PSEUDORIEMANNIAN METRICS ON SPACES
OF BILINEAR STRUCTURES

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ABSTRACT. The space of all non degenerate bilinear structures on a manifold M carries a one parameter family of pseudo Riemannian metrics. We determine the geodesic equation, covariant derivative, curvature, and we solve the geodesic equation explicitly. Each space of pseudo Riemannian metrics with fixed signature is a geodesically closed submanifold. The space of non degenerate 2-forms is also a geodesically closed submanifold. Then we show that, if we fix a distribution on M, the space of all Riemannian metrics splits as the product of three spaces which are everywhere mutually orthogonal, for the usual metric. We investigate this situation in detail.

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0. INTRODUCTION

If M is a (not necessarily compact) smooth finite dimensional manifold, the space $\mathcal{B} = C^\infty(GL(TM, T^*M))$ of all non degenerate $\binom{0}{2}$-tensor fields on it can be endowed with a structure of an infinite dimensional smooth

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manifold modeled on the space $C_c^\infty(L(TM, T^*M))$ of $\binom{0}{2}$ -tensor fields with compact support, in the sense of [Michor, 1980]. The tangent bundle of $\mathcal{B}$ is $T\mathcal{B} = \mathcal{B} \times C_c^\infty(L(TM, T^*M))$ and we consider on $\mathcal{B}$ the one parameter family of smooth pseudo Riemannian metrics G^α defined in section 1.

Section 2 is devoted to the study of the geometry of $(\mathcal{B}, G^\alpha)$. We start by computing the geodesic equation, using variational methods, and from it the covariant derivative and the curvature of the Levi-Civita connection are obtained. We explicitly solve the geodesic equation and we find the domain of definition of the exponential mapping which is open for the topology considered on $\mathcal{B}$. We show that the exponential mapping is a real analytic diffeomorphism from an open neighborhood of zero onto its image.

Pseudo Riemannian metrics of fixed signature and non degenerate 2-forms are splitting submanifolds of $\mathcal{B}$. In section 3 we show that they are geodesically closed. In particular this applies to the manifold $\mathcal{M}$ of all Riemannian metrics on M, with its usual metric, the geometry of which has been studied in [Ebin, 1970], [Freed, Groisser, 1989], [Gil-Medrano, Michor, 1990]. If M is compact, the space of symplectic structures on M is also a splitting submanifold, it seems that it should admit a connection for the induced metric, but we have not been able to find it because the exterior derivative has a complicated expression in the terms in which the metric is simple.

Finally in section 4 we investigate the splitting of $\mathcal{M}$ induced by fixing a distribution $V \subset TM$: it turns out to be the product of the space $\mathcal{M}(V)$ of all fiber metrics on the distribution, the space $\mathcal{M}(TM/V)$ of all fiber metrics on the normal, and the space $\mathcal{P}_V(M)$ of all almost product structures having V as the vertical distribution. Each slice $\mathcal{M}(V) \times \mathcal{M}(TM/V)$ is a geodesically closed submanifold and the induced metric on each slice $\mathcal{P}_V(M)$ is flat. As a Riemannian manifold $\mathcal{M}$ turns out to be what is usually known as a product with varying metric on the fibers. If the distribution V is the vertical bundle of a fiber bundle (E, p, B) then $\mathcal{P}_V(E)$ is just the space of all connections and we expect that results of this section could be used for studying the moduli space of connections modulo the gauge group. This paper is a sequel to [Gil-Medrano, Michor, 1990] and essentially the same techniques and ideas are used. We will refer to it frequently.

1. The general setup

1.1. Bilinear structures. Let M be a smooth second countable finite dimensional manifold. Let $\otimes^2 T^*M$ denote the vector bundle of all $\binom{0}{2}$-tensors on M which we canonically identify with the bundle $L(TM, T^*M)$. Let $GL(TM, T^*M)$ denote the non degenerate ones. For any $b : T_xM \to T_x^*M$ we let the *transposed* be given by $b^t : T_xM \to T_x^{**}M \xrightarrow{b'} T_x^*M$. As a bilinear

structure b is skew symmetric if and only if $b^t = -b$, and b is symmetric if and only if $b^t = b$. In the latter case a frame (e_j) of $T_x M$ can be chosen in such a way that in the dual frame (e^j) of $T_x^* M$ we have

$$b = e^1 \otimes e^1 + \cdots + e^p \otimes e^p - e^{p+1} \otimes e^{p+1} - e^{p+q} \otimes e^{p+q};$$

b has signature (p, q) and is non degenerate if and only if $p + q = n$, the dimension of M. In this case q alone will be called the signature.

A section $b \in C^\infty(GL(TM, T^*M))$ will be called a non degenerate bilinear structure on M and we will denote the space of all such structures by $\mathcal{B}(M) = \mathcal{B} := C^\infty(GL(TM, T^*M))$. It is open in the space of sections $C^\infty(L(TM, T^*M))$ for the Whitney C^∞-topology, in which the latter space is, however, not a topological vector space, since $\frac{1}{n}h$ converges to 0 if and only if h has compact support. So the space $\mathcal{B}_c = C_c^\infty(L(TM, T^*M))$ of sections with compact support is the largest topological vector space contained in the topological group $(C^\infty(L(TM, T^*M)), +)$, and the trace of the Whitney C^∞-topology on it coincides with the inductive limit topology

$$C_c^\infty(L(TM, T^*M)) = \varinjlim_K C_K^\infty(L(TM, T^*M)),$$

where $C_K^\infty(L(TM, T^*M))$ is the space of all sections with support contained in K and where K runs through all compact subsets of M.

So we declare the path components of $\mathcal{B} = C^\infty(GL(TM, T^*M))$ for the Whitney C^∞-topology also to be open. We get a topology which is finer than the Whitney topology, where each connected component is homeomorphic to an open subset in $\mathcal{B}_c = C_c^\infty(L(TM, T^*M))$. So $\mathcal{B} = C^\infty(GL(TM, T^*M))$ is a smooth manifold modeled on nuclear (LF)-spaces, and the tangent bundle is given by $T\mathcal{B} = \mathcal{B} \times \mathcal{B}_c$.

1.2. Remarks. The main reference for the infinite dimensional manifold structures is [Michor, 1980]. But the differential calculus used there is not completely up to date, the reader should consult [Frölicher, Kriegl, 1988], whose calculus is more natural and much easier to apply. There a mapping between locally convex spaces is smooth if and only if it maps smooth curves to smooth curves. See also [Kriegl, Michor, 1990] for a setting for real analytic mappings along the same lines and applications to manifolds of mappings.

As a final remark let us add that the differential structure on the space $\mathcal{B}$ of non degenerate bilinear structures is not completely satisfying, if M is not compact. In fact $C^\infty(L(TM, T^*M))$ is a topological vector space with the compact C^∞-topology, but the space $\mathcal{B} = C^\infty(GL(TM, T^*M))$ of non degenerate bilinear structures is not open in it. Nevertheless, we will see

later that the exponential mapping for some pseudo riemannian metrics on $\mathcal{B}$ is defined also for some tangent vectors which are not in $\mathcal{B}_c$. This is an indication that the most natural setting for manifolds of mappings is based on the compact C^∞-topology, but that one loses existence of charts. In [Michor, 1984] a setting for infinite dimensional manifolds is presented which is based on an axiomatic structure of smooth curves instead of charts.

1.3. The metrics. The tangent bundle of the space

$$\mathcal{B} = C^\infty(GL(TM, T^*M))$$

of bilinear structures is

$$T\mathcal{B} = \mathcal{B} \times \mathcal{B}_c = C^\infty(GL(TM, T^*M)) \times\ C_c^\infty(L(TM, T^*M)).$$

Then $b \in \mathcal{B}$ induces two fiberwise bilinear forms on $L(TM, T^*M)$ which are given by $(h, k) \mapsto \operatorname{tr}(b^{-1}hb^{-1}k)$ and $(h, k) \mapsto \operatorname{tr}(b^{-1}h)\operatorname{tr}(b^{-1}k)$. We split each endomorphism $H = b^{-1}h : TM \to TM$ into its trace free part $H_0 := H - \frac{\operatorname{tr}(H)}{\dim M}Id$ and its trace part which simplifies some formulas later on. Thus we have $\operatorname{tr}(b^{-1}hb^{-1}k) = \operatorname{tr}((b^{-1}h)_0(b^{-1}k)_0) + \frac{1}{\dim M}\operatorname{tr}(b^{-1}h)\operatorname{tr}(b^{-1}k)$. The structure b also induces a volume density on the base manifold M by the local formula

$$\operatorname{vol}(b) = \sqrt{|\det(b_{ij})|}\,|dx_1 \wedge \cdots \wedge dx_n|.$$

