Version 5, November 14, 2002

POLARIZATIONS IN CLASSICAL INVARIANT THEORY

Mark Losik, Peter W. Michor, Vladimir L. Popov

Abstract. We give a weak version of the first main theorem of invariant theory, namely, we describe a class of representations of a reductive algebraic group on a vector space V such that $\mathbb{C}[V^q]^G$ is the integral closure of $\mathbb{C}[V^q]^G_{\mathrm{pol}}$, the subring generated by all generalized polarizations, in $\mathbb{C}[V^q]$. For finite groups we have stronger results.

1. Introduction

Let G be a reductive complex algebraic group, V a finite dimensional complex vector space, and $\rho : G \to \mathrm{GL}(V)$ a regular representation (later called a representation of a reductive group for shortness' sake). Consider the corresponding diagonal action of G on the product V^q and the algebra $\mathbb{C}[V^q]^G$ of G-invariant polynomials on V^q. The problem of finding a finite system of generators for the algebra $\mathbb{C}[V^q]^G$ is called the first main theorem of invariant theory. This theorem was proved for the standard representations of all classical groups and for the standard representation of the symmetric group S_n in $\mathbb{C}^n$ by Weyl (see [7]), for the Weyl groups of the types $B_n = C_n$ (see [2] and [4]) and the dihedral groups (see [4]). For the above finite groups the algebra $\mathbb{C}[V^q]^G$ is generated by the polarizations of a system of basic invariants of $\mathbb{C}[V]^G$. However Wallach (see [6]) proved that this is not true for the Weyl group D_n $(n \geq 4)$ and $q = 2$. Then Wallach and Hunziker (see [6] and [4]) introduced generalized polarizations and proved the first main theorem for D_n using the usual and the generalized polarizations of the basic invariants of $\mathbb{C}[V]^{D_n}$. But for other Weyl groups the above problem is open till now.

The aim of this paper is to indicate the representations for which there is a close relationship between the algebra $\mathbb{C}[V^q]^G$ and its subalgebra generated by polarizations of basic invariants of $\mathbb{C}[V]^G$. We consider this problem in the following more general setting.

Let G be a reductive group and X an affine G-variety, i.e., an affine variety X with an action of G by regular automorphisms of X. Let Y be an affine variety

1991 *Mathematics Subject Classification.* 14L24, 14L30.

Key words and phrases. invariants, reductive groups, polarizations.

M.L. and P.W.M. were supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 14195 MAT'.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

and let Z be a G-stable affine variety of regular morphisms from Y to X with the natural action of G. We define generalized polarizations (distinct from the generalized polarizations of Wallach and Hunziker) of a regular G-invariant function on X which are regular G-invariant functions on Z. In particular, for the linear case, i.e. when $X = V$ is a vector space, G is a subgroup of $\mathrm{GL}(V)$, and Z is the affine variety (isomorphic to $\mathbb{V}^q$) of linear morphisms from $Y = \mathbb{C}^q$ to V, the generalized polarizations are the usual polarizations of homogeneous G-invariant polynomials on V.

Let $Z//G$ be a categorical quotient, i.e. the affine variety with coordinate ring $\mathbb{C}[Z]^G$. Under some assumptions on Z the coordinate ring $\mathbb{C}[Z]^G$ contains $\mathbb{C}[X]^G$. Let $(Z//G)_{\mathrm{pol}}$ be an affine variety whose coordinate ring is the subring $\mathbb{C}[Z]^G_{\mathrm{pol}}$ of $\mathbb{C}[Z]^G$ generated by $\mathbb{C}[X]^G$ and the generalized polarizations of generators of $\mathbb{C}[X]^G$.

The aim of this paper is to find the cases when the morphism $p_{q,Z} : Z//G \to (Z//G)_{\mathrm{pol}}$ induced by the inclusion $\mathbb{C}[Z]^G_{\mathrm{pol}} \subset \mathbb{C}[Z]^G$ is finite, i.e. the ring $\mathbb{C}[Z]^G$ is integral over $\mathbb{C}[Z]^G_{\mathrm{pol}}$.

In particular, we indicate when this is true in the linear case $Z = V^q$, where $\rho : G \to \mathrm{GL}(V)$ is a representation. Moreover, in this case the ring $\mathbb{C}[Z]^G = \mathbb{C}[V^q]^G$ is the integral closure of $\mathbb{C}[Z]^G_{\mathrm{pol}}$ in $\mathbb{C}[V^q]$ whenever the group G is connected, see corollary 3.5.

For the linear case and a finite group G we prove that the morphism $p_{q,Z}$ is a bijective normalization of $(V^q//G)_{\mathrm{pol}}$, in particular, for each $f \in \mathbb{C}[V^q]^G$ there are $F \in \mathbb{C}[V^q]^G_{\mathrm{pol}}$ and an integer $k > 0$ such that $f = F^k$, see 4.2. Moreover, a generalization of the above constructions shows that for $q > 2$ the same is true for a subset of polarizations of a system of homogeneous generators of $\mathbb{C}[V]^G$, see 4.5 and 4.6.

Throughout the paper an affine variety means a complex affine variety endowed with the Zarisky topology. We shall deal only algebraic groups and varieties over $\mathbb{C}$. But note that by Lefschetz' principle all results remain true over any algebraically closed field of characteristic 0.

2. Generalized polarizations

2.1. Generalized polarizations. Let G be a reductive group and X an affine G-variety. Consider the set $\mathfrak{F}(\mathbb{C}^q, X)$ of regular morphisms from $\mathbb{C}^q$ to X and the pointwise action of G on $\mathfrak{F}(\mathbb{C}^q, X)$: $(gf)(z) = g(f(z))$ for $g \in G$ and $f \in \mathfrak{F}(\mathbb{C}^q, X)$).

Let Z be a G-stable subset of $\mathfrak{F}(\mathbb{C}^q, X)$ equipped with the structure of an affine variety such that the restriction of the action of each $g \in G$ to Z is an automorphism of Z and the evaluation map $\mathrm{ev}_Z : Z \times \mathbb{C}^q \to X$ (i.e. for $f \in Z$ and $y \in \mathbb{C}^q$ $\mathrm{ev}_Z(f, y) = f(y)$) is a regular morphism). The group G acts on $Z \times \mathbb{C}^q$ in the following way: $g(f, y) = (gf, y)$. By definition the map ev_Z is G-equivariant. By the Hilbert-Nagata theorem (see, for example, [5]) the algebras $\mathbb{C}[X]^G$, $\mathbb{C}[Z]^G$, and $\mathbb{C}[Z \times \mathbb{C}^q]^G$ are finitely generated. Then the categorical quotients $X//G$, $Z//G$, and $(Z \times \mathbb{C}^q)//G = (Z//G) \times \mathbb{C}^q$ are affine varieties.

Denote by π_X the projection $X \to X//G$. For later needs we recall the following interpretation of the categorical quotient $X//G$ and the projection π_X. Let $\sigma_1, \dots, \sigma_m$ be a system of generators of the algebra $\mathbb{C}[X]^G$ and let $\sigma = (\sigma_1, \dots, \sigma_m)$:

$X \to \mathbb{C}^m$ the corresponding morphism. Then it is known (see, for example [5]) that $\sigma(X)$ is an irreducible closed subset of $\mathbb{C}^m$ and one can identify $\sigma(X)$ as an affine variety with $X//G$ and the morphism $X \to \sigma(X)$ induced by σ (and denoted again by σ) with the projection π_X.

The morphism ev_Z induces a morphism $\overline{\mathrm{ev}}_Z : (Z//G) \times \mathbb{C}^q \to X//G$ and the corresponding homomorphism of algebras $\overline{\mathrm{ev}}_Z^* : \mathbb{C}[X//G] \to \mathbb{C}[Z//G] \otimes \mathbb{C}[\mathbb{C}^q]$.

Let $\tau \in \mathbb{C}[X]^G$. Since $\mathbb{C}[\mathbb{C}^q]$ equals the polynomial ring $\mathbb{C}[t_1, \dots, t_q]$ in the variables $t_1, \dots, t_q$, and since the powers $t_1^{i_1} \dots t_q^{i_q}$ for $i_1, \dots, i_p = 0, 1, \dots$ form a basis of $\mathbb{C}[\mathbb{C}^p]$ as a complex vector space, we have the following unique decomposition

$$\overline{\mathrm{ev}}_Z^*(\tau) = \sum_{i_1,\dots,i_q} \tau_{i_1 \dots i_q} \otimes (t_1^{i_1} \dots t_q^{i_q}), \tag{1}$$

where $\tau_{i_1 \dots i_q} \in \mathbb{C}[Z]^G$ and only finitely many of the $\tau_{i_1 \dots i_q}$ are nonzero. The nonzero G-invariant regular functions $\tau_{i_1 \dots i_q}$ on Z are called the *generalized polarizations* of τ with respect to Z.

2.2. Affine varieties with graded coordinate rings. In the sequel we shall consider generalized polarizations in the following special case when the structure of affine variety on Z can be defined in a canonical way.

