THE GENERALIZED CAYLEY MAP FROM AN ALGEBRAIC GROUP TO ITS LIE ALGEBRA

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For Alexandre Kirillov, on the occasion of his 65-th anniversary

ABSTRACT. Each infinitesimally faithful representation of a reductive complex connected algebraic group G induces a dominant morphism Φ from the group to its Lie algebra $\mathfrak{g}$ by orthogonal projection in the endomorphism ring of the representation space. The map Φ identifies the field $Q(G)$ of rational functions on G with an algebraic extension of the field $Q(\mathfrak{g})$ of rational functions on $\mathfrak{g}$. For the spin representation of $\mathrm{Spin}(V)$ the map Φ essentially coincides with the classical Cayley transform. In general, properties of Φ are established and these properties are applied to deal with a separation of variables (Richardson) problem for reductive algebraic groups: Find $\mathrm{Harm}(G)$ so that for the coordinate ring $A(G)$ of G we have $A(G) = A(G)^G \otimes \mathrm{Harm}(G)$. As a consequence of a partial solution to this problem and a complete solution for $SL(n)$ one has in general the equality $[Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G]$ of the degrees of extension fields. Among other results, Φ yields (for the complex case) a generalization, involving generic regular orbits, of the result of Richardson showing that the Cayley map, when G is semisimple, defines an isomorphism from the variety of unipotent elements in G to the variety of nilpotent elements in $\mathfrak{g}$. In addition if G is semisimple the Cayley map establishes a diffeomorphism between the real submanifold of hyperbolic elements in G and the space of infinitesimal hyperbolic elements in $\mathfrak{g}$. Some examples are computed in detail.

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1. Introduction

Let G be a connected complex reductive algebraic group and let $\mathfrak{g} = \mathrm{Lie}(G)$. To any rational locally faithful representation $\pi : G \to \mathrm{Aut}(V)$ one can associate a dominant morphism

$$\Phi : G \to \mathfrak{g}$$

which (see example (3) below) we refer to as a generalized Cayley map. If $\pi' : \mathfrak{g} \to \mathrm{End}(V)$ is the differential of π the bilinear form $(\alpha, \beta) = \mathrm{tr}(\alpha\beta)$ on $\mathrm{End}(V)$ defines a projection $pr_\pi : \mathrm{End}(V) \to \pi'(\mathfrak{g})$. The generalized Cayley map arises from the restriction of pr_π to $\pi(G)$. This paper establishes a number of striking properties of the map Φ.

The map Φ is conjugation-equivariant and consequently Φ *carries conjugacy classes in G to adjoint orbits in* $\mathfrak{g}$ and the corresponding cohomomorphism Φ^* defines a conjugation-equivariant injection

$$0 \longrightarrow A(\mathfrak{g}) \longrightarrow A(G)$$

of affine rings. On the level of quotient fields Φ^* defines $Q(G)$ as a finite algebraic extension of $Q(\mathfrak{g})$. We will write $\deg \pi$ for the degree of this extension.

Examples.

(1) $G = Gl(n, \mathbb{C})$, π is the standard representation, $\deg(\pi) = 1$.
(2) $G = Sl(n, \mathbb{C})$, π is the standard representation, $\deg(\pi) = n$, see (6.2).
(3) $G = \mathrm{Spin}(n, \mathbb{C})$, π is the Spin representation, $\deg(\pi) = n$ for n even and $\deg(\pi) = n - 1$ for n odd, see (7.15).

Theorem. (7.15) *In the above example (3) the map Φ is, on a Zariski open subset and up to scalar multiplication, given by the Cayley transform.*

Let $A(G)^G$ and $A(\mathfrak{g})^G$ be the subalgebras of G-invariants in $A(G)$ and $A(\mathfrak{g})$ respectively. The space $\mathrm{Harm}(g) \subset A(\mathfrak{g})$ of harmonic polynomials on $\mathfrak{g}$ was defined in [9] and the decomposition

$$(4) \qquad\qquad A(\mathfrak{g}) = A(\mathfrak{g})^G \otimes \mathrm{Harm}(\mathfrak{g})$$

was proved in [9]. As a generalization of (4), using Quillen's proof of a conjecture of Serre, Richardson in [16] proved that a G-stable subspace H of $A(G)$ exists such that

