ON A CONSTRUCTION CONNECTING LIE ALGEBRAS WITH GENERAL ALGEBRAS

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In this paper we introduce a general construction which associates an algebra $A(\mathcal{L},b)$ with every pair $(\mathcal{L},b)$, where $\mathcal{L}$ is a Lie algebra and b is an invariant symmetric bilinear form on $\mathcal{L}$. By virtue of this construction several well-known (associative and non-associative) algebras can be dealt with under a unified view. We give characterizations of those pairs $(\mathcal{L},b)$ which generate associative algebras $A(\mathcal{L},b)$ and of those algebras which can be represented in the form $A(\mathcal{L},b)$.

1. Passing from Lie algebras to algebras

1.1. DEFINITION. Let $\mathcal{L}$ be a Lie algebra over a (commutative) field k and let $b: \mathcal{L}\times\mathcal{L} \longrightarrow k$ be an invariant (i.e. $b([X,Y],Z) = b(X,[Y,Z])$ for all $X,Y,Z \in \mathcal{L}$) symmetric bilinear form on $\mathcal{L}$. Then we define an algebra $A(\mathcal{L},b)$, associated with the pair $(\mathcal{L},b)$ as follows: As a vector space, $A(\mathcal{L},b)$ is just the direct sum $\mathcal{L} + k$. The multiplication of $A(\mathcal{L},b)$ is defined by the formula:

$$(X,s)(Y,t) = ([X,Y] + sY + tX,\ st + b(X,Y)).$$

Obviously, $A(\mathcal{L},b)$ is an algebra and $(0,1)$ is its identity.

1.2. PROPOSITION. (i) If char $k \neq 2$ then the algebra $A(\mathcal{L},b)$ is commutative if and only if $\mathcal{L}$ is abelian. If char k = 2 then $A(\mathcal{L},b)$ is always commutative.

(ii) Suppose that char $k \neq 2$. Then $(\mathcal{L},b)$ is isomorphic with $(\mathcal{L}',b')$ (i.e. there is a Lie algebra isomorphism $\phi: \mathcal{L} \longrightarrow \mathcal{L}'$ with $b(X,Y) = b(\phi(X), \phi(Y))$) if and only if $A(\mathcal{L},b)$ is isomorphic with $A(\mathcal{L},b)$. For char $k = 2$ there are non-isomorphic pairs $(\mathcal{L},b)$, $(\mathcal{L}',b')$ generating isomorphic algebras $A(\mathcal{L},b)$ and $A(\mathcal{L}',b')$.

(iii) $A(\mathcal{L},b)$ is always power associative, i.e. we have $x^2x = xx^2$ for all $x \in A(\mathcal{L},b)$.

(iv) We write $\mathrm{Ass}(x,y,z)$ for the associator $x(yz) - (xy)z$ of three elements x,y,z. In $A(\mathcal{L},b)$ we have

$$\mathrm{Ass}((X,s),(Y,t),(Z,u)) = (\alpha_b(X,Y,Z),0),$$

where

$$\alpha_b(X,Y,Z) = -b(X,Y)Z + b(Y,Z)X + [[Z,X],Y].$$

In particular, $A(\mathcal{L},b)$ is associative if and only if $\alpha_b(X,Y,Z) = 0$ for all $X,Y,Z \in \mathcal{L}$.

(v) The map α_b satisfies the identity

$$\alpha_b(X,Y,Z) + \alpha_b(Y,Z,X) + \alpha_b(Z,X,Y) = 0.$$

(vi) If char $k \neq 2,3$ and $A(\mathcal{L},b)$ is alternative (i.e. $x(xy) = x^2y$ and $(xy)y = xy^2$) then it is associative.

Proof. Assertion (i) follows from the identity $(X,s)(Y,s) - (Y,s)(X,t) = (2[X,Y],0)$.

(ii) Obviously, any isomorphism $\phi: (\mathcal{L},b) \longrightarrow (\mathcal{L}',b')$ induces an isomorphism $A(\mathcal{L},b) \longrightarrow A(\mathcal{L}',b')$, $(X,s) \longrightarrow (\phi(X),s)$. Suppose now that char $k \neq 2$ and that $\psi: A(\mathcal{L},b) \longrightarrow A(\mathcal{L}',b')$ is an isomorphism. Let $X \in \mathcal{L}\setminus\{0\}$ and write $\psi(X,s) = (X',s')$. Since ψ preserves units, $X' \neq 0$. From $\psi((X,0)^2) = (\psi(X,0))^2$ we conclude that $2s'X' = 0$ and $b(X,X) = s'^2 + b'(X',X')$, thence $s' = 0$ and $b(X,X) = b'(X',X')$. Thus we get the isomorphism we need by defining $\psi^*: \mathcal{L} \longrightarrow \mathcal{L}'$, $\psi^*(X) = X'$ if $X \neq 0$ and $\psi^*(0) = 0$.

