Annals of Global Analysis and Geometry
Vol. **8**, No. 3 (1990), 299–313

THE MOMENT MAPPING FOR UNITARY REPRESENTATIONS

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Abstract. For any unitary representation of an arbitrary Lie group I construct a moment mapping from the space of smooth vectors of the representation into the dual of the Lie algebra. This moment mapping is equivariant and smooth. For the space of analytic vectors the same construction is possible and leads to a real analytic moment mapping.

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1. Introduction

With the help of the cartesian closed calculus for smooth mappings as explained in [F-K] we can show, that for any Lie group and for any unitary representation its restriction to the space of smooth vectors is smooth. The imaginary part of the hermitian inner product restricts to a "weak" symplectic structure on the vector space of smooth vectors. This gives rise to the Poisson bracket on a suitably chosen space of smooth functions on the space of smooth vectors. The derivative of the representation on the space of smooth vectors is a symplectic action of the Lie algebra, which can be lifted to a Hamiltonian action, i.e. a Lie algebra homomorphism from the Lie algebra into the function space with the Poisson bracket. This in turn gives rise to the moment mapping from the space of smooth vectors into the dual of the Lie algebra.

1991 *Mathematics Subject Classification.* 22E45.
Key words and phrases. Unitary representation, moment mapping.

In [K-M] the cartesian closed setting for real analytic mappings in infinite dimensions is fully developed. With its help it can be shown that the moment mapping restricts to a real analytic mapping from the subspace of analytic vectors into the dual of the Lie algebra.

For an irreducible representation which is constructed by geometric quantization of an coadjoint orbit (the Kirillov method), the restriction of the moment mapping to the intersection of the unit sphere with the space of smooth vectors takes values has as image exactly the convex hull of the orbit one started with, if the construction is suitably normalized. This has been proved by Wildberger [Wil]. I thank J. Hilgert for bringing this paper to my attention.

Let me add some thoughts on the rôle of the moment mapping in the study of unitary representations. I think that its restriction to the intersection of the unit sphere with the space of smooth vectors maps to the convex hull of one coadjoint orbit, if the representation is irreducible (I was unable to prove this). It is known that not all irreducible representations come from line bundles over coadjoint orbits (alias geometric quantization), but there might be a higher dimensional vector bundle over this coadjoint orbit, whose space of sections contains the space of smooth vectors as subspace of sections which are covariantly constant along some complex polarization.

For the convenience of the reader I have added three sections on the smooth, holomorphic and real analytic setting, which is used in the rest of the paper. These sections are of review character.

2. Calculus of smooth mappings

2.1. The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces a whole flock of different theories were developed, each of them rather complicated and none really convincing. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. This was the original motivation for the development of a whole new field within general topology, convergence spaces.

Then in 1982, Alfred Frölicher and Andreas Kriegl presented independently the solution to the question for the right differential calculus in infinite dimensions. They joined forces in the further development of the theory and the (up to now) final outcome is the book [F-K].

In this section I will sketch the basic definitions and the most important results of the Frölicher-Kriegl calculus.

2.2. The c^∞-topology. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).

(3) $\{E_B \to E : B \text{ bounded absolutely convex in } E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

2.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent (completeness) conditions is satisfied:

(1) Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges. This is also called c^∞-complete.
(2) If B is bounded closed absolutely convex, then E_B is a Banach space.
(3) Any Lipschitz curve in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1' = c_2$ (existence of antiderivative).

2.4. Lemma. *Let E be a locally convex space. Then the following properties are equivalent:*

(1) *E is c^∞-complete.*
(2) *If $f : \mathbb{R}^k \to E$ is scalarwise* Lip^k*, then f is* Lip^k*, for $k > 1$.*
(3) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.*
(4) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.*

Here a mapping $f : \mathbb{R}^k \to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}^n$. f scalarwise C^∞ means that $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

This lemma says that a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

2.5. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mapping from E to F.

For E and F finite dimensional this gives the usual notion of smooth mappings: this has been first proved in [Bo]. Constant mappings are smooth. Multilinear mappings are smooth if and only if they are bounded. Therefore we denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

2.6. Structure on $C^\infty(E, F)$. We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$.

2.7. Lemma. *For locally convex spaces E and F we have:*

(1) *If F is convenient, then also $C^\infty(E,F)$ is convenient, for any E. The space $L(E,F)$ is a closed linear subspace of $C^\infty(E,F)$, so it also convenient.*
(2) *If E is convenient, then a curve $c:\mathbb{R}\to L(E,F)$ is smooth if and only if $t\mapsto c(t)(x)$ is a smooth curve in F for all $x\in E$.*

2.8. Theorem. *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E\times F,G)\cong C^\infty(E,C^\infty(F,G)),$$

which is even a diffeomorphism.

Of coarse this statement is also true for c^∞-open subsets of convenient vector spaces.

2.9. Corollary. *Let all spaces be convenient vector spaces. Then the following canonical mappings are smooth.*

$$\begin{aligned}
&\operatorname{ev}: C^\infty(E,F)\times E\to F,\quad \operatorname{ev}(f,x)=f(x)\\
&\operatorname{ins}: E\to C^\infty(F,E\times F),\quad \operatorname{ins}(x)(y)=(x,y)\\
&(\quad)^\wedge : C^\infty(E,C^\infty(F,G))\to C^\infty(E\times F,G)\\
&(\quad)^\vee : C^\infty(E\times F,G)\to C^\infty(E,C^\infty(F,G))\\
&\operatorname{comp}: C^\infty(F,G)\times C^\infty(E,F)\to C^\infty(E,G)\\
&C^\infty(\quad,\quad): C^\infty(F,F')\times C^\infty(E',E)\to C^\infty(C^\infty(E,F),C^\infty(E',F'))\\
&\qquad (f,g)\mapsto (h\mapsto f\circ h\circ g)\\
&\prod : \prod C^\infty(E_i,F_i)\to C^\infty(\prod E_i,\prod F_i)
\end{aligned}$$