Let X be an irreducible affine variety such that the coordinate ring $\mathbb{C}[X]$ is graded:

$$\mathbb{C}[X] = \oplus_{i \geq 0} \mathbb{C}^i[X]$$

and $\mathbb{C}^0(X) = \mathbb{C}$. Let G be a reductive group acting on X by automorphisms of X preserving the above grading of $\mathbb{C}[X]$. Then the algebra $\mathbb{C}[X]^G$ has a natural grading: $\mathbb{C}[X]^G = \bigoplus_{i \geq 0} (\mathbb{C}_i[X] \cap \mathbb{C}[X]^G)$.

2.3. Example. Consider the linear case, i.e. $X = V$ is a finite dimensional vector space and $G \subset \mathrm{GL}(V)$. We have the natural grading of $\mathbb{C}[V]$ of a polynomial ring. It is clear that the G-variety V satisfies the above conditions.

2.4. Example. Let V be a finite dimensional vector space, let G be a reductive subgroup of $\mathrm{GL}(V)$, and let H be an algebraic normal subgroup of G. Then the quotient group G/H is reductive. Since the group H is reductive, the categorical quotient $X = V//H$ is an affine variety and the group G/H acts on X by automorphisms. It is clear that the algebra $\mathbb{C}[X] = \mathbb{C}[V]^H$ is a graded subalgebra of the graded algebra $\mathbb{C}[V]$ and the induced action of the group G/H on $\mathbb{C}[X]$ preserves the structure of graded algebra of $\mathbb{C}[X]$.

In particular, one can take for the normal subgroup H the component G_0 of the identity of G. Then the quotient group G/G_0 is finite.

2.5. The affine G-variety $\mathfrak{F}(\mathbb{C}^q, X)$. Let first $X = \mathbb{C}^n$ with a fixed grading on its coordinate ring $\mathbb{C}[\mathbb{C}^n] = \mathbb{C}[u_1, \dots, u_n]$ so that $\deg(u_i) = d_i \in \mathbb{N}$. For example, the u_i might correspond to homogeneous generators of an algebra $\mathbb{C}[V]^G$ of invariant polynomials on a G-module V. For each $h \in \mathbb{C}[\mathbb{C}^n]$ we have a map $\hat{h} : \mathfrak{F}(\mathbb{C}^q, \mathbb{C}^n) \to \mathbb{C}[\mathbb{C}^q, \mathbb{C}] = \mathbb{C}[C^q] = \mathbb{C}[t_1, \dots, t_q]$, given by $\hat{h}(f) = h \circ f$. For each polynomial $P \in \mathbb{C}[t_1, \dots, t_q]$ we write $P = \sum_{i_1,\dots,i_q} P_{i_1 \dots i_q} t_1^{i_1} \dots t_q^{i_q}$. Then for each

$h = \sum_{j_1,\dots,j_n} h_{j_1\dots j_n} u_1^{j_1}\dots u_n^{j_n} \in \mathbb{C}[\mathbb{C}^n]$ we have a set of complex valued functions $\hat{h}_{i_1\dots i_q}$ on $\mathfrak{F}(\mathbb{C}^q, X)$ such that for $f = (f^1,\dots,f^n) \in \mathbb{C}[\mathbb{C}^q, \mathbb{C}^n]$ with

$$f^k = \sum_{l_{k,1},\dots,l_{k,q}} f^k_{l_{k,1}\dots l_{k,q}} t_1^{l_{k,1}}\dots t_q^{l_{k,q}}$$

we have

$$h \circ f = \hat{h}(f) = \sum_{i_1,\dots,i_q \geq 0} \hat{h}_{i_1\dots i_q}(f) t_1^{i_1}\dots t_q^{i_q} = \\ = \sum_{j_1,\dots,j_n\geq 0} h_{j_1\dots j_n} \prod_{k=1}^{q} \Big(\sum_{l_{k,1},\dots,l_{k,q}\geq 0} f^k_{l_{k,1}\dots l_{k,q}} t_1^{l_{1,1}}\dots t_q^{l_{k,q}} \Big)^{j_k}$$

Note that each $\hat{h}_{i_1\dots i_q}(f)$ is a polynomial in the coefficients $f^k_{l_{k,1}\dots l_{k,q}}$ of f of degree $\deg h$ in terms of the grading fixed above.

Let now X be an irreducible affine variety such that the coordinate ring $\mathbb{C}[X]$ is graded: We consider X as closed subset of $\mathbb{C}^n$ with a fixed grading on $\mathbb{C}[\mathbb{C}^n]$ as above, such that $\mathbb{C}[X] = \mathbb{C}[\mathbb{C}^n]/I_X$ where I_X is a graded ideal. Then $\mathfrak{F}(\mathbb{C}^q, X)$ is the set of all f in $\mathfrak{F}(\mathbb{C}^q, \mathbb{C}^n)$ such that $\hat{h}(f) = 0$ for all $h \in I_X$. Thus for each $h \in \mathbb{C}[X]$ we have a map $\hat{h} : \mathfrak{F}(\mathbb{C}^q, X) \to \mathbb{C}[\mathbb{C}^q]$ given by $\hat{h}(f) = h \circ f$ and a set of complex valued functions $\hat{h}_{i_1\dots i_q}$ on $\mathfrak{F}(\mathbb{C}^q, X)$ such that $\hat{h}_{i_1\dots i_q}(f) = \hat{h}(f)_{i_1\dots i_q}$. Let μ be a minimal number such that the ideal I_X is generated by functions $g_1,\dots,g_m$ of degree $\leq \mu$ in $\mathbb{C}[\mathbb{C}^n]$ in the grading from above. We have seen that $\mathfrak{F}(\mathbb{C}^q, X)$ is the common zero set of the finitely many polarized functions functions $(\widehat{g}_i)_{i_1,\dots,i_q}$ which are homeogeneous. Thus the ideal $\hat{I}_X$ of $\mathbb{C}[\mathfrak{F}(\mathbb{C}^q, \mathbb{C}^n)]$ generated by the functions $(\widehat{g}_i)_{i_1,\dots,i_q}$ is a graded ideal in the induced grading. By Hilbert's Nullstellensatz the ideal of all functions vanishing on $\mathfrak{F}(\mathbb{C}^q, X)$ is the radical of the ideal $\hat{I}_X$ and is thus also a graded ideal; only finitely many of the coordinates $f^k_{l_{k,1}\dots l_{k,q}}$ of $\mathbb{C}[\mathfrak{F}(\mathbb{C}^q, \mathbb{C}^n)]$ are involved at the same time in the application of Hilbert's Nullstellensatz.

For any $d > \mu$, denote by $\mathfrak{F}_d(\mathbb{C}^q, X)$ the set of all morphisms $f \in \mathfrak{F}(\mathbb{C}^q, X)$ such that for all $h \in \mathbb{C}[X]$ of degree $\leq \mu$ the degree of $\hat{h}(f)$ does not exceed d. It follows from the argument above that $\mathfrak{F}_d(\mathbb{C}^q, X)$ is an affine subvariety of $\mathfrak{F}_d(\mathbb{C}^q, \mathbb{C}^n)$ with graded coordinate ring.

Proposition. *In this situation we have:*

(1) $\mathfrak{F}_d(\mathbb{C}^q, \mathbb{C}^n)$ *is an affine variety with grading on* $\mathbb{C}[\mathfrak{F}_d(\mathbb{C}^q, \mathbb{C}^n)]$ *induced from the grading of* $\mathbb{C}[\mathbb{C}^n]$.

(2) *The polynomials* $\hat{h}_{i_1\dots i_q}$ *for* $h \in I_X \subset \mathbb{C}[\mathbb{C}^n]$ *form a graded ideal* $\hat{I}_X$ *in* $\mathbb{C}[\mathfrak{F}_d(\mathbb{C}^q, \mathbb{C}^n)]$. *Its radical is the ideal* $I_{\mathfrak{F}_d(\mathbb{C}^q,X)}$ *which describes* $\mathfrak{F}_d(\mathbb{C}^q, X)$ *as affine variety with graded coordinate ring.*

(3) *If* G *is a reductive group of automorphisms of* X *preserving the graded structure of* $\mathbb{C}[X]$ *then* $\mathfrak{F}_d(\mathbb{C}^q, X)$ *is a* G*-stable subset of* $\mathfrak{F}(\mathbb{C}^q, X)$. *The group* G *induces an action of* G *on* $\mathfrak{F}_d(\mathbb{C}^q, X)$ *by automorphisms of* $\mathfrak{F}_d(\mathbb{C}^q, X)$ *and the corresponding evaluation map* $\mathfrak{F}_d(\mathbb{C}^q, X) \times \mathbb{C}^q \to X$ *is regular.*

Proof. (1) and (2) are clear from the discussion above.

(3) We can find an equivariant embedding $X \to \mathbb{C}^n$ of X into an affine G-space $\mathbb{C}^n$ with induced G-invariant grading: Take for example the complex dual of the G-invariant subspace $\bigoplus_{i=1}^{\mu} \mathbb{C}^i[X]$, where μ is as above. Then all assertions are an easy consequence of the definitions. $\square$

2.6. The zero fiber. In the situation of 2.2, let Z be an irreducible G-stable closed subset of $\mathbb{C}[\mathfrak{F}_d(\mathbb{C}^q, X)]$ defined by an ideal of $\mathbb{C}[\mathfrak{F}_d(\mathbb{C}^q, X)]$ which is homogeneous with respect to the grading described above. The structures on $\mathfrak{F}_d(\mathbb{C}^q, X)$ induce on Z a structure of an affine variety, a grading of $\mathbb{C}[Z]$, and an action of the group G by automorphisms of Z preserving the grading.