To construct a counterexample in case char $k = 2$, let $k = \mathbb{Z}/2$ and choose a basis for k^2, say $\{X,Y\}$. Then we take $\mathcal{L}$ to be k^2 with the trivial Lie structure and $b = 0$; for $\mathcal{L}'$ we take k^2 with the Lie structure defined by $[X,Y] = X + Y$; b' is defined by stipulating $b'(X,X) = b'(Y,Y) = b'(X,Y) = 1$. Then $\mathcal{L}$ is not isomorphic with $\mathcal{L}'$, but $A(\mathcal{L},b) \cong A(\mathcal{L}',b')$, via the morphism $\Psi: A(\mathcal{L},b) \longrightarrow A(\mathcal{L}',b')$ given by $\Psi(X,0) = (X,1)$, $\Psi(Y,0) = (Y,1)$; $\Psi(X,1) = (X,0)$, $\Psi(Y,1) = (Y,0)$.

The proof of assertions (iii)-(v) rests on simple computations and is therefore left to the reader.

(vi) By Bourbaki [2], p. 612, an algebra is alternative if and only if its associator is skew-symmetric. Thus if $A(\mathcal{L},b)$ is alternative then α_b is skew-symmetric and hence (v) takes on the form $3\,\alpha_b(X,Y,Z) = 0$, so (iv) implies the assertion.

Remark. Note that in the proof of (v) and (vi) we did not use the assumption that b is symmetric.

1.3. NOTATION. We write κ for the Cartan-Killing form, $\kappa(X,Y) = \mathrm{Tr}(\mathrm{ad}\,X\ \mathrm{ad}\,Y)$. The set $\{X \in \mathcal{L} \mid b(X,\mathcal{L}) = 0\}$ is denoted with $\mathcal{L}^\perp$, and $\{X \in \mathcal{L} \mid b(X,Y) = 0\}$ with $Y^\perp$.

Throughout the rest of this section we always assume that char $k = 0$ and that $\mathcal{L}$ is finite-dimensional.

1.4. LEMMA. Assume that $A(\mathcal{L},b)$ is associative. Then

(i) $\kappa(X,Y) = (n-1)\,b(X,Y)$, where $n = \dim \mathcal{L}$;

(ii) every commutative subalgebra $\mathfrak{C}$ of $\mathcal{L}$ with $\dim \mathfrak{C} > 1$ lies in the ideal $\mathcal{L}^\perp$.

(iii) $[\mathcal{L}^\perp,[\mathcal{L},\mathcal{L}]] = 0$;

(iv) $(\mathrm{ad}\,U)^2 V = b(U,U)V$ for all $U \in \mathcal{L}$, $V \in \mathcal{L}^\perp$.

Proof. We infer from 1.2(iv) that

(*) $[X,[Y,Z]] = b(X,Y)Z - b(Z,X)Y$ for all $X,Y,Z \in \mathcal{L}$.

Thus $\kappa(X,Y) = \mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y) = \mathrm{Tr}(b(X,Y)\mathrm{Id} - b(X,.)Y) = nb(X,Y) - b(X,Y) = (n-1)b(X,Y)$, which establishes (i). If in (*) we put $X = Y = U$, $Z = V$, then we get (iv).

(ii) Let A,B be two linearly independent elements of $\mathfrak{C}$. Then by (*) we have for any $X \in \mathcal{L}$

$$0 = [X,[A,B]] = b(X,A)B - b(B,X)A$$

and hence $b(X,A) = b(X,B) = 0$; that is, $A,B \in \mathcal{L}^\perp$. Thus $\mathfrak{C} \subset \mathcal{L}^\perp$.

(iii) The right hand side of (*) vanishes whenever $X \in \mathcal{L}^\perp$, thus $[\mathcal{L}^\perp,[\mathcal{L},\mathcal{L}]] = 0$.