2.10. Theorem. *Let E and F be convenient vector spaces. Then the differential operator*

$$\begin{aligned}
d: C^\infty(E,F)&\to C^\infty(E,L(E,F)),\\
df(x)v&:=\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t},
\end{aligned}$$

exists and is linear and bounded (smooth). Also the chain rule holds:

$$d(f\circ g)(x)v=df(g(x))dg(x)v.$$

2.11. Remarks. Note that the conclusion of theorem 2.8 is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more.

If one wants theorem 2.8 to be true and assumes some other obvious properties, then the calculus of smooth functions is already uniquely determined.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

For Fréchet spaces smoothness in the sense described here coincides with the notion C^∞_c of [Ke]. This is the differential calculus used by [Mic], [Mil], and [P-S].

3. Calculus of holomorphic mappings

3.1. Along the lines of thought of the Frölicher-Kriegl calculus of smooth mappings, in [K-N] the cartesian closed setting for holomorphic mappings was developed. We will now sketch the basics and the main results. It can be shown that again convenient vector spaces are the right ones to consider. Here we will start with them for the sake of shortness.

3.2. Let E be a complex locally convex vector space whose underlying real space is convenient – this will be called convenient in the sequel. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and let us denote by $\mathcal{H}(\mathbb{D}, E)$ the space of all mappings $c : \mathbb{D} \to E$ such that $\lambda \circ c : \mathbb{D} \to \mathbb{C}$ is holomorphic for each continuous complex-linear functional λ on E. Its elements will be called the holomorphic curves.

If E and F are convenient complex vector spaces (or c^∞-open sets therein), a mapping $f : E \to F$ is called *holomorphic* if $f \circ c$ is a holomorphic curve in F for each holomorphic curve c in E. Obviously f is holomorphic if and only if $\lambda \circ f : E \to \mathbb{C}$ is holomorphic for each complex linear continuous functional λ on F. Let $\mathcal{H}(E, F)$ denote the space of all holomorphic mappings from E to F.

3.3. Theorem (Hartog's theorem). *Let E_k for $k = 1, 2$ and F be complex convenient vector spaces and let $U_k \subset E_k$ be c^∞-open. A mapping $f : U_1 \times U_2 \to F$ is holomorphic if and only if it is separately holomorphic (i. e. $f(\quad, y)$ and $f(x, \quad)$ are holomorphic for all $x \in U_1$ and $y \in U_2$).*

This implies also that in finite dimensions we have recovered the usual definition.

3.4 Lemma. *If $f : E \supset U \to F$ is holomorphic then $df : U \times E \to F$ exists, is holomorphic and $\mathbb{C}$-linear in the second variable.*

A multilinear mapping is holomorphic if and only if it is bounded.

3.5 Lemma. *If E and F are Banach spaces and U is open in E, then for a mapping $f : U \to F$ the following conditions are equivalent:*

(1) *f is holomorphic.*
(2) *f is locally a convergent series of homogeneous continuous polynomials.*
(3) *f is $\mathbb{C}$-differentiable in the sense of Fréchet.*

3.6 Lemma. *Let E and F be convenient vector spaces. A mapping $f : E \to F$ is holomorphic if and only if it is smooth and its derivative is everywhere $\mathbb{C}$-linear.*

An immediate consequence of this result is that $\mathcal{H}(E, F)$ is a closed linear subspace of $C^\infty(E_\mathbb{R}, F_\mathbb{R})$ and so it is a convenient vector space if F is one, by 2.7. The chain rule follows from 2.10. The following theorem is an easy consequence of 2.8.

3.7 Theorem. *The category of convenient complex vector spaces and holomorphic mappings between them is cartesian closed, i. e.*

$$\mathcal{H}(E\times F,G)\cong\mathcal{H}(E,\mathcal{H}(F,G)).$$

An immediate consequence of this is again that all canonical structural mappings as in 2.9 are holomorphic.

4. Calculus of real analytic mappings

4.1. In this section we sketch the cartesian closed setting to real analytic mappings in infinite dimension following the lines of the Frölicher-Kriegl calculus, as it is presented in [K-M]. Surprisingly enough one has to deviate from the most obvious notion of real analytic curves in order to get a meaningful theory, but again convenient vector spaces turn out to be the right kind of spaces.

4.2. Real analytic curves. Let E be a real convenient vector space with dual E'. A curve $c:\mathbb{R}\to E$ is called *real analytic* if $\lambda\circ c:\mathbb{R}\to\mathbb{R}$ is real analytic for each $\lambda\in E'$. It turns out that the set of these curves depends only on the bornology of E.

In contrast a curve is called *topologically real analytic* if it is locally given by power series which converge in the topology of E. They can be extended to germs of holomorphic curves along $\mathbb{R}$ in the complexification $E_{\mathbb{C}}$ of E. If the dual E' of E admits a Baire topology which is compatible with the duality, then each real analytic curve in E is in fact topologically real analytic for the bornological topology on E.

4.3. Real analytic mappings. Let E and F be convenient vector spaces. Let U be a c^∞-open set in E. A mapping $f:U\to F$ is called *real analytic* if and only if it is smooth (maps smooth curves to smooth curves) and maps real analytic curves to real analytic curves.