For each real α we have a smooth symmetric bilinear form on $\mathcal{B}$, given by

$$G_b^\alpha(h, k) = \int_M (\operatorname{tr}((b^{-1}h)_0(b^{-1}k)_0) + \alpha\operatorname{tr}(b^{-1}h)\operatorname{tr}(b^{-1}k))\operatorname{vol}(b).$$

It is invariant under the action of the diffeomorphism group $\operatorname{Diff}(M)$ on the space $\mathcal{B}$ of bilinear structures. The integral is defined since h, k have compact support. For $n = \dim M$ we have

$$G_b(h, k) := G_b^{1/n}(h, k) = \int_M \operatorname{tr}(b^{-1}hb^{-1}k)\operatorname{vol}(b),$$

which for positive definite b is the usual metric on the space of all Riemannian metrics considered by [Ebin, 1970], [Freed, Groisser, 1989], and [Gil-Medrano, Michor, 1990]. We will see below in 1.4 that for $\alpha \neq 0$ it is weakly non degenerate, i.e. G_b^α defines a linear injective mapping from the tangent space $T_b\mathcal{B} = \mathcal{B}_c = C_c^\infty(L(TM, T^*M))$ into its dual $C_c^\infty(L(TM, T^*M))'$, the space of distributional densities with values in the dual bundle. This linear mapping is, however, never surjective. So we have a one parameter family of pseudo Riemannian metrics on the infinite dimensional space $\mathcal{B}$. The use of the calculus of [Frölicher, Kriegl, 1988] makes it completely obvious that it is smooth in all appearing variables.

1.4. Lemma. *For $h, k \in T_b\mathcal{B}$ we have*

$$G_b^\alpha(h, k) = G_b(h + \frac{\alpha n - 1}{n} \operatorname{tr}(b^{-1}h)b, k),$$

$$G_b(h, k) = G_b^\alpha(h - \frac{\alpha n - 1}{\alpha n^2} \operatorname{tr}(b^{-1}h)b, k), \ \ if \ \alpha \neq 0,$$

where $n = \dim M$. The pseudo Riemannian metric G^α is weakly non degenerate for all $\alpha \neq 0$.

Proof. The first equation is an obvious reformulation of the definition, the second follows since $h \mapsto h - \frac{\alpha n - 1}{\alpha n^2} \operatorname{tr}(b^{-1}h)b$ is the inverse of the transform $h \mapsto h + \frac{\alpha n - 1}{n} \operatorname{tr}(b^{-1}h)b$. Since $\operatorname{tr}(b_x^{-1}h_x(b_x^{-1}h_x)^{t,g}) > 0$ if $h_x \neq 0$, where $\ell^{t,g}$ is the transposed of a linear mapping with respect to an arbitrary fixed Riemannian metric g, we have

$$G_b(h, b(b^{-1}h)^{t,g}) = \int_M \operatorname{tr}(b^{-1}h(b^{-1}h)^{t,g}) \operatorname{vol}(b) > 0$$

if $h \neq 0$. So G is weakly non degenerate, and by the second equation G^α is weakly non degenerate for $\alpha \neq 0$. $\square$

1.5. Remark. Since G^α is only a weak pseudo Riemannian metric, all objects which are only implicitly given a priori lie in the Sobolev completions of the relevant spaces. In particular this applies to the formula

$$2G^\alpha(\xi, \nabla_\eta^\alpha \zeta) = \xi G^\alpha(\eta, \zeta) + \eta G^\alpha(\zeta, \xi) - \zeta G^\alpha(\xi, \eta)$$
$$+ G^\alpha([\xi, \eta], \zeta) + G^\alpha([\eta, \zeta], \xi) - G^\alpha([\zeta, \xi], \eta),$$

which a priori gives only uniqueness but not existence of the Levi Civita covariant derivative. But we refer to [Gil-Medrano, Michor, 1990, 2.1] for a careful explanation of the role of covariant derivatives etc.

1.6. Lemma. *For $x \in M$ the pseudo metric on $GL(T_xM, T_x^*M)$ given by*

$$\gamma_{b_x}^\alpha(h_x, k_x) := \operatorname{tr}((b_x^{-1}h_x)_0(b_x^{-1}k_x)_0) + \alpha \operatorname{tr}(b_x^{-1}h_x) \operatorname{tr}(b_x^{-1}k_x)$$

has signature (the number of negative eigenvalues) $\frac{n(n-1)}{2}$ for $\alpha > 0$ and has signature $(\frac{n(n-1)}{2} + 1)$ for $\alpha < 0$.

Proof. In the framing $H = b_x^{-1}h_x$ and $K = b_x^{-1}k_x$ we have to determine the signature of the symmetric bilinear form $H, K \mapsto \operatorname{tr}(H_0K_0) + \alpha \operatorname{tr}(H) \operatorname{tr}(K)$. Since the signature is constant on connected components we have to determine it only for $\alpha = \frac{1}{n}$ and $\alpha = \frac{1}{n} - 1$.

For $\alpha = \frac{1}{n}$ we note first that on the space of matrices $H, K \mapsto \operatorname{tr}(HK^t)$ is positive definite, and since the linear isomorphism $K \mapsto K^t$ has the space of symmetric matrices as eigenspace for the eigenvalue 1, and has the space of skew symmetric matrices as eigenspace for the eigenvalue -1, we conclude that the signature is $\frac{n(n-1)}{2}$ in this case.

For $\alpha = \frac{1}{n} - 1$ we proceed as follows: On the space of matrices with zeros on the main diagonal the signature of $H, K \mapsto \operatorname{tr}(HK)$ is $\frac{n(n-1)}{2}$ by the argument above and the form $H, K \mapsto -\operatorname{tr}(H)\operatorname{tr}(K)$ vanishes. On the space of diagonal matrices which we identify with $\mathbb{R}^n$ the whole bilinear form is given by $\langle x, y \rangle = \sum_i x^i y^i - (\sum_i x^i)(\sum_i y^i)$. Let (e_i) denote the standard basis of $\mathbb{R}^n$ and put $a_1 := \frac{1}{n}(e_1 + \cdots + e_n)$ and $a_i := \frac{1}{\sqrt{i-1+(i-1)^2}}(e_1 + \cdots + e_{i-1} - (i-1)e_i)$ for $i > 1$. Then $\langle a_1, a_1 \rangle = -1 + \frac{1}{n}$ and for $i > 1$ we get $\langle a_i, a_j \rangle = \delta_{i,j}$. So the signature there is 1. $\square$

2. Geodesics, Levi Civita connection, and curvature

2.1. Let $t \mapsto b(t)$ be a smooth curve in $\mathcal{B}$: so $b : \mathbb{R} \times M \to GL(TM, T^*M)$ is smooth and by the choice of the topology on $\mathcal{B}$ made in 1.1 the curve $b(t)$ varies only in a compact subset of M, locally in t, by [Michor, 1980, 4.4.4, 4.11, and 11.9]. Then its energy is given by

$$E_a^b(b) := \frac{1}{2} \int_a^b G_b^\alpha(b_t, b_t) dt$$

$$= \frac{1}{2} \int_a^b \int_M \left(\operatorname{tr}((b^{-1}b_t)_0 (b^{-1}b_t)_0) + \alpha \operatorname{tr}(b^{-1}b_t)^2 \right) \operatorname{vol}(b) \, dt,$$

where $b_t = \frac{\partial}{\partial t} b(t)$.

Now we consider a variation of this curve, so we assume now that $(t, s) \mapsto b(t, s)$ is smooth in all variables and locally in (t, s) it only varies within a compact subset in M — this is again the effect of the topology chosen in 1.1. Note that $b(t, 0)$ is the old $b(t)$ above.