1.5. LEMMA. *Suppose that* $A(\mathcal{L},b)$ *is associative. Then the following assertions hold:*

(i) $\mathcal{L}$ *is either solvable or simple of* rank 1.

(ii) *If* $0 \neq \mathcal{L}^{\perp} \neq \mathcal{L}$ *then* $\mathcal{L}^{\perp} = [\mathcal{L},\mathcal{L}] = [\mathcal{L},[\mathcal{L},\mathcal{L}]]$ *and* $\mathcal{L}^{\perp}$ *is commutative. Moreover,* $X \in \mathcal{L}^{\perp}$ *if and only if* $b(X,X) = 0$.

(iii) *If* $\mathcal{L}$ *is solvable then* $\dim \mathcal{L}/\mathcal{L}^{\perp} \leq 1$.

Proof. The assertions are obvious for $\dim \mathcal{L} \leq 1$, so let us assume that $n = \dim \mathcal{L} > 1$. Then we have $b = \frac{1}{n-1}\kappa$, by 1.4(i) (and hence $\mathcal{L}^{\perp} = 0$ if and only if $\mathcal{L}$ is semisimple).

(i) If $\mathcal{L}$ is semisimple then by 1.4(ii) every Cartan subalgebra of $\mathcal{L}$ has dimension 1, so $\mathcal{L}$ is actually simple of rank 1. Assume now that $\mathcal{L}$ is not semisimple. Then by our assumption above, $\mathcal{L}^{\perp} \neq 0$. Suppose that $\mathcal{S}$ is a semisimple subalgebra of $\mathcal{L}$. Since $\mathcal{S} = [\mathcal{S},\mathcal{S}] \subseteq [\mathcal{L},\mathcal{L}]$, 1.4(iii) yields that $[\mathcal{L}^{\perp},\mathcal{L}] = 0$. Now any non-zero $Y \in \mathcal{L}^{\perp}$ together with any non-zero $S \in \mathcal{S}$ generates a two-dimensional commutative Lie subalgebra $\mathfrak{C}$ of $\mathcal{L}$, which by 1.4(ii) is contained in $\mathcal{L}^{\perp}$, so $[S,\mathcal{S}] \subseteq [\mathcal{L}^{\perp},\mathcal{L}] = 0$, a contradiction. This establishes (i).

(ii) Assume that $0 \neq Z \in \mathcal{L}^{\perp}$. Then formula (*) of the proof of 1.4 implies that $[X,[Y,Z]] = b(X,Y)Z$ for all $X,Y \in \mathcal{L}$. By 1.4 (iii) $[Y,Z] = 0$, and hence $b(X,Y) = 0$, whenever $Y \in [\mathcal{L},\mathcal{L}]$, $X \in \mathcal{L}$. Thus $[\mathcal{L},\mathcal{L}] \subseteq \mathcal{L}^{\perp}$. Conversely, let $X,Y \in \mathcal{L}$ with $b(X,Y) \neq 0$. Then $Z = b(X,Y)^{-1}[X,[Y,Z]] \in [\mathcal{L},[\mathcal{L},\mathcal{L}]]$. Thus $[\mathcal{L},\mathcal{L}] \subseteq \mathcal{L}^{\perp} \subseteq [\mathcal{L},[\mathcal{L},\mathcal{L}]] \subseteq [\mathcal{L},\mathcal{L}]$; the commutativity of $\mathcal{L}^{\perp}$ follows from 1.4(iii).

To show the second part of (ii), suppose that $b(X,Y) \neq 0$, but $b(X,X) = 0$. Then $[X,[X,Y]] = -b(Y,X)X$, hence $X \in [\mathcal{L},\mathcal{L}] = \mathcal{L}^{\perp}$, a contradiction.

(iii) Suppose that $\mathcal{L}$ is solvable and that there are elements $X,Y \in \mathcal{L}$ such that $X + \mathcal{L}^{\perp}$ and $Y + \mathcal{L}^{\perp}$ are linearly independent in $\mathcal{L}/\mathcal{L}^{\perp}$. Then we get

$$b(X,X)X - b(X,Y)Y = [X,[X,Y]] \in [\mathcal{L},\mathcal{L}] = \mathcal{L}^{\perp}.$$

Thus $b(Y,Y) = 0$ and therefore, by (ii), $Y \in [\mathcal{L},\mathcal{L}] = \mathcal{L}^{\perp}$, a contradiction.