Let $C^\omega(U,F)$ denote the space of all real analytic mappings. We equip the space $C^\omega(U,\mathbb{R})$ of all real analytic functions with the initial topology with respect to the families of mappings

$$C^\omega(U,\mathbb{R})\xrightarrow{c^*}C^\omega(\mathbb{R},\mathbb{R}),\ \text{for all } c\in C^\omega(\mathbb{R},U)$$
$$C^\omega(U,\mathbb{R})\xrightarrow{c^*}C^\infty(\mathbb{R},\mathbb{R}),\ \text{for all } c\in C^\infty(\mathbb{R},U),$$

where $C^\infty(\mathbb{R},\mathbb{R})$ carries the topology of compact convergence in each derivative separately as in section 2, and where $C^\omega(\mathbb{R},\mathbb{R})$ is equipped with the final locally convex topology with respect to the embeddings (restriction mappings) of all spaces of holomorphic mappings from a neighborhood V of $\mathbb{R}$ in $\mathbb{C}$ mapping $\mathbb{R}$ to $\mathbb{R}$, and each of these spaces carries the topology of compact convergence.

Furthermore we equip the space $C^\omega(U,F)$ with the initial topology with respect to the family of mappings

$$C^\omega(U,F)\xrightarrow{\lambda_*}C^\omega(U,\mathbb{R}),\ \text{for all } \lambda\in F'.$$

It turns out that this is again a convenient space.

4.4. Theorem. *In the setting of 4.3 a mapping $f : U \to F$ is real analytic if and only if it is smooth and is real analytic along each affine line in E.*

4.5. Lemma. *The space $L(E, F)$ of all bounded linear mappings is a closed linear subspace of $C^\omega(E, F)$. A mapping $f : U \to L(E, F)$ is real analytic if and only if $\mathrm{ev}_x \circ f : U \to F$ is real analytic for each point $x \in E$.*

4.6. Theorem. *The category of convenient spaces and real analytic mappings is cartesian closed. So the equation*

$$C^\omega(U, C^\omega(V, F)) \cong C^\omega(U \times V, F)$$

is valid for all c^∞-open sets U in E and V in F, where E, F, and G are convenient vector spaces.

This implies again that all structure mappings as in 2.9 are real analytic. Furthermore the differential operator

$$d : C^\omega(U, F) \to C^\omega(U, L(E, F))$$

exists, is unique and real analytic. Multilinear mappings are real analytic if and only if they are bounded. Powerful real analytic uniform boundedness principles are available.

5. The Space of Smooth Vectors

5.1. Let G be any (finite dimensional second countable) real Lie group, and let $\rho : G \to U(\mathbf{H})$ be a unitary representation on a Hilbert space $\mathbf{H}$. Then the associated mapping $\hat\rho : G \times \mathbf{H} \to \mathbf{H}$ is in general *not* jointly continuous, it is only separately continuous, so that $g \mapsto \rho(g)x$, $G \to \mathbf{H}$, is continuous for any $x \in \mathbf{H}$.

Definition. A vector $x \in \mathbf{H}$ is called *smooth* (or *real analytic*) if the mapping $g \mapsto \rho(g)x$, $G \to \mathbf{H}$ is smooth (or real analytic). Let us denote by $\mathbf{H}_\infty$ the linear subspace of all smooth vectors in $\mathbf{H}$. Then we have an embedding $j : \mathbf{H}_\infty \to C^\infty(G, \mathbf{H})$, given by $x \mapsto (g \mapsto \rho(g)x)$. We equip $C^\infty(G, \mathbf{H})$ with the compact C^∞-topology (of uniform convergence on compact subsets of G, in all derivatives separately). Then it is easily seen (and proved in [Wa, p 253]) that $\mathbf{H}_\infty$ is a closed linear subspace. So with the induced topology $\mathbf{H}_\infty$ becomes a Frèchet space. Clearly $\mathbf{H}_\infty$ is also an invariant subspace, so we have a representation $\rho : G \to L(\mathbf{H}_\infty, \mathbf{H}_\infty)$. For more detailed information on $\mathbf{H}_\infty$ see [Wa, chapt. 4.4.] or [Kn, chapt. III.].

5.2. Theorem. *The mapping $\hat\rho : G \times \mathbf{H}_\infty \to \mathbf{H}_\infty$ is smooth in the sense of Frölicher-Kriegl.*

Proof. By cartesian closedness 2.8 it suffices to show that the canonically associated mapping

$$\hat\rho^\vee : G \to C^\infty(\mathbf{H}_\infty, \mathbf{H}_\infty)$$

is smooth; but it takes values in the closed subspace $L(\mathbf{H}_\infty, \mathbf{H}_\infty)$ of all bounded linear operators. So by it suffices to show that the mapping $\rho : G \to L(\mathbf{H}_\infty, \mathbf{H}_\infty)$ is smooth.

But for that, since $\mathbf{H}_\infty$ is a Frèchet space and thus convenient, by 2.7(2) it suffices to show that

$$G \xrightarrow{\rho} L(\mathbf{H}_\infty, \mathbf{H}_\infty) \xrightarrow{ev_x} \mathbf{H}_\infty$$

is smooth for each $x \in \mathbf{H}_\infty$. This requirement means that $g \mapsto \rho(g)x$, $G \to \mathbf{H}_\infty$, is smooth. For this it suffices to show that

$$\begin{aligned}
&G \to \mathbf{H}_\infty \xrightarrow{j} C^\infty(G, \mathbf{H}),\\
&g \mapsto \rho(g)x \mapsto (h \mapsto \rho(h)(g)x),
\end{aligned}$$

is smooth. But again by cartesian closedness it suffices to show that the associated mapping

$$\begin{aligned}
&G \times G \to \mathbf{H},\\
&(g, h) \mapsto \rho(h)(g)x = \rho(hg)x,
\end{aligned}$$

is smooth. And this is the case since x is a smooth vector. □

6. The model for the moment mapping

6.1. We now consider $\mathbf{H}_\infty$ as a "weak" symplectic Frèchet manifold, equipped with the symplectic structure Ω, the restriction of the imaginary part of the Hermitian inner product $\langle \quad , \quad \rangle$ on $\mathbf{H}$. Then $\Omega \in \Omega^2(\mathbf{H}_\infty)$ is a closed 2-form which is non degenerate in the sense that

$$\check{\Omega} : T\mathbf{H}_\infty = \mathbf{H}_\infty \times \mathbf{H}_\infty \to T^*\mathbf{H}_\infty = \mathbf{H}_\infty \times \mathbf{H}_\infty{}'$$

is injective (but not surjective), where $\mathbf{H}_\infty{}' = L(\mathbf{H}_\infty, \mathbb{R})$ denotes the real topological dual space. This is the meaning of "weak" above.