2.2. Lemma. *In the setting of 2.1 we have the first variation formula*

$$\frac{\partial}{\partial s}|_0 E(G^\alpha)^{a_1}_{a_0}(b(\quad,s)) = G^\alpha_b(b_t, b_s)|^{t=a_1}_{t=a_0} +$$

$$+ \int_{a_0}^{a_1} G(-b_{tt} + b_t b^{-1} b_t + \frac{1}{4}\,\mathrm{tr}(b^{-1}b_t b^{-1}b_t)b - \frac{1}{2}\,\mathrm{tr}(b^{-1}b_t)b_t +$$

$$+ \alpha\,(-\,\mathrm{tr}(b^{-1}b_{tt}) - \frac{1}{4}\,\mathrm{tr}(b^{-1}b_t)^2 + \mathrm{tr}(b^{-1}b_t b^{-1}b_t))b, b_s)\,dt =$$

$$= G^\alpha_b(b_t, b_s)|^{t=a_1}_{t=a_0} +$$

$$+ \int_{a_0}^{a_1} G^\alpha(-b_{tt} + b_t b^{-1} b_t - \frac{1}{2}\,\mathrm{tr}(b^{-1}b_t)b_t + \frac{1}{4\alpha n}\,\mathrm{tr}(b_t^{-1}b^{-1}b_t)b +$$

$$+ \frac{\alpha n - 1}{4\alpha n^2}\,\mathrm{tr}(b^{-1}b_t)^2 b, b_s)\,dt$$

Proof. We may interchange $\frac{\partial}{\partial s}|_0$ with the first integral describing the energy in 2.1 since this is finite dimensional analysis, and we may interchange it with the second one, since $\int_M$ is a continuous linear functional on the space of all smooth densities with compact support on M, by the chain rule. Then we use that tr_* is linear and continuous, $d(\mathrm{vol})(b)h = \frac{1}{2}\,\mathrm{tr}(b^{-1}h)\,\mathrm{vol}(b)$, and that $d((\quad)^{-1})_*(b)h = -b^{-1}hb^{-1}$ and partial integration. □

2.3. The geodesic equation. By lemma 2.2 the curve $t \mapsto b(t)$ is a geodesic if and only if we have

$$b_{tt} = b_t b^{-1} b_t - \frac{1}{2}\,\mathrm{tr}(b^{-1}b_t)b_t + \frac{1}{4\alpha n}\,\mathrm{tr}(b^{-1}b_t b^{-1}b_t)b + \frac{\alpha n - 1}{4\alpha n^2}\,\mathrm{tr}(b^{-1}b_t)^2 b.$$

$$= \Gamma_b(b_t, b_t),$$

where the G^α-*Christoffel symbol* $\Gamma^\alpha\colon \mathcal{B} \times \mathcal{B}_c \times \mathcal{B}_c \to \mathcal{B}_c$ is given by symmetrisation

$$\Gamma^\alpha_b(h, k) = \frac{1}{2}hb^{-1}k + \frac{1}{2}kb^{-1}h - \frac{1}{4}\,\mathrm{tr}(b^{-1}h)k - \frac{1}{4}\,\mathrm{tr}(b^{-1}k)h +$$

$$+ \frac{1}{4\alpha n}\,\mathrm{tr}(b^{-1}hb^{-1}k)b + \frac{\alpha n - 1}{4\alpha n^2}\,\mathrm{tr}(b^{-1}h)\,\mathrm{tr}(b^{-1}k)b.$$

The sign of Γ^α is chosen in such a way that the horizontal subspace of $T^2\mathcal{B}$ is parameterized by $(x, y; z, \Gamma_x(y, z))$. If instead of the obvious framing we use $T\mathcal{B} = \mathcal{B} \times \mathcal{B}_c \ni (b, h) \mapsto (b, b^{-1}h) =: (b, H) \in \{b\} \times C^\infty_c(L(TM, TM))$, the Christoffel symbol looks like

$$\overline{\Gamma^\alpha}_b(H, K) = \frac{1}{2}(HK + KH) - \frac{1}{4}\,\mathrm{tr}(H)K - \frac{1}{4}\,\mathrm{tr}(K)H$$

$$+ \frac{1}{4\alpha n}\,\mathrm{tr}(HK)Id + \frac{\alpha n - 1}{4\alpha n^2}\,\mathrm{tr}(H)\,\mathrm{tr}(K).$$

and the G^α-geodesic equation for $B(t) := b^{-1}b_t$ becomes

$$B_t = \frac{\partial}{\partial t}(b^{-1}b_t) = \frac{1}{4\alpha n}\operatorname{tr}(BB)Id - \frac{1}{2}\operatorname{tr}(B)B + \frac{\alpha n - 1}{4\alpha n^2}\operatorname{tr}(B)^2 Id.$$

2.4. The curvature. For vector fields X, $Y \in \mathfrak{X}(\mathcal{N})$ and a vector field $s: \mathcal{N} \to T\mathcal{M}$ along $f: \mathcal{N} \to \mathcal{M}$ we have

$$R(X,Y)s = (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y])s = (K \circ TK - K \circ TK \circ \kappa_{T\mathcal{M}}) \circ T^2 s \circ TX \circ Y,$$

where $K: TTM \to M$ is the connector (see [Gil-Medrano, Michor, 2.1]) and where the second formula in local coordinates reduces to the usual formula

$$R(h,k)\ell = d\Gamma(h)(k,\ell) - d\Gamma(k)(h,\ell) - \Gamma(h, \Gamma(k,\ell)) + \Gamma(k, \Gamma(h,\ell)).$$

A global derivation of this formula can be found in [Kainz, Michor, 1987].

2.5. Theorem. *The curvature for the pseudo Riemannian metric G^α on the manifold $\mathcal{B}$ of all non degenerate bilinear structures is given by*

$$b^{-1}R_b^\alpha(h,k)l = \frac{1}{4}[[H,K],L] + \frac{1}{16\alpha}(-\operatorname{tr}(HL)K + \operatorname{tr}(KL)H +$$

$$+ \frac{4\alpha n - 3\alpha n^2 + 4n - 4}{16\alpha n^2}(\operatorname{tr}(H)\operatorname{tr}(L)K -$$

$$- \operatorname{tr}(K)\operatorname{tr}(L)H +$$

$$+ \frac{4\alpha^2 n^2 - 4\alpha n + \alpha n^2 + 3}{16\alpha n^2}(\operatorname{tr}(HL)\operatorname{tr}(K)Id$$

$$- \operatorname{tr}(KL)\operatorname{tr}(H)Id),$$

where $H = b^{-1}h, K = b^{-1}k$ and $L = b^{-1}l$.

Proof. This is a long but direct computation. $\quad\square$

The geodesic equation can be solved explicitly and we have

2.6. Theorem. *Let $b^0 \in \mathcal{B}$ and $h \in T_{b^0}\mathcal{B} = \mathcal{B}_c$. Then the geodesic for the metric G^α in $\mathcal{B}$ starting at b^0 in the direction of h is the curve*

$$\operatorname{Exp}_{b^0}^\alpha(th) = b^0 e^{(a(t)Id + b(t)H_0)},$$

where H_0 is the traceless part of $H := (b^0)^{-1}h$ (i.e. $H_0 = H - \frac{\operatorname{tr}(H)}{n}Id$) and where $a(t) = a_{\alpha,H}(t)$ and $b(t) = b_{\alpha,H}(t)$ in $C^\infty(M)$ are defined as follows:

$$a_{\alpha,H}(t) = \frac{2}{n}\log\left((1 + \frac{t}{4}\operatorname{tr}(H))^2 + t^2\frac{\alpha^{-1}}{16}\operatorname{tr}(H_0^2)\right)$$

$$b_{\alpha,H}(t) = \begin{cases} \dfrac{4}{\sqrt{\alpha^{-1}\operatorname{tr}(H_0^2)}}\arctan\left(\dfrac{t\sqrt{\alpha^{-1}\operatorname{tr}(H_0^2)}}{4 + t\operatorname{tr}(H)}\right) & \text{for } \alpha^{-1}\operatorname{tr}(H_0^2) > 0 \\[3mm] \dfrac{4}{\sqrt{-\alpha^{-1}\operatorname{tr}(H_0^2)}}\operatorname{Artanh}\left(\dfrac{t\sqrt{-\alpha^{-1}\operatorname{tr}(H_0^2)}}{4 + t\operatorname{tr}(H)}\right) & \text{for } \alpha^{-1}\operatorname{tr}(H_0^2) < 0 \\[3mm] \dfrac{t}{1 + \frac{t}{4}\operatorname{tr}(H)} & \text{for } \operatorname{tr}(H_0^2) = 0 \end{cases}$$

Here arctan *is taken to have values in* $(-\frac{\pi}{2}, \frac{\pi}{2})$ *for the points of the basis manifold, where* $\mathrm{tr}(H) \geq 0$, *and on a point where* $\mathrm{tr}(H) < 0$ *we define*

$$
\arctan\left(\frac{t\sqrt{\alpha^{-1}\,\mathrm{tr}(H_0^2)}}{4 + t\,\mathrm{tr}(H)} \right) = \begin{cases} \arctan\ in\ [0, \frac{\pi}{2}) & for\ t \in [0, -\frac{4}{\mathrm{tr}(H)}) \\[2mm] \frac{\pi}{2} & for\ t = -\frac{4}{\mathrm{tr}(H)} \\[2mm] \arctan\ in\ (\frac{\pi}{2}, \pi) & for\ t \in (-\frac{4}{\mathrm{tr}(H)}, \infty). \end{cases}
$$