Let 0_X be the point of X corresponding to the maximal ideal $\mathfrak{m}_0 = \oplus_{i>0}\mathbb{C}_i[X]$. and $\pi_X : X \to X/G$ the projection. The *zero fiber* of π is $X^0 = \pi_X^{-1} \circ \pi_X(0_X)$.

Similarly, let $O_{q,X} : \mathbb{C}^q \to X$ be the morphism given by $O_{q,X}(t) = 0_X$ for each $t \in \mathbb{C}^q$. Clearly $O_{q,X} \in \mathfrak{F}_d(\mathbb{C}^q, X)$ for each d and the maximal ideal corresponding to $O_{q,X}$ consists of all elements of $\mathbb{C}[F_d(\mathbb{C}^q, X)]$ of positive degrees. By definition the point $O_{q,X}$ belongs to each irreducible G-stable closed subset Z of $\mathbb{C}[\mathfrak{F}_d(\mathbb{C}^q, X)]$ defined by a homogeneous ideal $\mathbb{C}[\mathfrak{F}_d(\mathbb{C}^q, X)]$. Denote by π_Z the projection $Z \to Z//G$. Then the *zero fiber* of π_Z is $Z^0 = \pi_Z^{-1} \circ \pi_Z(O_X)$.

Examples. In the following examples we consider an n-dimensional vector space V with the natural action of a reductive subgroup G of $\mathrm{GL}(V)$ and the standard graded algebra structure on the coordinate ring $\mathbb{C}[V]$. By definition the algebra $\mathbb{C}[V]$ is generated by homogeneous elements of degree 1, i.e. $\mu = 1$. Then we can take for the basis h_i the coordinates x_i in V with respect to some basis $e_1, \dots, e_n$ of V. Since the algebra $\mathbb{C}[V]$ is free, for each d the affine variety $\mathfrak{F}_d(\mathbb{C}^q, V)$ is an affine space $\mathbb{C}^N$ with standard grading on $\mathbb{C}(\mathfrak{F}_d(\mathbb{C}^q, V)) = \mathbb{C}[\mathbb{C}^N]$ and with a linear action of the group G.

2.7. Example. Consider the set $\mathfrak{F}_1(\mathbb{C}^{q-1}, V)$. For each $f \in \mathfrak{F}_1(\mathbb{C}^{q-1}, V)$ and $t = (t_1, \dots, t_{q-1})$ we have $f(t) = v_1 + \sum_{i=2}^{q} t_i v_i$, where $v_1, \dots, v_q$ are uniquely given vectors in V. This mapping $f \to (v_1, \dots, v_q)$ is an isomorphism $\mathfrak{F}_1(\mathbb{C}^{q-1}, V) \to V^q$ of affine varieties which is G-equivariant for the natural action of G on $\mathfrak{F}_1(\mathbb{C}^{q-1}, V)$ and the diagonal action of G on V^q. We can identify $\mathfrak{F}_1(\mathbb{C}^{q-1}, V)$ with V^q via this map. It is easily checked that, for $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}, V)$ and a homogeneous polynomial $\tau \in \mathbb{C}[V]^G$, the generalized polarizations of τ coincide with the standard polarizations of τ on V^q.

2.8. Example. Let $Z = L(\mathbb{C}^q, V)$ be the set of linear morphisms from $\mathbb{C}^q$ to V^q. It is clear that Z is a closed G-stable subset of $\mathfrak{F}_1(\mathbb{C}^q, V) = V^{q+1}$. It is identified with the subspace V^q of V^{q+1} defined by the equation $v_1 = 0$. It is easily seen that, for $Z = L(V^q, V)$ and for each homogeneous polynomial $\tau \in \mathbb{C}[V]^G$, the generalized polarizations of σ coincide with the standard polarizations of σ on V^q. It is easily checked that, if $\deg \sigma = p$, the number of distinct polarizations of τ equals $\binom{p+q-1}{p}$.

Remark that, for $Z = \mathfrak{F}_1(\mathbb{C}^q, V)$ and $Z = L(\mathbb{C}^q, V)$, the generalized polarizations are multi-homogeneous functions on V^{q+1} and V^q respectively.

2.9. Example. Consider the set $\mathfrak{F}_{q-1}(\mathbb{C}, V)$. For each $f \in \mathfrak{F}_{q-1}(\mathbb{C}, V)$ and $t \in \mathbb{C}$ we have $f(t) = \sum_{i=1}^{q} t^{i-1} v_i$, where $v_1, \dots, v_q$ are uniquely given vectors in V. It is easily checked that the map $f \to (v_1, \dots, v_q)$ is an isomorphism $\mathfrak{F}_{q-1}(\mathbb{C}, V) \to V^q$ of affine varieties, which again is G-equivariant for the natural action of G on $\mathfrak{F}_{q-1}(\mathbb{C}, V)$ and the diagonal action of G on V^q. Then we can identify $\mathfrak{F}_{q-1}(\mathbb{C}, V)$ with V^q via this map.

Consider the morphism $j_{q-1} : \mathbb{C} \to \mathbb{C}^{q-1}$ defined as follows:

$$j_{q-1}(t) = (t, t^2, \dots, t^{q-1}) \quad (t \in \mathbb{C})$$

and the corresponding homomorphism $j_{q-1}^* : \mathbb{C}[\mathbb{C}^{q-1}] \to \mathbb{C}[\mathbb{C}]$. Then $f \to f \circ j_{q-1}$ is an isomorphism $\mathfrak{F}(j_{q-1}, V) : \mathfrak{F}_1(\mathbb{C}^{q-1}, V) \to \mathfrak{F}_{q-1}(\mathbb{C}, V)$ of affine varieties. We consider the evaluation mappings $\mathrm{ev}_1 : \mathfrak{F}_1(\mathbb{C}^{q-1}, V) \times \mathbb{C}^{q-1} \to V$ and by ev_{q-1} : $\mathfrak{F}_{q-1}(\mathbb{C}, V) \times \mathbb{C} \to V$. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}[V]^G \arrow[r, equal] \arrow[d, "\mathrm{ev}_1^*"'] & \mathbb{C}[V]^G \arrow[d, "\mathrm{ev}_{q-1}^*"'] \\
\mathbb{C}[\mathfrak{F}_1(\mathbb{C}^{q-1}, V)] \otimes \mathbb{C}[\mathbb{C}^{q-1}] \arrow[r, "\mathfrak{F}(j_{q-1}, V)^* \otimes j_{q-1}^*"] & \mathbb{C}[\mathfrak{F}_{q-1}(\mathbb{C}, V)] \otimes \mathbb{C}[\mathbb{C}].
\end{tikzcd}$$

Let $\tau \in \mathbb{C}[V]^G$ be a homogeneous polynomial, $\tau'_{i_1,\dots,i_{q-1}}$ the generalized polarization of τ for $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}, V)$, and τ''_i the generalized polarization of τ for $Z = \mathfrak{F}_{q-1}(\mathbb{C}, V)$. Since $j_{q-1}^*(t_1^{i_1} \dots t_{q-1}^{i_{q-1}}) = t^{i_1+2i_2+\dots+(q-1)i_{q-1}}$, applying the above commutative diagram we have

$$\tau''_i = \sum_{i_1+2i_2+\dots+(q-1)i_{q-1}=i} \tau'_{i_1\dots i_{q-1}}.$$

If $\deg \tau = p$, the number of distinct generalized polarizations of τ''_i equals $p(q-1)+1$.

Remark that the difference $\binom{p+q-1}{p} - p(q-1) - 1$ between numbers of the standard polarizations of τ ($\deg \sigma = p > 1$) and the above generalized polarizations of τ vanishes for $q = 1, 2$ and is strictly positive for $q > 2$.

Consider, for example, the action of the group $G = \mathbb{Z}_2$ on $V = \mathbb{C}$ generated by the morphism $z \to -z$ for $z \in \mathbb{C}$. Then $\mathbb{C}[V]^G = \mathbb{C}[z^2]$.

Let $q = 3$ and $\tau = z^2$. Then for $Z = \mathfrak{F}_1(\mathbb{C}^2, V)$ we have the polarizations $\tau'_{00} = z_0^2$, $\tau'_{10} = 2z_0z_1$, $\tau'_{01} = 2z_0z_2$, $\tau'_{20} = z_1^2$, $\tau'_{02} = z_2^2$, and $\tau'_{11} = 2z_1z_2$. For $Z = \mathfrak{F}_2(\mathbb{C}, V)$ we have $\tau''_0 = \tau'_{00}$, $\tau''_1 = \tau'_{10}$, $\tau''_2 = \tau'_{01} + \tau'_{20}$, $\tau_3 = \tau'_{11}$, and $\tau''_4 = \tau_{02}$.

2.10. The algebra $\mathbb{C}[Z]^G_{\mathrm{pol}}$. We assume that we have the following data: an irreducible affine G-variety X, where G is a reductive group, such that algebra $\mathbb{C}[X]$ is graded and the action of the group G preserves this grading, and a G-stable irreducible closed subset Z of $\mathfrak{F}_d(\mathbb{C}^q, X)$.