6.2. Review. For a finite dimensional symplectic manifold (M, Ω) we have the following exact sequence of Lie algebras:

$$0 \to H^0(M) \to C^\infty(M) \xrightarrow{\operatorname{grad}^\Omega} \mathfrak{X}_\Omega(M) \xrightarrow{\gamma} H^1(M) \to 0$$

Here $H^*(M)$ is the real De Rham cohomology of M, the space $C^\infty(M)$ is equipped with the Poisson bracket $\{ \quad , \quad \}$, $\mathfrak{X}_\Omega(M)$ consists of all vector fields ξ with $\mathcal{L}_\xi \Omega = 0$ (the locally Hamiltonian vector fields), which is a Lie algebra for the Lie bracket. Also $\operatorname{grad}^\Omega f$ is the Hamiltonian vector field for $f \in C^\infty(M)$ given by $i(\operatorname{grad}^\Omega f)\Omega = df$, and $\gamma(\xi) = [i_\xi \Omega]$. The spaces $H^0(M)$ and $H^1(M)$ are equipped with the zero bracket.

Given a symplectic left action $\ell : G \times M \to M$ of a connected Lie group G on M, the first partial derivative of ℓ gives a mapping $\ell' : \mathfrak{g} \to \mathfrak{X}_\Omega(M)$ which sends each element X of the Lie algebra $\mathfrak{g}$ of G to the fundamental vector field. This is a Lie algebra homomorphism.

$$\begin{array}{ccccccc}
H^0(M) & \xrightarrow{i} & C^\infty(M) & \xrightarrow{\operatorname{grad}^\Omega} & \mathfrak{X}_\Omega(M) & \xrightarrow{\gamma} & H^1(M)\\
 & & {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle\ell'} & & \\
 & & \mathfrak{g} & = & \mathfrak{g} & &
\end{array}$$

A linear lift $\sigma : \mathfrak{g} \to C^\infty(M)$ of ℓ' with $\operatorname{grad}^\Omega \circ \sigma = \ell'$ exists if and only if $\gamma \circ \ell' = 0$ in $H^1(M)$. This lift σ may be changed to a Lie algebra homomorphism if and only if the 2-cocycle $\bar\sigma : \mathfrak{g} \times \mathfrak{g} \to H^0(M)$, given by $(i \circ \bar\sigma)(X,Y) = \{\sigma(X), \sigma(Y)\} - \sigma([X,Y])$, vanishes in $H^2(\mathfrak{g}, H^0(M))$, for if $\bar\sigma = \delta\alpha$ then $\sigma - i\circ\alpha$ is a Lie algebra homomorphism.

If $\sigma : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism, we may associate the *moment mapping* $\mu : M \to \mathfrak{g}' = L(\mathfrak{g}, \mathbb{R})$ to it, which is given by $\mu(x)(X) = \sigma(X)(x)$ for $x \in M$ and $X \in \mathfrak{g}$. It is G-equivariant for a suitably chosen (in general affine) action of G on $\mathfrak{g}'$. See [We] or [L-M] for all this.

7. Hamiltonian Mechanics on $\mathbf{H}_\infty$

7.1. We now want to carry over to the setting of 5.1 and 5.2 the procedure of 6.2. The first thing to note is that the hamiltonian mapping $\operatorname{grad}^\Omega : C^\infty(\mathbf{H}_\infty) \to \mathfrak{X}_\Omega(\mathbf{H}_\infty)$ does not make sense in general, since $\check\Omega : \mathbf{H}_\infty \to \mathbf{H}_\infty{}'$ is not invertible: $\operatorname{grad}^\Omega f = \check\Omega^{-1} df$ is defined only for those $f \in C^\infty(\mathbf{H}_\infty)$ with $df(x)$ in the image of $\check\Omega$ for all $x \in \mathbf{H}_\infty$. A similar difficulty arises for the definition of the Poisson bracket on $C^\infty(\mathbf{H}_\infty)$.

Let $\langle x, y\rangle = Re\langle x,y\rangle + \sqrt{-1}\Omega(x,y)$ be the decomposition of the hermitian inner product into real and imaginary parts. Then $Re\langle x,y\rangle = \Omega(\sqrt{-1}x, y)$, thus the real linear subspaces $\check\Omega(\mathbf{H}_\infty) = \Omega(\mathbf{H}_\infty, \quad)$ and $Re\langle \mathbf{H}_\infty, \quad\rangle$ of $\mathbf{H}_\infty{}' = L(\mathbf{H}_\infty, \mathbb{R})$ coincide.

7.2 Definition. Let $\mathbf{H}_\infty^*$ denote the real linear subspace

$$\mathbf{H}_\infty^* = \Omega(\mathbf{H}_\infty, \quad) = Re\langle \mathbf{H}_\infty, \quad\rangle$$

of $\mathbf{H}_\infty{}' = L(\mathbf{H}_\infty, \mathbb{R})$, and let us call it the *smooth dual* of $\mathbf{H}_\infty$ in view of the embedding of test functions into distributions. We have two canonical isomorphisms $\mathbf{H}_\infty^* \cong \mathbf{H}_\infty$ induced by Ω and $Re\langle\quad,\quad\rangle$, respectively. Both induce the same Fréchet topology on $\mathbf{H}_\infty^*$, which we fix from now on.

