as

$$(-1)^{l^- - k} \pi_1^* \tilde{\tau}^- \wedge \pi_2^* i_1^* \omega_z.$$

Next we have to understand the orientation that is given by $\pi_1^* \tilde{\tau}^- \wedge \pi_2^* i_1^* \omega_z$. If we apply $(\tilde{\varphi}_+)^*$ to (2.10.) and make use of

$$(p_1 \circ \tilde{\varphi}_+)(\theta^+, \theta^-, t) = \theta^+ = \pi_1(\theta^+, \theta^-, t)$$

and

$$(p_2 \circ \tilde{\varphi}_+)(\theta^+, \theta^-, t) = t\theta^- = t(i_1 \circ \pi_2)(\theta^+, \theta^-, t)$$

we obtain that

$$\begin{aligned} \pi_1^* \tilde{\tau}^- \wedge \lambda_3 \pi_2^* i_1^* \omega_z &= (\tilde{\varphi}_+)^* p_1^* \tilde{\tau}^- \wedge (\tilde{\varphi}_+)^* p_2^* \omega_z \\ &= \lambda_1 (\tilde{\varphi}_+)^* \tilde{\mu}^-. \end{aligned}$$

So, the two orientations given by $\tilde{\tau}$ and $-d\pi_3 \wedge \pi_1^* \tilde{\tau}^- \wedge \pi_2^* \tilde{\tau}^+$ coincide up to the factor $(-1)^{l^- - k}$.

□

Chapter 3

Morse Homology

3.1 Morse Homology

3.1. Definition $C_k(f;\mathbb{Z})$, $C_*(f;\mathbb{Z})$
Given a Morse–Smale pair (f,g) on a smooth compact manifold M with boundary. We define

$$C_k(f;\mathbb{Z}) := \mathbb{Z}[Cr_k(f)]$$
$$C_*(f;\mathbb{Z}) := \mathbb{Z}[Cr(f)].$$

So, $C_k(f;\mathbb{Z})$ is the free $\mathbb{Z}$-module generated by $Cr_k(f)$ and $C_*(f;\mathbb{Z})$ is the free $\mathbb{Z}$-module generated by $Cr(f)$.

Remark: the definition of $C_k(f;\mathbb{Z})$ and $C_*(f;\mathbb{Z})$
By non-degeneracy of critical points, one knows that if we consider compact manifolds, there are only finitely many critical points and so all these modules are finitely generated.
Clearly $C_*(f;\mathbb{Z})$ possesses a natural $\mathbb{Z}$-grading, as indicated by *: $C_*(f;\mathbb{Z}) := \bigoplus_{k\in\mathbb{Z}} C_k(f;\mathbb{Z})$ and it is obvious that $C_k(f;\mathbb{Z}) = 0$ for $k < 0$ and for $k > \dim M$. So we obtain a finite sequence of freely-generated modules for every Morse–Smale pair (f,g) on a compact manifold M.
Instead of looking at critical points on the whole manifold, we can restrict ourselves to critical points on the boundary or on any other sub-manifold. As critical points on the boundary are critical points on the whole manifold, $C_*(f_{\partial M};\mathbb{Z})$ — the complex of critical points where ∂M is considered as a manifold in its own right — can be regarded as a sub-module of $C_*(f;\mathbb{Z})$. Because the indices must not shift we can consider the inclusions $C_k(f_{\partial M};\mathbb{Z}) \hookrightarrow C_k(f;\mathbb{Z})$.

As usually, we can define the dual complexes $C^k(f;\mathbb{Z})$ and $C^*(f;\mathbb{Z})$ by considering homomorphisms $C_k(f;\mathbb{Z}) \to \mathbb{Z}$ respectively $C_*(f;\mathbb{Z}) \to \mathbb{Z}$. As $C^k(f;\mathbb{Z})$ and $C^*(f;\mathbb{Z})$ are freely generated and we know a base, homeomorphisms can be identified with maps from elements of the base, i.e. the critical points, to $\mathbb{Z}$:

$$C^k(f;\mathbb{Z}) \cong \mathrm{Maps}(Cr_k(f);\mathbb{Z}).$$

We can also form the modules generated by the critical points over Abelian groups different from $\mathbb{Z}$ by setting

$$C_k(f;G) := C_k(f;\mathbb{Z}) \otimes G,$$

$$C^k(f;G) := \mathrm{Hom}(C_k(f;\mathbb{Z}),G).$$

<u>Remark</u>: the differential
Given a Morse–Smale pair (f,g), we fix orientations of $W^-(x)$ for every $x \in Cr(f)$. In section 5 of the last chapter we have seen that by orienting all unstable manifolds, orientations on $\mathcal{T}(x,y)$ are defined too. Consider

$$I_k : Cr_k(f) \times Cr_{k-1}(f) \to \mathbb{Z}, \quad (p,q) \mapsto I_k(p,q)$$

where I_k is defined as follows: If $\mathcal{T}(p,q) = \emptyset$ we set $I_k(p,q) = 0$. If $\mathcal{T}(p,q) \neq \emptyset$, $\mathcal{T}(p,q)$ is zero-dimensional and compact and consequently it consists of a finite collection of oriented points $\gamma \in \mathcal{T}(p,q)$. We define $n(\gamma)$ to be the sign given by the orientation of γ and set

$$I_k(p,q) := \sum_{\gamma \in \mathcal{T}(p,q)} n(\gamma).$$

3.2. Definition *the differential*
The differential $\partial_k : C_k(f;\mathbb{Z}) \to C_{k-1}(f;\mathbb{Z})$ *is defined by setting*

$$\partial_k(x) := \sum_{\mathrm{ind}(y)=k-1} I_k(x,y) \cdot y$$

on the generators and extending this $\mathbb{Z}$-linearly to the whole module.

3.3. Proposition
$(C_*(f;\mathbb{Z}),\partial_*)$ *is a graded differential complex.*

<u>Proof</u>:
What remains to show is that $\partial_{k-1} \circ \partial_k = 0$: let $x \in Cr_k(f)$ be arbitrary,

then

$$\begin{aligned}
\partial_{k-1}(\partial_k(x)) &= \partial_{k-1}\left(\sum_{\mathrm{ind}(y)=k-1} I_k(x,y)\cdot y\right) \\
&= \sum_{\mathrm{ind}(y)=k-1} I_k(x,y)\cdot \partial_{k-1}(y) \\
&= \sum_{\mathrm{ind}(y)=k-1} I_k(x,y)\left(\sum_{\mathrm{ind}(z)=k-2} I_{k-1}(y,z)\cdot z\right) \\
&= \sum_{\mathrm{ind}(z)=k-2}\left(\sum_{\mathrm{ind}(y)=k-1} I_k(x,y)I_{k-1}(y,z)\right)\cdot z
\end{aligned}$$

so $\partial_{k-1}\circ\partial_k = 0$ if

$$\sum_{\mathrm{ind}(y)=k-1} I_k(x,y)I_{k-1}(y,z)$$

vanishes for all $x \in Cr_k(f)$ and $z \in Cr_{k-2}(f)$.

$$\begin{aligned}
\sum_{\mathrm{ind}(y)=k-1} I_k(x,y)I_{k-1}(y,z) &= \sum_{\mathrm{ind}(y)=k-1}\left(\sum_{\gamma\in\mathcal{T}(x,y)} n(\gamma)\sum_{\sigma\in\mathcal{T}(y,z)} n(\sigma)\right) \\
&= \sum_{\mathrm{ind}(y)=k-1}\left(\sum_{(\gamma,\sigma)\in\mathcal{T}(x,y)\times\mathcal{T}(y,z)} n(\gamma)n(\sigma)\right) \\
&= \sum_{(\gamma,\sigma)\in\bigsqcup_{\mathrm{ind}(y)=k-1}\mathcal{T}(x,y)\times\mathcal{T}(y,z)} n(\gamma)n(\sigma) \\
&= -\sum_{(\gamma,\sigma)\in\partial_1\hat{\mathcal{T}}(x,z)} [\gamma\times\sigma]
\end{aligned}$$

because the orientation on $\mathcal{T}(x,y)\times\mathcal{T}(y,z)$ is the same as the one obtained when regarding $\mathcal{T}(x,y)\times\mathcal{T}(y,z)$ as a subset of $\partial_1\hat{\mathcal{T}}(x,z)$ up to a factor $(-1)^{\mathrm{ind}(x)-\mathrm{ind}(y)} = (-1)^1 = -1$, see Proposition (2.35.). As $\partial_1\hat{\mathcal{T}}(x,z) = \partial\hat{\mathcal{T}}(x,z)$ is 0-dimensional and compact, it is a collection of finitely many oriented points, denoted by $[\gamma\times\sigma]$. $\hat{\mathcal{T}}(x,z)$ is 1-dimensional, so we can apply:

3.4. Theorem *classification of 1-dimensional smooth manifolds*
Any smooth, connected 1-dimensional manifold is diffeomorphic either to the circle S^1 or to some interval of real numbers.

Proof:
A proof can be found in [11].

□

As $\hat{\mathcal{T}}(x,z)$ is compact, it can only be a union of finitely many copies of S^1 and of closed, bounded intervals. Consequently, the sum over the orientations of the boundary-points is zero, and so the claim follows.

□

3.5. Definition *Morse homology*
The homology

$$H_k(f;\mathbb{Z}) := \frac{ker(\partial_k : C_k(f;\mathbb{Z}) \to C_{k-1}(f;\mathbb{Z}))}{im(\partial_{k+1} : C_{k+1}(f;\mathbb{Z}) \to C_k(f;\mathbb{Z}))}$$

of the differential complex $(C_*(f), \partial_*)$ *is called* Morse homology.