To describe the domain of definition of the exponential mapping we consider the sets

$$
Z^h : = \{ x \in M : \frac{1}{\alpha}\,\mathrm{tr}_x(H_0^2) = 0\ and\ \mathrm{tr}_x(H) < 0 \},
$$

$$
\begin{aligned}
G^h : &= \{ x \in M : 0 > \frac{1}{\alpha}\,\mathrm{tr}_x(H_0^2) > -\mathrm{tr}_x(H)^2\ and\ \mathrm{tr}_x(H) < 0 \} \\
&= \{ x \in M : \alpha\gamma(h,h) \lessgtr \gamma^\alpha(h,h) \lessgtr 0\ for\ \alpha \lessgtr 0, \mathrm{tr}_x(H) < 0 \},
\end{aligned}
$$

$$
\begin{aligned}
E^h : &= \{ x \in M : -\mathrm{tr}_x(H)^2 = \frac{1}{\alpha}\,\mathrm{tr}_x(H_0^2)\ and\ \mathrm{tr}_x(H) < 0 \} \\
&= \{ x \in M : \gamma^\alpha(h,h) = 0\ and\ \mathrm{tr}_x(H) < 0 \},
\end{aligned}
$$

$$
\begin{aligned}
L^h : &= \{ x \in M : -\mathrm{tr}_x(H)^2 > \frac{1}{\alpha}\,\mathrm{tr}_x(H_0^2) \} \\
&= \{ x \in M : \gamma^\alpha(h,h) \gtrless 0\ for\ \alpha \lessgtr 0 \},
\end{aligned}
$$

where $\gamma(h,h) = \mathrm{tr}_x(H^2)$, *and* $\gamma^\alpha(h,h) = \mathrm{tr}_x(H_0^2) + \alpha\,\mathrm{tr}_x(H)^2$, *see 1.6, are the integrands of* $G_{b^0}(h,h)$ *and* $G_{b^0}^\alpha(h,h)$, *respectively. Then we consider the numbers*

$$
z^h : = \inf\{ -\frac{4}{\mathrm{tr}_x(H)} : x \in Z^h \},
$$

$$
g^h : = \inf\{ 4\frac{-\alpha\,\mathrm{tr}_x(H) - \sqrt{-\alpha\,\mathrm{tr}_x(H_0^2)}}{\mathrm{tr}_x(H_0^2) + \alpha\,\mathrm{tr}(H)^2} : x \in G^h \},
$$

$$
e^h : = \inf\{ -\frac{2}{\mathrm{tr}_x(H)} : x \in E^h \},
$$

$$
l^h : = \inf\{ 4\frac{-\alpha\,\mathrm{tr}_x(H) - \sqrt{-\alpha\,\mathrm{tr}_x(H_0^2)}}{\mathrm{tr}_x(H_0^2) + \alpha\,\mathrm{tr}(H)^2} : x \in L^h \},
$$

if the corresponding set is not empty, with value ∞ *if the set is empty. Put* $m^h : = \inf\{ z^h, g^h, e^h, l^h \}$. *Then* $\mathrm{Exp}_{b^0}^\alpha(th)$ *is maximally defined for* $t \in [0, m^h)$.

The second representations of the sets G^h, L^h, and E^h clarifies how to take care of timelike, spacelike, and lightlike vector, respectively.

Proof. The geodesical equation is very similar to that of the metric G on the space of all Riemannian metrics, whose solution can be found in [Freed, Groisser, 1989], see also [Gil-Medrano, Michor, 1990]. The difference now is essentially that one should control the sign of various appearing constants. Here we use a slightly simpler method that unable us to deal only with scalar equations. Using $X(t) := g^{-1}g_t$ the geodesic equation reads as

$$X' = -\frac{1}{2}\operatorname{tr}(X)X + \frac{1}{4\alpha n}\operatorname{tr}(X^2)Id + \frac{\alpha n - 1}{4\alpha n^2}\operatorname{tr}(X)^2 Id,$$

and it is easy to see that a solution X satisfies

$$X_0' = -\frac{1}{2}\operatorname{tr}(X)X_0.$$

Then $X(t)$ is in the plane generated by H_0 and Id for all t and the solution has the form $g(t) = b^0 \exp(a(t)Id + b(t)H_0)$. Since $g_t = g(t)(a'(t)Id + b'(t)H_0)$ we have

$$X(t) = a'(t)Id + b'(t)H_0 \text{ and}$$
$$X'(t) = a''(t)Id + b''(t)H_0,$$

and the geodesic equation becomes

$$a''(t)Id + b''(t)H_0 = -\frac{1}{2}na'(t)(a'(t)Id + b'(t)H_0)+$$
$$+ \frac{1}{4\alpha n}(na'(t)^2 + b'(t)^2\operatorname{tr}(H_0^2))Id+$$
$$+ \frac{\alpha n - 1}{4\alpha n^2}(n^2 a'(t)^2)Id.$$

We may assume that Id and H_0 are linearly independent; if not $H_0 = 0$ and $b(t) = 0$. Hence the geodesic equation reduces to the differential equation

$$\begin{cases} a'' = -\dfrac{n}{4}(a')^2 + \dfrac{\operatorname{tr}(H_0^2)}{4\alpha n}(b')^2 \\ b'' = -\dfrac{n}{2}a'\,b' \end{cases}$$

with initial conditions $a(0) = b(0) = 0$, $a'(0) = \frac{\operatorname{tr}(H)}{n}$, and $b'(0) = 1$.

If we take $p(t) = \exp(\frac{n}{2}a)$ it is easy to see that then p should be a solution of $p''' = 0$ and from the initial conditions

$$p(t) = 1 + \frac{t}{2}\operatorname{tr}(H) + \frac{t^2}{16}(\operatorname{tr}(H)^2 + \alpha^{-1}\operatorname{tr}(H_0^2)).$$

Using that the second equation becomes $b' = p^{-1}$, and then b is obtained just by computing the integral. The solutions are defined in $[0, m^h)$ where m^h is the infimum over the support of h of the first positive root of the polynomial p, if it exists, and ∞ otherwise. The description of m^h is now a technical fact. $\square$

2.7. The exponential mapping. For $b^0 \in GL(T_xM, T_x^*M)$ and $H = (b^0)^{-1}h$ let C_{b^0} be the subset of $L(T_xM, T_x^*M)$ given by the union of the sets (compare with Z^h, G^h, E^h, L^h from 2.6)

$$\{h : \operatorname{tr}(H_0^2) = 0, \operatorname{tr}(H) \le -4\},$$

$$\left\{h : 0 > \frac{1}{\alpha}\operatorname{tr}(H_0^2) > -\operatorname{tr}(H)^2, 4\frac{-\alpha\operatorname{tr}(H) - \sqrt{-\alpha\operatorname{tr}(H_0^2)}}{\operatorname{tr}(H_0^2) + \alpha\operatorname{tr}(H)^2} \le 1, \operatorname{tr}(H) < 0\right\},$$

$$\left\{h : -\operatorname{tr}(H)^2 = \frac{1}{\alpha}\operatorname{tr}(H_0^2), \operatorname{tr}(H) < -2\right\},$$

$$\left\{h : -\operatorname{tr}(H)^2 > \frac{1}{\alpha}\operatorname{tr}(H_0^2), 4\frac{-\alpha\operatorname{tr}(H) - \sqrt{-\alpha\operatorname{tr}(H_0^2)}}{\operatorname{tr}(H_0^2) + \alpha\operatorname{tr}(H)^2} \le 1\right\}^{\text{closure}}$$

which by some limit considerations coincides with the union of the following two sets:

$$\left\{h : 0 > \frac{1}{\alpha}\operatorname{tr}(H_0^2) > -\operatorname{tr}(H)^2,\right.$$

$$\left. 4\frac{-\alpha\operatorname{tr}(H) - \sqrt{-\alpha\operatorname{tr}(H_0^2)}}{\operatorname{tr}(H_0^2) + \alpha\operatorname{tr}(H)^2} \le 1, \operatorname{tr}(H) < 0\right\}^{\text{closure}},$$

$$\left\{h : -\operatorname{tr}(H)^2 > \frac{1}{\alpha}\operatorname{tr}(H_0^2), 4\frac{-\alpha\operatorname{tr}(H) - \sqrt{-\alpha\operatorname{tr}(H_0^2)}}{\operatorname{tr}(H_0^2) + \alpha\operatorname{tr}(H)^2} \le 1\right\}^{\text{closure}}.$$

So C_{b^0} is closed. We consider the open sets $U_{b^0} := L(T_xM, T_x^*M) \setminus C_{b^0}$, $U'_{b^0} := \{(b^0)^{-1}h : h \in U_{b^0}\} \subset L(T_xM, T_xM)$, and finally the open sub fiber bundles over $GL(TM, T^*M)$

$$U := \bigcup\{\{b^0\} \times U_{b^0} : b^0 \in GL(TM, T^*M)\} \subset$$
$$\subset GL(TM, T^*M) \times_M L(TM, T^*M),$$

$$U' := \bigcup\{\{b^0\} \times U'_{b^0} : b^0 \in GL(TM, T^*M)\} \subset$$
$$\subset GL(TM, T^*M) \times_M L(TM, TM).$$

Then we consider the mapping $\Phi : U \to GL(TM, T^*M)$ which is given by the following composition

$$U \xrightarrow{\sharp} U' \xrightarrow{\varphi} GL(TM, T^*M) \times_M L(TM, TM) \xrightarrow{Id \times_M \exp}$$

$$\xrightarrow{Id \times_M \exp} GL(TM, T^*M) \times_M GL(TM, TM) \xrightarrow{\flat} GL(TM, T^*M),$$

where $\sharp(b^0, h) := (b^0, (b^0)^{-1}h)$ is a fiber respecting diffeomorphism, $\flat(b^0, H) := b^0H$ is a diffeomorphism for fixed b^0, and where the other two mappings will be discussed below.