7.3 Definition. Let $C_*^\infty(\mathbf{H}_\infty, \mathbb{R}) \subset C^\infty(\mathbf{H}_\infty, \mathbb{R})$ denote the linear subspace consisting of all smooth functions $f : \mathbf{H}_\infty \to \mathbb{R}$ such that each iterated derivative $d^k f(x) \in L^k_{\text{sym}}(\mathbf{H}_\infty, \mathbb{R})$ has the property that

$$d^k f(x)(\quad, y_2, \dots, y_k) \in \mathbf{H}_\infty{}^*$$

is actually in the smooth dual $\mathbf{H}_\infty^* \subset \mathbf{H}_\infty{}'$ for all $x, y_2, \dots, y_k \in \mathbf{H}_\infty$, and that the mapping

$$\prod^k \mathbf{H}_\infty \to \mathbf{H}_\infty$$
$$(x, y_2, \dots, y_k) \mapsto \check\Omega^{-1}(df(x)(\quad, y_2, \dots, y_k))$$

is smooth. Note that we could also have used $Re\langle\quad,\quad\rangle$ instead of Ω. By the symmetry of higher derivatives this is then true for all entries of $d^k f(x)$, for all x.

7.4 Lemma. *For $f \in C^\infty(\mathbf{H}_\infty, \mathbb{R})$ the following assertions are equivalent:*

(1) *$df : \mathbf{H}_\infty \to {\mathbf{H}_\infty}'$ factors to a smooth mapping $\mathbf{H}_\infty \to \mathbf{H}_\infty^*$.*
(2) *f has a smooth Ω-gradient $\operatorname{grad}^\Omega f \in \mathfrak{X}(\mathbf{H}_\infty) = C^\infty(\mathbf{H}_\infty, \mathbf{H}_\infty)$ such that $df(x)y = \Omega(\operatorname{grad}^\Omega f(x), y)$.*
(3) *$f \in C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$.*

Proof. Clearly (3) $\Longrightarrow$ (2) $\Longleftrightarrow$ (1). We have to show that (2) $\Longrightarrow$ (3).

Suppose that $f : \mathbf{H}_\infty \to \mathbb{R}$ is smooth and $df(x)y = \Omega(\operatorname{grad}^\Omega f(x), y)$. Then

$$\begin{aligned} d^k f(x)(y_1, \dots, y_k) &= d^k f(x)(y_2, \dots, y_k, y_1) \\ &= (d^{k-1}(df)(x)(y_2, \dots, y_k)(y_1) \\ &= \Omega(d^{k-1}(\operatorname{grad}^\Omega f)(x)(y_2, \dots, y_k), y_1). \quad \square \end{aligned}$$

7.5. Theorem. *The mapping $\operatorname{grad}^\Omega : C^\infty_*(\mathbf{H}_\infty, \mathbb{R}) \to \mathfrak{X}_\Omega(\mathbf{H}_\infty)$, given by $\operatorname{grad}^\Omega f := \check{\Omega}^{-1} \circ df$, is well defined; also the Poisson bracket*

$$\begin{gathered} \{\ \ ,\ \ \} : C^\infty_*(\mathbf{H}_\infty, \mathbb{R}) \times C^\infty_*(\mathbf{H}_\infty, \mathbb{R}) \to C^\infty_*(\mathbf{H}_\infty, \mathbb{R}), \\ \{f, g\} := i(\operatorname{grad}^\Omega f) i(\operatorname{grad}^\Omega g)\Omega = \Omega(\operatorname{grad}^\Omega g, \operatorname{grad}^\Omega f) = \\ = (\operatorname{grad}^\Omega f)(g) = dg(\operatorname{grad}^\Omega f) \end{gathered}$$

is well defined and gives a Lie algebra structure to the space $C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$.

We also have the following long exact sequence of Lie algebras and Lie algebra homomorphisms:

$$0 \to H^0(\mathbf{H}_\infty) \to C^\infty_*(\mathbf{H}_\infty, \mathbb{R}) \xrightarrow{\operatorname{grad}^\Omega} \mathfrak{X}_\Omega(\mathbf{H}_\infty) \xrightarrow{\gamma} H^1(\mathbf{H}_\infty) = 0$$

Proof. It is clear from lemma 7.4, that the hamiltonian mapping is defined, and thus also the Poisson bracket is defined as a mapping $\{\ \ ,\ \ \} : C^\infty_*(\mathbf{H}_\infty, \mathbb{R}) \times C^\infty_*(\mathbf{H}_\infty, \mathbb{R}) \to C^\infty(\mathbf{H}_\infty, \mathbb{R})$, and it only remains to check that it has values in the subspace $C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$.