Remark: Morse cohomology
We define the differential of $(C^*(f;\mathbb{Z}), \partial^*)$ by

$$\partial^k : C^k(f;\mathbb{Z}) \to C^{k+1}(f;\mathbb{Z}), \quad \phi \mapsto \partial^k \phi$$

where

$$(\partial^k \phi)(x) := \sum_{\text{ind}(y)=k} I_k(x,y)\phi(y), \quad \text{for} \quad x \in Cr_{k+1}(f)$$

is $\mathbb{Z}$-linearly extended.
$(C^*(f;\mathbb{Z}), \partial^*)$ is the complex dual to $(C_*(f;\mathbb{Z}), \partial_*)$ and so it must be a differential complex too ($\partial^{k+1} \circ \partial^k = 0$).
So we can define the cohomology of the differential complex $(C^*(f;\mathbb{Z}), \partial^*)$,

$$H^k(f;\mathbb{Z}) = \frac{\ker(\partial^k : C^k(f;\mathbb{Z}) \to C^{k+1}(f;\mathbb{Z}))}{\mathrm{im}(\partial^{k-1} : C^{k-1}(f;\mathbb{Z}) \to C^k(f;\mathbb{Z}))}$$

called the *Morse cohomology.*

Remark:
We show that for a given Morse–Smale pair (f,g) the homology is independent from the fixed orientations next
A priori, the Morse homology depends on the chosen Morse–Smale pair (f,g) and on the fixed orientations of all the unstable manifolds. In section 3 of this chapter it is shown that the Morse homology is isomorphic to the singular homology for any Morse–Smale pair. Consequently the Morse homology is independent from the specific Morse–Smale pair and the orientations.

If we would have the aim to build a homology theory with the help of the Morse–Smale complex that is as intrinsic as possible, this approach would not be totally satisfying, as the isomorphism to another homology is exploited to get these results. In [16] Morse homology is developed as a full-fledged homology theory.

3.6. Proposition
Different choices of orientations of the unstable manifolds lead to isomorphic differential complexes and consequently to isomorphic homologies (respectively cohomologies).

Proof:
We consider two collections of orientations of all unstable manifolds $(\theta_x)_{x\in Cr(f)}$ and $(\tilde{\theta}_x)_{x\in Cr(f)}$ and define

$$\psi : C_*(f;\mathbb{Z}) \to C_*(f;\mathbb{Z})$$

by setting $x \mapsto \varepsilon(x)\cdot x$, where $\varepsilon(x) = +1$ if $\theta_x = \tilde{\theta}_x$ and -1 if $\theta_x = -\tilde{\theta}_x$ and extending this linearly. Obviously this is a module-automorphism.
We claim that ψ is a chain map. In the following $\sim$ will indicate when we are working with the orientations $\tilde{\theta}_{x\in Cr(f)}$. First we compute

$$\tilde{I}_k(x,y) = \sum_{\gamma\in\mathcal{T}(x,y)} \tilde{n}(\gamma) = \sum_{\gamma\in\mathcal{T}(x,y)} \varepsilon(x)\cdot n(\gamma)\cdot\varepsilon(y) = \varepsilon(x)\cdot I_k(x,y)\cdot\varepsilon(y)$$

where the last equation follows from Proposition (2.34.). Consequently

$$\begin{aligned}
(\psi\circ\tilde{\partial}_k)(x) &= \psi\left(\sum_{\mathrm{ind}(y)=k-1} \tilde{I}_k(x,y)\cdot y\right) \\
&= \sum_{\mathrm{ind}(y)=k-1} \varepsilon(x)I_k(x,y)\varepsilon(y)\cdot\psi(y) \\
&= \sum_{\mathrm{ind}(y)=k-1} \varepsilon(x)I_k(x,y)\varepsilon(y)\varepsilon(y)\cdot y \\
&= \varepsilon(x)\sum_{\mathrm{ind}(y)=k-1} I_k(x,y)\cdot y \\
&= \varepsilon(x)\cdot\partial_k(x) = \partial_k(\varepsilon(x)\cdot x) = (\partial_k\circ\psi)(x)
\end{aligned}$$

and so ψ is an isomorphism of chain complexes.

□

3.2 Spectral Sequences

We introduce spectral sequences of filtered differential complexes, explain convergence and state two results we will apply to the Morse–Smale complex in order to make contact with standard homology respectively cohomology theories in the last two chapters. For a general introduction we refer to [20] and [6] and we follow the approach presented in the last one.

Remark: exact couples
First of all, we introduce the purely algebraic notation of an *exact couple.* An exact couple consists of two Abelian groups A, B and group homomorphisms $i : A \to A$, $j : A \to B$, $k : B \to A$ such that the following diagram is exact:

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

and we define $d : B \to B$ by $d := j \circ k$. We calculate $d^2 = (j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = 0$ since $k \circ j = 0$ by exactness. Hence we can compute the homology of d: $H(B) := \ker d/\mathrm{im}\, d$.
We obtain the *derived couple*

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

by setting $A' := i(A)$, $B' := H(B)$ and

$$i' : A' \to A', \quad i(a) \mapsto i(i(a))$$

$$j' : A' \to B', \quad i(a) \mapsto [j(a)]$$
$$k' : B' \to A', \quad [b] \mapsto k(b).$$

A' and B' are Abelian groups. j' is well-defined because $j(a)$ is a cycle — $d(j(a)) = ((j \circ k) \circ j)(a) = (j \circ (k \circ j))(a) = 0$ — and because $[j(a)]$ is independent from the particular choice of a: Suppose $i(a) = i(\overline{a})$ hence $i(a - \overline{a}) = 0$ and by exactness of the exact couple there is $b \in B$ with $k(b) = a - \overline{a}$ and consequently $j(a) - j(\overline{a}) = (j \circ k)(b) = d(b)$ and so $[j(a)] = [j(\overline{a})]$.
That k' is well-defined can be checked as follows: First we have $0 = d(b) = (j \circ k)(b)$ for $[b] \in H(B)$, i.e. $k(b)$ is in the kernel of j, hence in the image of i and so $k(b) = i(a) \in i(A) = A'$. Pick another representative of the homology class $[b]$: $[b] = [\overline{b}] \Longrightarrow \overline{b} - b = d(e)$ for some $e \in B$. But then $k'([\overline{b}]) - k'([b]) = k(\overline{b} - b) = (k \circ d)(e) = ((k \circ j) \circ k)(e) = 0$.

3.7. Lemma
The derived couple of an exact couple is an exact couple.

Proof:
A' and B' are again Abelian groups, i', j', k' are obviously group homomorphisms. Next we simply demonstrate exactness at every group in the triangle.

1.) Exactness at $A' \xrightarrow{j'} B' \xrightarrow{k'} A'$:
im $j' \subset$ ker k': $k'(j'(a')) = k'(j'(i(a))) = k'(j(a)) = (k \circ j)(a) = 0$.
ker $k' \subset$ im j': $k'(b) = k(b) = 0 \Longrightarrow b = j(a) = j'(i(a))$.

2.) Exactness at $B' \xrightarrow{k'} A' \xrightarrow{i'} A'$:
im $k' \subset$ ker i': $i'(k'[b]) = i'(k(b)) = i(k(b)) = 0$.
ker $i' \subset$ im k': suppose $i'(i(a)) = i(i(a)) = 0$, hence $i(a) \in$ ker i and by exactness there is an $b \in B$ with $k(b) = i(a)$. Consequently $k'[b] = k(b) = i(a)$.

3.)Exactness at $A' \xrightarrow{i'} A' \xrightarrow{j'} B'$:
im $i' \subset$ ker j': $(j' \circ i')(i(a)) = j'(i(i(a))) = [(j \circ i)(a)] = [0]$.
ker $j' \subset$ im i': $j'(i(a)) = [j(a)] = 0 \Longrightarrow j(a) = d(b) = (j \circ k)(b)$ for some $b \in B$. Hence $j(a - k(b)) = 0$ and by exactness there is an $c \in A$ such that $i(c) = a - k(b) \Longrightarrow a = i(c) + k(b) \Longrightarrow i(a) = i(i(c)) + i(k(b)) = i(i(c)) \Longrightarrow$ $i(a) = i'(i(c))$.

□

Remark: spectral sequence of filtered complexes
Let K be a differential complex with differential operator d, i.e. K is an Abelian group and $d : K \to K$ is a group homomorphism with $d^2 = 0$. A subgroup $K' \subset K$ is a *sub-complex* of K if $d(K') \subset K'$.
A *finite filtration* of a differential complex K is a sequence of sub-complexes $K_0, \dots, K_n$ such that

$$K = K_0 \supset K_1 \supset \ K_2 \supset \dots \supset K_n = \emptyset \tag{3.1}$$

and we set $K_p = K$ for $p < 0$. A differential complex K equipped with a filtration is called a *filtered complex.* We call

$$GK := \bigoplus_{p=0}^{\infty} K_p/K_{p+1} \tag{3.2}$$

the *associated graded complex.* Next we set $A := \oplus_{p \in \mathbb{Z}} K_p$ and A is a differential complex with differential operator d. Define $i : A \to A$ by the inclusion $K_{p+1} \hookrightarrow K_p$ for all $p \in \mathbb{Z}$. B shall be the quotient defined by the exact sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0 \tag{3.3}$$

and observe that B is the associated graded complex GK of K.
The sequence of sub-complexes (3.1) induces a sequence in homology

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K_0) \longleftarrow H(K_1) \longleftarrow \ldots \longleftarrow H(K_n) = 0$$

and $H(K)$ inherits a filtration given by $F_pH(K) := \operatorname{im}(H(K_p) \to H(K))$ and one obtains a sequence of inclusions

$$H(K) = F_0H(K) \supset F_1H(K) \supset F_2H(K) \supset \ldots \supset F_nH(K) = 0 \qquad (3.4)$$

making $H(K)$ into a filtered complex and this filtration is called the *induced filtration* on $H(K)$.
From (3.3) we obtain an exact couple

$$\begin{tikzcd} A_1 := H(A) \arrow[rr, "i_1"] & & A_1 := H(A) \arrow[dl, "j_1"] \\ & B_1 := H(B) \arrow[ul, "k_1"] & \end{tikzcd}$$

and because the derived couples of exact couples are exact again we can iterate the process of building derived couples and obtain

$$\begin{tikzcd} A_r \arrow[rr, "i_r"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd}$$

after $(r-1)$ steps.
A_1 is the direct sum of

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K_0) \longleftarrow H(K_1) \longleftarrow H(K_2) \longleftarrow \ldots \longleftarrow H(K_n) = 0$$

A_2 is the direct sum of

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K_0) \supset F_1(H(K)) \longleftarrow i(H(K_2)) \longleftarrow \ldots \longleftarrow i(H(K_n)) = 0$$

A_3 is the direct sum of

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K_0) \supset F_1(H(K)) \supset F_2(H(K)) \longleftarrow \ldots \longleftarrow i(i(H(K_n))) = 0$$

etc.
After n steps one obtains that A_{n+1} is the direct sum of

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K_0) \supset F_1(H(K)) \supset F_2(H(K)) \supset \ldots \supset F_n(H(K)) = 0$$

and hence $A_{n+1} = A_{n+2} = \dots$. It is costumery to write $A_\infty := A_{n+1}$ in this case and one sees that

$$A_\infty = \bigoplus_{i=0}^{n} F_i H(B). \tag{3.5}$$

Since

$$\begin{tikzcd} A_{n+1} \arrow[rr, "i_{n+1}"] & & A_{n+1} \arrow[dl, "j_{n+1}"] \\ & B_{n+1} \arrow[ul, "k_{n+1}"] & \end{tikzcd}$$

is an exact couple and $i_{n+1} : A_{n+1} \to A_{n+1}$ is the inclusion, k_{n+1} is trivial. Hence $B_{n+1} = B_{n+2} = \dots$ and we write $B_\infty := B_{n+1}$ and obtain

$$\begin{tikzcd} A_\infty \arrow[rr, "i_\infty"] & & A_\infty \arrow[dl, "j_\infty"] \\ & B_\infty \arrow[ul, "k_\infty=0"] & \end{tikzcd}$$

B_∞ is the quotient of i_∞, so B_∞ is the associated graded complex $GH(K)$ of the differential complex $H(K)$ filtered by (3.4).