The usual fiberwise exponential mapping

$$\exp : L(TM, TM) \to GL(TM, TM)$$

is a diffeomorphism near the zero section, on the ball of radius π centered at zero in a norm on the Lie algebra for which the Lie bracket is sub multiplicative, for example. If we fix a symmetric positive definite inner product g, then exp restricts to a global diffeomorphism from the linear subspace of g-symmetric endomorphisms onto the open subset of matrices which are positive definite with respect to g. If g has signature this is no longer true since then g-symmetric matrices may have non real eigenvalues.

On the open set of all matrices whose eigenvalues λ satisfy $|\Im\lambda| < \pi$, the exponential mapping is a diffeomorphism, see [Varadarajan, 1977].

The smooth mapping $\varphi : U' \to GL(TM, T^*M) \times_M L(TM, TM)$ is given by $\varphi(b^0, H) := (b^0, a_{\alpha,H}(1)Id + b_{\alpha,H}(1)H_0)$ (see theorem 2.6). It is a diffeomorphism onto its image with the following inverse:

$$\psi(H) := \begin{cases} \dfrac{4}{n}\left(e^{\frac{\operatorname{tr}(H)}{4}}\cos\left(\dfrac{\sqrt{\alpha^{-1}\operatorname{tr}(H_0^2)}}{4}\right) - 1\right)Id + \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \operatorname{tr}(H_0^2) \neq 0 \\ \dfrac{4}{\sqrt{\alpha^{-1}\operatorname{tr}(H_0^2)}}e^{\frac{\operatorname{tr}(H)}{4}}\sin\left(\dfrac{\sqrt{\alpha^{-1}\operatorname{tr}(H_0^2)}}{4}\right)H_0 \\ \dfrac{4}{n}\left(e^{\frac{\operatorname{tr}(H)}{4}} - 1\right)Id \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where cos is considered as a complex function, $\cos(iz) = i\cosh(z)$.

The mapping $(pr_1, \Phi) : U \to GL(TM, T^*M) \times_M GL(TM, T^*M)$ is a diffeomorphism on an open neighborhood of the zero section in U.

2.8. Theorem. *In the setting of 2.7 the exponential mapping* $\operatorname{Exp}_{b^0}^{\alpha}$ *for the metric* G^{α} *is a real analytic mapping defined on the open subset*

$$\mathcal{U}_{b^0} := \{h \in C_c^{\infty}(L(TM, T^*M)) : (b^0, h)(M) \subset U\}$$

and it is given by

$$\operatorname{Exp}_{b^0}(h) = \Phi \circ (b^0, h).$$

The mapping $(\pi_{\mathcal{B}}, \operatorname{Exp}) : T\mathcal{B} \to \mathcal{B} \times \mathcal{B}$ *is a real analytic diffeomorphism from an open neighborhood of the zero section in* $T\mathcal{B}$ *onto an open neighborhood of the diagonal in* $\mathcal{B} \times \mathcal{B}$. $\mathcal{U}_{b_0}$ *is the maximal domain of definition for the exponential mapping.*

Proof. Most assertions are easy consequences of the considerations above. For real analyticity of Exp the proof of [Gil-Medrano, Michor, 1990, 3.4] applies which made use of deep results from [Kriegl, Michor, 1990] $\square$

3. Some submanifolds of $\mathcal{B}$

3.1. Submanifolds of pseudo Riemannian metrics. We denote by $\mathcal{M}^q$ the space of all pseudo Riemannian metrics on the manifold M of signature (the dimension of a maximal negative definite subspace) q. It is an open set in a closed locally affine subspace of $\mathcal{B}$ and thus a splitting submanifold of it with tangent bundle $T\mathcal{M}^q = \mathcal{M}^q \times C_c^\infty(S^2 T^*M)$.

We consider a geodesic $b(t) = b^0 e^{(a(t)Id + b(t)H_0)}$ for the metric G^α in $\mathcal{B}$ starting at b^0 in the direction of h as in 2.6. If $b^0 \in \mathcal{M}^q$ then $h \in T_{b^0}\mathcal{M}^q$ if and only if $H = (b^0)^{-1}h \in L_{\mathrm{sym},b^0}(TM, TM)$ is symmetric with respect to the pseudo Riemannian metric b^0. But then $e^{(a(t)Id + b(t)H_0)} \in L_{\mathrm{sym},b^0}(TM, TM)$ for all t in the domain of definition of the geodesic, so $b(t)$ is a curve of pseudo Riemannian metrics and thus of the same signature q as b^0. Thus we have

3.2. Theorem. *For each $q \le n = \dim M$ the submanifold $\mathcal{M}^q$ of pseudo Riemannian metrics of signature q on M is a geodesically closed submanifold of $(\mathcal{B}, G^\alpha)$ for each $\alpha \ne 0$.*

Remark. The geodesics of $(\mathcal{M}^0, G^\alpha)$ have been studied, for $\alpha = \frac{1}{n}$, in [Freed, Groisser, 1989], [Gil-Medrano, Michor, 1990] and from 3.2 and 2.6 we see that they are completely analogous for every positive α.

For fixed $x \in M$ there exists a family of homothetic pseudo metrics on the finite dimensional manifold $S_+^2 T_x^*M$ whose geodesics are given by the evaluation of the geodesics of $(\mathcal{M}^0, G^\alpha)$ (see [Gil-Medrano, Michor, 1990] for more details). When α is negative, it is not difficult to see, from 3.2 and 2.6 again, that a geodesic of $(\mathcal{M}^0, G^\alpha)$ is defined for all t if and only if the initial velocity h satisfies $\gamma^\alpha(h, h) \le 0$ and $\operatorname{tr} H > 0$ at each point of M and then the same is true for all the above pseudo metrics on $S_+^2 T_x^*M$. These results appear already in [DeWitt, 1967] for $n = 3$.

3.3. The local signature of G^α. Since G^α operates in infinite dimensional spaces, the usual definition of signature is not applicable. But for fixed $g \in \mathcal{M}^q$ the signature of

$$\gamma_{g_x}^\alpha(h_x, k_x) = \operatorname{tr}((g_x^{-1}h_x)_0 (g_x^{-1}k_x)_0) + \alpha \operatorname{tr}(g_x^{-1}h_x)\operatorname{tr}(g_x^{-1}k_x)$$

on $T_g(S_q^2 T_x^*M) = S^2 T_x^*M$ is independent of $x \in M$ and the special choice of $g \in \mathcal{M}^q$. We will call it the *local signature* of G^α.

3.4. Lemma. *The signature of the quadratic form of 3.3 is*

$$Q(\alpha, q) = q(q - n) + \begin{cases} 0 & \text{for } \alpha > 0 \\ 1 & \text{for } \alpha < 0. \end{cases}$$

This result is due to [Schmidt, 1989].

Proof. Since the signature is constant on connected components we have to determine it only for $\alpha = \frac{1}{n}$ and $\alpha = \frac{1}{n} - 1$. In a basis for TM and its dual basis for T^*M the bilinear form $h \in S^2 T_x^* M$ has a symmetric matrix. If the basis is orthonormal for g we have (for $A^t = A$ and $C^t = C$)

$$H = g^{-1}h = \begin{pmatrix} -Id_q & 0 \\ 0 & Id_{n-q} \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} = \begin{pmatrix} -A & -B \\ B^t & C \end{pmatrix},$$

which describes the typical matrix in the space $L_{\mathrm{sym},g}(T_xM, T_xM)$ of all $H \in L(T_xM, T_xM)$ which are symmetric with respect to g_x.

