Longer version of the paper (including a proof from [6]):
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Some Remarks on the Plücker Relations

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1. The Plücker relations

Let V denote a finite-dimensional vector space. An s-vector $P \in \Lambda^s V$ is called *decomposable* or *simple* if it can be written in the form

$$P = u \wedge v \wedge \cdots \wedge w \quad \text{for } u, v, \ldots, w \in V.$$

We shall use in the following both Penrose's abstract index notation and exterior calculus with the conventions of [3].

Theorem 1. *Let $P \in \Lambda^s V$ be an s-vector. Then P is decomposable if and only if one of the following conditions holds:*

(1) $i(\Phi)P \wedge P = 0$ *for all* $\Phi \in \Lambda^{s-1}V^*$. *In index notation* $P_{[abc\cdots d}P_{e]fg\cdots h} = 0$.
(2) $i(i_P\Psi)P = 0$ *for all* $\Psi \in \Lambda^{s+1}V^*$.
(3) $i_{\alpha_1\wedge\cdots\wedge\alpha_{s-k}}P$ *is decomposable for all* $\alpha_i \in V^*$, *for any fixed* $k \geq 2$.
(4) $i(\Psi)P \wedge P = 0$ *for all* $\Psi \in \Lambda^{s-2}V^*$ *In index notation* $P_{[abc\cdots d}P_{ef]g\cdots h} = 0$.
(5) $i(i_P\Psi)P = 0$ *for all* $\Psi \in \Lambda^{s+2}V^*$.

Proof. (1) These are the well known classical Plücker relations. For completeness' sake we include a proof. Let $P \in \Lambda^n V$ and consider the induced linear mapping $\sharp_P : \Lambda^{s-1}V^* \to V$. Its image, W, is contained in each linear subspace U of V with $P \in \Lambda^s U$. Thus W is the minimal subspace with this property. P is decomposable if and only if $\dim W = s$, and this is the case if and only if $w \wedge P = 0$ for each $w \in W$. But $i_\Phi P$ for $\phi \in \Lambda^{s-1}V^*$ is the typical element in W.

(2) This well known variant of the Plücker relations follows by duality (see [4]):

$$\begin{aligned}\langle P \wedge i(\Phi)P, \Psi\rangle &= \langle i(\Phi)P, i_P\Psi\rangle = \langle P, \Phi \wedge i_P\Psi\rangle = \\ &= (-1)^{(s-1)}\langle P, i_P\Psi \wedge \Phi\rangle = (-1)^{(s-1)}\langle i(i_P\Psi)P, \Phi\rangle.\end{aligned}$$

(3) This is due to [6]. There it is proved using exterior algebra. Appearently, this result is included in formula (4), page 116 of [7]. For completenes's sake we include here the proof from [6]. It is enough to prove that P is decomposable if and only if $i_\alpha P$ is decomposable for all $\alpha \in V^*$.

If P is decomposable then by 1 we have $i_{\alpha\wedge\Phi}P \wedge P = 0$ for all α and all $\Phi \in \Lambda^{s-2}V^*$, so also $0 = i_\alpha(i_\Phi i_\alpha P \wedge P) = -i_\Phi i_\alpha P \wedge i_\alpha P$; thus $i_\alpha P$ is decomposable by 1.

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If all $i_\alpha P$ are decomposable we take $\epsilon^1 \in V^*$ such that $i(\epsilon^1)P \neq 0$; then

$$i(\epsilon^1)P = e_2 \wedge ... \wedge e_s. \qquad (i)$$

for $e_a \in ker\,\epsilon^1 \subseteq V$. Let us also take $e_1 \in V$ with $\epsilon^1(e_1) = 1$, and denote by V_1 the s-dimensional subspace spanned by the $e_1, ..., e_s$, and by V_2 an arbitrary complement of the span of $e_2, ..., e_s$ in $ker\,\epsilon^1$. Then $V = V_1 \oplus V_2$, and we have

$$P = \rho e_1 \wedge ... \wedge e_s + \sum_{i=1}^{s-1} P'_i \wedge P''_i + P''_s,$$

where $\rho \in \mathbf{R}$, $P'_i \in \wedge^{s-i} V_1$, $P''_i \in \wedge^i V_2$, $P''_s \in \wedge^s V_2$. Moreover, (i) implies $\rho = 1$ and

$$i(\epsilon^1)P'_i = 0 \qquad (i = 1, ..., s-1). \qquad (ii)$$

(If some $P''_i = 0$ we will also assume $P'_i = 0$.)