<u>Remark</u>: some terminology
Usually, one denotes $H(B)$ by E^1 which is a differential complex with differential operator $d_1 := j_1 \circ k_1$ and inductively defines $E^{r+1} := H(E^r)$ with differential $d_{r+1} := j_{r+1} \circ k_{r+1}$. A sequence of differential groups (E^r, d_r) in which each E^r is the homology of E^{r-1} is called a *spectral sequence*. If E^r becomes stationary for r sufficiently large, we denote the stationary value by E^∞ and if E^∞ is equal to the associated graded complex of some filtered group G we say that the sequence *converges* to G.
Now we can cite two results from [20] we will make use of in the next two sections. The statements given in [20] are slightly more general than the versions we will use because in [20] the filtrations are not assumed to be finite but must satisfy a weaker condition.

3.8. Theorem
Let K be a filtered differential complex with a finite filtration, see (3.1). Then there is a convergent spectral sequence with

$$E^1_{r,t} \cong H_{r+t}(K_r/K_{r+1}) \tag{3.6}$$

where the differential operator d_1 of the E^1-term is given by the connecting homomorphism of the triple (K_r, K_{r+1}, K_{r+2}) and E^∞ is isomorphic to the associated graded complex of $H(K)$ with the filtration given by (3.4).

Proof:
We have already given the main arguments.

□

Remark: induced maps in spectral sequences
Given a chain map τ between two differential complexes K and K' preserving the filtration, τ induces a homomorphism τ^1 between E^1 and E'^1 because the spectral sequences are defined in terms of the homologies. Step–by–step, τ induced homomorphisms $\tau^r : E^r \to E'^r$. The induced map between E^∞ and E'^∞ is denoted by τ^∞. If τ^r is an isomorphism, so is τ^s for $s \geq r$.

3.9. Theorem
Let K and K' be differential complexes equipped with finite filtrations, see (3.1). Given a chain map $\tau : K \to K'$ preserving the filtrations.
If for some $r \geq 1$ the induced map $\tau^r : E^r \to E'^r$ is an isomorphism, then τ induces an isomorphism

$$\tau_* : H_*(K) \xrightarrow{\cong} H_*(K').$$

Proof:
This follows by exploiting the fact that if τ^r is an isomorphism for some r, so is τ^∞. By Theorem (3.8.), τ induces an isomorphism between the associated graded complex of $H(K)$ respectively of $H(K')$. After applying the five lemma several times one obtains that $H(K)$ and $H(K')$ themselves are isomorphic.

□

3.3 Isomorphism to Singular Homology

The main aim of this section is to establish the connection between Morse homology and singular homology:

3.10. Theorem $H_*(f;\mathbb{Z}) \cong H_*(M;\mathbb{Z})$
Given a compact smooth manifold M, possibly with boundary ∂M and a Morse–Smale pair (f, g) on M (see Definition 1.15.). Then the homology of the Morse–Smale complex $(C_*(f;\mathbb{Z}), \partial_*)$ and the one of the singular complex $(S_*(M), d)$ of M are isomorphic.

We start with the following proposition:

3.11. Proposition
Given a Morse–Smale pair (f,g) and let $p \in M$ be a critical point of f of index k. Then $\hat{W}^-(p)/\partial\hat{W}^-(p)$ is homeomorphic to S^k.

Proof:
In Proposition (1.14.) we proved that $W^-(p) = \hat{W}^-(p) \setminus \partial\hat{W}^-(p)$ is diffeomorphic to $\mathbb{R}^k$, and in particular homeomorphic to Euclidean space. From the parametrisation $S^{k-1}\times]0,\infty[\to W^-(p)$ we used there we obtain a parametrisation $S^{k-1}\times]0,1[\to W^-(p) \hookrightarrow \hat{W}^-(p) \to \hat{W}^-(p)/\partial\hat{W}^-(p)$ which extends continuously to

$$S^{k-1} \times [0,1] \to \hat{W}^-(p)/\partial\hat{W}^-(p).$$

To see this, let U be a neighbourhood of $\partial\hat{W}^-(p)$ in $\hat{W}^-(p)$ and suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in $W^-(p)$ with $r(x_n) \to 1$, with $r : S^{k-1}\times[0,1[\to [0,1[$ being the projection to the second factor. Now we have to show that $x_n \in U$ for sufficiently large $n \in \mathbb{N}$. This follows since the complement of U is compact and hence r is bounded away from 1 on this complement.
Hence we obtain a continuous, bijective map

$$\Sigma S^{k-1} \to \hat{W}^-(p)/\partial\hat{W}^-(p)$$

where ΣS^{k-1} denotes the suspension of S^{k-1} which is homeomorphic to S^k. The compactness of $\hat{W}^-(p)$ and $\partial\hat{W}^-(p)$ now implies that we have found a homeomorphism

$$S^k \xrightarrow{\cong} \hat{W}^-(p)/\partial\hat{W}^-(p). \tag{3.7}$$

□

3.12. Corollary
Let (f,g) be a Morse–Smale pair on M and assume p is a critical point of f. Then

$$H_r(\hat{W}^-(p), \partial\hat{W}^-(p); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad r = \operatorname{ind}(p) \\ 0 & \text{if} \quad r \neq \operatorname{ind}(p). \end{cases}$$

Remark: a convenient filtration of M
We intend to apply Theorem (3.8.) and hence we define a finite filtration on M:

$$M_r := \bigcup_{\operatorname{ind}(x)\leq r} W^-(x) = \bigcup_{\operatorname{ind}(x)\leq r} \hat{\imath}_x(\hat{W}^-(x))$$

for all $r \in \mathbb{N}$ and $M_r := \emptyset$ for $r < 0$. Obviously $M_{r-1} \subset M_r$ for all $r \in \mathbb{Z}$ and $M_r = M$ for $r \geq \dim M$. Furthermore all M_r are compact because they are finite unions of images of compact sets (the $\hat{W}^-(x)$'s) under continuous

maps (the $\hat{i}_x$'s).
The filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$$

induces a filtration on the singular complex $S_*(M)$ given by $F_r S_*(M) = S_*(M_r)$. Now we can apply Theorem (3.8.):

<u>the E^1-term</u>:
$E^1_{r,t} = H_{r+t}(M_r, M_{r-1}; \mathbb{Z})$ and this can be computed with the help of Proposition (3.11.). Consider the map of pairs

$$\bigsqcup_{\operatorname{ind}(x)=k} (\hat{W}^-(x), \partial\hat{W}^-(x)) \stackrel{\sqcup(\hat{i}_x)}{\to} (M_k, M_{k-1})$$

which is continuous. This induces a bijective continuous map

$$\bigvee_{\operatorname{ind}(x)=k} \hat{W}^-(x)/\partial\hat{W}^-(x) \to M_k/M_{k-1} \tag{3.8}$$

where $\bigvee$ denotes the wedge, and by compactness of $\hat{W}^-(p)$ this is a homeomorphism. Hence

$$H_{r+t}(M_r, M_{r-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z}[Cr_r(f)] & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t \neq 0. \end{cases}$$

<u>the E^∞-term</u>:
We observe that for $k \geq 2$ all the differential in the E^k-terms are trivial because the E^1-term is non–trivial only in one row. Consequently, the E^2-term is concentrated in one row and because the spectral sequence does not change after the E^2-term, the E^∞-term is non-trivial only in the row $t = 0$, too. We want to calculate

$$E^\infty_{r,0} \quad = \quad GH(M)_{r,t} = \frac{F_r H_r(M)}{F_{r-1} H_r(M)} = \frac{\operatorname{im}(H_r(M_r) \to H_r(M))}{\operatorname{im}(H_r(M_{r-1}) \to H_r(M))}.$$

We claim that $\operatorname{im}(H_r(M_{r-1}) \to H_r(M)) = 0$. Consider the long exact sequence

$$\ldots \to H_{r+1}(M_{r-k}, M_{r-k-1}) \to H_r(M_{r-k-1}) \to H_r(M_{r-k}) \to H_r(M_{r-k}, M_{r-k-1}) \to \ldots$$

for $k > 0$. We know that $H_r(M_{r-k}, M_{r-k-1}) = 0$ and $H_{r+1}(M_{r-k}, M_{r-k-1}) = 0$, hence

$$H_r(M_{r-k-1}) \cong H_r(M_{r-k})$$

for $k > 0$. But then $H_r(M_{r-1}) \cong H_r(M_{r-2}) \cong \ldots \cong H_r(M_{-1}) = 0$ and the claim follows.