$$(5) \qquad\qquad A(G) = A(G)^G \otimes H$$

holds. He also raised the question as to whether one can give an explicit construction of such a subspace H along the lines of (4). Although this problem is not solved in the present paper we do solve a weakened version of the problem. Let π be given and let $\mathrm{Sing}_\pi G$ be the subvariety of G where the differential $d\Phi$ is not invertible. Then $\mathrm{Sing}_\pi G$ is a hypersurface given as the zero set of the Jacobian $\Psi \in A(G)^G$ of the mapping Φ, see (2.1.2). Let $\mathrm{Harm}(G) = \Phi^*(\mathrm{Harm}(\mathfrak{g}))$. We localize with respect to Ψ and prove the following

Theorem. (3.2) *One has*

$$A(G)_\Psi = A(G)_\Psi^G \otimes \mathrm{Harm}(G).$$ (6)

The statement without localization,

$$A(G) = A(G)^G \otimes \mathrm{Harm}(G),$$ (7)

holds if and only if Φ maps regular orbits to regular orbits. This is the case in example (2).

Moreover (6) readily induces the following

Corollary. (3.3) *For the G-equivariant extension of the rational function fields $\Phi^* : Q(\mathfrak{g}) \to Q(G)$ the degrees satisfy*

$$[Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G].$$

Let $U \subset G$ be the unipotent variety and $N \subset \mathfrak{g}$ the nilcone. If G is semisimple and π is suitable then Richardson and Bardsley in [1] used Φ, even in the finite characteristic case (for good primes), to establish that

$$\Phi : U \to N$$ (8)

is an isomorphism of algebraic varieties. We consider here the complex case and generalize this for reductive algebraic groups to

Theorem. (4.5) *Let $a \in G$ be regular and assume that $d\Phi(a)$ is invertible. Then Φ restricts to an isomorphism of affine varieties*

$$\Phi : \overline{\mathrm{Conj}_G(a)} \to \overline{\mathrm{Ad}_G(\Phi(a))}.$$ (9)

Note that (8) is certainly not an isomorphism in Example (1).

Any $a \in G$ has a (multiplicative) Jordan decomposition $a = a_s a_u$ where a_s and a_u are respectively the semisimple and unipotent components of a. Analogously any $X \in \mathfrak{g}$ has an (additive) Jordan decomposition $X = X_s + X_n$ where X_s and X_n are the semisimple and nilpotent components. In contrast to (8) the map Φ always carries semisimple elements to semisimple elements. In fact one has

Theorem. (4.11) *For any $a \in G$ one has*

$$\Phi(a_s) = \Phi(a)_s.$$ (10)

An element $a \in G$ is called elliptic (resp. hyperbolic) if a is semisimple and the eigenvalues of $\pi(a)$ are of norm 1 (resp. positive) for all π. Expanding the multiplicative Jordan decomposition, every element $a \in G$ has a unique decomposition

$$a = a_e a_h a_u$$ (11)

where a_e and a_h are respectively elliptic and hyperbolic and all three components commute. We will say that a is of *positive type* if a_e is the identity. Let G_{pos} be the space of all elements of positive type. Analogously an element $X \in \mathfrak{g}$ is called elliptic (resp. hyperbolic) if X is semisimple and the eigenvalues of $\pi'(X)$ are pure imaginary (resp. real) for all π. Expanding the additive Jordan decomposition every element $X \in \mathfrak{g}$ has a unique decomposition

$$(12) \qquad\qquad X = X_e + X_h + X_n$$

where x_e and x_h are respectively elliptic and hyperbolic and all three components commute. We will say that X is of real type if $X_e = 0$. Let $\mathfrak{g}_{real}$ be the space of all elements of real type in $\mathfrak{g}$. Given π let G_{nonsing} be the (Zariski open) complement of G_{sing} in G.

Theorem. (5.5) *If $a \in G$ is hyperbolic then $\Phi(a)$ is hyperbolic. Furthermore one has $G_{\mathrm{pos}} \subset G_{\mathrm{nonsing}}$ and, extending (8), one has $\Phi(G_{\mathrm{pos}}) \subset \mathfrak{g}_{\mathrm{real}}$ and in fact*

$$(13) \qquad\qquad \Phi : G_{pos} \to \mathfrak{g}_{\mathrm{real}}$$

is a diffeomorphism. In particular Φ defines a bijection of the set of all hyperbolic elements in G to the set of all hyperbolic elements in $\mathfrak{g}$.

