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A QUANTUM GROUP LIKE STRUCTURE ON NON COMMUTATIVE 2–TORI

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Introduction

This paper grew out of a project which aims at a general theory of bundles, and in particular principal bundles in non commutative differential geometry. The first important question in this direction is: What should replace the cartesian product, expressed in terms of the algebras representing the factors. Usually one takes the (topological) tensor product of the two algebras. The tensor product is the categorical coproduct in the category of commutative algebras, but does not have a good categorical interpretation in the non-commutative case: (coordinates on) the two factors commute; why should they?

One can as well consider algebras which have the same underlying vector space as the tensor product but are equipped with a deformation of the usual multiplication.

Clearly the question of cartesian products is also relevant for studying analogs of Lie groups in non commutative geometry, since the comultiplication mappings on the corresponding algebras represent a map from the cartesian product of the 'group' with itself to the 'group'.

In this paper we show that in the case of non commutative two tori one gets in a natural way simple structures which have analogous formal properties as Hopf algebra structures but with a deformed multiplication on the tensor product.

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1. Non commutative 2–tori

The non commutative 2–tori are probably the simplest example of non commutative algebras which are commonly thought to describe non commutative spaces. They are quite well studied and arise in several applications of non commutative geometry to physics, e.g. in J. Bellisard's interpretation of the quantum hall effect. We deal here with the smooth version of this algebras, so we consider the space $\mathcal{S}(\mathbb{Z}^2,\mathbb{C})$ of all complex valued Schwartz sequences (i.e. sequences which decay faster than any polynomial) $(a_{m,n})$ on $\mathbb{Z}^2$. Now for a complex number q of modulus 1 we define a multiplication on this space as follows: Let $\delta^{m,n}$ be the sequence which is one on (m,n) and zero on all other points. Then any element of $\mathcal{S}(\mathbb{Z}^2,\mathbb{C})$ can be written as $\sum a_{m,n}\delta^{m,n}$ (this is a convergent sum in the natural Fréchet topology on $\mathcal{S}(\mathbb{Z}^2,\mathbb{C})$), and we define the multiplication by $\delta^{m,n}\delta^{k,\ell} = q^{-kn}\delta^{m+k,n+\ell}$. Now we write $U := \delta^{1,0}$ and $V = \delta^{0,1}$, then by definition $\delta^{m,n} = U^mV^n$, and the only relevant relation is $UV = qVU$. Let us write T_q for the resulting algebra.

2. The comultiplication $\Delta_q : C^\infty(S^1,\mathbb{C}) \to T_q$

First it should be noted that the non commutative 2–tori themselves are deformations of the topological tensor product of the algebra $C^\infty(S^1,\mathbb{C}) \cong \mathcal{S}(\mathbb{Z},\mathbb{C})$ with itself, which corresponds to the case $q = 1$. In fact the natural comultiplication on $\mathcal{S}(\mathbb{Z},\mathbb{C})$ induced by the group structure on S^1 given by $(a_n) \mapsto \sum a_n(UV)^n$ makes sense as an algebra homomorphism $\Delta_q : \mathcal{S}(\mathbb{Z},\mathbb{C}) \to T_q$ for any q, so there is some kind of 'multiplication mapping' from any non commutative 2–torus to S^1. It can be even shown that this comultiplication is coassociative in a similar sense as we will use below, but there are no counit and antipode mappings fitting to this comultiplication.

3. The quantum group like structure on T_q

3.1. By the general principles of non commutative geometry the algebra T_q should be a description of the orbit space $S^1/\mathbb{Z}$, where the action of $\mathbb{Z}$ on S^1 is induced by multiplication by q. But this space clearly is a group so this should be reflected by some kind of coalgebra structure on the algebra T_q.

First we construct the algebra which we consider as a representative of the cartesian product of the non commutative torus described by T_q with itself. As indicated in the introduction we take as the underlying vector space of this algebra the tensor product, i.e. the space $\mathcal{S}(\mathbb{Z}^4,\mathbb{C})$ of Schwartz sequences on $\mathbb{Z}^4$. As above we write $\delta^{k,\ell,m,n}$ for the sequence which is one on (k,ℓ,m,n) and zero everywhere else. Then we define a multiplication on $\mathcal{S}(\mathbb{Z}^4,\mathbb{C})$ by:

$$\delta^{k_1,\ell_1,m_1,n_1}\delta^{k_2,\ell_2,m_2,n_2} = q^{\frac{k_2n_1}{2}-k_2\ell_1-\frac{m_1\ell_2}{2}-m_1n_2}\delta^{k_1+k_2,\ell_1+\ell_2,m_1+m_2,n_1+n_2}$$