Next we show that $H_r(M_r) \to H_r(M)$ is surjective. Consider the long sequence

$$\ldots \to H_r(M_r) \to H_r(M_{r+1}) \to H_r(M_{r+1}, M_r)$$

and because of $H_r(M_{r+1}, M_r) = 0$ surjectivity of the map $H_r(M_r) \to H_r(M_{r+1})$ follows. Now consider the part

$$\ldots \to H_{r+1}(M_{r+k+1}, M_{r+k}) \to H_r(M_{r+k}) \to H_r(M_{r+k+1}) \to H_r(M_{r+k+1}, M_{r+k}) \to \ldots$$

of the long exact sequence and observe that $H_{r+1}(M_{r+k+1}, M_{r+k}) = 0$ and $H_r(M_{r+k+1}, M_{r+k}) = 0$ for $k > 0$. Hence $H_r(M_{r+k}) \cong H_r(M_{r+k+1})$ and so $H_r(M_{r+1}) \cong H_r(M_{r+2}) \cong \ldots \cong H_r(M_n) = H_r(M)$ and this implies that $H_r(M_r) \to H_r(M)$ is indeed surjective.
So we obtain $E^\infty_{r,0} = H_r(M)$.

Applying Theorem (3.8.) leads to

$$H_r(M;\mathbb{Z}) = E^\infty_{r,0} = E^2_{r,0} = \frac{\ker(d^1 : H_r(M_r, M_{r-1}) \to H_{r-1}(M_{r-1}, M_{r-2})}{\operatorname{im}(d^1 : H_{r+1}(M_r, M_{r-1}) \to H_r(M_{r-1}, M_{r-2})}$$

where $d^1 : H_r(M_r, M_{r-1};\mathbb{Z}) \to H_{r-1}(M_{r-1}, M_{r-2};\mathbb{Z})$ is the differential of the E^1-term and is given by the boundary operator of the triple (M_r, M_{r-1}, M_{r-2}). Consequently Theorem (3.10.) follows if we can show that $(H_r(M_r, M_{r-1};\mathbb{Z}), d^1)$ and $(C_r(f;\mathbb{Z}), \partial_r)$ are isomorphic chain complexes because then

$$H_r(f;\mathbb{Z}) \cong E^2_{r,0} = E^\infty_{r,0} = H_r(M;\mathbb{Z}).$$

To show that these chain complexes are isomorphic is the task of the rest of this section.

<u>Remark</u>: fundamental classes and manifolds with corners
Let X be an n-dimensional topological manifold. Then it is known that

$$\bigsqcup_{x \in X} H_n(X, X \setminus \{x\};\mathbb{Z}) \to X$$

is a covering space of X with fibre $\mathbb{Z}$. An orientation of X is equivalent to a continuous section in $\bigsqcup_{x\in X} H_n(X, X \setminus \{x\};\mathbb{Z}) \to X$ that maps every point $x \in X$ to a generator of $H_n(X, X \setminus \{x\};\mathbb{Z})$. For closed, oriented X there is a unique class in $H_n(X;\mathbb{Z})$ — written $[X]$ and called the *fundamental class* of X — that restricts to the distinguished generator of $H_n(X, X \setminus \{x\};\mathbb{Z})$ for all $x \in X$. For a detailed introduction to fundamental classes of topological manifolds, see [7] for instance.
Now assume X is a compact, oriented n-dimensional smooth manifold with corners. As remarked before $(X, \partial X)$ is a topological manifold with boundary. Consider a connected component $F \subset \partial_1 X$. Then there is a canonically compact smooth manifold with corners $\tilde{F}$ with interior F which satisfies a

compatibility-condition:

3.13. Lemma
Given a compact connected smooth manifold with corners X. Consider a connected component F of $\partial_1 X$ with topological closure $\overline{F} \subset \partial X$. Then there is a smooth compact manifold with corners $\tilde{F}$ and a surjective map $p : \tilde{F} \to \overline{F}$ such that
2.) $p : \tilde{F} \to X$ is a smooth extension of the inclusion $F \hookrightarrow X$.
1.) $p : \tilde{F} \to \overline{F}$ is a locally injective and a local homeomorphism.

Proof:
We start by choosing a Riemannian metric g on X which induces a Riemannian metric on F. This Riemannian metric on F gives rise to a metric d_F on F (by geodesic distance) which induces a topology on F. Be aware that this topology may differ from the one F inherits as a subset of X. We denote the metric on X which is induced by g by d_X. Now consider the topological completion $(\tilde{F}, d_{\tilde{F}})$ of (F, d_F).
The inclusion $iF \to X$ is uniformly convergent (this follow from $d_X(i(x), i(y)) \leq d_F(x, y)$) and consequently $i : F \hookrightarrow X$ can be extended to a (uniformly) continuous map $p : \tilde{F} \to \tilde{X} = X$ (because X is complete). The map $p : \tilde{F} \to X$ is surjective on $\overline{F}$: given $z \in \overline{F}$ arbitrary. Choose a sequence in F with $i(z_n) \to z$ in X. But $(z_n)_{n\in\mathbb{N}}$ is convergent in $\tilde{F}$ too and we denote the limit by y. But then

$$p(y) = p(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} p(z_n) = \lim_{n\to\infty} i(z_n) = z.$$

Next we check that $p : \tilde{F} \to X$ is locally injective. Let x be a point in $\tilde{F}$. By definition of $\tilde{F}$ x can be represented by $(x_n)_{n\in\mathbb{N}}$, a Cauchy sequence with respect to d_F. For $p(x) \in \overline{F}$ there is a small open neighbourhood of $p(x)$ which is contained in a chart neighbourhood of $p(x)$ in X. But if we use a chart we see that $V \cap F$ consists of finitely many connected components $V_1, \ldots, V_s$ (with $s \leq n$) and if V was sufficiently small, in these components (looking at each of these components separated) the topology induced by d_F and the one induced by d_X coincide. There is exactly one $i \in \{1, \ldots, s\}$ such that there is an $N \in \mathbb{N}$ such that $x_n \in V_i$ for all $n \geq N$. Without loss of generality we can assume $(x_n)_{n\in\mathbb{N}} \subset V_i$. But in V_i from $z_n \to z \in \overline{F}$ and $y_n \to z \in \overline{F}$ it follows that $d_F(x_n, y_n) \to 0$. Consequently for every $x \in \tilde{F}$ there is an open neighbourhood U of x such that if $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are two Cauchy sequences in $U \cap F$ that represent z_1 and z_2 with $p(z_1) = p(z_2)$ then $d_F(x_n, y_n) \to 0$ and hence $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ represent the same point in $\tilde{F}$.
Furthermore, we can use the constructed neighbourhood U to show that p is a local homeomorphism. We want to construct an inverse of $p_U : U \to p_U(U)$. Let $z \in p_U(U)$, and choose a sequence $(z_n)_{n\in\mathbb{N})}$ in $p(U) \cap F$ that

converges to z in X. Set $p^{-1}(z) = (z_n)_{n\in\mathbb{N}} \in \tilde{F}$. This is a well-defined map which is inverse to p_U and which can be checked to be continuous.
Now we can equip $\tilde{F}$ with a smooth structure such that $p : \tilde{F} \to X$ is smooth. For an arbitrary $x \in \tilde{F}$ we find an open neighbourhood U such that p restricted to this neighbourhood is a local homeomorphism. Then we choose a chart for $p(x) \in X$ which is denoted by (A, α). We want $(U, \alpha \circ p_U)$ to be a chart neighbourhood of x. So we have to check whether chart-changes are smooth: Let V be another neighbourhood of a point $y \in \tilde{F}$ as described before and (B, β) a chart for $p_V(y)$. Assume $U \cap V \neq \emptyset$. The coordinate change from $(U, \alpha \circ p_V)$ to $(V, \beta \circ p_V)$ computes to

$$(\beta \circ p_V) \circ (\alpha \circ p_U)^{-1} = \beta \circ \alpha^{-1}$$

what is smooth by construction. Additionally, the expression for p in local coordinates is given by

$$\beta \circ p \circ (\alpha \circ p_U)^{-1} = \beta \circ \alpha^{-1}$$

and consequently p is smooth.
To show that $\tilde{F}$ is compact one proceeds as follows: One checks that p is a closed mapping and that $p^{-1}(y)$ consists of finitely many points for every $p \in \overline{F}$. But then p is proper and because $\overline{F}$ is compact (it is a closed set in the compact space X), $p^{-1}(\overline{F}) = \tilde{F}$ is compact too. Points in $p^{-1}(y)$ are represented by Cauchy series converging to y between which the distance goes to zero with respect to d_F. It might happen that d_X goes to zero but d_F does not for such two series. However, if we consider a small neighbourhood U of y as used to demonstrate local injectivity of p we see that on every V_j the topology induced by d_X and the one induced by d_F is the same, so if $d_X(x_n, y_n) \to 0$ it follows that $d_F(x_n, y_n) \to 0$. Consequently, there are at most $s \leq n$ different points in $p^{-1}(y)$.

□

3.14. Lemma
Given a compact smooth oriented manifold with corners X. Let F be a connected component of $\partial_1 X$. F inherits an orientation from X as an open subset of ∂X, hence $\tilde{F}$ is canonically oriented too. Let $[\tilde{F}] \in H_{n-1}(\tilde{F}, \partial\tilde{F})$ be the fundamental class of $\tilde{F}$. Consider
$p : (\tilde{F}, \partial\tilde{F}) \to (\partial X, \partial_{\geq 2} X)$ *and*
$\iota_* : H_*(\partial X; \mathbb{Z}) \to H_*(\partial X, \partial_{\geq 2} X; \mathbb{Z})$.
Denote the set of connected components of $\partial_1 X$ by $\pi_0(\partial_1 X)$.
Then

$$\sum_{F\in\pi_0(\partial_1 X)} p_*([\tilde{F}]) = \iota_*([\partial X]) \in H_{n-1}(\partial X, \partial_{\geq 2} X; \mathbb{Z})$$

holds.