1.6. THEOREM. $A(\mathfrak{L},b)$ *is associative if and only if one of the following assertions hold:*

(i) $\mathfrak{L}$ *is a simple Lie algebra of* rank 1 *and* $b = \frac{1}{n-1}\kappa$, *where* $n = \dim \mathfrak{L}$.

(ii) $\mathfrak{L}$ *is nilpotent of step* 2 (i.e. $[\mathfrak{L},[\mathfrak{L},\mathfrak{L}]] = 0$) *and* $b = 0$.

(iii) $\dim \mathfrak{L} \leq 1$ (and b is arbitrary).

(iv) $\mathfrak{L}^{\perp} = [\mathfrak{L},\mathfrak{L}]$ *and there is an element* $X \in \mathfrak{L}$ *such that* $\mathfrak{L}$ *is the split extension* $\mathfrak{L}^{\perp} \oplus kX$ *of* $\mathfrak{L}^{\perp}$ *with the one-dimensional subspace* kX. *Moreover,* $\mathfrak{L}^{\perp}$ *is commutative and* $(\operatorname{ad} X)^2 Y = b(X,X)Y$ *for all* $Y \in [\mathfrak{L},\mathfrak{L}]$; $b = \frac{1}{n-1}\kappa$.

Proof. Suppose first that $A(\mathfrak{L},b)$ is associative and that $\dim \mathfrak{L} > 1$. If $\mathfrak{L}^{\perp} = 0$ then assertion (i) holds, by 1.4(i) and 1.5(i). If $\mathfrak{L}^{\perp} \neq 0$ then, by 1.4(iii),(iv) and 1.5(ii),(iii), either $\mathfrak{L}^{\perp} = \mathfrak{L}$ (which implies (ii)) or $\dim \mathfrak{L}/\mathfrak{L}^{\perp} = 1$ and hence (iv) holds.

Conversely, it is immediate that each of the assertions (ii) -(iv) implies that the condition in 1.2(iv), $\alpha_b = 0$, is satisfied, so that $A(\mathfrak{L},b)$ is associative. (Note that in case (iv) every product $[A,[B,C]]$ vanishes unless A and B, or A and C, are contained in $kX \setminus \{0\}$.) In the case of (i) we first remark that we may assume that $k = \mathbb{C}$, since the condition $\alpha_b = 0$ of 1.2(iv) naturally extends to the complexification $(\mathfrak{L}\otimes\mathbb{C},b_{\mathbb{C}})$ and $A(\mathfrak{L},b)$ can be considered as a subalgebra of the algebra $A(\mathfrak{L}\otimes\mathbb{C},b_{\mathbb{C}})$, taken as algebra over k (cf. Bourbaki [3], p. 21). Thus we are left to show that $A(\mathfrak{sl}(2,\mathbb{C}),\frac{1}{2}\kappa)$ is associative; this will be done in Example 2.5 of the next section.

2. Examples.

2.1. The trivial cases:

If $\dim \mathcal{L} = 0$, then $b = 0$ and $A(0,0) \cong k$.

If $\dim \mathcal{L} = 1$, then $\mathcal{L} \cong k$. Let $b(X,Y) := \alpha XY$ for some $\alpha \in k$.

Then $A(\mathcal{L},b) \cong {}^{k[X]}/{<X^2-\alpha>}$ (the isomorphism is given by $X \mapsto (1,0)$).

If $k = \mathbb{R}$, we get for

- (i) $\alpha<0$ the algebra $\mathbb{C}$ of complex numbers.
- (ii) $\alpha=0$ the commutative associative algebra generated by 1 and δ with $\delta^2 = 0$, sometimes called the algebra of dual numbers.
- (iii) $\alpha>0$ the commutative associative algebra generated by 1 and ε with $\varepsilon^2 = 1$.

These are all quadratic algebras over $\mathbb{R}$ in the sense of Bourbaki

2.2. Let $\mathcal{L} = so(3,\mathbb{R})$ and let $b = \kappa$, its Cartan - Killing form. Let $\mathbb{E}^3$ be the oriented Euclidean 3 - space with inner product $<\cdot,\cdot>$ and normed determinant function D.