Now we treat the case $\alpha = \frac{1}{n}$. The standard inner product $\mathrm{tr}(HK^t)$ is positive definite on $L_{\mathrm{sym},g}(T_xM, T_xM)$ and the linear mapping $K \mapsto K^t$ has an eigenspace of dimension $q(n-q)$ for the eigenvalue -1 in it, and a complementary eigenspace for the eigenvalue 1. So $\mathrm{tr}(HK)$ has signature $q(n-q)$.

For the case $\alpha = \frac{1}{n} - 1$ we again split the space $L_{\mathrm{sym},g}(T_xM, T_xM)$ into the subspace with 0 on the main diagonal, where $\gamma_g^\alpha(h,k) = \mathrm{tr}(HK)$ and where $K \mapsto K^t$ has again an eigenspace of dimension $q(n-q)$ for the eigenvalue -1, and the space of diagonal matrices. There γ_g^α has signature 1 as determined in the proof of 1.6. $\square$

3.5. The submanifold of almost symplectic structures.
A 2-form $\omega \in \Omega^2(M) = C^\infty(\Lambda^2 T^*M)$ can be non degenerate only if M is of even dimension $\dim M = n = 2m$. Then ω is non degenerate if and only if $\omega \wedge \cdots \wedge \omega = \omega^m$ is nowhere vanishing. Usually this latter $2m$-form is regarded as the volume form associated with ω, but a short computation shows that we have

$$\mathrm{vol}(\omega) = \tfrac{1}{m!}|\omega^m|.$$

This implies $m\varphi \wedge \omega^{m-1} = \frac{1}{2}\mathrm{tr}(\omega^{-1}\varphi)\omega^m$.

3.6. Theorem. *The space $\Omega^2_{\mathrm{nd}}(M)$ of non degenerate 2-forms is a splitting geodesically closed submanifold of $(\mathcal{B}, G^\alpha)$ for each $\alpha \neq 0$.*

Proof. We consider a geodesic $b(t) = b^0 e^{(a(t)Id + b(t)H_0)}$ for the metric G^α in $\mathcal{B}$ starting at b^0 in the direction of h as in 2.6. If $b^0 = \omega \in \Omega^2_{\mathrm{nd}}(M)$ then $h \in \Omega^2_c(M)$ if and only if $H = \omega^{-1}h$ is symmetric with respect to ω, since we have $\omega(HX, Y) = \langle \omega\omega^{-1}hX, Y \rangle = \langle hX, Y \rangle = h(X, Y) = -h(Y, X) = -\omega(HY, X) = \omega(X, HY)$. At a point $x \in M$ we may choose a Darboux frame (e_i) such that $\omega(X, Y) = Y^t J X$ where $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$. Then h is skew if and only if JH is a skew symmetric matrix in the Darboux frame,

or $JH = H^t J$. Since $(e^A)^t = e^{A^t}$ the matrix $e^{a(t)Id+b(t)H_0}$ has then the same property, $b(t)$ is skew for all t. So $\Omega^2_{nd}(M)$ is a geodesically closed submanifold. $\square$

3.7. Lemma. *For a non degenerate 2-form ω the signature of the quadratic form $\varphi \mapsto \text{tr}(\omega^{-1}\varphi\omega^{-1}\varphi)$ on $\Lambda^2 T_x^* M$ is $m^2 - m$ for $\alpha > 0$ and $m^2 - m + 1$ for $\alpha < 0$.*

Proof. Use the method of 1.6 and 3.4; the description of the space of matrices can be read of the proof of 3.6. $\square$

3.8. Symplectic structures. The space $\text{Symp}(M)$ of all symplectic structures is a closed submanifold of $(\mathcal{B}, G^\alpha)$. For a compact manifold M it is splitting by the Hodge decomposition Theorem. For $\dim M = 2$ we have $\text{Symp}(M) = \Omega^2_{nd}(M)$, so it is geodesically closed. But for $\dim M \geq 4$ the submanifold $\text{Symp}(M)$ is not geodesically closed. For $\omega \in \text{Symp}(M)$ and $\varphi, \psi \in T_\omega \text{Symp}(M)$ the Christoffel form $\Gamma^\alpha_\omega(\varphi, \psi)$ is not closed in general. The direct approach would need variational calculus with a partial differential equation as constraint. This will be treated in another paper.

4. Splitting the manifold of metrics

The developments in this sections were ignited by a question posed by Maria Christina Abbati. The second author wants to thank her for her question.

4.1. Almost product structures with given vertical distribution. Let M be a smooth finite dimensional manifold, connected for simplicity's sake, and let V be a distribution on it. We will denote by (TM, π, M) the tangent bundle, by (V, π_V, M) the vector subbundle determined by V, and by $(N = TM/V, \pi_N, M)$ the normal bundle. Let $i : V \hookrightarrow TM$ denote the embedding of V and $p : TM \twoheadrightarrow N$ the epimorphism onto the normal bundle.

Let us recall that an almost product structure on a manifold M is a (1,1)-tensor field P (i.e. $P \in C^\infty(L(TM, TM))$) such that $P^2 = Id$. It is evident that an almost product structure P on M induces a decomposition of TM of the form $TM = \ker(P - Id) \oplus \ker(P + Id)$. These subbundles are called vertical and horizontal and will be denoted by V^P, H^P respectively . We also have in a natural way two projectors $v^P = \frac{1}{2}(P + Id)$ and $h^P = \frac{1}{2}(Id - P)$, the vertical (over V^P), and the horizontal (over H^P) projections. The almost product structure P also determines a monomorphism $C_P : N \to TM$, called the horizontal lifting, given by $C_P \circ p = h^P$; it is an isomorphism onto H^P inverse of $p|H^P$.

For a given distribution V in M we will denote by $\mathcal{P}_V(M)$ the space of all almost product structures with vertical V (i.e. such that $V = V^P$). So,

giving an element of $\mathcal{P}_V(M)$ is equivalent to choosing a subbundle of TM supplementary of V, this subbundle is given then by $\ker(P + Id)$.

4.2. Proposition. *The space $\mathcal{P}_V(M)$ of almost product structures with vertical distribution V is a real analytic manifold with trivial tangent bundle whose fiber is $\{\xi \in C_c^\infty(L(TM, TM)) : \operatorname{im}\xi \subset V \subset \ker\xi\}$*

Proof. We topologize $C^\infty(L(TM, TM))$ in such a way that it becomes a topological locally affine space whose model vector space is the space

$$C_c^\infty(L(TM, TM))$$

of sections with compact support. Then

$$\mathcal{P}_V(M) = \{P \in C^\infty(L(TM, TM)) : P^2 = Id \ , \ \ker(P - Id) = V\}$$

is a closed locally affine subspace of $C^\infty(L(TM, TM))$, and thus a real analytic manifold.

The tangent space at P is given by

$$\{\xi \in C_c^\infty(L(TM, TM)) : V \subset \ker\xi \ , \ \xi P + P\xi = 0\}.$$

Now, for a $(1, 1)$-tensor field ξ the conditions $V \subset \ker\xi$ and $\xi P + P\xi = 0$ are equivalent to $h^P\xi = 0$ and $\xi v^P = 0$. The last couple of conditions can be written only in term of V as $\operatorname{im}\xi \subset V \subset \ker\xi$. $\square$

4.3. For each metric g on M we have a canonical choice of a complementary of V, just by taking the orthogonal with respect to that metric, $V^{\perp,g}$, that defines an almost product structure given by $P|V = Id$ and $P|V^{\perp,g} = -Id$ (This structure is such that (g, P) is an almost product Riemannian structure, i.e. $g(P\cdot, P\cdot) = g(\cdot, \cdot)$). g also determines a metric on the bundle (V, π_V, M) simply by restriction and a metric on the normal bundle (N, π_N, M) as the restriction to $V^{\perp,g}$ via the isomorphism given by the horizontal lifting.

Conversely, given an element P of $\mathcal{P}_V(M)$ and metrics $g_1 \in \mathcal{M}(N)$ and $g_2 \in \mathcal{M}(V)$ a metric on M can be defined by $g(\cdot, \cdot) = g_1(p\cdot, p\cdot) + g_2(v^P\cdot, v^P\cdot)$. It is easy to see that a bijection is then established between $\mathcal{M}(M)$ and $\mathcal{P}_V(M) \times \mathcal{M}(N) \times \mathcal{M}(V)$.