Proof:
For every point $x \in \partial X \setminus \partial_{\geq 2} X$ the identity induces an homomorphism

$$(l_x)_* : H_*(\partial X, \partial_{\geq 2} X; \mathbb{Z}) \to H_*(\partial X, \partial X \setminus \{x\}; \mathbb{Z}).$$

Let $x \in \partial X \setminus \partial_{\geq 2} X$, then there is an $F \in \pi_0(\partial_1 X)$ such that $x \in \tilde{F} \setminus \partial \tilde{F}$. By excision one has

$$H_{n-1}(\tilde{F}, \tilde{F} \setminus \{x\}; \mathbb{Z}) = H_{n-1}(\partial X, \partial X \setminus \{x\}; \mathbb{Z}).$$

By definition, if $[\tilde{F}]$ is restricted to $H_{n-1}(\tilde{F}, \tilde{F} \setminus \{x\}; \mathbb{Z})$ it equals the distinguished generator of $H_{n-1}(\tilde{F}, \tilde{F} \setminus \{x\}; \mathbb{Z})$ and because the orientation of $\tilde{F}$ is induced by the orientation on ∂X, $p_*([\tilde{F}])$ equals the generator of $H_{n-1}(\partial X, \partial X \setminus \{x\}; \mathbb{Z})$ if it is restricted to it. Consequently,

$$(l_x)_* \left(\sum_{F \in \pi_0(\partial_1 X)} p_*([\tilde{F}]) \right)$$

restricts to the distinguished generator of $H_{n-1}(\partial X, \partial X \setminus \{x\}; \mathbb{Z})$ for all $x \in \partial X \setminus \partial_{\geq 2} X$. But on the other hand, $(l_x)_*(\iota_*([\partial X]))$ is also the uniquely determined generator of $H_{n-1}(\partial X, \partial X \setminus \{x\}; \mathbb{Z})$ for every $x \in \partial X \setminus \partial_{\geq 2} X$, see [20]. Hence in $H_{n-1}(\partial X, \partial X \setminus \{x\}; \mathbb{Z})$ we have

$$(l_x)_* \left(\sum_{F \in \pi_0(\partial_1 X)} p_*([\tilde{F}]) \right) = (l_x)_* \left(\iota_*[\partial X] \right)$$

for all $x \in \partial X \setminus \partial_{\geq 2} X$.
Additionally

$$\begin{aligned}
H_{n-1}(\partial X, \partial_{\geq 2} X; \mathbb{Z}) &= H_{n-1}(\partial X / \partial_{\geq 2} X, *; \mathbb{Z}) \\
&= H_{n-1}\Big(\bigvee_{F \in \pi_0(\partial_1 X)} \tilde{F} / \partial \tilde{F}, *; \mathbb{Z}\Big) \\
&= \bigoplus_{F \in \pi_0(\partial_1 X)} H_{n-1}(\tilde{F} / \partial \tilde{F}, *; \mathbb{Z}) \\
&= \bigoplus_{F \in \pi_0(\partial_1 X)} H_{n-1}(\tilde{F}, \partial \tilde{F}; \mathbb{Z}) \\
&= \bigoplus_{F \in \pi_0(\partial_1 X)} \mathbb{Z}.
\end{aligned}$$

Now suppose we have $\alpha \in H_{n-1}(\partial X, \partial_{\geq 2} X; \mathbb{Z})$ such that $(l_x)_*(\alpha) = 0 \in H_{n-1}(\partial X, \partial X \setminus \{x\}; \mathbb{Z})$ for all $x \in \partial X \setminus \partial_{\geq 2} X$. By the decomposition of

$H_{n-1}(\partial X, \partial_{\geq 2} X; \mathbb{Z})$ from above, one sees that α can be written as a linear combination of generators of $H_{n-1}(\tilde{F}; \partial\tilde{F}; \mathbb{Z})$ with $F \in \pi_0(\partial_1 X)$. From the general theory we know that for $x \in F$ the generator of $H_{n-1}(\tilde{F}, \partial\tilde{F}; \mathbb{Z})$ restricts to the distinguished generator of $H_{n-1}(\partial X; \partial X \setminus \{x\}; \mathbb{Z})$. Hence, if $(l_x)_*(\alpha) = 0$ for $x \in \tilde{F} \setminus \partial\tilde{F} = F$, then the coefficient for the generator $[\tilde{F}]$ of $H_{n-1}(\tilde{F}, \partial\tilde{F}; \mathbb{Z})$ in the linear combination of α must vanish. We can choose an x in every $F \in \pi_0(\partial_1 X)$ and obtain that all coefficients must vanish and consequently $\alpha = 0$.
If we apply this observations to

$$\alpha = \sum_{F \in \pi_0(\partial_1 X)} p_*([\tilde{F}]) - \iota_*([\partial X])$$

we obtain that

$$\sum_{F \in \pi_0(\partial_1 X)} p_*([\tilde{F}]) = \iota_*([\partial X]).$$

□

<u>Remark</u>: the isomorphism
We define

$$\Phi_r : C_r(f; \mathbb{Z}) \to H_r(M_r, M_{r-1}; \mathbb{Z}) \tag{3.9}$$

by setting $\Phi_r(x) := (\hat{i}_x)_*([\hat{W}^-(x)])$ for $x \in Cr_r(f)$ and extending this linearly to $C_r(f; \mathbb{Z})$. Φ_r is the composition of

$$T_r : C_r(f; \mathbb{Z}) \to H_r(\bigsqcup_{\mathrm{ind}(x)=r} (\hat{W}^-(x), \partial\hat{W}^-(x)); \mathbb{Z}),$$

$$T_r\left(\sum_{\mathrm{ind}(x)=r} \lambda_x \cdot x\right) = \sum_{\mathrm{ind}(x)=r} \lambda_x \cdot [\hat{W}^-(x)]$$

and

$$\sqcup(\hat{i}_x)_* : H_r(\bigsqcup_{\mathrm{ind}(x)=r} (\hat{W}^-(x), \partial\hat{W}^-(x)); \mathbb{Z}) \to H_r(M_r, M_{r-1}; \mathbb{Z}).$$

3.15. Proposition
The map

$$\Phi_* : C_*(f; \mathbb{Z}) \to H_*(M_*, M_{*-1}; \mathbb{Z})$$

defined in (3.9) is an isomorphism.

<u>Proof</u>:
T_r is obviously an isomorphism. Furthermore, we have seen before that $\sqcup(\hat{i}_x)$ induces an isomorphism in homology. Hence Φ_r is an isomorphism.

□

3.16. Proposition
The map

$$\Phi_* : C_*(f;\mathbb{Z}) \to H_*(M_*, M_{*-1};\mathbb{Z})$$

defined in (3.9) is a chain map, i.e. the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_r(M_r, M_{r-1}) \arrow[r, "d^1"] & H_{r-1}(M_{r-1}, M_{r-2}) \arrow[r] & \cdots \\
\cdots \arrow[r] & C_r(f;\mathbb{Z}) \arrow[u, "\Phi_r"] \arrow[r, "\partial_r"] & C_{r-1}(f;\mathbb{Z}) \arrow[u, "\Phi_{r-1}"] \arrow[r] & \cdots
\end{tikzcd}$$

is commutative.

Proof:
We have to show that the following large diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & H_r(M_r, M_{r-1}) \arrow[r, "d^1"] & H_{r-1}(M_{r-1}, M_{r-2}) \arrow[r] & \cdots \\
 & H_r(\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x))) \arrow[u, "\sqcup(\hat{i}_x)_*"] & H_{r-1}(\bigsqcup_{\mathrm{ind}(y)=r-1}(\hat{W}^-(y), \partial\hat{W}^-(y))) \arrow[u, "\sqcup(\hat{i}_y)_*"] & \\
\cdots \arrow[r] & C_r(f;\mathbb{Z}) \arrow[u, "T_r"] \arrow[r, "\partial_r"] & C_{r-1}(f;\mathbb{Z}) \arrow[u, "T_{r-1}"] \arrow[r] & \cdots
\end{tikzcd}$$

We have a map between triples

$$\bigsqcup_{\mathrm{ind}(x)=r}(\hat{i}_x) : \bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x)) \to (M_r, M_{r-1}, M_{r-2})$$

and this map induces the following commutative diagram

$$\begin{tikzcd}
H_r(M_r, M_{r-1};\mathbb{Z}) \arrow[r, "d^1"] & H_{r-1}(M_{r-1}, M_{r-2};\mathbb{Z}) \\
H_r(\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x));\mathbb{Z}) \arrow[u, "\sqcup(\hat{i}_x)_*"] \arrow[r, "\partial"] & H_{r-1}(\bigsqcup_{\mathrm{ind}(x)=r}(\partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x));\mathbb{Z}) \arrow[u, "\sqcup(\hat{i}_y)_*"]
\end{tikzcd}$$

where

$$\partial : H_r(\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x));\mathbb{Z}) \to H_{r-1}(\bigsqcup_{\mathrm{ind}(x)=r}(\partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x));\mathbb{Z})$$

is the boundary-operator of $\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x))$. For convenience sake we introduce the following abbreviations:

$$A_r := \bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x)),$$

$$B_r := \bigsqcup_{\mathrm{ind}(x)=r} (\partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x)).$$

To show that the large diagram is commutative it suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
 & & H_{r-1}(M_{r-1}, M_{r-2}; \mathbb{Z}) \\
H_r(A_r;\mathbb{Z}) \arrow[r, "\partial"] & H_{r-1}(B_r;\mathbb{Z}) \arrow[ur, "\sqcup(\hat{i}_x)_*"] & \\
C_r(f;\mathbb{Z}) \arrow[u, "T_r"] \arrow[rr, "\partial_r"] & & C_{r-1}(f;\mathbb{Z}) \arrow[uu, "\Phi_{r-1}"']
\end{tikzcd}$$

and by linearity it even suffices to assure that

$$(\Phi_r \circ \partial_{r-1})(z) = (\sqcup(\hat{i}_x)_* \circ \partial \circ T_r)(z)$$

holds for all $z \in Cr_r(f)$.
To show this equality, we calculate the left side first:

$$\begin{aligned}
\Phi_r(\partial_{r-1}(z)) &= \Phi_r\left(\sum_{\mathrm{ind}(y)=r-1} I_r(z,y)\cdot y\right) \\
&= \sum_{\mathrm{ind}(y)=r-1} I_r(z,y)\Phi_{r-1}(y) \\
&= \sum_{\mathrm{ind}(y)=r-1} I_r(z,y)(\hat{i}_y)_*([\hat{W}^-(y)])
\end{aligned}$$

On the other hand we obtain $T_r(z) = [\hat{W}^-(z)]$ and we have to calculate its image under

$$\partial : H_r(\bigsqcup_{\mathrm{ind}(x)=r} (\hat{W}^-(x), \partial\hat{W}^-(x)); \mathbb{Z}) \to H_{r-1}(\bigsqcup_{\mathrm{ind}(x)=r} (\partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x)); \mathbb{Z}).$$