So let f, $g \in C^\infty_*(\mathbf{H}_\infty)$, then $\{f, g\}(x) = dg(x)(\operatorname{grad}^\Omega f(x))$ and by the symmetry of $dg(x)$ we have

$$\begin{aligned} d(\{f, g\})(x)y &= d^2 g(x)(y, \operatorname{grad}^\Omega f(x)) + dg(x)(d(\operatorname{grad}^\Omega f)(x)y) \\ &= \Omega\Big(d(\operatorname{grad}^\Omega g)(x)(\operatorname{grad}^\Omega f(x)), y\Big) \\ &\qquad + \Omega\Big(\operatorname{grad}^\Omega g(x), d(\operatorname{grad}^\Omega f)(x)y\Big) \\ &= \Omega\Big(d(\operatorname{grad}^\Omega g)(x)(\operatorname{grad}^\Omega f(x)) - d(\operatorname{grad}^\Omega f)(x)(\operatorname{grad}^\Omega g(x)), y\Big), \end{aligned}$$

since $\operatorname{grad}^\Omega f \in \mathfrak{X}_\Omega(\mathbf{H}_\infty)$ and for any $X \in \mathfrak{X}_\Omega(\mathbf{H}_\infty)$ the condition $\mathcal{L}_X \Omega = 0$ implies $\Omega(dX(x)y_1, y_2) = -\Omega(y_1, dX(x)y_2)$. So (2) of lemma 7.4 is satisfied and thus $\{f, g\} \in C^\infty_*(\mathbf{H}_\infty)$.

For the rest any coordinate free finite dimensional proof works. $\square$

8. The moment mapping for a unitary representation

8.1. We consider now again as in 5.1 a unitary representation $\rho : G \to U(\mathbf{H})$. By theorem 5.2 the associated mapping $\hat\rho : G \times \mathbf{H}_\infty \to \mathbf{H}_\infty$ is smooth, so we have the infinitesimal mapping $\rho' : \mathfrak{g} \to \mathfrak{X}(\mathbf{H}_\infty)$, given by $\rho'(X)(x) = T_e(\hat\rho(\quad, x))X$ for $X \in \mathfrak{g}$ and $x \in \mathbf{H}_\infty$. Since ρ is a unitary representation, the mapping ρ' has values in the Lie subalgebra of all linear hamiltonian vector fields $\xi \in \mathfrak{X}(\mathbf{H}_\infty)$ which respect the symplectic form Ω, i.e. $\xi : \mathbf{H}_\infty \to \mathbf{H}_\infty$ is linear and $\mathcal{L}_\xi \Omega = 0$.

Now let us consider the mapping $\check\Omega \circ \rho'(X) : \mathbf{H}_\infty \to T(\mathbf{H}_\infty) \to T^*(\mathbf{H}_\infty)$. We have $d(\check\Omega \circ \rho'(X)) = d(i_{\rho'(X)}\Omega) = \mathcal{L}_{\rho'(X)}\Omega = 0$, so the linear 1-form $\check\Omega \circ \rho'(X)$ is closed, and since $H^1(\mathbf{H}_\infty) = 0$, it is exact. So there is a function $\sigma(X) \in C^\infty(\mathbf{H}_\infty, \mathbb{R})$ with $d\sigma(X) = \check\Omega \circ \rho'(X)$, and $\sigma(X)$ is uniquely determined up to addition of a constant. If we require $\sigma(X)(0) = 0$, then $\sigma(X)$ is uniquely determined and is a quadratic function. In fact we have $\sigma(X)(x) = \int_{c_x} \check\Omega \circ \rho'(X)$, where $c_x(t) = tx$. Thus

$$\begin{aligned}\sigma(X)(x) &= \textstyle\int_0^1 \Omega(\rho'(X)(tx), \frac{d}{dt}tx)dt = \\ &= \Omega(\rho'(X)(x), x)\textstyle\int_0^1 t dt \\ &= \tfrac12 \Omega(\rho'(X)(x), x).\end{aligned}$$

8.2. Lemma. *The mapping*

$$\sigma : \mathfrak{g} \to C^\infty_*(\mathbf{H}_\infty, \mathbb{R}), \qquad \sigma(X)(x) = \tfrac12 \Omega(\rho'(X)(x), x)$$

for $X \in \mathfrak{g}$ and $x \in \mathbf{H}_\infty$, is a Lie algebra homomorphism and $\operatorname{grad}^\Omega \circ \sigma = \rho'$.

For $g \in G$ we have $\rho(g)^*\sigma(X) = \sigma(X) \circ \rho(g) = \sigma(Ad(g^{-1})X)$, so σ is G-equivariant.

Proof. First we have to check that $\sigma(X) \in C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$. Since $\rho'(X) : \mathbf{H}_\infty \to \mathbf{H}_\infty$ is smooth and linear, i.e. bounded linear, this follows from the formula for $\sigma(X)$. Furthermore

$$\begin{aligned}\operatorname{grad}^\Omega(\sigma(X))(x) &= \check\Omega^{-1}(d\sigma(X)(x)) = \\ &= \tfrac12 \check\Omega^{-1}\left(\Omega(\rho'(X)(\quad), x) + \Omega(\rho'(X)(x), \quad)\right) = \\ &= \check\Omega^{-1}\left(\Omega(\rho'(X)(x), \quad)\right) = \rho'(X)(x),\end{aligned}$$

since $\Omega(\rho'(X)(x), y) = \Omega(\rho'(X)(y), x)$.

Clearly $\sigma([X,Y]) - \{\sigma(X), \sigma(Y)\}$ is a constant function by 7.5; since it also vanishes at $0 \in \mathbf{H}_\infty$, the mapping $\sigma : \mathfrak{g} \to C^\infty_*(\mathbf{H}_\infty)$ is a Lie algebra homomorphism.

For the last assertion we have

$$\begin{aligned}\sigma(X)(\rho(g)x) &= \tfrac12 \Omega(\rho'(X)(\rho(g)x), \rho(g)x) \\ &= \tfrac12 (\rho(g)^*\Omega)(\rho(g^{-1})\rho'(X)(\rho(g)x), x) \\ &= \tfrac12 \Omega(\rho'(Ad(g^{-1})X)x, x) = \sigma(Ad(g^{-1})X)(x). \quad \square\end{aligned}$$

8.3. The moment mapping. For a unitary representation $\rho : G \to U(\mathbf{H})$ we can now define the *moment mapping*

$$\begin{aligned}\mu : \mathbf{H}_\infty &\to \mathfrak{g}' = L(\mathfrak{g}, \mathbb{R}),\\ \mu(x)(X) &:= \sigma(X)(x) = \tfrac{1}{2}\Omega(\rho'(X)x, x),\end{aligned}$$

for $x \in \mathbf{H}_\infty$ and $X \in \mathfrak{g}$.