4.4. Proposition. *There is a real analytic diffeomorphism*

$$\mathcal{M}(M) \cong \mathcal{M}(N) \times \mathcal{M}(V) \times \mathcal{P}_V(M)$$

Proof. In order to show that the above bijection is in fact a real analytic diffeomorphism it will be convenient to write the maps in the following way:

Let Φ be the map from $\mathcal{M}(M)$ to $\mathcal{M}(N) \times \mathcal{M}(V) \times \mathcal{P}_V(M)$ and let Π_1, Π_2, Π_3 the projections. We identify each metric g with its associated mapping $g : TM \to T^*M$, so that $\mathcal{M}(M) \subset C^\infty(L(TM, T^*M))$.

We let g_V denote the restriction of the metric g to the subbundle V, associated to it is the vector bundle isomorphism $g_V = i^* g i : V \to V^*$, where $i : V \to TM$ is the injection and $i^* : T^*M \to V^*$ is its adjoint. Then $\Pi_2 \circ \Phi(g) = g_V$.

It is easy to see that the associated almost product structure described above is given by $\Pi_3 \circ \Phi(g) = 2ig_V^{-1}i^*g - Id$.

Let us denote C_g the horizontal lifting determined by $\Pi_3 \circ \Phi(g)$ (then, $C_g p = Id - ig_V^{-1}i^*g$) and C_g^* its adjoint. Then, $\Pi_1 \circ \Phi(g) = C_g^* g C_g$.

Now, the inverse of Φ is given by $\Psi(g_1, g_2, P) = p^* g_1 p + (v^P)^* g_2 v^P$.

Thus both Φ and Ψ are the push forward of sections by a fiber respecting smooth mapping which is fiberwise quadratic, so it extends to a fiberwise holomorphic mapping in a neighborhood between the complexifications of the affine bundles in question. By the argument used in the proof of [Gil-Medrano, Michor, 3.4] they are real analytic. $\square$

4.5. We have seen that any distribution on M induces a product structure on $\mathcal{M}$. We consider now the Riemannian manifold $(\mathcal{M}, G)$, and we are going to see that there exists a metric on $\mathcal{M}(N) \times \mathcal{M}(V)$ and a family of metrics on $\mathcal{P}_V(M)$ such that G is what is usually called a product manifold with varying metric on the fibers, although it is not the product metric.

To show that we will need some formulas which are obtained by straightforward computations.

For each $(g_1, g_2) \in \mathcal{M}(N) \times \mathcal{M}(V)$ we have the immersion $\Psi_{(g_1, g_2)} : \mathcal{P}_V(M) \to \mathcal{M}$. The tangent map, at a point $P \in \mathcal{P}_V(M)$,

$$T_P \Psi_{(g_1, g_2)} : \{\xi \in C_c^\infty(L(TM, TM)) : \operatorname{im} \xi \subset V \subset \ker \xi\} \to C_c^\infty(S^2 T^*M)$$

is given by $T_P \Psi_{(g_1, g_2)}(\xi) = \frac{1}{2}\{(v^P)^* g_2 \xi + \xi^* g_2 v^P\}$.

For each $P \in \mathcal{P}_V(M)$ we have the immersion $\Psi_P : \mathcal{M}(N) \times \mathcal{M}(V) \to \mathcal{M}$. The tangent map, at a point $(g_1, g_2) \in \mathcal{M}(N) \times \mathcal{M}(V)$,

$$T_{(g_1, g_2)} \Psi_P : C_c^\infty(S^2 N^*) \times C_c^\infty(S^2 V^*) \to C_c^\infty(S^2 T^*M)$$

is given by $T_{(g_1, g_2)} \Psi_P(h_1, h_2) = p^* h_1 p + (v^P)^* h_2 v^P$.

For each $g \in \mathcal{M}$ the tangent map, at g, of the submersion $\Pi_3 \circ \Phi$ is given by $T_g(\Pi_3 \circ \Phi)(h) = 2v^P g^{-1} h h^P$ and the tangent map, at g, of the submersion $(\Pi_1, \Pi_2) \circ \Phi$ is given by $(T_g(\Pi_1, \Pi_2) \circ \Phi)(h) = (C_g^* h C_g, i^* h i)$.

4.6. If P is an almost product structure on a manifold M, each element $H \in C^\infty(L(TM, TM))$ can be written as $H = H_1 + H_2$ where we have $H_1 = v^P H v^P + h^P H h^P$ and $H_2 = v^P H h^P + h^P H v^P$. That gives a decomposition of

$$C_c^\infty(L(TM, TM)) = \mathcal{D}_1(P) \oplus \mathcal{D}_2(P)$$

where

$$\mathcal{D}_1(P) = \{H \in C_c^\infty(L(TM, TM)) : H(V^P) \subset V^P \text{ and } H(H^P) \subset H^P\},$$
$$\mathcal{D}_2(P) = \{H \in C_c^\infty(L(TM, TM)) : H(V^P) \subset H^P \text{ and } H(H^P) \subset V^P\}.$$

4.7. Let us assume now that a distribution V has been fixed on M, for $g \in \mathcal{M}$ and $i = 1, 2$, let us denote $\mathcal{D}_i(g) = \{h \in C_c^\infty(S^2 T^* M) \; ; \; g^{-1} h \in \mathcal{D}_i((\Pi_3 \circ \Phi)(g))\}$. It is clear that if $h \in C_c^\infty(S^2 T^* M)$ and if we take $H = g^{-1} h$ then $h = h_1 + h_2$ with $h_i = g H_i$, $i = 1, 2$. It is straightforward to see that if H is g-symmetric, then H_1 and H_2 are also g-symmetric. We have in that way two complementary distributions in $\mathcal{M}$.

4.8. Proposition. *These distributions are mutually orthogonal with respect to the metric G on $\mathcal{M}$. They are both integrable, more precisely the leaves are the slices of the product.*

Proof. $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal to each other because for any $h, k \in T_g(\mathcal{M})$ we have $\operatorname{tr}(H_1 K_2) = \operatorname{tr}(H_2 K_1) = 0$ and then, by the definition of G,

$$G_g(h, k) = G_g(h_1, k_1) + G_g(h_2, k_2).$$

The tangent space $\Psi_{(g_1, g_2)}(\mathcal{P}_V(M))$ at the point $g = \Psi_{(g_1, g_2)}(P)$ is the kernel of the tangent mapping $T_g((\Pi_1, \Pi_2) \circ \Phi)$ which, by 4.5, is exactly $\mathcal{D}_2(g)$. Analogously, the slice $\Psi_P(\mathcal{M}(N) \times \mathcal{M}(V))$ has as tangent at g the space $\ker T_g(\Pi_3 \circ \Phi)$ which, again by 4.5, is equal to $\mathcal{D}_1(g)$. $\square$

4.9. Proposition. *The distribution $\mathcal{D}_1$ gives rise to a totally geodesic foliation. A non constant geodesic of $\mathcal{M}$ issuing from g in the direction of a vector in $\mathcal{D}_2(g)$ has the property that it never meets again the leaf of $\mathcal{D}_2$ passing through g and that its tangent vector is never again in $\mathcal{D}_2$.*

Proof. Let $g \in \mathcal{M}$ and let $(g_1, g_2, P) = \Phi(g)$. The geodesic starting from g in direction of $h \in T_g \mathcal{M}$ is given by [Gil-Medrano, Michor, 3.2] or 2.6:

$$g(t) := \operatorname{Exp}_g^G(th) = g e^{(a(t) Id + b(t) H_0)}.$$

For the first assertion, if $h \in \mathcal{D}_1(g)$ we have $e^{(a(t) Id + b(t) H_0)}(H^P) \subset H^P$, and $e^{(a(t) Id + b(t) H_0)}(V) \subset V$; since $e^{(a(t) Id + b(t) H_0)}$ is non singular both inclusions

are in fact equalities and, consequently, $V^{\perp, g(t)} = H^P$. So $g(t) \in \Psi_P(\mathcal{M}(N) \times \mathcal{M}(V))$ for all t.

Let us suppose now that $h \in \mathcal{D}_2(g)$ then $\operatorname{tr} H = 0$ and consequently $H_0 = H$. From [Gil-Medrano, Michor, 3.2] or 2.6 it is easy to see that $a(t) \neq 0$ unless the geodesic is constant. For $v_1, v_2 \in V$ we have $g(t)(v_1, v_2) = e^{a(t)} g(v_1, v_2)$ and then $g(t) \in \Psi_{(g_1, g_2)}(\mathcal{P}_V(M))$ only for $t = 0$.