Obvious questions arise with regard to the restriction of the generalized Cayley map to subgroups of G. In this connection one readily establishes

Theorem. (2.6) (2.7) *We have*

$$(14) \qquad\qquad \Phi_\pi | K = \Phi_{\pi | K}$$

in the following cases: If K is any reductive subgroup of a reductive G which is the connected centralizer of a subset $A \subset G$. Or if K is a subgroup corresponding to a simple ideal in the Lie algebra of a semisimple group G.

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2. The generalized Cayley mapping and its basic properties

2.1. The setting. Let G be a (real or complex) Lie group with Lie algebra $\mathfrak{g}$, and let $\pi : G \to \mathrm{Aut}(V)$ be a finite dimensional representation with $\pi' : \mathfrak{g} \to \mathrm{End}(V)$ injective. We say that G *admits a Cayley mapping* if the inner product $(A, B) \mapsto \mathrm{tr}(AB)$ on $\mathrm{End}(V)$ restricts to a non-degenerate inner product B_π on $\pi'(\mathfrak{g})$. Thus the orthogonal projection $\mathrm{pr}_\pi : \mathrm{End}(V) \to \pi'(\mathfrak{g})$ is well defined. We consider the real analytic mapping

$$(1) \quad \Phi = \Phi_\pi : G \to \mathfrak{g} \quad \text{given by} \quad \pi' \circ \Phi_\pi = \mathrm{pr}_\pi \circ \pi : G \to \mathrm{End}(V) \to \pi'(\mathfrak{g}),$$
$$B_\pi(\Phi_\pi(g), X) = \mathrm{tr}(\pi'(\Phi_\pi(g))\pi'(X)) = \mathrm{tr}(\pi(g)\pi'(X)), \qquad g \in G, X \in \mathfrak{g},$$

which we call the *generalized Cayley mapping* of the representation π. The choice of the name is justified by the fact that for the Spin representation of $\mathrm{Spin}(n)$ on $\mathbb{C}^n$ it coincides, up to a scalar, with the Cayley transform, see (7.15).

By $d\Phi = \mathrm{pr}_2 \circ T\Phi : TG \to T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ we denote the differential of Φ, so that $d\Phi(g) = \mathrm{pr}_2 \circ T_g\Phi : T_gG \to \mathfrak{g}$. Let $X_1, \ldots, X_n$ be a linear basis of $\mathfrak{g}$ and let $L_{X_1}, \ldots, L_{X_n}$ be the corresponding left invariant vector fields on G. Let

$$(2) \qquad\qquad \Psi_\pi(g) = \Psi(g) := \det(d\Phi(g)), \qquad g \in G$$

be the *Cayley determinant function* of the representation π, where the determinant is computed with respect to the bases $L_{X_i}(g)$ of T_gG and X_i of $\mathfrak{g}$, respectively. Note that Ψ does not depend on the choice of the basis (X_i) of $\mathfrak{g}$. We get the function Ψ multiplied by the modular function if we choose right invariant vector fields.

In the following we shall use the notation $\mu : G \times G \to G$ for the multiplication, $\mu(g,h) = gh = \mu_g(h) = \mu^h(g)$ for left and right translations, $T(\mu_g)$ and $T(\mu^h)$ for the corresponding tangent mappings.

2.2. Proposition. *In the following cases the infinitesimally faithful representation* $\pi : G \to \mathrm{Aut}(V)$ *admits a Cayley mapping:*

    (1) *if G is a reductive complex Lie group and π is a holomorphic representation.*
    (2) *if G is a real reductive Lie group and π is a real or complex representation which maps each element in the connected center to a semisimple transformation (in the complexification of V). In particular if G is a real compact Lie group.*

By a real reductive Lie group we mean one where the complexification of the Lie algebra is reductive. For abelian Lie groups there are representations acting by unipotent matrices only, and we have to exclude these.