8.4 Theorem. *The moment mapping $\mu : \mathbf{H}_\infty \to \mathfrak{g}'$ has the following properties:*

(1) $(d\mu(x)y)(X) = \Omega(\rho'(X)x, y)$ *for* $x, y \in \mathbf{H}_\infty$ *and* $X \in \mathfrak{g}$, *so* $\mu \in C^\infty_*(\mathbf{H}_\infty, \mathfrak{g}')$.

(2) *For* $x \in \mathbf{H}_\infty$ *the image of* $d\mu(x) : \mathbf{H}_\infty \to \mathfrak{g}'$ *is the annihilator* $\mathfrak{g}_x^\Omega$ *of the Lie algebra* $\mathfrak{g}_x = \{X \in \mathfrak{g} : \rho'(X)(x) = 0\}$ *of the isotropy group* $G_x = \{g \in G : \rho(g)x = x\}$ *in* $\mathfrak{g}'$.

(3) *For* $x \in \mathbf{H}_\infty$ *the kernel of* $d\mu(x)$ *is*

$$(T_x(\rho(G)x))^\Omega = \{y \in \mathbf{H}_\infty : \Omega(y, T_x(\rho(G)x)) = 0\},$$

the Ω*-annihilator of the tangent space at* x *of the* G*-orbit through* x.

(4) *The moment mapping is equivariant:* $Ad'(g) \circ \mu = \mu \circ \rho(g)$ *for all* $g \in G$, *where* $Ad'(g) = Ad(g^{-1})' : \mathfrak{g}' \to \mathfrak{g}'$ *is the coadjoint action.*

(5) *The pullback operator* $\mu^* : C^\infty(\mathfrak{g}, \mathbb{R}) \to C^\infty(\mathbf{H}_\infty, \mathbb{R})$ *actually has values in the subspace* $C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$. *It also is a Lie algebra homomorphism for the Poisson brackets involved.*

Proof. (1). Differentiating the defining equation we get

(a) $$(d\mu(x)y)(X) = \tfrac{1}{2}\Omega(\rho'(X)y, x) + \tfrac{1}{2}\Omega(\rho'(X)x, y) = \Omega(\rho'(X)x, y).$$

From lemma 7.4 we see that $\mu \in C^\infty_*(\mathbf{H}_\infty, \mathfrak{g}')$.

(2) and (3) are immediate consequences of this formula.

(4). We have

$$\begin{aligned}\mu(\rho(g)x)(X) &= \sigma(X)(\rho(g)x) = \sigma(Ad(g^{-1})X)(x) \text{ by lemma 8.2}\\ &= \mu(x)(Ad(g^{-1})X) = (Ad(g^{-1})'\mu(x))(X).\end{aligned}$$

(5). Let $f \in C^\infty(\mathfrak{g}', \mathbb{R})$, then we have

(b) $$\begin{aligned}d(\mu^* f)(x)y &= d(f \circ \mu)(x)y = df(\mu(x))d\mu(x)y\\ &= (d\mu(x)y)(df(\mu(x))) = \Omega(\rho'(df(\mu(x)))x, y)\end{aligned}$$

by (a), which is smooth in x as a mapping into $\mathbf{H}_\infty \cong \mathbf{H}^*_\infty \subset \mathbf{H}'_\infty$ since $\mathfrak{g}'$ is finite dimensional. From lemma 7.4 we have that $f \circ \mu \in C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$.

$$\Omega(\operatorname{grad}^\Omega(\mu^* f)(x), y) = d(\mu^* f)(x)y = \Omega(\rho'(df(\mu(x)))x, y)$$

by (b), so $\operatorname{grad}^\Omega(\mu^* f)(x) = \rho'(df(\mu(x)))x$. The Poisson structure on $\mathfrak{g}'$ is given as follows. We view the Lie bracket on $\mathfrak{g}$ as a linear mapping $\Lambda^2\mathfrak{g} \to \mathfrak{g}$; its adjoint

$P : \mathfrak{g}' \to \Lambda^2\mathfrak{g}'$ is then a section of the bundle $\Lambda^2 T\mathfrak{g}' \to \mathfrak{g}'$, which is called the Poisson structure on $\mathfrak{g}'$. If for $\alpha \in \mathfrak{g}'$ we view $df(\alpha) \in L(\mathfrak{g}', \mathbb{R})$ as an element in $\mathfrak{g}$, the Poisson bracket for $f_i \in C^\infty(\mathfrak{g}', \mathbb{R})$ is given by $\{f_1, f_2\}_{\mathfrak{g}'}(\alpha) = (df_1 \wedge df_2)(P)|_\alpha = \alpha([df_1(\alpha), df_2(\alpha)])$. Then we may compute as follows.

$$\begin{aligned}
(\mu^*\{f_1, f_2\}_{\mathfrak{g}'})(x) &= \{f_1, f_2\}_{\mathfrak{g}'}(\mu(x)) \\
&= \mu(x)([df_1(\mu(x)), df_2(\mu(x))]) \\
&= \sigma([df_1(\mu(x)), df_2(\mu(x))])(x) \\
&= \{\sigma(df_1(\mu(x))), \sigma(df_2(\mu(x)))\}(x) && \text{by lemma 8.2} \\
&= \Omega(\operatorname{grad}^\Omega \sigma(df_2(\mu(x)))(x), \operatorname{grad}^\Omega \sigma(df_1(\mu(x)))(x)) \\
&= \Omega(\rho'(df_2(\mu(x)))x, \rho'(df_1(\mu(x)))x) \\
&= \Omega(\operatorname{grad}^\Omega(\mu^* f_2)(x), \operatorname{grad}^\Omega(\mu^* f_1)(x)) && \text{by (b)} \\
&= \{\mu^* f_1, \mu^* f_2\}_{\mathbf{H}_\infty}(x). \quad \square
\end{aligned}$$

9. The real analytic moment mapping

9.1. Let again $\rho : G \to U(\mathbf{H})$ be a unitary representation of a Lie group G on a Hilbert space $\mathbf{H}$.