Consider the map $m_Z : Z \to X$ given by $m_Z(f) = f(0)$ for each $f \in Z$. It is easily seen that the map m_Z is a G-equivariant morphism of affine varieties. Then the ring homomorphism $m_Z^* : \mathbb{C}[X] \to \mathbb{C}[Z]$ induces the homomorphism $\mathbb{C}[X]^G \to \mathbb{C}[Z]^G$.

Consider the subalgebra $\mathbb{C}[Z]^G_{\mathrm{pol}}$ of the algebra $\mathbb{C}[Z]^G$ generated by the subalgebra $m_Z^*(\mathbb{C}[X]^G)$ and all generalized polarizations of a system of generators for the algebra $\mathbb{C}[X]^G$. Since the algebra $\mathbb{C}[X]^G$ is finitely generated, the algebra $\mathbb{C}[Z]^G_{\text{pol}}$ is also finitely generated. Let $(Z//G)_{\text{pol}}$ be the affine variety with the coordinate ring $\mathbb{C}[Z]^G_{\text{pol}}$. Denote by $p_{q,Z}$ the dominant morphism from $Z//G$ to $(Z//G)_{\text{pol}}$ induced by the inclusion $\mathbb{C}[Z]^G_{\text{pol}} \subset \mathbb{C}[Z]^G$.

Our aim is to find the cases when the morphism $p_{q,Z}$ establishes a close relationship between $Z//G$ and $(Z//G)_{\text{pol}}$, or, equivalently, a close relationship between the algebra $\mathbb{C}[Z]^G$ and its subalgebra $\mathbb{C}[Z]^G_{\text{pol}}$.

Lemma. *Let the morphism $m_Z : Z \to X$ be dominant. Then the algebra $\mathbb{C}[X]^G$ is a subalgebra of $\mathbb{C}[Z]^G$ and the algebra $\mathbb{C}[Z]^G_{\text{pol}}$ is generated by the generalized polarizations of the basic invariants of $\mathbb{C}[X]^G$. Moreover, each homogeneous element $\tau \in \mathbb{C}[X]^G$ defines uniquely the indexed set of the generalized polarizations $\tau_{i_1\dots i_q}$ of τ.*

Proof. If the morphism $m_Z : Z \to X$ is dominant it defines the inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[Z]^G$ which identify each homogeneous element $\tau \in \mathbb{C}[X]^G$ of degree p with the generalized polarization $m_Z^*(\tau) = \tau_{q0\dots0}$. This proves the first statement of the lemma. The last statement follows from the definition of generalized polarizations since by the above assumptions the algebra $\mathbb{C}[X]^G$ is given as a subalgebra of the algebra $\mathbb{C}[Z]$. $\square$

2.11. Note that, for $Z = \mathfrak{F}_d(\mathbb{C}^q, X)$, the morphism m_Z is surjective and thus dominant. Recall Examples 2.7 and 2.9 where we have $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}, V) = \mathbb{V}^q$ and $Z = \mathfrak{F}_{q-1}(\mathbb{C}, V) = V^q$. In both cases the morphism m_Z coincides with the projection of $\mathbb{V}^q$ onto its first factor V. Therefore, the algebra $\mathbb{C}[V]^G$ is identified with the subalgebra of $\mathbb{C}[V^q]^G$ consisting of polynomials $f(v_1, \dots, v_q) \in \mathbb{C}[V^q]^G$ (where $v_1, \dots, v_q \in V$) which do not depend on $v_2, \dots, v_q$. Considering a basic invariant $\sigma_i \in \mathbb{C}[X]^G$ as an element of the algebra $\mathbb{C}[V^q]$ we can construct the indexed set $\sigma_{i,i_1,\dots,i_q}$ of its generalized polarizations. Thus the algebra $\mathbb{C}[Z]^G_{\text{pol}}$ is generated by the generalized polarizations of basic invariants of $\mathbb{C}[X]^G$ and these polarizations are naturally indexed.

Now we prove the following simple generalization of a theorem due to Hilbert (see [1], Kap. 1, §4).

2.12. Theorem. *Let G be a reductive group and let X be an irreducible affine G-variety such that the coordinate ring $\mathbb{C}[X]$ is graded and the action of G preserves the grading. Assume that $f_1, \dots, f_s$ are homogeneous G-invariant regular functions on X such that the ideal $I(f_1, \dots, f_s)$ generated by $f_1, \dots, f_s$ defines the zero fiber X^0. Then we have:*

(1) *The algebra $\mathbb{C}[X]^G$ is integral over its subalgebra $\mathbb{C}[f_1, \dots, f_s]$ generated by $f_1, \dots, f_s$;*
(2) *If the group G is connected, the algebra $\mathbb{C}[X]^G$ is the integral closure of $\mathbb{C}[f_1, \dots, f_s]$ in the algebra $\mathbb{C}[X]$.*

Proof. (1) It suffices to prove that $R = \mathbb{C}[X]^G$ is a finitely generated module over $\mathbb{C}[f_1, \dots, f_s]$.

Since the group G preserves the grading of $\mathbb{C}[X]$, the algebra $\mathbb{C}[X]^G$ is graded as well. Let $R := \oplus_{i\geq 0} R_i$ be the corresponding grading of $R = \mathbb{C}[X]^G$ and $\mathfrak{m} :=$

$\oplus_{i>0}R_i$. By Hilbert's Nullstellen Satz for affine varieties we have

$$\sqrt{\oplus_{j=1}^s Rf_j} = \mathfrak{m}.$$

Then there is a positive integer N such that $\mathfrak{m}^N \subset \oplus_{j=1}^s Rf_j$. Therefore, for $n \geq N$ we have

$$R_n \subset \oplus_{j=1}^s f_j R_{n-d_j},$$

where $d_j = \deg f_j$. Thus, for the finite dimensional vector space $B := \oplus_{i=0}^{N-1} R_i$ and for any $n \geq N$, by induction we get

$$R_n \subset \mathbb{C}[f_1, \dots, f_s]B.$$

(2) Let the group G be connected and let f be a root of a polynomial $x^k + a_{k-1}x^{k-1} + \cdots + a_0$ whose coefficients $a_{k-1}, \dots, a_0$ belong to $\mathbb{C}[f_1, \dots, f_s]$. Then the set of roots of this polynomial is G-invariant and, since the group G is connected, it acts trivially on this set. Thus $f \in \mathbb{C}[X]^G$. $\square$

3. Polarizations in invariant theory for reductive groups

3.1. The class of representations $R_{q,\mathrm{pol}}(G)$. Let G be a reductive group and $\rho : G \to \mathrm{GL}(V)$ a representation of G in a finite dimensional complex vector space V. Applying the constructions of Section 2 to $X = V$, the group $\rho(G) \subset \mathrm{GL}(V)$, and $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}, V) = V^q$ we get the algebras $\mathbb{C}[V]^G$, $\mathbb{C}[V^q]^G$, and $\mathbb{C}[V^q]^G_{\mathrm{pol}}$, the affine varieties $V^q//G$, $(V^q//G)_{\mathrm{pol}}$, and the morphisms $\pi_{V^q} : V^q \to V^q//G$ and $p_{q-1,V^q} : V^q//G \to (V^q//G)_{\mathrm{pol}}$. It is clear that the zero fiber $(V^q)^0$ from 2.6 equals the usual zero fiber of the G-module V^q with the diagonal action of G. Remark that in this case the morphism $m_{V^q} : V^q \to V$, the projection onto the first factor, is surjective.

The aim of this section is to define a class of representations of reductive groups such that the polarizations of the basic invariants define the zero fiber $(V^q)^0$ and one can apply Theorem 2.12.

Let $\sigma_1, \dots, \sigma_m$ be a system of generators of the algebra $\mathbb{C}[V]$. Recall that the morphism $\sigma = (\sigma_1, \dots, \sigma_m) : V \to \sigma(V)$ can be considered as the projection $\pi_X : X \to X//G$.

Consider the map $P_{q,V} : \mathfrak{F}_1(\mathbb{C}^{q-1}, V) \to \mathfrak{F}_1(\mathbb{C}^{q-1}, V//G)$ defined by $P_{q,V}(f) = \pi_X \circ f$. For $f = v_1 + \sum_{i=2}^q t_i v_i$ the value $P_{q,V}(f)$ is uniquely defined by the values of the polarizations of the basic invariants σ_i at $(v_1, \dots, v_q)$. Thus $P_{q,V}(\mathfrak{F}_1(\mathbb{C}^q, V))$ may be identified with $(V^q//G)_{\mathrm{pol}}$ naturally. Denote also by $P_{q,V}$ the defined by $P_{q,V}$ map from $L(\mathbb{C}^q, V)$ onto $P_{q,V}(L(\mathbb{C}^q, V))$. Then by definition $P_{q,V}$ is equal to the composition of the projection $\pi_{q,V^q} : V^q \to V^q//G$ and the morphism $p_{q,V^q} : V^q//G \to (V^q//G)_{\mathrm{pol}}$.

Let G be a reductive group. Let us denote by

$$R_{q,\mathrm{pol}}(G)$$

the set of isomorphism classes of representations $\rho : G \to GL(V)$ such that the ideal I_{pol} generated by the polarizations of the basic invariants defines the zero fiber $(V^q)^0$.