Let $\epsilon^i \in V^*$ be covectors which vanish on V_2, and are such that $\epsilon^i(e_j) = \delta^i_j$ $(i, j = 1, ..., s)$. According to our hypothesis, the $(s-1)$-vectors

$$i(\epsilon^a)P = (-1)^{a-1} e_1 \wedge ... \wedge \hat{e}_a \wedge ... \wedge e_s + \sum_{i=1}^{s-1} (i(\epsilon^a)P'_i) \wedge P''_i$$

$(a = 2, ..., s)$, where the hat denotes the absence of the factor, must also be decomposable. In view of (ii), for $\lambda = \epsilon^1 \wedge ... \wedge \hat{\epsilon}^a \wedge ... \wedge \hat{\epsilon}^b \wedge ... \wedge \epsilon^s$ with $(b \neq a)$, we have $i(\lambda)i(\epsilon^a)P = \pm e_b$, where $b = 2, ..., s$, and the sign depends on whether $a < b$ or $b < a$; the Plücker relation (1) yields

$$e_b \wedge (i(\epsilon^a)P) = \sum_{i=1}^{n-1} e_b \wedge (i(\epsilon^a)P'_i) \wedge P''_i = 0.$$

This implies $e_b \wedge (i(\epsilon^a)P'_i) = 0$ for $i = 1, ..., s-1$, and the $(n-i-1)$-vector $i(\epsilon^a)P'_i$ belongs to the ideal generated by $e_2 \wedge ... \wedge \hat{e}_a \wedge ... \wedge e_s$. Therefore, $i(\epsilon^a)P'_i = 0$, except for $i = 1$, and, using again (ii), $i(\epsilon^a)P'_1 = \kappa e_2 \wedge ... \wedge \hat{e}_a \wedge ... \wedge e_s$ for some real κ. Accordingly, $P'_1 = (-1)^{a-1}\kappa e_2 \wedge ... \wedge e_a \wedge ... \wedge e_n$, $P'_2 = 0, \ldots, P'_{s-1} = 0$, and we deduce

$$P = e_2 \wedge ... \wedge e_s \wedge ((-1)^{s-1} e_1 + (-1)^{a-1} P''_1) + P''_s. \qquad (iii)$$

In other words, P is reducible. But, then, if we take $\alpha = \beta + \gamma \in V^*$, where β vanishes on the second term of (iii) but not on the first, and γ vanishes on the first term but not on the second, we see that $i(\alpha)P$ is not decomposable unless $P''_s = 0$. Hence, P is decomposable.

(4) Another proof using representation theory will be given below. Here we prove it by induction on s. Let $s = 3$. Suppose that $i_\alpha P \wedge P = 0$ for all $\alpha \in V^*$. Then for all $\beta \in V^*$ we have $0 = i_\beta(i_\alpha P \wedge P) = i_{\alpha\wedge\beta} P \wedge P + i_\alpha P \wedge i_\beta P$. Interchange α and β in the last expression and add it to the original, then we get $0 = 2 i_\alpha P \wedge i_\beta P$ and in turn $i_{\alpha\wedge\beta} P \wedge P = 0$ for all α and β, which are the original Plücker relations, so P is decomposable. Now the induction step. Suppose that $P \in \Lambda^s V$ and that $i_{\alpha_1\wedge\cdots\wedge\alpha_{s-2}} P \wedge P = 0$ for all $\alpha_i \in V^*$. Then we have

$$0 = i_{\alpha_1}(i_{\alpha_1\wedge\cdots\wedge\alpha_{s-2}} P \wedge P) = i_{\alpha_1\wedge\cdots\wedge\alpha_{s-2}} P \wedge i_{\alpha_1} P = i_{\alpha_2\wedge\cdots\wedge\alpha_{s-2}}(i_{\alpha_1}P) \wedge (i_{\alpha_1}P)$$

for all α_i, so that by induction we may conclude that $i_{\alpha_1}P$ is decomposable for all α_1, and then by (3) P is decomposable.

(5) Again this follows by duality. □

Let us note that the following result (Lemma 1 in [2]), a version of the 'three plane lemma' also implies (3):

Let $\{P_i : i \in I\}$ be a family of decomposable non-zero k-vectors in V such that each $P_i + P_j$ is again decomposable. Then

(a) either the linear span W of the linear subspaces $W(P_i) = \mathrm{Im}(\sharp_{P_i})$ is at most $(k+1)$-dimensional
(b) or the intersection $\bigcap_{i \in I} W(P_i)$ is at least $(k-1)$-dimensional.

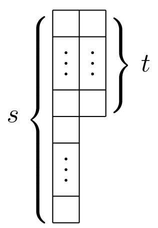
Finally note that (1) and (4) are both invariant under $\mathrm{GL}(V)$. In the next section we shall decompose (1) into its irreducible components in this representation.

If $\dim V$ is high enough in comparison with s, then (4) seemingly comprises less equations.