Proof. (1) A connected reductive complex Lie group G has a compact real form, so the Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup $K \subset G_o$. Since $\pi' : \mathfrak{g} \to \mathrm{End}(V)$ is complex linear it suffices to show that the trace form is non-degenerate on $\pi'(\mathfrak{k})$. Let us choose a K-invariant Hermitian product $(\ ,\ )$ on V by integration. Then $\pi'(X)$ is skew Hermitian with respect to this inner product for $X \in \mathfrak{k}$, so $0 \geq (\pi'(X)v, \pi'(X)v) = -(\pi'(X)^2v, v)$ and $\pi'(X)^2$ is negative definite Hermitian, so $\mathrm{tr}(\pi'(X)^2)$ is the sum of the negative eigenvalues of $\pi'(X)^2$. Thus B_π is non-degenerate on $\mathfrak{k}$.

(2) The trace form is non-degenerate on the semisimple part of $\mathfrak{g}$, and on the center since it is mapped to semisimple endomorphisms. For a compact group one can repeat the argument from the proof of (1) with a G-invariant positive definite inner product on V. $\square$

2.3. Remark. Most of the time (when not stated explicitly otherwise) G will denote a connected reductive complex algebraic group and π will be a rational representation; in particular in sections (4), (3), and (7) below.

Note that the center $\mathfrak{c}$ of the Lie algebra $\mathfrak{g}$ then belongs to any Cartan subalgebra, and its Lie group (the connected center) to the corresponding Cartan subgroup.

2.4. Proposition. *Let G be a complex or real Lie group and let Φ be the generalized Cayley mapping of a representation. The Cayley mapping Φ has the following simple properties:*

    (1) $\Phi \circ \mathrm{Conj}_b = \mathrm{Ad}_b \circ \Phi$.

(2) *For all $g \in G$ we have (where $Z_{\mathfrak{g}}(\mathfrak{g}^g)$ denotes the centralizer of $\mathfrak{g}^g = \{X \in \mathfrak{g} : \mathrm{Ad}_g(X) = X\}$ in $\mathfrak{g}$)*

$$G^g \subseteq G^{\Phi(g)}, \qquad \mathfrak{g}^a \subseteq \mathfrak{g}^{\Phi(g)},$$
$$\Phi(G^g) \subseteq \mathfrak{g}^g, \qquad \Phi(g) \in \mathrm{Cent}(\mathfrak{g}^g) \subset Z_{\mathfrak{g}}(\mathfrak{g}^g).$$

Moreover, if G is a connected reductive complex algebraic group then we even have $\mathrm{Cent}(\mathfrak{g}^g) = Z_{\mathfrak{g}}(\mathfrak{g}^g)$.

(3) *$d\Phi(e) : \mathfrak{g} \to \mathfrak{g}$ is the identity mapping. Thus $d\Phi(g)$ is invertible for g in the non-empty Zariski open dense subset $\{h \in G : \Psi(h) \neq 0\}$ of G, and is not invertible on the hypersurface $\{h \in G : \Psi(h) = 0\}$.*

(4) *Ψ is invariant under conjugation.*

(5) *Let $H \subset G$ be a Cartan subgroup with Cartan Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then $\Phi(H) \subset \mathfrak{h}$.*

(6) *Let χ_π be the character of the representation π, given by $\chi_\pi(g) = \mathrm{tr}(\pi(g))$. Then $d\chi_\pi(g)(T_e(\mu_g)X) = \mathrm{tr}(\pi'(\Phi_\pi(g))\pi'(X)) = B_\pi(\Phi_\pi(g), X)$.*

(7) *The differential $d\Phi_\pi(g).T_e(\mu_g).X \in \mathfrak{g}$ is given by the implicit equation $\mathrm{tr}(\pi'(d\Phi_\pi(g)\, T_e(\mu_g)\, X)\pi'(Y)) = \mathrm{tr}(\pi(g)\pi'(X)\pi'(Y))$ for $Y \in \mathfrak{g}$.*

(8) *If $\Phi(e) = 0$ and $a \in G$ is such that $\pi(a) \in \pi'(\mathfrak{g})$ then $d\Phi(a^{-1})$ is not invertible.*

Proof. (1) follows by the invariance of the trace.

(2) Most of it is obvious. Let G be a connected reductive complex algebraic group. We claim that then g lies in the identity component of G^g; this is immediate if g is semisimple (see (4.1) below) since then g lies in a Cartan subgroup. Using the Jordan decomposition (see (2.6) below) it is true in general. But then if $X \in \mathfrak{g}$ commutes with $\mathfrak{g}^g$ it commutes with g, hence $X \in \mathfrak{g}^g$ and consequently $X \in \mathrm{Cent}(\mathfrak{g}^g)$, establishing $Z_{\mathfrak{g}}(\mathfrak{g}^g) = \mathrm{Cent}(\mathfrak{g}^g)$.