3.2. Theorem.

(1) *The trivial representation of G in a vector space V belongs to $R_{q,\mathrm{pol}}(G)$ for any q.*

(2) *The standard representations of the classical groups $G = \mathrm{GL}_n, \mathrm{O}_n,$ and SO_n in $V = \mathbb{C}^n$ and the group $G = \mathrm{Sp}_n$ in $V = \mathbb{C}^{2n}$ belong to the corresponding classes $R_{q,\mathrm{pol}}(G)$ for any q.*

(3) *The standard representation of SL_n in $\mathbb{C}^n$ belongs to $R_{q,\mathrm{pol}}(\mathrm{SL}_n)$ for $q < n$ and does not belong to $R_{q,\mathrm{pol}}(\mathrm{SL}_n)$ for $q \geq n$.*

Proof. (1) is evident. (2) follows from the classical results of Weyl (see [7]).

(3) It is clear that, for the standard representation ρ of SL_n in $\mathbb{C}^n$,

$$\mathbb{C}[(\mathbb{C}^n)^q]^{\mathrm{SL}_n} = \mathbb{C}$$

for $q = 1, \dots, n-1$ since we have a dense orbit. The diagonal action of SL_n in $(\mathbb{C}^n)^n$ coincides with the action of SL_n on the space M_n of $n \times n$ matrices by the left multiplication. Thus we have a nontrivial invariant of this action, namely, the determinant of the matrix. Thus the zero fiber $((\mathbb{C}^n)^n)^0$ equals the set of matrices with zero determinant, whereas the set of polarizations of basic invariants is empty and the ideal I_{pol} defines the whole of M_n. $\square$

3.3. Lemma. *Let G be a reductive group and let G_0 be the component of the identity of G. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and $\rho_0 : G_0 \to \mathrm{GL}(V)$ the restriction of ρ to G_0. Then the zero fiber of ρ coincides with the zero fiber of ρ_0.*

Proof. Let $\pi_V : V \to V//G$ and $\rho_{V,0} : V \to V//G_0$ be the projections. Consider the natural action of the finite group G/G_0 on $V//G_0$ and the corresponding projection $\overline{\pi}_V : V//G_0 \to (V//G_0)/(G/G_0) = V//G$. It is evident that $\pi_V = \overline{\pi}_V \circ \pi_{V,0}$. Since the affine variety $V//G$ is normal and the projection $\overline{\pi}_V$ is a finite morphism (see, for example, [5]), the ring $\mathbb{C}[V//G_0]$ is integral over its subring $\overline{\pi}_V^* \mathbb{C}[V//G]$. Thus the zero fiber of the action of G/G_0 on $V//G_0$ coincides with $\pi_{V,0}(0)$ and $V^0 = \pi_{V,0}^{-1} \circ \pi_{V,0}(0)$. $\square$

3.4. Theorem. *A representation $G \to \mathrm{GL}(V)$ belongs to $R_{q,\mathrm{pol}}(G)$ iff the following condition is satisfied: Whenever the linear span $L(v_1, \dots, v_q)$ of $v_1, \dots, v_q \in V$ is contained in the zero fiber V^0, then the vector $(v_1, \dots, v_q) \in V^q$ belongs to the zero fiber $(V^q)^0$ for the diagonal action of G on V^q.*

Proof. By definition $(V^q)^0 = P_{q,V}^{-1} \circ P_{q,V}(0)$.

Let $P_{q,V}^{-1} \circ P_{q,V}(0) = (V^q)^0$ and $L(v_1, \dots, v_q) \subset V^0$. Then $\pi_V \circ f(t) = 0$, i.e., $f \in P_{q,V}^{-1} \circ P_{q,V}(0)$, for each $t = (t_1, \dots, t_q)$ and $f(t) = \sum_{i=0}^q t_i v_i$.

Let the condition of the theorem be satisfied and $(v_1, \dots, v_q) \in (V^q)^0$. By Lemma 3.3 it suffices to assume that the group G is connected. By the Hilbert-Mumford criterion for the zero fiber (see, for example, [5]) there is a one-parameter subgroup $\lambda : (\mathbb{C}^*, \cdot) \to G$ such that $\lim_{t\to 0} \lambda(t)(v_1, \dots, v_q) = 0$. Then, for each $i = 1, \dots, q$, $\lim_{t\to 0} \lambda(t) v_i = 0$. Thus for each $(t_1, \dots, t_q) \in \mathbb{C}^q$ we have $\lim_{t\to 0} \lambda(t) \sum_{i=0}^q t_i v_i = 0$ and consequently $\sum_{i=0}^q t_i v_i \in V^0$. $\square$

Theorems 2.12 and 3.4 imply immediately the following three corollaries.

3.5. Corollary. *For each representation $\rho : G \to \mathrm{GL}(V)$ in $R_{q,\mathrm{pol}}(G)$ the morphism $p_q : V^q//G \to (V^q//G)_{\mathrm{pol}}$ is finite. Moreover, if the group G is connected, the ring $\mathbb{C}[V^q//G] = \mathbb{C}[V^q]^G$ is the integral closure of $\mathbb{C}[V^q]^G_{\mathrm{pol}}$ in $\mathbb{C}[V^q]$.*

3.6. Corollary. *If $\rho \in R_{q,\mathrm{pol}}(G)$ then $\rho \in R_{q-1,\mathrm{pol}}(G)$.*

3.7. Corollary. *If all orbits of a representation $\rho : G \to \mathrm{GL}(V)$ are closed then $\rho \in \rho \in R_{q,\mathrm{pol}}(G)$ for each q. In particular, this is true for a finite group G.*

3.8.. Now we indicate several representations belonging to $R_{q,\mathrm{pol}}(G)$. The space R_n of homogeneous polynomials in two variables x and y of degree n is called the space of binary forms of degree n.

Theorem. *The canonical representation of the group SL_2 on the space R_n of binary forms of degree n belongs to $R_{q,\mathrm{pol}}(\mathrm{SL}_2)$ for $n \neq 1$ and any q. The representation of SL_2 on R_1 belongs to $R_{q,\mathrm{pol}}(\mathrm{SL}_2)$ for $q < n$ and does not belong to $R_{q,\mathrm{pol}}(\mathrm{SL}_2)$ for $q \geq n$.*

Proof. Recall that the representation of SL_2 on R_n and the contragradient representation of SL_2 on R_n^* are isomorphic (see, for example, [5]). The representation of SL_2 on $R_0 = \mathbb{C}$ is trivial and the representation of SL_2 on R_1 is isomorphic to the standard representation of SL_2 on $\mathbb{C}^2$, so by Theorem 3.2 they belong $R_{q,\mathrm{pol}}(\mathrm{SL}_2)$. Thus it suffices to consider the case $n > 1$.

It is known (see, for example, [5]) that a form $f \in R_n$ belongs to the zero fiber R_n^0 iff the decomposition of f into the product of linear forms contains a factor of multiplicity $> \frac{n}{2}$ or if $f = 0$. Consider the subspace $R_{n,l}^0$ of R_n consisting of all forms of type $l^r h$, where $r = \left[\frac{n}{2}\right] + 1$, $l \in R_1$ is a nonzero linear form, and $h \in R_{n-r}$. By a linear transformation of variables x and y one can assume that $l = x$. Then, for the one-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

of SL_2, we have $\lim_{t\to 0} \lambda(t) R_{n,l}^0 = 0$. By the Hilbert-Mumford criterion for the zero fiber this implies $R_{n,l}^0 \subset R_n^0$, and by 3.4 we have $(f_1, \dots, f_q) \in (R_n^q)^0$ for any $f_1, \dots, f_q \in R_{n,q}^0$. Moreover, R_n^0 is the union of the subspaces $R_{n,l}^0$ for distinct linearly independent l. Then by Theorem 3.4 the proof of the theorem follows from the following lemma.

3.9. Lemma. *If the linear span of $f_1, \dots, f_q$ is contained in R_n^0 then there is a linear form l such that $f_1, \dots, f_q \in R_{n,l}^0$.*

Proof. It suffices to consider the case $q = 2$.

Assume that $f_1 = l_1^r h_1$ and $f_2 = l_2^r h_2$, where $r = \left[\frac{n}{2}\right] + 1$, l_1, l_2 are linear forms, $h_1, h_2 \in R_{n-r}$, and $t_1 f_1 + t_2 f_2 \in R_n^0$ for any $t_1, t_2 \in \mathbb{C}$. One need to prove that the forms l_1 and l_2 are linearly dependent.

For contradiction suppose that l_1 and l_2 are linearly independent. By a coordinate change one can assume that $l_1 = x$ and $l_2 = y$. By assumption, for any $t_1, t_2 \in \mathbb{C}$ there is a linear form $l = \alpha x + \beta y$ ($\alpha, \beta \in \mathbb{C}$) and $h \in R_{n-r}$ such that

$$t_1 x^r h_1 + t_2 y^r h_2 = (\alpha x + \beta y)^r h \tag{1}$$

where α, β and h depend on t_1 and t_2. We may assume also that $h_1, h_2 \neq 0$ and, for any $t_1, t_2 \neq 0$, α, β, and h are nonzero.
(2) *Remark: $\frac{\alpha}{\beta}$ cannot be constant in t_1, t_2.* Namely, if $\frac{\alpha}{\beta}$ is constant in $t_1, t_2 \in \mathbb{C}\setminus 0$, by continuity of both sides of (1) it is constant in $t_1, t_2 \in \mathbb{C}$. Thus, for $t_1 = 0$ and $t_2 \neq 0$, y^r divides h which contradicts the degree assumptions. In particular, this is true if $\frac{\alpha}{\beta}$ is a root of a polynomial whose coefficients do not depend on t_1, t_2.