Define a cross product "$\times$" in $\mathbb{E}^3$ by stipulating $<X\times Y,Z> = D(X,Y,Z)$. Then $so(3,\mathbb{R})$ is isomorphic to $(\mathbb{E}^3,\times)$ in such way that $[X,Y] = X\times Y$ and $\kappa(X,Y) = -2<X,Y>$. To see this, put

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and notice that $[X_i,X_{i+1}] = X_{i+2}$, where we compute the indices modulo 3.

The product formula in $A(so(3,\mathbb{R}),1/2\kappa)$ is then

(1) $(X,s)(Y,t) = (X\times Y + sY + tX,\ st - <X,Y>)$, which yields exactly the algebra $\mathbb{H}$ of quaternions : choose a positively oriented orthonormal basis i,j,k in $\mathbb{E}^3$ and check that the multiplication - table is :

(2)

	(i,0)	(j,0)	(k,0)
(i,0)	(0,−1)	(k,0)	(−j,0)
(j,0)	(−k,0)	(0,−1)	(i,0)
(k,0)	(j,0)	(−i,0)	(0,−1)

Then obviously in the algebra $A(so(3,\mathbb{R}),\alpha\kappa)$, $\alpha\in\mathbb{R}$, we get the multiplication - table :

(3)

	(i,0)	(j,0)	(k,0)
(i,0)	(0,−2α)	(k,0)	(−j,0)
(j,0)	(−k,0)	(0,−2α)	(i,0)
(k,0)	(j,0)	(−i,0)	(0,−2α)

This is associative if and only if $\alpha = 1/2$.

2.3. Let $\mathfrak{L} = so(3,\mathbb{C})$ and let $b = \kappa$ be again its (complex) Cartan - Killing form. Then $\mathfrak{L} \cong \mathbb{C}^3$, $[X,Y] = X\times_{\mathbb{C}}Y$ (the "complexified vector product" with the same coordinate formula as the real one), and $\kappa_{\mathbb{C}}(X,Y) = -2\Sigma_{i=1}^{3}X^iY^i$. As we just take the product formula 2.2.1 with complex scalars, we get $A(so(3,C),1/2\ \kappa_{\mathbb{C}}) \cong \mathbb{H}\times_{\mathbb{R}}\mathbb{C}$ (cf. 2.5.). Likewise the algebra $A(so(3,C),\alpha\kappa_{\mathbb{C}})$ for $\alpha\in\mathbb{C}$ is given by the multiplication - table 2.2.3., but now over $\mathbb{C}$. $A(so(3,\mathbb{C}),\alpha\kappa_{\mathbb{C}})$ is associative if and only if $\alpha = 1/2$.

2.4. Let $\mathfrak{L} = sl(2,\mathbb{R})$ and let $b = \kappa$, the Cartan - Killing form. Then $\mathfrak{L}$ is the Lie algebra of traceless 2×2 - matrices. Choose the following basis of $\mathfrak{L}$:

$$X_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $[X_0,X_1] = X_2$, $[X_1,X_2] = -X_0$, $[X_2,X_0] = X_1$, and $\frac{1}{2}\kappa(\Sigma x^iX_i,\Sigma y^iY_i) = -x^0y^0 + x^1y^1 + x^2y^2$. Now let $\mathbb{L}^3$ be the Lorentzian 3 - space with inner product $\langle .,. \rangle_L$, with signature +,−,−. Define the Lorentzian vector product $\times_L$ on $\mathbb{L}^3$ by $\langle X\times_L Y,Z\rangle_L = -\det(X,Y,Z)$. For the standard basis e_0, e_1, e_2 on $\mathbb{L}^3$ we get

$$e_0\times_L e_1 = e_2 \qquad e_1\times_L e_2 = -e_0 \qquad e_2\times_L e_0 = e_1$$

Thus $(sl(2,\mathbb{R}),\langle .,. \rangle,\frac{1}{2}\kappa)$ is isomorphic to $(\mathbb{L}^3,\times_L,-\langle .,. \rangle_L)$ and the multiplication formula of 1.1. becomes on $\mathbb{L}^3\times\mathbb{R}$:

(1) $$(X,s)(Y,t) = (X\times_L Y + sY + tX,\ st - \langle X,Y\rangle_L)$$

This gives an associative algebra, sometimes called the algebra of pseudoquaternions (see Yaglom, [8]) : check the multiplication - table

(2)

	$(e_0,0)$	$(e_1,0)$	$(e_2,0)$
$(e_0,0)$	$(0,-1)$	$(e_2,0)$	$(-e_1,0)$
$(e_1,0)$	$(-e_2,0)$	$(0,1)$	$(-e_0,0)$
$(e_2,0)$	$(e_1,0)$	$(e_0,0)$	$(0,1)$