To do this, we consider the triple $\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x), \emptyset)$ and the map of triples

$$i : \bigsqcup_{\mathrm{ind}(x)=r} (\hat{W}^-(x), \partial\hat{W}^-(x), \emptyset) \to \bigsqcup_{\mathrm{ind}(x)=r} (\hat{W}^-(x), \partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x)).$$

which induces a commutative diagram

$$\begin{tikzcd}
H_r(\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x))) \arrow[r, "\partial"] \arrow[dr, "\overline{\partial}"'] & H_{r-1}(\bigsqcup_{\mathrm{ind}(x)=r}(\partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x))) \\
 & H_{r-1}(\bigsqcup_{\mathrm{ind}(x)=r}(\partial\hat{W}^-(x), \emptyset)) \arrow[u, "i_*"']
\end{tikzcd}$$

where ∂ denotes the boundary-operator of the triple again $\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x), \partial_{\geq 2}\hat{W}^-(x))$ and $\overline{\partial}$ is the boundary-operator of the triple $\bigsqcup_{\mathrm{ind}(x)=r}(\hat{W}^-(x), \partial\hat{W}^-(x), \emptyset)$. Hence we obtain

$$\partial(T_r(z)) = i_*(\overline{\partial}(T_r(z))).$$

Furthermore $\overline{\partial}(T_r(z)) = \overline{\partial}([\hat{W}^-(z)]) = [\partial\hat{W}^-(z)]$ — the last equality is valid because the fundamental class of a closed oriented topological manifold with boundary is mapped to the fundamental class of the boundary by $\overline{\partial}$ if the orientation of the boundary is induced by the orientation of the manifold, see [20].
By Lemma (3.14.)

$$i_*([\partial\hat{W}^-(z)]) = \sum_{F\in\pi_0(\partial_1\hat{W}^-(z))} p_*([\tilde{F}]).$$

holds. From Theorem (2.30.) we know that

$$\partial_1 W^-(z) = \bigsqcup_{y\in Cr(f)} \mathcal{T}(z,y) \times W^-(y)$$

and if $\mathrm{ind}(y) < r$ the class of the connected component in $\mathcal{T}(z,y) \times W^-(y)$ is mapped to zero by $\sqcup(\hat{i}_x)_*$ because $W^-(y)$ would have dimension smaller than $r-1$. So we are only interested in the connected components of the form $F = \{\gamma\} \times W^-(y)$ where $y \in Cr_{r-1}(f)$ and $\gamma \in \mathcal{T}(z,y)$ and in this case $\tilde{F} = \overline{F}$ which is equal to $\{\gamma\} \times \hat{W}^-(y)$. Now $\sqcup(\hat{i}_x)_*((i_*(\overline{\partial}(T_r(z))))) =$

$$\begin{aligned}
&= \sqcup(\hat{i}_x)_* \left(\sum_{\mathrm{ind}(y)=r-1} \sum_{\gamma\in\mathcal{T}(z,y)} (i_{\{\gamma\}\times\hat{W}^-(y)})_*([\{\gamma\} \times \hat{W}^-(y)]) \right) \\
&= \sqcup(\hat{i}_x)_* \left(\sum_{\mathrm{ind}(y)=r-1} \sum_{\gamma\in\mathcal{T}(z,y)} n(\gamma) \times (i_{\hat{W}^-(y)})_*([\hat{W}^-(y)]) \right) \\
&= \sum_{\mathrm{ind}(y)=r-1} \left(\sum_{\gamma\in\mathcal{T}(z,y)} n(\gamma) \right) \sqcup (\hat{i}_x)_*(i_{\hat{W}^-(y)})_*([\hat{W}^-(y)]) \\
&= \sum_{\mathrm{ind}(y)=r-1} I_r(z,y) \cdot \sqcup(\hat{i}_x)_*(i_{\hat{W}^-(y)})_*([\hat{W}^-(y)]) \\
&= \sum_{\mathrm{ind}(y)=r-1} I_r(z,y)(\hat{i}_y)_*([\hat{W}^-(y)])
\end{aligned}$$

and this is equal to $\Phi_r(\partial_{r-1}(z))$.

$\square$

Remark: Morse inequalities
Next we use Theorem (3.10.) to deduce the *Morse inequalities*. In the classical approach to Morse Theory these inequalities are deduced in a different way, see [10] for instance. Thom presented the idea to use a decomposition of the manifold obtained with the help of the negative gradient flow of an appropriate function to deduce Morse inequalities in [21]. See [18] for a similar treatment.

3.17. Corollary
Let (f, g) be a Morse–Smale pair on M. Set $c_j := |Cr_j(f)|$, i.e. the number of critical points with index j, $b_j := \dim_{\mathbb{R}} H_j(M; \mathbb{R}) = rank H_j(M; \mathbb{Z})$ shall denote the j'th Betti-number. Then one has
1.) $b_j \leq c_j$ for all $j = 0, \ldots, n$.
2.) $b_r - b_{r-1} + \ldots \pm b_0 \leq c_r - c_{r-1} + \ldots \pm c_0$ for $r = 0, \ldots, n$ and
3.) $b_n - b_{n-1} + \ldots \pm b_0 = c_n - c_{n-1} + \ldots \pm c_0 = (-1)^n \chi(M)$.

Proof:
By Theorem (3.10.) we know that

$$\begin{aligned} b_j &= \dim_{\mathbb{R}} H_j(M; \mathbb{R}) = \dim_{\mathbb{R}}(H_j(M; \mathbb{Z}) \otimes \mathbb{R}) \\ &= \dim_{\mathbb{R}}(H_j(f; \mathbb{Z}) \otimes \mathbb{R}) = \dim_{\mathbb{R}} H_j(f; \mathbb{R}) \end{aligned}$$

It is well-known that

$$\dim_{\mathbb{R}} C_j(f; \mathbb{R}) = \dim_{\mathbb{R}} \mathrm{im}(\partial_j) + \dim_{\mathbb{R}} \ker(\partial_j)$$

furthermore

$$\begin{aligned} \dim_{\mathbb{R}} H_j(f; \mathbb{R}) &= \dim_{\mathbb{R}} \left(\ker(\partial_j)/\mathrm{im}(\partial_{j+1})\right) \\ &= \dim_{\mathbb{R}} \ker(\partial_j) - \dim_{\mathbb{R}} \mathrm{im}(\partial_{j+1}) \end{aligned}$$

and consequently

$$c_j = \dim_{\mathbb{R}} \mathrm{im}(\partial_j) + \dim_{\mathbb{R}} \mathrm{im}(\partial_{j+1}) + b_j.$$

Hence $b_j \leq c_j$ for all $j = 1, \ldots, n$ and

$$\dim_{\mathbb{R}} \mathrm{im}(\partial_{r+1}) + b_r - b_{r-1} + \ldots \pm b_0 = c_r - c_{r-1} + \ldots \pm c_0.$$

But 2.) and 3.) follow directly from this.

□

3.18. Corollary
Given a Morse function on a compact manifold M. Then the number of critical points of this function is at least $|\chi(M)|$, where $\chi(M)$ denotes the Euler-number of M.

Proof:
Given a Morse function f we can find a Riemannian metric g such that (f, g) is a Morse–Smale pair, see section 3 of the first chapter. Define c_j and b_j for $j = 0, \ldots, n$ as before. We can apply Corollary (3.17.) and obtain

$$\sum_{s=0}^{n} c_s \geq \sum_{s=0}^{n} (-1)^{n-s} c_s = \sum_{s=0}^{n} (-1)^{n-s} b_s = \chi(M),$$

and

$$\sum_{s=0}^{n} c_s \geq \sum_{s=0}^{n} (-1)^{n-s+1} c_s = -\sum_{s=0}^{n} (-1)^{n-s} b_s = -\chi(M).$$

□

3.4 Isomorphism to deRham Cohomology

The main aim of this section is

3.19. Theorem $H^*(f; \mathbb{R}) \cong H^*_{dR}(M; \mathbb{R})$
Given a Morse–Smale pair on a compact smooth manifold, possibly with boundary. Then the cohomology of the Morse–Smale complex $(C^*(f; \mathbb{Z}), \partial^*)$ is isomorphic to the deRham cohomology of this manifold.

Remark: integration
We denote the space of smooth real-valued k-forms on M by $\Omega^k(M)$ and define a map

$$\mathrm{Int}^k : \Omega^k(M) \to C^k(f; \mathbb{R}), \quad \omega \mapsto \mathrm{Int}^k(\omega)$$

where $\mathrm{Int}^k(\omega)$ is given by $x \mapsto \int_{\hat{W}^-(x)} (\hat{i}_x)^*(\omega)$ on $Cr_k(f)$ and by extending this linearly to $C_k(f; \mathbb{Z})$. $\hat{W}^-(x)$ is a compact space and hence the integral is well-defined. Furthermore $W^-(x)$ is the interior of $\hat{W}^-(x)$ and differs from $\hat{W}^-(x)$ only by a set of measure zero. So we could also define $\mathrm{Int}^k(x)$ by

$$x \mapsto \int_{W^-(x)} \omega.$$

To prove that Int^* is a chain-map we will use the following adaptation of Stokes Theorem:

3.20. Theorem *Stokes Theorem for manifolds with corners*
Let X be a compact n-dimensional oriented smooth manifold with corners. Denote by $\iota_1 : \partial_1 X \hookrightarrow X$ the inclusion of the 1-boundary into X. The 1-boundary inherits an orientation from X.

If $\omega \in \Omega^{n-1}(X)$ is a smooth differential form on X, then $\int_{\partial_1 X} \iota_1^*(\omega)$ exists and

$$\int_{\partial_1 X} \iota_1^*(\omega) = \int_X d\omega$$

holds.