From [Gil-Medrano, Michor, 4.5], or 2.6 (for $\alpha = \frac{1}{n}$) we have that

$$P(t) := g(t)^{-1} g'(t) = e^{-\frac{1}{2} na(t)} \left(\frac{4 \operatorname{tr}(H) + nt \operatorname{tr}(H^2)}{4n} Id + H_0 \right),$$

where $n = \dim M$. Now, for $h \in \mathcal{D}_2(g)$, we have $g'(t) \in \mathcal{D}_2(g(t))$ if and only if the coefficient of Id is zero. If this happens for some $t \neq 0$ then $\operatorname{tr}(H^2) = 0$ which implies that $H = 0$. $\square$

4.10. For each $P \in \mathcal{P}_V(M)$, one can define a metric on $\mathcal{M}(N) \times \mathcal{M}(V)$ by $\check{G}^P = \Psi_P^* G$ and for each $(g_1, g_2) \in \mathcal{M}(N) \times \mathcal{M}(V)$ a metric on $\mathcal{P}_V(M)$ can be defined by $\hat{G}^{g_1, g_2} = \Psi_{(g_1, g_2)}^* G$. The next propositions are devoted to the study of these metrics.

4.11. Proposition. *All the metrics $\check{G}^P$ on $\mathcal{M}(N) \times \mathcal{M}(V)$ are the same.*

Proof. From 4.5, for $(h_1, h_2), (k_1, k_2) \in T_{(g_1, g_2)}\mathcal{M}(N) \times \mathcal{M}(V)$ we have

$$\check{G}^P_{(g_1, g_2)}((h_1, h_2), (k_1, k_2)) = G_g(h, k)$$

where $g = \Psi_{(g_1, g_2)}(P) = p^* g_1 p + (v^P)^* g_2(v^P)$, $h = p^* h_1 p + (v^P)^* h_2(v^P)$ and $k = p^* k_1 p + (v^P)^* k_2(v^P)$.

It is easy to see that $H = g^{-1} h$ is given by $H = C_P H_1 p + i H_2 v^P$ with $H_1 = g_1^{-1} h_1$, $H_2 = g_2^{-1} h_2$ and then, $HK = C_P H_1 K_1 p + i H_2 K_2 v^P$.

Now, if we take, at each point, a base of TM which is obtained from basis of V and N via the maps i and C_P we see that $\operatorname{tr} HK = \operatorname{tr} H_1 K_1 + \operatorname{tr} H_2 K_2$ and then

$$G_g(h, k) = \int_M \{\operatorname{tr} H_1 K_1 + \operatorname{tr} H_2 K_2\} \operatorname{vol}(g).$$

The integrand does not depend on P and $\operatorname{vol}(g)$ is also independent of P because if we have a curve $g(t)$ in $\Psi_{(g_1, g_2)}(\mathcal{P}_V(M))$ then $g'(t) \in \mathcal{D}_2(g(t))$ by the proof of 4.8, and so $\operatorname{tr}(g(t)^{-1} g'(t)) = 0$. From the expression of $(\operatorname{vol}(g(t)))'$ (see 2.2) we conclude that $\operatorname{vol}(g(t))$ is constant. $\square$

We will denote this metric on $\mathcal{M}(N) \times \mathcal{M}(V)$ by $\check{G}$.

4.12. Proposition. *For each $(g_1, g_2) \in \mathcal{M}(N) \times \mathcal{M}(V)$ the metric $\hat{G}^{g_1,g_2}$ on $\mathcal{P}_V(M)$ is flat. Its exponential map, at each point, is then given by*

$$\mathrm{Exp}_P^{\hat{G}^{g_1,g_2}}(\xi) = P + \xi.$$

Proof. From 4.5 we see that for $\xi, \eta \in T_P \mathcal{P}_V(M)$ we have

$$\hat{G}_P^{g_1,g_2}(\xi, \eta) = G_g(h, k)$$

where $g = \Psi_{(g_1,g_2)}(P) = p^* g_1 p + (v^P)^* g_2(v^P)$, $2h = (v^P)^* g_2 \xi + \xi^* g_2(v^P)$ and $2k = (v^P)^* g_2 \eta + \eta^* g_2(v^P)$.

Now, $g_2 = i^* gi$ and then $2h = (v^P)^* g\xi + \xi^* g(v^P)$. Having in mind the definition of v^P and the facts that $P^* g = gP$ and that, for $\xi \in T_P \mathcal{P}_V(M)$, $P\xi = \xi$ we have that $2h = g\xi + \xi^* g$ and then $2H = \xi + g^{-1}\xi^* g$; analogously $2K = \eta + g^{-1}\eta^* g$. Consequently $4HK = \xi\eta + \xi g^{-1}\eta^* g + g^{-1}\xi^* g\eta + g^{-1}\xi^*\eta^* g = \xi g^{-1}\eta^* g + g^{-1}\xi^* g\eta$, the last equality because $\xi\eta = 0$.

The distribution V is contained in the kernel of the mapping $\xi^* g_2 \eta : TM \to T^* M$; the annihilator of V contains the image of this mapping. So there is a unique mapping $\widetilde{\xi^* g_2 \eta} : N \to N^*$ such that $p^* \widetilde{\xi^* g_2 \eta} p = \xi^* g_2 \eta$.

Then $g^{-1}\xi^* g\eta = g^{-1}\xi^* g_2 \eta = C_P g_1^{-1} \widetilde{\xi^* g_2 \eta} p$ and by an argument similar to that in 4.11 $\mathrm{tr}(g^{-1}\xi^* g_2 \eta) = \mathrm{tr}(g_1^{-1}\widetilde{\xi^* g_2 \eta})$. So, we conclude that

$$\hat{G}_P^{g_1,g_2}(\xi, \eta) = \frac{1}{4}\int_M \{\mathrm{tr}(g_1^{-1}\widetilde{\xi^* g_2 \eta}) + \mathrm{tr}(g_1^{-1}\widetilde{\eta^* g_2 \xi})\} \mathrm{vol}(g)$$

which is independent of P because $\mathrm{vol}(g)$ does not depend on P as we have shown in 4.11.

So, all the metrics are flat and then it is immediate that geodesics are just straight lines. $\square$

Remark. For an element $\xi \in T_P \mathcal{P}_V(M)$, $\xi^2 = 0$ and then $e^\xi = Id + \xi$ and geodesics can also be written in the form $P(t) = Pe^{t\xi}$.

4.13. Proposition. *In the submanifold $\Psi_{(g_1,g_2)}(\mathcal{P}_V(M))$ the geodesic starting at $g = \Psi_{(g_1,g_2)}(P)$ in the direction of $h \in \mathcal{D}_2(g)$ is given by*

$$g(t) = g(Id + tH + t^2 H^2 h^P).$$

Proof. The splitting submanifold $\Psi_{(g_1,g_2)}(\mathcal{P}_V(M))$ of $(\mathcal{M}, G)$ with the restricted metric is isometric to $(\mathcal{P}_V(M), \hat{G}^{g_1,g_2})$. The geodesic of the submanifold $\Psi_{(g_1,g_2)}(\mathcal{P}_V(M))$ starting at $g = \Psi_{(g_1,g_2)}(P)$ in the direction of $h \in \mathcal{D}_2(g)$ is given by

$$g(t) = \Psi_{(g_1,g_2)}(P(t)) = p^* g_1 p + (v^{P(t)})^* g_2(v^{P(t)}),$$

where $P(t) = P + 2tv^p g^{-1}hh^P$, by 4.5 and 4.12. Using 4.1 and 4.3 we have $g_1 = C_g^* g C_g$, $g_2 = i^* gi$, $C_g p = h^P$ and then

$$
\begin{aligned}
g(t) &= (h^P)^* g h^P + (v^{P(t)})^* g(v^{P(t)}) \\
&= (h^P)^* g h^P + (v^P)^* g(v^P) + t^2 (v^P H h^P)^* g(v^P H h^P) \\
&\quad + t\{(v^P H h^P)^* g v^P + (v^P)^* g(v^P H h^P)\} \\
&= g(Id + t\{h^P H v^P + v^P H h^P\} + t^2 h^P H v^P H h^P),
\end{aligned}
$$

the last equality because h^P, v^P, H are g-symmetric. Finally, recalling that $h \in \mathcal{D}_2(g)$ we have

$$
g(t) = g + th + t^2 g H^2 h^P. \quad \square
$$

4.14. Theorem. *The map $(\Pi_1, \Pi_2) \circ \Phi : (\mathcal{M}, G) \to (\mathcal{M}(N) \times \mathcal{M}(V), \check{G})$ is a Riemannian submersion. In fact, $(\mathcal{M}, G)$ is a product manifold with varying metric on the fibers.*

Proof. It follows by straightforward computation that

$$
G_g = ((\Pi_1, \Pi_2) \circ \Phi)^* \check{G}_{(g_1, g_2)} + (\Pi_3 \circ \Phi)^* \hat{G}_P^{g_1, g_2}. \quad \square
$$

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