Writing $U_1 := \delta^{1,0,0,0}$, $V_1 := \delta^{0,1,0,0}$, $U_2 := \delta^{0,0,1,0}$ and $V_2 := \delta^{0,0,0,1}$ we see that $\delta^{k,\ell,m,n} = U_1^k V_1^\ell U_2^m V_2^n$ and the relations between these generators are:

$$\begin{aligned} U_1V_1 &= qV_1U_1 & U_2V_2 &= qV_2U_2 \\ U_1V_2 &= q^{-1/2}V_2U_1 & V_1U_2 &= q^{1/2}U_2V_1 \\ U_1U_2 &= U_2U_1 & V_1V_2 &= V_2V_1 \end{aligned}$$

Let us remark a little on these relations. Clearly we should have two canonical copies of the original algebra T_q contained, so the relation between U_1 and V_1 as well as the one between U_2 and V_2 is clear. Then it turns out that all other relations fall out of the further development if one assumes that any product of two of the generators is a scalar multiple of the opposite product. We write P_q^2 for the resulting algebra in the sequel.

3.2. Now we define the comultiplication $\Delta : T_q \to P_q^2$ by $\Delta(U) = U_1U_2$ and $\Delta(V) = V_1V_2$. Then this induces an algebra homomorphism, which is obviously continuous, since

$$U_1U_2V_1V_2 = q^{-1/2}U_1V_1U_2V_2 = q^{3/2}V_1U_1V_2U_2 = qV_1V_2U_1U_2.$$

To formulate coassociativity we need a representative of the three fold product of a non commutative torus with itself. Again we want this to be a deformation of the third tensor power of T_q, so the underlying vector space is $\mathcal{S}(\mathbb{Z}^6, \mathbb{C})$. We write down this algebra only in terms of generators and relations, so we write a Schwartz sequence as $\sum a_{i,j,k,\ell,m,n} U_1^i V_1^j U_2^k V_2^\ell U_3^m V_3^n$. The choice of relations in this algebra is dictated by requiring the same deformation which led to P_q^2 in the first and second factor as well as in the second and third factor and no additional deformations. This gives the following relations:

$$\begin{aligned} U_1V_1 &= qV_1U_1 & U_2V_2 &= qV_2U_2 & U_3V_3 &= qV_3U_3 \\ U_1V_2 &= q^{-1/2}V_2U_1 & U_2V_3 &= q^{-1/2}V_3U_2 & U_3V_1 &= V_1U_3 \\ V_1U_2 &= q^{1/2}U_2V_1 & V_2U_3 &= q^{1/2}U_3V_2 & V_3U_1 &= U_1V_3 \\ U_1U_2 &= U_2U_1 & U_2U_3 &= U_3U_2 & U_3U_1 &= U_1U_3 \\ V_1V_2 &= V_2V_1 & V_2V_3 &= V_3V_2 & V_3V_1 &= V_1V_3 \end{aligned}$$

We write P_q^3 for the resulting algebra.

3.3. Taking into account the canonical vector space isomorphisms $P_q^2 \cong T_q \hat{\otimes} T_q$ and $P_q^3 \cong T_q \hat{\otimes} T_q \hat{\otimes} T_q$, where $\hat{\otimes}$ denotes the projective tensor product, we get continuous linear maps (Δ, Id) and (Id, Δ) from P_q^2 to P_q^3 which are induced by $\Delta\hat{\otimes}Id$ and $Id\hat{\otimes}\Delta$, respectively. Now it turns out that in this setting these maps are even algebra homomorphisms. We show this only for (Δ, Id), the proof for (Id, Δ) is completely analogous.