But in fact this algebra is isomorphic to the full algebra of 2x2 - matrices :

$$(0,1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 \qquad (e_0,0) \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

$$(e_1,0) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \qquad (e_2,0) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

gives the same multiplication - table for the matrix - multiplication. Here the σ_i are the Pauli matrices, very dear to physicists. Thus $A(\mathfrak{sl}(2,\mathbb{R}),\frac{1}{2}\kappa) \cong L(\mathbb{R}^2,\mathbb{R}^2)$, the algebra of all real 2x2 - matrices.

$A(\mathfrak{sl}(2,\mathbb{R}),\alpha\kappa)$ gives the multiplication - table (2) with $(0,-2\alpha)$, $(0,2\alpha)$, $(0,2\alpha)$ in the main diagonal, associative if and only if $\alpha = 1/2$.

2.5. Let $\mathcal{L} = \mathfrak{sl}(2,\mathbb{C})$, $\kappa_{\mathbb{C}}$ its Cartan - Killing form. Then we can apply the discussion of 2.4. with complex scalars and conclude that $A(\mathfrak{sl}(2,\mathbb{C}),\frac{1}{2}\kappa_{\mathbb{C}}) \cong A(\mathfrak{sl}(2,\mathbb{R}),\frac{1}{2}\kappa) \times_{\mathbb{R}} \mathbb{C}$ equals the algebra of complex 2x2 - matrices. This is well known to physicists via the formula $\sigma_i\sigma_j = \delta_{ij} + \sqrt{-1}\ \varepsilon_{ijk}\sigma_k$ for the Pauli matrices.

2.6. Let $\mathcal{L}$ be the real 2 - dimensional Lie algebra satisfying $[X,Y] = X$. (This is the Lie algebra of the "ax + b" - group) Then the Cartan - Killing form κ is given by $\kappa(X,\mathcal{L}) = 0$ and $\kappa(Y,Y) = 1$. This gives an associative algebra $A(\mathcal{L},\kappa)$ which is isomorphic to the real algebra of all upper triangular 2x2 - matrices :

$$(0,1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (X,0) \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (Y,0) \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives the correct multiplication - table.

2.7. The algebra of Cayley numbers is not of the form $A(\mathcal{L},b)$ since it is alternative but not associative (cf. 1.2.6). But it can be represented in a similar form : we use the isomorphism $so(3,\mathbb{C}) \cong (\mathbb{C}^3,\times_{\mathbb{C}})$ of 2.3. and consider the usual hermitian inner product $(.,.)$ on $\mathbb{C}^3$. Then $\mathbb{C}^3\times\mathbb{C}$, with multiplication

$$(X,s)(Y,t) = (\overline{X\times_{\mathbb{C}}Y} + sY + \bar{t}X,\ st - (X,Y))$$

is the algebra of Cayley numbers (see Greub, [3]).
In chapter 4 we define a concept generalising this product.
In char $k = 2$ the Cayley numbers are associative.

2.8. Let $\mathcal{L}$ be a nilpotent Lie algebra of step 2. Then $\mathcal{L} = V \oplus W$ as a vector space, and $[\mathcal{L},W] = \{0\}$, $[X,Y] =: \omega(X,Y) \in W$ for $X,Y \in V$, where $\omega: V\times V \to W$ is an arbitrary skew - symmetric bilinear map. If we want an associative algebra, then $b = 0$ and $A(\mathcal{L},0) = V\times W\times k$ as a vector space with product

$$(v,w,0)(v',w',0) = (0,\omega(v,v'),0)$$

and $(0,0,1)$ as unit.

3. Passing from algebras to Lie algebras

3.1. PROPOSITION. Let A be an algebra with unit over a commutative base field k. Then the commutator $[x,y] = xy - yx$ of two elements in A satisfies the Jacobi identity if and only if the associator $\mathrm{Ass}(x,y,z) = x(yz)-(xy)z$ satisfies

$$(°) \qquad \sum_{\sigma\in \mathfrak{S}_3} \mathrm{sgn}(\sigma)\mathrm{Ass}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}) = 0$$

for all triplets x_1,x_2,x_3 of elements in A. If char $k \neq 2,3$ and A is alternative then (°) implies that A is associative.