(3) $T_e\pi(G) = \pi'(\mathfrak{g})$.

(4) follows from (1) and the form of the determinant (2.1.2).

(5) Let $a \in H$ be regular in G so that $G^a = H$ and $\mathfrak{g}^a = \mathfrak{h}$. Then use (2).

(6) Insert the definitions.

(7) This follows from

$$\mathrm{tr}(\pi'(d\Phi_\pi(g).T_e(\mu_g).X)\pi'(Y)) = \tfrac{d}{dt}|_0\, \mathrm{tr}(\pi'(\Phi_\pi(g\exp(tX)))\pi'(Y))$$
$$= \tfrac{d}{dt}|_0\, \mathrm{tr}(\pi(g\exp(tX))\pi'(Y)) = \tfrac{d}{dt}|_0\, \mathrm{tr}(\pi(g)\pi(\exp(tX))\pi'(Y))$$
$$= \mathrm{tr}(\pi(g)\pi'(X)\pi'(Y)).$$

(8) Take $X \in \mathfrak{g}$ with $\pi(a) = \pi'(X)$ so that $\pi(a^{-1})\pi'(X) = \mathrm{Id}_V$. Then we have $\mathrm{tr}(\pi(a^{-1})\pi'(X)\pi'(Y)) = \mathrm{tr}(\pi'(Y)) = 0$ for all $Y \in \mathfrak{g}$ so that $d\Phi(a^{-1})$ has a non-trivial kernel by (7). $\square$

2.5. Proposition. *Let G be a (real or complex) Lie group and let $\pi : G \to \mathrm{Aut}(V)$*

be a representation which admits a Cayley mapping.

(1) *For $a \in G$ we have*

$$d\Phi(a)(T_a(\mathrm{Conj}_G(a))) = T_{\Phi(a)}(\mathrm{Ad}_G(\Phi(a))) = [\mathfrak{g}, \Phi(a)],$$
$$d\Phi(a)(T_a(G^a)) \subseteq \mathfrak{g}^a \subseteq \mathfrak{g}^{\Phi(a)},$$
$$T_a(\mathrm{Conj}_G(a)) = \{T_e(\mu^a)X - T_e(\mu_a)X : X \in \mathfrak{g}\}$$
$$= \{T_e(\mu^a)(X - \mathrm{Ad}_a X) : X \in \mathfrak{g}\},$$
$$T_a(G^a) = T_e(\mu_a)\mathfrak{g}^a = T_e(\mu^a)\mathfrak{g}^a.$$

Moreover, if G is a reductive complex algebraic group and a is a semisimple element in G then we have

$$T_a G = T_a(\mathrm{Conj}_G(a)) \oplus T_a(G^a), \qquad \mathfrak{g} = [\mathfrak{g}, \Phi(a)] \oplus \mathfrak{g}^{\Phi(a)}.$$

(2) *Suppose that $d\Phi(a) : T_a G \to \mathfrak{g}$ is invertible for $a \in G$. Then $\mathfrak{g}^a = \mathfrak{g}^{\Phi(a)}$. In particular, the G-orbit $\mathrm{Conj}_G(a)$ of a in G has the same dimension as its Φ-image, $\mathrm{Ad}_G(\Phi(a))$; consequently, $\Phi : \mathrm{Conj}_G(a) \to \mathrm{Ad}_G(\Phi(a))$ is a covering. If in addition G is connected reductive and a is semisimple, then the centralizer $G_{\Phi(a)}$ is connected, so Φ is a diffeomorphism between the orbits.*

Proof. (1) Most of it follows easily from (2.4). Let us now suppose G is a reductive complex algebraic group and that $a \in G$ is semisimple (4.1) so that a is contained in a Cartan subgroup with Cartan Lie subalgebra $\mathfrak{h}$. By (2.4.5) we get $\Phi(a) \in \mathfrak{h}$. We claim that then $\mathfrak{g} = [\mathfrak{g}, \Phi(a)] \oplus \mathfrak{g}^{\Phi(a)}$. By dimension it suffices to check that $[\mathfrak{g}, \Phi(a)] \cap \mathfrak{g}^{\Phi(a)} = 0$. This follows from the root space decomposition: Put $\Phi(a) = H \in \mathfrak{h}$, let $Y = [X, H] \in \mathfrak{g}^H$, let $X = X_0 + \sum_{\alpha \in R} X_\alpha$ be the root space decomposition. Then $Y = [X, H] = 0 + \sum_{\alpha \in R} \alpha(H) X_\alpha$ and $0 = [Y, H] = \sum_{\alpha \in R} \alpha(H)^2 X_\alpha$ whence either $\alpha(H) = 0$ or $X_\alpha = 0$ so that $Y = 0$.