On the generators of P_q^2 the map (Δ, Id) is given by

$$U_1 \mapsto U_1U_2 \quad V_1 \mapsto V_1V_2 \quad U_2 \mapsto U_3 \quad V_2 \mapsto V_3,$$

and this induces an algebra homomorphism since

$$\begin{aligned} U_1U_2V_3 &= q^{-1/2}U_1V_3U_2 = q^{-1/2}V_3U_1U_2 \\ V_1V_2U_3 &= q^{1/2}V_1U_3V_2 = q^{1/2}U_3V_1V_2 \\ U_1U_2U_3 &= U_3U_1U_2 \\ V_1V_2V_3 &= V_3V_1V_2 \end{aligned}$$

But then coassociativity is obvious since both $(\Delta, Id)\circ\Delta$ and $(Id, \Delta)\circ\Delta$ are continuous algebra homomorphisms $T_q \to P_q^3$ which map U to $U_1U_2U_3$ and V to $V_1V_2V_3$.

3.4. Let us next turn to the counit $\varepsilon : T_q \to \mathbb{C}$. We cannot expect to get a counit which is an algebra homomorphism since there are no nonzero homomorphisms from T_q to a commutative algebra. This can be interpreted as the fact that the non commutative torus has no classical points. So we define ε as a linear map by $\varepsilon(U^kV^\ell) := q^{\frac{k\ell}{2}}$. Clearly this defines a continuous linear mapping. Again using the canonical vector space isomorphism $P_q^2 \cong T_q\hat{\otimes}T_q$ we get continuous linear mappings (ε, Id) and (Id, ε) from P_q^2 to T_q. These maps are given by $(\varepsilon, Id)(U_1^kV_1^\ell U_2^mV_2^n) = q^{\frac{k\ell}{2}}U^mV^n$ and by $(Id, \varepsilon)(U_1^kV_1^\ell U_2^mV_2^n) = q^{\frac{mn}{2}}U^kV^\ell$. To prove that ε is in fact a counit for the comultiplication Δ we have to show that both $(\varepsilon, Id)\circ\Delta$ and $(Id, \varepsilon)\circ\Delta$ are the identity map. Since these are continuous linear maps it suffices to check this on elements of the form U^kV^ℓ. But

$$\Delta(U^kV^\ell) = (U_1U_2)^k(V_1V_2)^\ell = U_1^kU_2^kV_1^\ell V_2^\ell = q^{-\frac{k\ell}{2}}U_1^kV_1^\ell U_2^kV_2^\ell,$$

so the result is obvious.

3.5. Finally we define the antipode map $S : T_q \to T_q$. The obvious choice is $S(U) := U^{-1}$ and $S(V) := V^{-1}$. Then this extends to a continuous algebra homomorphism since $U^{-1}V^{-1} = qV^{-1}U^{-1}$. So in contrast to the case of Hopf algebras we get an antipode which is a homomorphism and not an anti homomorphism. This seems rather positive since algebra homomorphisms should correspond to mappings of the underlying 'spaces' while the meaning of anti homomorphisms is rather unclear.

As before we get continuous linear mappings (S, Id) and (Id, S) from P_q^2 to itself using the vector space isomorphism with the tensor product. These maps are given by $U_1^k V_1^\ell U_2^m V_2^n \mapsto U_1^{-k} V_1^{-\ell} U_2^m V_2^n$ and $U_1^k V_1^\ell U_2^m V_2^n \mapsto U_1^k V_1^\ell U_2^{-m} V_2^{-n}$, respectively, and they are not algebra homomorphisms (since the multiplication on P_q^2 is twisted). Moreover the same method leads to a continuous linear map $\mu : P_q^2 \to T_q$ which represents the multiplication of T_q and is given by $\mu(U_1^k V_1^\ell U_2^m V_2^n) = U^k V^\ell U^m V^n = q^{-\ell m} U^{k+m} V^{\ell+n}$.

To prove that the antipode is really an analog of a group inversion we have to show that $(\mu \circ (S, Id) \circ \Delta)(x) = \varepsilon(x) \cdot 1$ and likewise with (S, Id) replaced by (Id, S). Since all these maps are linear and continuous it suffices to check this on elements of the form $U^k V^\ell$. But for these we get:

$$\begin{aligned}
(\mu \circ (S, Id) \circ \Delta)(U^k V^\ell) &= (\mu \circ (S, Id))(q^{-\frac{k\ell}{2}} U_1^k V_1^\ell U_2^k V_2^\ell) \\
&= \mu(q^{-\frac{k\ell}{2}} U_1^{-k} V_1^{-\ell} U_2^k V_2^\ell) \\
&= q^{-\frac{k\ell}{2}} q^{k\ell} = q^{\frac{k\ell}{2}} = \varepsilon(U^k V^\ell) \cdot 1
\end{aligned}$$

and likewise with (Id, S).

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