Proof:
We adapt the proof of the classical Stokes Theorem given in [13].
To prove existence of $\int_{\partial_1} \iota_1^* \omega$ it suffices to prove existence of $\int_F \iota_1^* \omega$ for any of the finitely many connected component F of $\partial_1 X$. By Lemma (3.13.) we can extend F to $\tilde{F}$ which is a compact smooth manifold with corners with interior F, so F differs from $\tilde{F}$ only by a set of measure zero. Looking at the proof of Lemma (3.13) one sees that one can extend all smooth forms on X to smooth forms on $\tilde{F}$ — $p^*\omega$ provides such an extension where $p : \tilde{F} \to M$ is the mapping defined in Lemma (3.13.). So, the integral of any smooth form on X over F is equal to the integral of this smooth form (extended to $\tilde{F}$) over $\tilde{F}$ and this last integral exists by compactness of $\tilde{F}$.
Let $(U_\alpha, u_\alpha)_{\alpha \in A}$ be an oriented atlas for X and $(g_\alpha)_{\alpha \in A}$ a smooth partition of unity subordinated to $(U_\alpha)_{\alpha \in A}$. Hence we can decompose ω and $d\omega$ as $\omega = \sum_{\alpha \in A}(f_\alpha \omega)$ and $d\omega = \sum_{\alpha \in A} d(f_\alpha \omega)$. For the integrals we obtain $\int_X d\omega = \sum_{\alpha \in A} \int_{U_\alpha} d(f_\alpha w)$ and $\int_{\partial_1 X} \omega = \sum_{\alpha \in A} \int_{U_\alpha \cap \partial_1 X} (f_\alpha \omega)$ and consequently it suffices to show that

$$\int_{U_\alpha} d(f_\alpha \omega) = \int_{U_\alpha \cap \partial_1 X} (f_\alpha \omega)$$

for all $\alpha \in A$. We will omit the indices from now on.
In the chart U where $f\omega$ is supported we can write

$$f\omega = \sum_{i=1}^{n} \omega_k dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n$$

where $\widehat{\ }$ indicates that this one-form is omitted. And we have

$$d(f\omega) = \sum_{i=1}^{n} (-1)^{k-1} \frac{\partial \omega_j}{\partial x^k} du^1 \wedge \ldots \wedge du^n.$$

Now we can calculate

$$\begin{aligned}
\int_U d(f\omega) &= \int_{u(U)} \sum_{k=1}^n (-1)^{k-1} \frac{\partial \omega_k}{\partial x^k} dx^1 \wedge \ldots \wedge dx^n \\
&= \sum_{k=1}^n (-1)^{k-1} \int_{u(U)} \frac{\partial \omega_k}{\partial x^k} dx^1 \wedge \ldots \wedge dx^n \\
&= \sum_{k=1}^n (-1)^{k-1} \int_{u(U)\cap\{x^k=0\}} \left(-\int_0^\infty (-1)^{k-1} \frac{\partial \omega_k}{\partial x^k} dx^k \right) dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n \\
&= \sum_{k=1}^n \int_{u(U)\cap\{x^k=0\}} \omega_k(x^1, \ldots, x^k = 0, \ldots, x^n) dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n \\
&= \sum_{k=1}^n \int_{u(U)\cap\partial_1 Q_n} \omega_k(x^1, \ldots, x^k = 0, \ldots, x^n) dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n
\end{aligned}$$

where the last equality follows from the fact that the set $\{x^1 > 0, \ldots, x^{k-1} > 0, x^k = 0, x^{k+1} > 0, \ldots, x^n > 0\}$ is a subset that differs from $\mathbb{R}^n \cap \{x^k = 0\}$ only be a set of measure 0.
On the other hand one has

$$\begin{aligned}
\int_{U\cap\partial_1 X} f\omega &= \int_{u(U)\cap\partial_1 Q_n} \sum_{k=1}^n \omega_k dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n \\
&= \sum_{k=1}^n \int_{u(U)\cap\partial_1 Q_n} \omega_k dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n.
\end{aligned}$$

□

3.21. Proposition
The map $\mathrm{Int}^* : \Omega^*(M) \to C^*(f; \mathbb{R})$ *is a chain map, i.e. the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^k(M) \arrow[r, "d"] \arrow[d, "\mathrm{Int}^k"] & \Omega^{k+1}(M) \arrow[r] \arrow[d, "\mathrm{Int}^{k+1}"] & \cdots \\
\cdots \arrow[r] & C^k(f;\mathbb{R}) \arrow[r, "\partial^k"] & C^{k+1}(f;\mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative.

Proof:
Let $\omega \in \Omega^k(M)$, $x \in Cr_{k+1}(f)$. By linearity it suffices to prove that

$$(\partial^k \circ \mathrm{Int}^k(\omega))(x) = (\mathrm{Int}^{k+1}(d\omega))(x).$$

On the one hand

$$(\partial^k \circ \mathrm{Int}^k(\omega))(x) = \sum_{\mathrm{ind}(y)=k} I_{k+1}(x, y)\, \mathrm{Int}^k(\omega)(y)$$

and on the other hand

$$(\mathrm{Int}^{k+1}(d\omega))(x) = \int_{\hat{W}^-(x)} (\hat{i}_x)^*(d\omega) = \int_{\hat{W}^-(x)} d((\hat{i}_x)^*\omega) = \int_{\partial_1 \hat{W}^-(x)} (\hat{i}_x)^*\omega.$$

From Theorem (2.33.) we know that $\partial_1 \hat{W}^-(x)$ is $\bigsqcup_{y\in Cr(f)} \mathcal{T}(x,y)\times W^-(y)$ and that their orientations fit, see Proposition (2.34.). The map $\mathcal{T}(x,y) \times W^-(y) \hookrightarrow \partial_1\hat{W}^-(x) \hookrightarrow \hat{W}^-(x) \overset{\hat{i}_x}{\to} M$ is equal to $\mathcal{T}(x,y) \times W^-(y) \overset{pr_2}{\to} W^-(y) \overset{\hat{i}_y}{\to} M$. $\hat{W}^-(y)$ differs from $W^-(y)$ only be a set of measure zero, so we obtain

$$\begin{aligned}
(\mathrm{Int}^{k+1}(d\omega))(x) &= \sum_{\mathrm{ind}(y)\le k} \int_{\mathcal{T}(x,y)\times \hat{W}^-(y)} (pr_2)^*(\hat{i}_y)^*\omega \\
&= \sum_{\mathrm{ind}(y)=k} \int_{\mathcal{T}(x,y)\times \hat{W}^-(y)} (pr_2)^*(\hat{i}_y)^*\omega
\end{aligned}$$

because for y with $\mathrm{ind}(y) < k$, $(\hat{i}_y)^*\omega$ would be the pull-back of a k-form to a manifold of dimension less than k and hence would vanish. Furthermore we get

$$\begin{aligned}
\sum_{\mathrm{ind}(y)=k} \int_{\mathcal{T}(x,y)\times \hat{W}^-(y)} (pr_2)^*(\hat{i}_y)^*\omega &= \sum_{\mathrm{ind}(y)=k} \sum_{\gamma\in\mathcal{T}(x,y)} \int_{\hat{W}^-(y)} n(\gamma)(\hat{i}_y)^*\omega \\
&= \sum_{\mathrm{ind}(y)=k} \left(\sum_{\gamma\in\mathcal{T}(x,y)} n(\gamma) \right) \int_{\hat{W}^-(y)} (\hat{i}_y)^*\omega \\
&= \sum_{\mathrm{ind}(y)=k} I_{k+1}(x,y) \int_{\hat{W}^-(y)} (\hat{i}_y)^*\omega \\
&= \sum_{\mathrm{ind}(y)=k} I_{k+1}(x,y)\, \mathrm{Int}^k(\omega)(y).
\end{aligned}$$

□

<u>Remark</u>: a filtration of M

We intend to apply Theorem (3.9.) to $\mathrm{Int}^k : \Omega^k(M) \to C^k(f;\mathbb{R})$.

Let $c_1 < c_2 < \ldots < c_r$ be the different critical values of f. Choose $\varepsilon > 0$ small enough such that $c_i + \varepsilon < c_{i+1} - \varepsilon$ for all $i = 1, \ldots, n-1$ and define $c_i^- := c_i - \varepsilon$ and $c_i^+ := c_i + \varepsilon$. So we obtain a finite sequence of regular values of f with

$$c_1^- < c_1^+ < c_2^- < \ldots < c_n^- < c_n^+$$

and set

$$\begin{aligned}
M_i^- &:= \{y \in M : f(y) \le c_i^-\}, \\
M_i^+ &:= \{y \in M : f(y) \le c_i^+\}.
\end{aligned}$$

We obtain a finite filtration of M:

$$\emptyset = M_1^- \subset M_1^+ \subset M_2^- \subset \ldots \subset M_n^- \subset M_n^+ = M \tag{3.10}$$

This induces a filtration on $\Omega^*(M)$ by

$$F_s^- \Omega^*(M) := \{\omega \in \Omega^*(M) : \omega \text{ vanishes on } M_s^-\},$$
$$F_s^+ \Omega^*(M) := \{\omega \in \Omega^*(M) : \omega \text{ vanishes on } M_s^+\}$$

We remark that another definition of $F_s^- \Omega^*(M)$ respectively of $F_s^* \Omega^*(M)$ would be to require that the pull-back of the forms to M_s^- respectively M_s^+ should vanish. Obviously

$$\{0\} = F_n^+ \Omega^*(M) \subset F_n^- \Omega^*(M) \subset F_{n-1}^+ \Omega^*(M) \subset \ldots \tag{3.11}$$
$$\ldots \subset F_1^+ \Omega^*(M) \subset F_1^- \Omega^*(M) = \Omega^*(M). \tag{3.12}$$

Furthermore, the subspaces $F_s^\pm \Omega^*(M)$ of $\Omega^*(M)$ form a sub-complex with respect to d. Indeed, if $\omega \in \Omega^*(M)$ vanishes on $M_i^\pm$, so does $d\omega$ as can be seen easily in charts.
Additionally, the filtration (3.10) induces a filtration on $C^*(f;\mathbb{R})$ by setting

$$F_s^- C^*(f;\mathbb{R}) := \{\varphi \in C^*(f;\mathbb{R}) : \varphi(x) = 0 \text{ for all } x \text{ with } f(x) \leq c_s^-\} \tag{3.13}$$
$$F_s^+ C^*(f;\mathbb{R}) := \{\varphi \in C^*(f;\mathbb{R}) : \varphi(x) = 0 \text{ for all } x \text{ with } f(x) \leq c_s^-\} \tag{3.14}$$

and one observes that

$$\{0\} = F_n^+ C^*(f;\mathbb{R}) \subset F_n^- C^*(f;\mathbb{R}) \subset F_{n-1}^+ C^*(f;\mathbb{R}) \subset \ldots$$
$$\ldots \subset F_1^+ C^*(f;\mathbb{R}) \subset F_1^- C^*(f;\mathbb{R}) = C^*(f;\mathbb{R}).$$

We verify that $F_s^\pm C^*(f;\mathbb{R})$ is a sub-complex of $C^*(f;\mathbb{R})$: Let $\varphi \in F_s^+ C^k(f;\mathbb{R})$ for instance, i.e. $\varphi(y) = 0$ for all y with $f(y) \leq c_s^+$. By definition $(\partial^k \varphi)(z) = \sum_{\mathrm{ind}(y)=k} I_{k+1}(z,y)\varphi(y)$ for $z \in Cr_{k+1}(f)$. If z is a critical point with value $f(z) \leq c_s^+$ and assume $\mathcal{T}(z,y) \neq \emptyset$. Then y is a critical point with $f(y) < f(z) \leq c_s^+$ simply because f decreases along flow lines. By assumption $\varphi(y) = 0$ and consequently $(\partial^k \varphi)(z) = 0$.