Similarly it suffices to see that $\mathfrak{g}^a \cap (\mathrm{Id} - \mathrm{Ad}_a)\mathfrak{g} = 0$. So let $a = \exp(H)$ for some $H \in \mathfrak{h}$ and suppose that $Y = X - \mathrm{Ad}_a X \in \mathfrak{g}^a$. Consider again the root space decomposition $X = X_0 + \sum_{\alpha \in R} X_\alpha$. Then we get $Y = X - \mathrm{Ad}_{\exp(H)} X = 0 + \sum_{\alpha \in R}(1 - e^{\alpha(H)})X_\alpha$ and $0 = Y - \mathrm{Ad}_{\exp(H)} Y = \sum_{\alpha \in R}(1 - e^{\alpha(H)})^2 X_\alpha$ whence either $1 - e^{\alpha(H)} = 0$ or $X_\alpha = 0$ so that $Y = 0$.

(2) By (2.4.2) we have $\mathfrak{g}^a \subseteq \mathfrak{g}^{\Phi(a)}$, so it suffices to prove the converse. Let $X \in \mathfrak{g}^{\Phi(a)}$ so that $[X, \Phi(a)] = 0$. Then

$$d\Phi(a)T_e(\mu^a)(X - \mathrm{Ad}_a X) = \tfrac{d}{dt}|_0 \Phi(\exp(tX)a\exp(-tX))$$
$$= \tfrac{d}{dt}|_0 \mathrm{Ad}_{\exp(tX)} \Phi(a) = [X, \Phi(a)] = 0$$

so that $X - \mathrm{Ad}_a X = 0$ since $d\Phi(a)$ is invertible. If G is connected reductive and a is semisimple (see (4.1)) then by (2.4.5) $\Phi(a)$ is also semisimple, thus $G^{\Phi(a)}$ is connected. $\square$

2.6. Theorem. *Let G be a (real or complex) Lie group and let $\pi : G \to \mathrm{Aut}(V)$ be a representation which admits a Cayley mapping. Let $H = (\bigcap_{a \in A} G^a)_o = (G^A)_o \subseteq G$ be a subgroup which is the connected centralizer of a subset $A \subseteq G$ and suppose that H is itself reductive.*

Then the representation $\pi|H : H \to \mathrm{End}(V)$ admits a Cayley mapping and we have

$$\Phi_\pi|H = \Phi_{\pi|H} : H \to \mathfrak{h}.$$

Note that this result proves again (2.4.5) by choosing A the Cartan subgroup. If we choose $A = G$ then H is the connected center of G. The assumption of this theorem holds in the following two cases:

(1) If $H = (G^g)_o$ is the connected centralizer of a semisimple element g in a complex algebraic group G (see (4.1) below), since then the connected centralizer $(G^g)_o = Z_G(g)_o$ is again reductive, by Proposition 13.19 in [2] on p. 321.

(2) If $H = (G^A)_o$ is the connected centralizer of a reductive subgroup $A \subseteq G$ of a complex reductive Lie group G, since it is well known that then H is itself reductive.