First consider the case when $n = 2m$. Then the equality (1) has the following form

$$t_1 x^{m+1} h_1 + t_2 y^{m+1} h_2 = (\alpha x + \beta y)^{m+1} h, \tag{3}$$

where $h, h_1, h_2 \in R_{m-1}$.

Let $h_1 = \sum_{i=0}^{m-1} a_i x^{m-i-1} y^i$ for $a_i \in \mathbb{C}$. Differentiating the equality (3) m times with respect to x, putting $y = -\frac{\alpha}{\beta}x$ for $\beta \neq 0$, and dividing by $t_1 x^m$ we get

$$\sum_{i=0}^{m-1} (-1)^i \frac{(2m-i)!}{(m-i)!} a_i \left(\frac{\alpha}{\beta}\right)^i = 0.$$

Since the coefficients a_i do not vanish simultaneously we have a contradiction by remark (2).

Consider now the case when $n = 2m - 1$. Then the equality has the following form

$$t_1 x^m h_1 + t_2 y^m h_2 = (\alpha x + \beta y)^m h, \tag{4}$$

where $h, h_1, h_2 \in R_{m-1}$. We assume till the end of the proof that $t_1, t_2 \neq 0$ and then $\alpha.\beta \neq 0$.

Let $h_1 = \sum_{i=0}^{m-1} a_i x^{m-i-1} y^i$ and $h_2 = \sum_{j=0}^{m-1} b_j x^i y^{m-j-1}$ for $(a_i, b_j \in \mathbb{C})$. We differentiate equality (4) $m-1$ times with respect to x, and put $y = -\frac{\alpha}{\beta}x$, and divide by x^m to get:

$$b_{m-1} \left(\frac{\alpha}{\beta}\right)^m = \frac{(-1)^{m-1} t_1}{(m-1)!\, t_2} \sum_{i=0}^{m-1} (-1)^i \frac{(2m-i+1)!}{(m-i)!} a_i \left(\frac{\alpha}{\beta}\right)^i. \tag{5}$$

Next we differentiate equality (4) $m-1$ times with respect to y, put $y = -\frac{\alpha}{\beta}x$, and divide by x^m to get:

$$(m-1)!\, t_1 a_{m-1} + t_2 \sum_{j=0}^{m-1} (-1)^{m-j} \frac{(2m-j+1)!}{(m-j)!} b_j \left(\frac{\alpha}{\beta}\right)^{m-j} = 0. \tag{6}$$

Multiply (6) by $b_{m-1} \left(\frac{\alpha}{\beta}\right)^{m-1}$, replace $b_{m-1} \left(\frac{\alpha}{\beta}\right)^m$ by its value from (5), and divide the result by t_1 we get

$$\begin{aligned} (m-1)!\, a_{m-1} b_{m-1} \left(\frac{\alpha}{\beta}\right)^{m-1} &+ \\ + \frac{1}{(m-1)!} \sum_{j,i=0}^{m-1} (-1)^{i-j-1} \frac{(2m-j+1)!}{(m-j)!} \frac{(2m-i+1)!}{(m-i)!} a_i b_j \left(\frac{\alpha}{\beta}\right)^{m-j+i-1} &= 0. \end{aligned} \tag{7}$$

Consider the left side of (7) as a polynomial in $\frac{\alpha}{\beta}$. By remark (2) all coefficients of this polynomial and, in particular, the constant term and the coefficient of $\left(\frac{\alpha}{\beta}\right)^{2(m-1)}$, vanish. Thus we have

$$a_0 b_{m-1} = a_{m-1} b_0 = 0.$$

If $b_{m-1} = 0$, by remark (2) equation (5) implies that $a_i = 0$ for $i = 0, \dots, m-1$, a contradiction. Similarly, if $a_{m-1} = 0$ then $b_i = 0$ for $i = 0, \dots, m-1$, a contradiction again. Thus $a_{m-1}, b_{m-1} \neq 0$, $a_0 = b_0 = 0$, and (4) has the following form

$$(8)\quad \left(t_1 x^{m-1} \sum_{i=1}^{m-1} a_i x^{m-i-1} y^{i-1} + t_2 y^{m-1} \sum_{i=1}^{m-1} b_i x^{i-1} y^{m-i-1} \right) xy = (\alpha x + \beta y)^m h.$$

Since $\alpha, \beta \neq 0$, xy divides h, and dividing (8) by xy we get

$$(9)\quad t_1 x^{m-1} \sum_{i=0}^{m-2} a_{i+1} x^{m-i-2} y^i + t_2 y^{m-1} \sum_{i=0}^{m-2} b_{i+1} x^i y^{m-i-2} = (\alpha x + \beta y)^{m-1} h',$$

where $h' = \frac{(\alpha x + \beta y)h}{xy} \in R_{m-2}$. Equality (9) is similar to (4) but for $n = 2m-3$. Proceeding this way we reduce our condition to the case $m = 2$. In this case either x or y divide $\alpha x + \beta y$, a contradiction.

This concludes the proof. $\square$

3.10. Corollary. *Each representation of the group* SL_2 *in a vector space* V *whose decomposition into irreducible representations does not contain a term isomorphic to* R_1 *belongs to* $R_{q,\mathrm{pol}}(\mathrm{SL}_2)$ *for any* q.

Proof. Recall that each irreducible representation of SL_2 is isomorphic to one of the canonical representations of SL_2 in R_n $(n = 0, 1, \dots)$.

Let $\rho : \mathrm{SL}_2 \to \mathrm{GL}(V)$ be a representation satisfying the condition of the corollary and let $V = \oplus_{i=1}^k V_i$ be a decomposition of V into the sum of SL_2-invariant subspaces such that the induced representation of SL_2 on each V_i is irreducible and thus isomorphic to one of the canonical representations in R_{n_i} $(n_i \neq 1)$. Suppose that the linear span of $v_1, \dots, v_q \in V$ is contained in V^0. Then by the Hilbert-Mumford criterion for zero fiber there is a one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{SL}_2$ such that

$$\lim_{t \to 0} \lambda(t)(t_1 v_1 + \dots + t_q v_q) = 0 \qquad \text{for all } t_1, \dots, t_q \in \mathbb{C}.$$

Denote by $v_{j,i}$ the component of the vector v_j in V_i. Then we have

$$\lim_{t \to 0} \lambda(t)(t_1 v_{1,i} + \dots + t_q v_{q,i}) = 0 \quad (i = 1, \dots, k).$$

By Lemma 3.9 there is a linear form l_i on $\mathbb{C}^2$ such that the linear span of $v_{1,i}, \dots, v_{q,i}$ is contained in $R^0_{n_i, l_i}$.

Claim: All l_i *are linearly dependent.* Suppose for contradiction that some l_1 and l_2, say, are linearly independent. One can take $x = l_1$ and $y = l_2$ for the coordinates in $\mathbb{C}^2$ so that $v_{1,1} = x^{r_1}h_1$ and $v_{1,2} = y^{r_2}h_2$ where $d_i := \deg h_i < r_i$. We have $\lim_{t\to 0}\lambda(t)(x^{r_1}h_1) = 0 = \lim_{t\to 0}\lambda(t)(y^{r_2}h_2)$. The one-parameter subgroup is given by

$$\lambda(t) = A^{-1}\begin{pmatrix} t^\lambda & 0 \\ 0 & t^{-\lambda}\end{pmatrix}A, \quad \text{where } \lambda\in\mathbb{N}, \quad A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \text{ for } ad-bc=1.$$

Since $A\binom{x}{y} = \binom{ax+by}{cx+dy}$ we have

$$\begin{aligned} Ah_1 &= \sum_{j=0}^{d_1} h_{1,j}x^jy^{d_1-j} \\ 0 &= \lim_{t\to 0}\begin{pmatrix} t^\lambda & 0 \\ 0 & t^{-\lambda}\end{pmatrix}A(x^{r_1}h_1) = \lim_{t\to 0}\begin{pmatrix} t^\lambda & 0 \\ 0 & t^{-\lambda}\end{pmatrix}((ax+by)^{r_1}(Ah_1)) \\ &= \lim_{t\to 0}\sum_{i=0}^{r_i}\binom{r_1}{i}a^ib^{r_1-i}x^iy^{r_1-i}\sum_{j=0}^{d_1}h_{1,j}x^jy^{d_1-j}t^{\lambda(2i-r_1+2j-d_1)} \end{aligned}$$

Let $h_{1,k}\neq 0$ for minimal k and consider the term with minimal degree in x

$$b^{r_1}h_{1,k}x^ky^{r_1+d_1-k}t^{\lambda(-r_1+2k-d_1)}.$$

Since this converges to 0 for $t\to 0$ but $2k-r_1-d_1<0$, we get $b=0$.