Proof. The proof of the first assertion is an easy computation and therefore left to the reader. For the second we only have to note that by Bourbaki [2], p. 612, A is associative if and only if Ass is skew-symmetric; if Ass is skew-symmetric then the left side of (°) is just $6\,\mathrm{Ass}(x_1,x_2,x_3)$.

3.2. Remarks. (i) It seems that up to now only conditions stronger than (°) have been dealt with in the literature; such as (cf. Nijenhuis and Richardson [5])

$$\mathrm{Ass}(x,y,z) = \mathrm{Ass}(y,x,z),$$
$$\mathrm{Ass}(x,y,z) = \mathrm{Ass}(x,z,y),$$
$$\mathrm{Ass}(x,y,z) = \mathrm{Ass}(z,y,x).$$

None of these conditions is satified for all of the algebras $A(\mathfrak{L},b)$ in section 1.

(ii) Proposition 3.1 has an obvious generalization to graded algebras and graded Lie algebras.

3.3. DEFINITION. Let $\mathfrak{G}$ be a subgroup of $\mathfrak{S}_3$. Then an algebra A is called $\mathfrak{G}$-associative if

$$\sum_{\sigma\in \mathfrak{G}} \mathrm{sgn}(\sigma)\mathrm{Ass}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}) = 0.$$

3.4. Remarks. (i) By 1.2(v) every algebra $A(\mathfrak{L},b)$ is $\mathfrak{A}_3$-associative, where $\mathfrak{A}_3$ denotes the alternating group in three elements.

(ii) The conditions in 3.2 correspond to $\mathfrak{G}$-associative algebras, where $\mathfrak{G}$ is a two-element subgroup of $\mathfrak{S}_3$.
(iii) The $\{1\}$-associative algebras are just the associative algebras.
(iv) If $\mathfrak{G} \subset \mathfrak{H}$ then every $\mathfrak{G}$-associative algebra is also $\mathfrak{H}$-associative.
(v) Note the formula

$$(\circ)\quad \mathrm{Ass}(x,y,z) + \mathrm{Ass}(y,x,z) + \mathrm{Ass}(z,x,y) = [x,yz] + [y,zx] + [z,xy].$$

Thus an algebra A is $\mathfrak{S}_3$-associative if and only if

$$[x,yz] + [y,zx] + [z,xy] = 0 \quad \text{for all } x,y,z \in A.$$

3.5. For the following, let char $k \neq 2$.

<u>DEFINITION</u>. A <u>Clifford trace</u> τ on a unital algebra A over k is a k-linear map $\tau: A \longrightarrow k$ such that for all $x,y \in A$:

(i) $\tau(1) = 1$,

(ii) $\frac{1}{2}(xy + yx) = \tau(xy)1 + \tau(x)y + \tau(y)x - 2\tau(x)\tau(y)1$.

Writing π for the complementary projection to τ, $\pi(x) = x - \tau(x)$, (ii) can be written also in the form

(ii') $\pi(x)\pi(y) + \pi(y)\pi(x) = 2\tau(\pi(x)\pi(y))1$,

that is, π satisfies the Clifford equation. (Note that this implies $\pi(xy) = \pi(yx)$ and $[\pi(x),\pi(y)] = [x,y]$.)

A <u>Clifford trace</u> τ is said to be <u>invariant</u> if for $x,y,z \in A$

$$\tau([\pi(x), \pi(y)]\pi(z)) = \tau(\pi(x)[\pi(y), \pi(z)]),$$

or, equivalently, if for all $x,y,z \in \ker\tau$ the equation

$$x(yz) - (xy)z + z(yx) - (zy)x = x(yz) - (yz)x + z(xy) - (yx)z$$

holds.

3.6. <u>THEOREM. Let A be a unital algebra over k with</u> char $k \neq 2$. <u>Then the following assertions are equivalent:</u>

(i) <u>A can be written in the form</u> $A = A(\mathfrak{L},b)$ <u>for some Lie algebra</u> $\mathfrak{L}$ <u>and invariant bilinear form</u> b.

(ii) <u>A is</u> $\mathfrak{S}_3$<u>-associative and admits an invariant Clifford trace.</u>

(ii) A is $\mathbb{A}_3$-associative and admits an invariant Clifford trace.