Next we want to show that Int^* preserves the filtrations on $\Omega^*(M)$ and $C^*(f;\mathbb{R})$, i.e.

$$\mathrm{Int}^k : F_s^\pm \Omega^*(M) \to F_s^\pm C^*(f;\mathbb{R}), \quad \omega \mapsto \mathrm{Int}^*(\omega)$$

for any $s = 1, \ldots, r$. To see this, let x be in $Cr_k(f)$ with $f(x) \leq c_s^\pm$. But then $\omega \in F_s^\pm \Omega^k(M)$ vanishes on $M_s^\pm$ and hence $\int_{\hat{W}^-(x)} (\hat{i}_x)^*(\omega)$ vanishes too. Hence $(\mathrm{Int}^k \omega)(x) = 0$ for every $x \in M_s^\pm$ and consequently $\mathrm{Int}^k \omega \in F_s^\pm C^*(f;\mathbb{R})$.

Remark: the spectral sequences
The filtrations on $\Omega^*(M)$ respectively $C^*(f;\mathbb{R})$ induce spectral sequences E and E' and Int^k induces maps between the terms of these spectral sequences. First consider the spectral sequence induced by the filtration on $\Omega^*(M)$: We know that the E_1 term is given by $H^*(M_s^+, M_s^-;\mathbb{R})$ alternating with terms of the form $H^*(M_{s+1}^-, M_s^+;\mathbb{R})$. The terms $H^*(M_{s+1}^-, M_s^+;\mathbb{R})$ can be calculated with the help of the following Theorem:

3.22. Theorem
Let f be a smooth real valued function on a manifold M that permits a Riemannian metric g such that $\mathrm{grad}_g(f)$ is tangential to the boundary. Let $a \leq b$ and suppose that the set $f^{-1}([a,b])$ consists of all $p \in M$ with $a \leq f(p) \leq b$, is compact, and contains no critical points of f.
Then $M^a := \{x \in M : f(x) \leq a\}$ is diffeomorphic to $M^b := \{x \in M : f(x) \leq b\}$. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

Proof:
One deforms the gradient vector field so that it vanishes on M^a. This vector field generates a family of diffeomorphisms which provide the diffeomorphism between M^a and M^b and the deformation retract. A detailed proof can be found in [10].

□

Consequently,

$$H^*(M_{s+1}^-, M_s^+;\mathbb{R}) = 0 \tag{3.15}$$

for all $s = 1, \ldots, r$ and we observe that the differential of the E_1-term must be trivial. Concerning the terms of the form $H^*(M_s^+, M_s^-;\mathbb{R})$ we will apply

3.23. Theorem
Let $f : M \to \mathbb{R}$ be a smooth function. Assume that there exists a Riemannian metric g such that $-\mathrm{grad}_g(f)$ is tangential to ∂M — in particular this holds if f is the function of a Morse–Smale pair (f,g). Let p be a non-degenerate critical point of index k. Setting $f(p) = c$, suppose that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical point of f other than p for some $\varepsilon > 0$.
Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ hat the homotopy type of $M^{c-\varepsilon}$ with a k-cell attached.

This Theorem is stated in [10], where also the following is remarked: More generally, suppose that there are m non-degenerate critical points $p_1, \ldots, p_m$

of indices $k_1, \ldots, k_m$. Then similar to the Theorem one can show that $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with $e^{k_1} \cup e^{k_2} \cup \ldots \cup e^{k_m}$ attached.
In the proof of Theorem (2.23.) as presented in [10] the following is shown: in a Morse chart for every critical point x with $f(x) = c$ a deformation retraction from $M^{c+\varepsilon}$ to $M^{c-\varepsilon} \cup \mathbb{R}^{\mathrm{ind}(x)}$ can be constructed. If we assume that this Morse chart is a convenient chart for the Morse–Smale pair that we consider, it follows that $M^{c+\varepsilon}$ is a deformation retract of $M^{c-\varepsilon} \cup W^-(x)$. Consequently

$$\bigvee_{x \in Cr(s)} W^-(x)/W^-(x)_{c_s} \to M_s^+/M_s^-,$$

where $W^-(x)_{c_s} := W^-(x) \cap f^{-1}(] - \infty, c_s^-])$, is a homotopy equivalence. Hence we obtain that

$$H^k(M_s^+, M_s^-; \mathbb{R}) \to H^k(\coprod_{x \in Cr(s)} (W^-(x), W^-(x)_{c_s}); \mathbb{R})$$

is an isomorphism.

Furthermore we will make use of the Universal Coefficient Theorem, see [6] for instance:

3.24. Theorem *Universal Coefficient Theorem*
For any space X and Abelian group G
a) the homology of X with coefficients in G has an unnatural splitting

$$H_k(X; G) \cong H_k(X; \mathbb{Z}) \otimes G \oplus Tor(H_{k-1}(X); G)$$

b) the cohomology of C with coefficients in G also has a spitting

$$H^k(X; G) \cong \mathrm{Hom}(H_k(X), G) \oplus Ext(H_{k-1}(X), G).$$

Now we can compute $H^*(M_s^+, M_s^-; \mathbb{R})$ with the help of Theorem (3.23.) and Theorem (3.24.):

$$\begin{aligned} H^k(M_s^+, M_s^-; \mathbb{R}) &= \mathrm{Hom}(H_k(M_s^+, M_s^-; \mathbb{Z}), \mathbb{R}) \\ &= \mathrm{Hom}(H_k(M_s^- \cup \bigcup_{x \in Cr(f): f(x) = c_s} e^{\mathrm{ind}(x)}, M_s^-; \mathbb{Z}), \mathbb{R}) \\ &= \mathrm{Hom}(H_k(\bigvee_{x \in Cr(f): f(x) = c_s} S^{\mathrm{ind}(x)}, *; \mathbb{Z}), \mathbb{R}) \\ &= \mathrm{Maps}(Cr_k(f) \cap f^{-1}(c_s), \mathbb{R}) \\ &=: \mathrm{Maps}(Cr_k(c_s), \mathbb{R}). \end{aligned}$$

Now we can calculate the spectral sequence E' that is induced by the filtration on $C^*(f;\mathbb{R})$. The E_1-term contains entries of the form $C^k(f;M^-_{s+1},M^+_s)$ and $C^k(f;M^+_s,M^-_s)$ where $C^k(f;M^-_{s+1},M^+_s)$ denotes maps from critical points x with value $c^+_s \le f(x) \le c^-_{s+1}$ to $\mathbb{R}$. But by construction of the filtration of M there are no such critical points, hence

$$C^k(f;M^-_{s+1},M^+_s) = 0 \tag{3.16}$$

and consequently the differential of the E'_1-term must be trivial. $C^k(f;M^+_s,M^-_s)$ consists of maps from critical points x with $c^-_s \le f(x) \le c^+_s$, but c_s is the only critical value in this range. Consequently we obtain

$$C^k(f;M^+_s,M^-_s) \quad = \quad \mathrm{Maps}(Cr_k(f)\cap f^{-1}(c_s),\mathbb{R}). \tag{3.17}$$

Now we must prove that the map $\mathrm{Int}^1 : E_1 \to E'_1$ induced by $\mathrm{Int}^* : \Omega^*(M) \to C^*(f;\mathbb{R})$ is an isomorphism. For the terms of the form (3.15) respectively (3.16) this is trivial. The interesting part is

$$\mathrm{Int}^1 : H^k(M^+_s,M^-_s;\mathbb{R}) \to \mathrm{Maps}(Cr_k(c_s),\mathbb{R}).$$

If we can show that this map is an isomorphism, we can apply Theorem (3.9.) and would obtain Theorem (3.18.).

We compose the isomorphism

$$H^k(M^+_s,M^-_s;\mathbb{R}) \to H^k(\bigsqcup_{x\in Cr(s)}(W^-(x),W^-(x)_{c_s});\mathbb{R})$$

with the isomorphism

$$H^k(\bigsqcup_{x\in Cr(s)}(W^-(x),W^-(x)_{c_s});\mathbb{R}) \to \mathrm{Maps}(Cr_k(c_s),\mathbb{R})$$

which is given by $[\omega] \mapsto (x \mapsto ([\omega],[W^-(x),W^-(x)_{c_s}]))$ where $([\omega],[W^-(x),W^-(x)_{c_s}])$ is the pairing of the cohomology class represented by ω with the homology class represented by the relative fundamental class $[W^-(x),W^-(x)_{c_s}]$ and obtain an isomorphism

$$H^k(M^+_s,M^-_s;\mathbb{R}) \to \mathrm{Maps}(Cr_k(c_s),\mathbb{R}).$$

Now we compare

$$\mathrm{Int}^1 : H^k(M^+_s,M^-_s;\mathbb{R}) \to \mathrm{Maps}(Cr_k(c_s),\mathbb{R})$$

with this isomorphism and see that the two maps coincide. This follows from the fact that the result of the paring between a cohomology class which is represented by a differential form ω and a homology class which is represented by the fundamental form of a submanifold is given by the integral of ω over the submanifold. Consequently Int^1 is an isomorphism too.

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Appendix A

CV

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