Similarly $\lim_{t\to 0}\lambda(t)(y^{r_2}h_2) = 0$ implies $d=0$. But $ad-bc=1$, a contradiction, and the claim follows.

Therefore, all l_i are linearly dependent. Choosing again one of l_i for the coordinate x in $\mathbb{C}^2$ we have for the diagonal action of SL_2 in V^q and

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix},$$

that $\lim_{t\to 0}(v_1,\dots,v_q) = 0$. Thus by the Hilbert-Mumford criterion for the zero fiber and Theorem 3.4 the representation ρ is in $R_{q,\mathrm{pol}}(\mathrm{SL}_2)$. □

3.11. Theorem. *Each representation ρ of the (complex) torus T^n in a vector space V belongs to $R_{q,\mathrm{pol}}(T^n)$.*

Proof. It is known that each representation of T^n is a sum of one-dimensional representations. Denote by ϵ_i the standard i-th character of the standard representation of T^n in $\mathbb{C}^n$. Then each character of T^n has a form $\chi = \sum_{i=1}^n \nu_i\epsilon_i$, where ν_i are integers. Let χ_j ($j=1,\dots,m$) be the weights of the representation ρ and let $V = \oplus_{j=1}^m V_{\chi_j}$ be the weight decomposition of V. Let $\lambda : \mathbb{C}^*\to T^n$ be a one parameter subgroup and let $v = \sum_j v_j \in V$ for $v_j\in V_{\chi_j}$. Then

$$\lambda(t)v = \sum_j t^{\langle\chi_j.\lambda\rangle}v_j,$$

where $\langle\chi_j.\lambda\rangle$ is an integer.

Suppose that the linear span $L(v_1,\dots,v_q)$ of $v_1,\dots,v_q\in V$ is contained in the zero fiber V^0. Let $v_k=\sum_{j=1}^m v_{k,j}$ be the weight decomposition, with $v_{k,j}\in V_{\chi,j}$. For each $k=1,\dots,q$ let $J_k:=\{j\in\{1,\dots,m\}:v_{k,j}\neq 0\}$. Put $J=\cup_k J_k$. Evidently, the set of $t=(t_1,\dots,t_q)\in\mathbb{C}^q$ such that the component $v_j\in V_{\chi_j})$ of the vector $V=\sum_{i=1}^q t_iv_i$ vanishes for some $j\in J$, has a codimension ≥ 1. Thus there exists $w=\sum_{j=1}^m w_j\in L(v_1,\dots,v_q)$ with $(w_j\in V_{\chi_j})$ such that $w_j\neq 0$ for each $j\in\cup_k J_k$. By the Hilbert-Mumford criterion there is a one-parameter subgroup $\lambda:\mathbb{C}^*\to T^n$ such that $\lim_{t\to 0}\lambda(t)w=0$. This implies that $\langle\chi_j.\lambda\rangle>0$ for each $j\in\cup_k J_k$. But then $\lim_{t\to 0}\lambda(t)(v_1,\dots,v_q)=0$, so that $(v_1,\dots,v_q)\in(V^q)^0$. Thus by Theorem 3.4 we may conclude that $\rho\in R_{q,\mathrm{pol}}(T^n)$. □

4. Polarizations in invariant theory for finite groups

In this section we show that for finite groups the above results can be generalized and strengthened.

4.1. Let G be a finite group and let X be an irreducible affine G-variety. Then the categorical quotient $X//G$ is the geometric one, i.e., $X//G=X/G$ is the orbit space. Consider the projection $\pi_X:X\to X/G$. Let Y be another irreducible affine variety, let $\mathfrak{F}(Y,X)$ be the set of regular morphisms from Y to X, and let $\mathfrak{F}(Y,X/G)$ be the set of regular morphisms from Y to X/G. Recall from 3.1 the map $P_{Y,X}:\mathfrak{F}(Y,X)\to\mathfrak{F}(Y,X/G)$ given by $P_{Y,X}(f)=\pi_X\circ f$ for $f\in\mathfrak{F}(Y,X)$, and the pointwise action $(gf)(y)=g(f(y))$ of the group G on $\mathfrak{F}(Y,X)$.

Theorem. *The map $P_{Y,X}$ induces an injective map of the set $\mathfrak{F}(Y,X)/G$ of orbits of G on $\mathfrak{F}(Y,X)$ to the set $\mathfrak{F}(Y,X/G)$.*

Proof. It suffices to prove that, if $f,f'\in\mathfrak{F}(Y,X)$ and $P(f)=P(f')$, there exists $g\in G$ with $f'=gf$. For each morphism $h:Y\to X$ the graph $\gamma(h)=\{(y,h(y))|\,y\in Y\}$ of h is an irreducible closed subset of $Y\times X$ which is isomorphic to Y.

Put $\Gamma=\{(y,x)\in Y\times X|\,\pi\circ f(y)=\pi(x)\}$. It is evident that Γ is a closed subset of $Y\times X$ and that Γ decomposes into irreducible components:

$$\Gamma=\bigcup_{g\in G}\gamma(gf). \tag{1}$$

Since the graph $\gamma(f')$ of the morphism f' is an irreducible component of Γ, there exists $g\in G$ such that $\gamma(f')=\gamma(gf)$, i.e. $f'=gf$. □

4.2. Let $\sigma_1,\dots,\sigma_m$ be a minimal system of generators of the algebra $\mathbb{C}[X]^G$ and let

$$\sigma=(\sigma_1,\dots,\sigma_m):X\to\mathbb{C}^m.$$

Recall from 2.1 that the map σ as a morphism of X to $\sigma(X)$ can be considered as the projection $\pi_X:X\to X/G$. By definition, for $f\in\mathfrak{F}(Y,X)$ its image $P_{Y,X}(f)$ is uniquely defined by the functions $\sigma_i\circ f$ $(i=1,\dots,m)$. Then by Theorem 4.1 the functions $\hat\sigma_i:\mathfrak{F}(Y,X)\to\mathbb{C}[Y]$ for $i=1,\dots,m$ have as level set of the values $\hat\sigma_i(f)=\sigma_i\circ f$ exactly the orbit Gf of f.

4.3. Consider the case when the algebra $\mathbb{C}[X]$ is graded, the action of the group G preserves this grading, the generators $\sigma_1,\dots,\sigma_m$ are homogeneous functions, and Z is a G-stable irreducible closed subset of $\mathfrak{F}_d(\mathbb{C}^q,X)$.

Theorem. *Let Z be a G-stable irreducible closed subset of $\mathfrak{F}_d(\mathbb{C}^q, X)$ and let the morphism m_Z be dominant. Then the morphism $p_{q,Z} : Z//G \to (Z//G)_{\mathrm{pol}}$ is finite and birational. If the variety Z is normal, the morphism $p_{q,Z}$ is a bijective normalization of $(Z//G)_{\mathrm{pol}}$ and then, in particular, the algebra $\mathbb{C}[Z]^G$ is the integral closure of the subalgebra $\mathbb{C}[Z]^G_{\mathrm{pol}}$ in its field of fractions.*

Proof. Let $Z = \mathfrak{F}_d(\mathbb{C}^q, X)$. Then the values of the generalized polarizations of $\sigma_1, \ldots, \sigma_m$ at $f \in \mathfrak{F}_d(\mathbb{C}^q, X)$ are exactly the coefficients of the polynomials $\sigma_i \circ f$ with respect to $t_1, \ldots, t_q$. Thus by Theorem 4.1 the orbit Gf is uniquely described by the values of the indexed generalized polarizations $(\sigma_i)_{i_1 \ldots i_q}$ at f. The same statement is true for each G-stable irreducible closed subset of $\mathfrak{F}_d(\mathbb{C}^q, X)$.

Since the morphism m_Z is dominant, by Lemma 2.10 the algebra $\mathbb{C}[Z]_{\mathrm{pol}}$ is generated by any minimal system of homogeneous generators σ_i and their generalized polarizations which are naturally indexed. By the above arguments, for each point $x \in (Z//G)_{\mathrm{pol}}$, there is a unique orbit $z \in Z/G$ such that $p_{q,Z}(z) = x$. Then the morphism $p_{q,Z}$ is bijective and, therefore, birational. Since $p_{q,Z}^{-1}(0) = \{0\}$, by Theorem 2.12 the morphism $p_{q,Z}$ is finite.

If the variety Z is normal, the orbit space Z/G is normal as well (see, for example, [5]). Then the morphism $p_{q,Z}$ is a bijective normalization of $(Z//G)_{\mathrm{pol}}$. The last statement follows from the definition of normalization. □

4.4. Corollary. *Let Z be a G-stable irreducible closed subset of $\mathfrak{F}_d(\mathbb{C}^q, X)$ such that the morphism $m_Z : Z \to X$ is dominant. Then for each $f \in \mathbb{C}[Z]^G$ there is some $F \in \mathbb{C}[Z]^G_{\mathrm{pol}}$ and an integer $k > 0$ such that $f = F^k$.*

Proof. Let $f \in \mathbb{C}[Z]^G$. It suffices to consider the case when the ideal (f) generated by f is prime. Since the morphism $p_{q,Z}$ is finite, it is closed. Since by Theorem 4.3 the morphism $p_{q,Z}$ is bijective, it is a homeomorphism of underlying topological spaces.