2. Representation theory

In order efficiently to analyse (1) and (4) it is necessary to take a small excursion through representation theory. An extensive discussion of Young tableau may be found in [1]. Here we shall just need

$Y^{s,t} \equiv$



regarded as irreducible representations of $\mathrm{GL}(V)$. Then, as special cases of the Littlewood-Richardson rules, we have

$$\begin{aligned} \Lambda^s V \otimes \Lambda^s V &= Y^{s,s} \oplus Y^{s+1,s-1} \oplus Y^{s+2,s-2} \oplus Y^{s+3,s-3} \oplus \cdots \oplus Y^{2s,0} \\ \Lambda^{s+1} \otimes \Lambda^{s-1} V &= Y^{s+1,s-1} \oplus Y^{s+2,s-2} \oplus Y^{s+3,s-3} \oplus \cdots \oplus Y^{2s,0} \\ \Lambda^{s+2} \otimes \Lambda^{s-2} V &= Y^{s+2,s-2} \oplus Y^{s+3,s-3} \oplus \cdots \oplus Y^{2s,0} \end{aligned}$$

and from the first two of these (1) says that $P \otimes P \in Y^{s,s}$. In fact,

$$(\star\star) \qquad \begin{aligned} \Lambda^s V \odot \Lambda^s V &= Y^{s,s} \oplus Y^{s+2,s-2} \oplus \cdots \\ \Lambda^s V \wedge \Lambda^s V &= Y^{s+1,s-1} \oplus Y^{s+3,s-3} \oplus \cdots \end{aligned}$$

so we can also see (by looking at the index expressions) the equivalence of (1) and (4) without any calculation. Having decomposed $\Lambda^s V \odot \Lambda^s V$ into irreducibles, it behoves one to investigate the consequences of having each irreducible component of $P \otimes P$ vanish separately. The first of these gives us another improvement on the classical Plücker relations:

Theorem 2. *An s-form P is simple if and only if the component of $P \otimes P$ in $Y^{s+2,s-2}$ vanishes.*

Proof. The representation $Y^{s+2,s-2}$ may be realised as those tensors

$$T_{a_1 b_1 a_2 b_2 \ldots a_{s-2} b_{s-2} cdef}$$

which are symmetric in the pairs a_jb_j for $j = 1, 2, \ldots s-2$, skew in $cdef$, and have the property that symmetrising over any three indices gives zero. The corresponding Young projection of

$$P_{a_1a_2\ldots a_{s-2}cd}P_{b_1b_2\ldots b_{s-2}ef}$$

is obtained by skewing over $cdef$ and symmetrising over each of the pairs a_jb_j for $j = 1, 2, \ldots, s-2$. Its vanishing, therefore, is equivalent to the vanishing of

$$Q_{[cd}Q_{ef]} \quad \text{where } Q_{cd} = \alpha^{a_1}\beta^{a_2}\cdots\gamma^{a_{s-2}}P_{a_1a_2\ldots a_{s-2}cd}$$

for all $\alpha^a, \beta^a, \ldots, \gamma^a \in V^*$. According to (4), this means that Q_{cd} is simple. Therefore, the theorem is equivalent to criterion (3) of Theorem 1. □

Notice that this generally cuts down further the number of equations needed to characterise the simple s-vectors. The simplest instance of this is for 4-forms: P is simple if and only if

$$P_{[abcd}P_{ef]gh} = P_{[abcd}P_{efgh]}.$$

Written in this way, it is slightly surprising that one can deduce the vanishing of each side of this equation separately. Theorem 2 is optimal in the sense that the vanishing of any other component or components in the irreducible decomposition $(\star\star)$ of $P \otimes P$ is either insufficient to force simplicity or causes P to vanish. In the case of four-forms, for example,

$$P_{[abcd}P_{efgh]} = 0$$

if $P = v \wedge Q$ for some vector v and three-form Q. On the other hand, if the $Y^{4,4}$ component of $P \otimes P$ vanishes, then arguing as in the proof of Theorem 2 shows that $P = 0$.

References

[1] W. Fulton, Young Tableau: with Applications to Representation Theory and Geometry, Cambridge University Press, 1997.

[2] J. Grabowski, G. Marmo, On Filippov algebroids and multiplicative Nambu-Poisson structures (to appear in Diff. Geom. Appl.), ESI preprint 668, math.DG/9902127.

[3] W. Greub, Multilinear Algebra, 2nd ed., Springer-Verlag, Berlin, 1978.

[4] P.A. Griffiths and J. Harris, Principles of Algebraic Geometry, 2nd ed., J. Willey & Sons, New York, 1994.

[5] W. Ślebodziński, Exterior forms and their applications, PWN-Polish Scientific Publishers, Warszawa, 1970.

[6] P.W. Michor and I. Vaisman, A note on n-ary Poisson brackets, ESI Preprint 663. math.SG/9901117.

[7] R. Weitzenböck, Invariantentheorie, P. Noordhoff, Groningen, 1923.

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