Proof. Suppose first that $A = A(\mathcal{L},b)$. Then A is $\mathbb{A}_3$-associative and $\tau : A \longrightarrow k$, $\tau(X,s) = s$, is an invariant Clifford trace. In fact, writing $\pi(X,s) = (X,0)$ (according to 3.5), we get

$$\pi(X,s)\pi(Y,t) + \pi(Y,t)\pi(X,s) = (X,0)(Y,0) + (Y,0)(X,0) = (0,2b(X,Y));$$

$$2\tau(\pi(X,s)\pi(Y,t)) = 2\tau((X,0)(Y,0)) = 2\tau(([X,Y],b(X,Y))) = 2b(X,Y),$$

which establishes our claim.

Suppose now that (ii) holds. Then $\mathfrak{m} = (A,[\ ,\ ]_A)$ is a Lie algebra (by 3.1). Let $\tau : A \longrightarrow k$ be the invariant Clifford trace. We consider k as one-dimensional (trivial) Lie algebra, so τ is a Lie homomorphism. We define $\mathcal{L}$ to be the Lie algebra ker τ, provided with the Lie bracket $[\ ,\ ] = \frac{1}{2}[\ ,\ ]_A$, and $b(X,Y) = \tau(XY)$, for all $X,Y \in \mathcal{L}$. Since τ is invariant, b is invariant, too. Let $\pi : A \longrightarrow \ker\tau = \mathcal{L}$ be the complementary projection, $\pi(x) = x - \tau(x)$; π is also a Lie algebra morphism.

Let $X,Y \in \mathcal{L}$. Then (XY denoting the product in A)

$$XY = \tfrac{1}{2}(XY - YX) + \tfrac{1}{2}(XY + YX) = \tfrac{1}{2}[X,Y]_A + \tau(XY)1 = [X,Y]_{\mathcal{L}} + b(X,Y)1.$$

For arbitrary $x,y \in A$ we have $x = \pi(x) + \tau(x)1$, $y = \pi(y) + \tau(y)$, and we get

$$\begin{aligned} xy &= (\pi(x) + \tau(x)1)(\pi(y) + \tau(y)1) = \\ &= \pi(x)\pi(y) + \tau(x)\pi(y) + \tau(y)\pi(x) + \tau(x)\tau(y)1 = \\ &= [\pi(x),\pi(y)]_{\mathcal{L}} + \tau(x)\pi(y) + \tau(y)\pi(x) + \tau(x)\tau(y)1 + \\ &\quad + \tau(\pi(x)\pi(y))1. \end{aligned}$$

Thus the map $A \longrightarrow A(\mathcal{L},b)$, $x \mapsto (\pi(x),\tau(x))$ is the required isomorphism.

Remark. If in the above Theorem we drop both the invariance of τ and the invariance of b then the arguments still work.

4. A final remark.

The following construction is presented here as a concept generalizing the ideas of Definition 1.1. Using this construction we can also cover the case of the Cayley algebra (cf. Example 2.7.)

4.1. Let k be a commutative field, and let A be a unital (commutative) k - algebra. Let $\alpha : A \to A$ be an algebra - antiautomorphism with $\alpha \circ \alpha = id$ (which we may view as conjugation).

Let $\mathcal{L}$ be an α - balanced Lie - module over A, i.e.

(1) $\mathcal{L}$ is a Lie algebra over k with bracket $[.,.]$

(2) $\mathcal{L}$ is an A - bimodule

(3) There is a representation D of $\mathcal{L}$ on A via (α - crossed) derivations, a Lie algebra homomorphism $D : \mathcal{L} \to Der^{(\alpha)} A$, $X \to D_X$, such that $D_X(ab) = D_X(a)b + \alpha(a) D_X(b)$

(4) $[aX,Y] = \alpha(a)[X,Y] - D_Y(a)X$ or equivalently

(4') $[X,aY] = \alpha(a)[X,Y] + D_X(a)Y$

(5) $[X,Ya] = [X,Y]a + YD_X(a)$

Furthermore let $b : \mathcal{L} \times \mathcal{L} \to A$ be k - linear and equivariant, i.e.

(6) $D_X(b(Y,Z)) = b([X,Y],Z) + b(Y,[X,Z])$

Then we consider the k - vector space $\mathcal{L} \times A$ with the following product:

(7) $(X,a)(Y,b) := ([X,Y] + aY + bX,\ ab + b(X,Y)).$

We leave it to the reader to verify that this definition yields an algebra.

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