Definition. A vector $x \in \mathbf{H}$ is called it real analytic if the mapping $g \mapsto \rho(g)x$, $G \to \mathbf{H}$ is a real analytic mapping, in the real analytic structure of the Lie group G, in the setting explained in section 4.

Let $\mathbf{H}_\omega$ denote the vector space of all real analytic vectors in $\mathbf{H}$. Then we have a linear embedding $j : \mathbf{H}_\omega \to C^\omega(G, \mathbf{H})$ into the space of real analytic mappings, given by $x \mapsto (g \mapsto \rho(g)x)$. We equip $C^\omega(G, \mathbf{H})$ with the convenient vector space structure described in [K-M, 5.4, see also 3.13]. Then $\mathbf{H}_\omega$ consists of all equivariant functions in $C^\omega(G, \mathbf{H})$ and is therefore a closed subspace. So it is a convenient vector space with the induced structure.

The space $\mathbf{H}_\omega$ is dense in the Hilbert space $\mathbf{H}$ by [Wa, 4.4.5.7] and an invariant subspace, so we have a representation $\rho : G \to L(\mathbf{H}_\omega, \mathbf{H}_\omega)$.

9.2. Theorem. *The mapping $\hat\rho : G \times \mathbf{H}_\omega \to \mathbf{H}_\omega$ is real analytic in the sense of* [K-M].

Proof. By cartesian closedness of the calculus 4.6 it suffices to show that the canonically associated mapping

$$\hat\rho^\vee : G \to C^\omega(\mathbf{H}_\omega, \mathbf{H}_\omega)$$

is real analytic. It takes values in the closed linear subspace $L(\mathbf{H}_\omega, \mathbf{H}_\omega)$ of all bounded linear operators. So it suffices to check that the mapping $\rho : G \to L(\mathbf{H}_\omega, \mathbf{H}_\omega)$ is real analytic. Since $\mathbf{H}_\omega$ is a convenient space, by 4.5 it suffices to show that

$$G \xrightarrow{\rho} L(\mathbf{H}_\omega, \mathbf{H}_\omega) \xrightarrow{\mathrm{ev}_x} \mathbf{H}_\omega$$

is real analytic for each $x \in \mathbf{H}_\omega$. Since the structure on $\mathbf{H}_\omega$ is induced by the embedding into $C^\omega(G, \mathbf{H})$, we have to check, that

$$\begin{aligned} &G \xrightarrow{\rho} L(\mathbf{H}_\omega, \mathbf{H}_\omega) \xrightarrow{\mathrm{ev}_x} \mathbf{H}_\omega \xrightarrow{j} C^\omega(G, \mathbf{H}), \\ &g \mapsto \rho(g) \mapsto \rho(g)x \mapsto (h \mapsto \rho(h)\rho(g)x), \end{aligned}$$

is real analytic for each $x \in \mathbf{H}_\omega$. Again by cartesian closedness 4.6 it suffices that the associated mapping

$$\begin{aligned} &G \times G \to \mathbf{H} \\ &(g, h) \mapsto \rho(h)\rho(g)x = \rho(hg)x \end{aligned}$$

is real analytic. And this is the case since x is a real analytic vector. $\square$

9.3. Again we consider now $\mathbf{H}_\omega$ as a "weak" symplectic real analytic Fréchet manifold, equipped with the symplectic structure Ω, the restriction of the imaginary part of the hermitian inner product $\langle \quad , \quad \rangle$ on $\mathbf{H}$. Then again $\Omega \in \Omega^2(\mathbf{H}_\omega)$ is a closed 2-form which is non degenerate in the sense that $\check{\Omega} : \mathbf{H}_\omega \to \mathbf{H}'_\omega = L(\mathbf{H}_\omega, \mathbb{R})$ is injective. Let

$$\mathbf{H}^*_\omega := \check{\Omega}(\mathbf{H}_\omega) = \Omega(\mathbf{H}_\omega, \quad) = Re\langle \mathbf{H}_\omega, \quad \rangle \subset \mathbf{H}'_\omega = L(\mathbf{H}_\omega, \mathbb{R})$$

again denote the *analytic dual* of $\mathbf{H}_\omega$, equipped with the topology induced by the isomorphism with $\mathbf{H}_\omega$.

9.4 Remark. All the results leading to the smooth moment mapping can now be carried over to the real analytic setting with *no* changes in the proofs. So all statements from 7.5 to 8.4 are valid in the real analytic situation. We summarize this in one more result:

9.5 Theorem. *Consider the injective linear continuous G-equivariant mapping $i : \mathbf{H}_\omega \to \mathbf{H}_\infty$. Then for the smooth moment mapping $\mu : \mathbf{H}_\infty \to \mathfrak{g}'$ from 8.4 the composition $\mu \circ i : \mathbf{H}_\omega \to \mathbf{H}_\infty \to \mathfrak{g}'$ is real analytic. It is called the real analytic moment mapping.*

Proof. It is immediately clear from 9.2 and the formula 8.3 for the smooth moment mapping, that $\mu \circ i$ is real analytic. $\square$

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