Consider the irreducible closed subset $V(f)$ of Z/G defined by the ideal (f). Then its image $p_{q,Z}(V(f))$ is an irreducible closed subset of $(Z/G)_{\mathrm{pol}}$. Since $\operatorname{codim} p_{q,Z}(V(f)) = \operatorname{codim}(V(f)) = 1$ there is $F \in k[Z]^G_{\mathrm{pol}}$ such that $p_{q,Z}(V(f)) = V(F)$, where (F) the principal ideal generated by F. By definition $V(p^*_{q,Z}F) = V(f)$, i.e. the radical $\sqrt{I(F)}$ of $I(F)$ equals (f). Therefore, there is an integer $k > 0$ such that $f = F^k$. □

4.5. Let V be a finite dimensional vector space and G a finite subgroup of $\mathrm{GL}(V)$. Consider the space V^q with the diagonal action of G. By Example 2.7 the G-modules V^q and $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}, V)$ are naturally isomorphic. For $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}, V)$ the generalized polarizations of homogeneous G-invariant polynomials on V are their standard polarizations. Then the algebra $\mathbb{C}[V^q]^G_{\mathrm{pol}}$, the variety $(V^q)^G_{\mathrm{pol}}$ and the morphism p_{q,V^q} are constructed with use of the standard polarizations. Note that in this case the morphism $m_Z : Z \to V$ is surjective and dominant.

Corollary. *Let V be a finite dimensional vector space, G a finite subgroup of* $\mathrm{GL}(V)$. *Consider the space V^q with the diagonal action of G. Then the morphism $p_{q,V^q} : V^q/G \to (V^q/G)_{\mathrm{pol}}$ is a bijective normalization of $(V^q/G)_{\mathrm{pol}}$. In particular, the ring $\mathbb{C}[V^q/G]$ is the integral closure of the subring $\mathbb{C}[V^q]^G_{\mathrm{pol}}$ in its field of frac-*

tions. Moreover, for each $f \in \mathbb{C}[V^q]^G$ there is some $F \in \mathbb{C}[V^q]^G_{\mathrm{pol}}$ and an integer $k > 0$ such that $F^k = f$.

Proof. Since for $Z = \mathfrak{F}_1(\mathbb{C}^{q-1}) = V^q$, the morphism $m_Z : Z \to V$ is dominant and Z is a smooth (and thus normal) variety, the statements of the corollary follows from Lemma 1.1, Theorem 3.2, and Corollary 4.4. □

4.6. Remark. Consider again the G-module V and put $Z = \mathfrak{F}_{q-1}(\mathbb{C}, V)$. By Example 2.9 the G-modules V^q and $Z = \mathfrak{F}_{q-1}(\mathbb{C}, V)$ are naturally isomorphic and for $Z = \mathfrak{F}_{q-1}(\mathbb{C}, V)$ the generalized polarizations of homogeneous G-invariant polynomials on V are sums of their standard polarizations.

Then Corollary 4.5 remains true if we replace the the standard polarizations by the above generalized polarizations in the construction of the algebra $\mathbb{C}[V^q]^G_{\mathrm{pol}}$, the variety $(V^q)^G_{\mathrm{pol}}$ and the morphism p_{q,V^q}. By the calculations of Examples 2.8 and 2.9 for $q > 2$ the dimension of the space generated by the polarizations of the basic invariants of the G-module V is strictly less than the dimension of the space generated by the above generalized polarizations of these invariants.

Examples. In the following examples we consider representations of finite groups in real vector spaces. But the results hold also for the complexifications of these representations.

4.7. Example. Let $V = \mathbb{R}^n$. The group B_n acts on $x = (x_1, \dots, x_n) \in V$ by permutations of $x_1, \dots, x_n$ and the sign changes $x_i \to -x_i$. The group D_n $(n \geq 4)$ acts on x by the above permutations and changes of an even number of signs. It is known (see, for example, [3]) that one can take for the basic invariants of $\mathbb{C}[V]^D_n$ the polynomials

$$\sigma_k = \sum_{i=1}^{n} x_i^{2k} \quad (k = 1, \dots, n-1), \quad \sigma_n = x_1 \dots x_n.$$

For odd $r \geq 1$ define the operator

$$P_r := \sum_{i=1}^{n} y_i^r \frac{\partial}{\partial x_i}.$$

Consider x_i, y_i as the standard coordinates in $V^2 = \mathbb{R}^{2n}$. The operator P_r commutes with the diagonal action of D_n and B_n on $\mathbb{R}[V^2]$ and then preserves $\mathbb{R}[V^2]^{D_n}$ and $k[V^2]^{B_n}$. It is known (see [6] and [4]) that the algebra $\mathbb{R}[V^2]^{D_n}$ is generated by the polarizations of the basic invariants σ_i and the polynomials

$$P_{r_1} \dots P_{r_k}(\sigma_n) \quad (r_i \geq 1 \quad \text{odd}, \quad \sum_{i=1}^{k} r_i \leq n - k).$$

Moreover, it is known that $P_3(\sigma_n)$ cannot be expressed in terms of the polarizations of σ_i's (see [6]). It is clear that the group B_n is generated by the group D_n and the reflection $w : (x_1, \dots, x_{n-1}, x_n) \to (x_1, \dots, x_{n-1}, -x_n)$. Then

$$w(\sigma_n) = -\sigma_n \quad \text{and} \quad w(P_{r_1} \dots P_{r_k}(\sigma_n)) = -P_{r_1} \dots P_{r_k}(\sigma_n).$$

This implies that $(P_{r_1} \dots P_{r_k}(\sigma_n))^2 \in \mathbb{R}[V^2]^{B_n}$. Since the polarizations of the basic invariants of B_n generate $\mathbb{R}[V^2]^{B_n}$ (see [2] and [4]), $(P_{r_1} \dots P_{r_k}(\sigma_n))^2$ is a polynomial in the polarizations of the basic invariants of B_n and, then, in the polarizations of the basic invariants of D_n.

4.8. Example. Let $V = \mathbb{R}^2$ and let the group G be the cyclic group $\mathbb{Z}_3$ whose action on V is generated by rotation over the origin by the angle $\frac{2\pi}{3}$. One can take for the basic generators of the algebra $\mathbb{R}[V]^G$ the polynomials

$$\sigma_1 = \frac{1}{2}(x_1^2 + y_1^2), \quad \sigma_2 = \frac{1}{3}(x_1^3 - 3x_1y_1^2), \quad \text{and} \quad \sigma_3 = \frac{1}{3}(3x_1^2y_1 - y_1^3),$$

where x_1, y_1 are the standard coordinates in $\mathbb{R}^2$.

Let x_1, y_1, x_2, y_2 be the standard coordinates in $V^2 = \mathbb{R}^4$. We have the polynomial $\sigma = x_1y_2 - y_1x_2 \in \mathbb{R}[V^2]^G$ which cannot be expressed in terms of the polarizations of the basic invariants σ_i. Consider the following polarizations of basic invariants

$$\begin{aligned}\sigma_{2,1} &= x_1^2x_2 - y_1^2x_2 - 2x_1y_1y_2, \quad & \sigma_{3,1} &= -(y_1^2y_2 - x_1^2y_2 - 2x_1y_1x_2)\\ \sigma_{2,2} &= x_1x_2^2 - x_1y_2^2 - 2y_1x_2y_2, \quad & \sigma_{3,2} &= -(y_1y_2^2 - y_1x_2^2 - 2x_1x_2y_2)\\ \sigma_{2,3} &= \frac{1}{3}(x_2^3 - 3x_2y_2^2), \quad & \sigma_{3,3} &= \frac{1}{3}(3x_2^2y_2 - y_2^3).\end{aligned}$$

It is easily checked that we have

$$\sigma^3 = \frac{3}{4}(3\sigma_3\sigma_{2,3} - 3\sigma_2\sigma_{3,3} + \sigma_{2,1}\sigma_{3,2} - \sigma_{3,1}\sigma_{2,2}).$$

References

[1] D. Hilbert, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), 313–373.

[2] H. Jergen, *Weitere Ergebnisse zur allgemeinen Vermutung in der Invariantentheorie von endlichen Spiegelgruppen*, Diplomarbeit, Universität Basel (1989), Basel, Switzerland.

[3] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge studies in advanced mathematics, vol. 29, Cambrige University Press, 1990.

[4] M. Hunziker, *Classical invariant theory for finite reflection groups*, Transformation Groups **2** (1997), no. 2, 147–163.

[5] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Vieweg, Braunschweig Wiesbaden, 1985.

[6] N.R. Wallach, *Invariant differential operators on a reductive Lie algebra and Weil group representations*, J. Amer. Math. Soc. **6** (1993), 779–816.

[7] H. Weyl, *The Classical Groups. Their invariants and representations*, Princeton University Press, 1946.

M. Losik: Saratov State University, ul. Astrakhanskaya, 83, 410026 Saratov, Russia

E-mail address: LosikMV@info.sgu.ru

P. W. Michor: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria; *and:* Erwin Schrödinger Institut für Mathematische Physik, Boltzmanngasse 9, A-1090 Wien, Austria

E-mail address: Peter.Michor@esi.ac.at

V.L. Popov: Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow, 117966, Russia

E-mail address: popov@ppc.msk.ru