Proof. By (2.2.1) both Cayley mappings Φ_π and $\Phi_{\pi|H}$ exist. By (2.1.2) we have $\Phi_\pi(G^a) \subseteq \mathfrak{g}^a$ for all $a \in A$, thus also

$$\Phi_\pi(H) \subseteq \bigcap_{a \in A} \Phi_\pi(G^a) \subseteq \bigcap_{a \in A} \mathfrak{g}^a = \mathfrak{h}.$$

Moreover, for $h \in H$ and $X \in \mathfrak{h}$ we have

$$\mathrm{tr}(\pi'(\Phi_\pi(h)\pi'(X)) = \mathrm{tr}(\pi(h)\pi'(X)) = \mathrm{tr}(\pi'(\Phi_{\pi|(G^g)_o}(h)\pi'(X))).$$

Thus $\Phi_\pi(h) = \Phi_{\pi|H}(h)$. □

2.7. Theorem. *Let G be a semisimple real or complex Lie group, let $\pi : G \to \mathrm{Aut}(V)$ be an infinitesimally effective representation. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition into the simple ideals $\mathfrak{g}_i$. Let $G_1, \ldots, G_k$ be the corresponding connected subgroups of G. Then*

$$\Phi_\pi|G_i = \Phi_{\pi|G_i} \quad \text{for } i = 1, \ldots, k.$$

This result cannot be extended to reductive groups as the standard representation of GL_n shows where Φ is the embedding $GL_n \to \mathfrak{gl}_n$.

Proof. Let $g_1 \in G_1$ and suppose that $\Phi_\pi(g_1) = \sum_{i=1}^k \Phi_\pi(g_1)_i \in \bigoplus_{i=1}^k \mathfrak{g}_i$ with $\Phi_\pi(g_1)_i \neq 0 \in \mathfrak{g}_i$ for some $i \neq 1$. Then for each $g_i \in G_i$ we have

$$\Phi_\pi(g_1) = \Phi_\pi(g_i \, g_1 \, g_i^{-1}) = \mathrm{Ad}_{g_i} \Phi_\pi(g_1)$$

so that $\Phi_\pi(g_1)$ contains a nontrivial G_i-orbit which is absurd. This shows that $\Phi_\pi(G_j) \subseteq \mathfrak{g}_j$ for $j = 1, \ldots, k$. Moreover for $g_j \in G_j$ and $X_j \in \mathfrak{g}_j$ we have

$$\mathrm{tr}(\pi'(\Phi_\pi(g_j))\pi'(X_j)) = \mathrm{tr}(\pi(g_j)\pi'(X_j)) = \mathrm{tr}((\pi|G_j)'(\Phi_{\pi|G_j}(g_j)(\pi|G_j)'(X_j))$$

which implies the result. □

2.8. Theorem. *Let G be a simple real or complex Lie group, and let $\pi_i : G \to \mathrm{End}(V_i)$ be nontrivial representations for $i = 1, 2$. The inner product B_{π_i} on $\mathfrak{g}$ from (2.1) is a multiple of the Cartan Killing form B, we write $B_{\pi_i} = j_{\pi_i} B$. Then we have.*

(1) *For the direct sum representation $\pi_1 \oplus \pi_2 : G \to \mathrm{End}(V_1 \oplus V_2)$ we have*

$$\Phi_{\pi_1 \oplus \pi_2}(g) = \frac{j_{\pi_1}}{j_{\pi_1 \oplus \pi_2}} \Phi_{\pi_1}(g) + \frac{j_{\pi_2}}{j_{\pi_1 \oplus \pi_2}} \Phi_{\pi_2}(g) \in \mathfrak{g}.$$

(2) *For the tensor product representation $\pi_1 \otimes \pi_2 : G \to \mathrm{End}(V_1 \otimes V_2)$ we have*

$$\Phi_{\pi_1 \otimes \pi_2}(g) = \frac{j_{\pi_1} \chi_{\pi_2}(g)}{j_{\pi_1 \otimes \pi_2}} \Phi_{\pi_1}(g) + \frac{\chi_{\pi_1}(g) j_{\pi_2}}{j_{\pi_1 \otimes \pi_2}} \Phi_{\pi_2}(g) \in \mathfrak{g}.$$

(3) *For the n-fold tensor product representation $\otimes^n \pi : G \to \mathrm{End}(\otimes^n V)$ we have*

$$\Phi_{\otimes^n \pi}(g) = \left(\frac{\chi_\pi(g)}{\dim(V)} \right)^{n-1} \Phi_\pi(g).$$

(4) *For the contragredient representation $\pi^\top : G \to \mathrm{End}(V^*)$ given by $\pi^\top(g) = \pi(g^{-1})^\top$ we have*

$$\Phi_{\pi^\top}(g) = -\Phi_\pi(g^{-1}).$$

For a complex simple Lie group the number j_{π_i} is a multiple of the *Dynkin index* of the representation π_i. It is non-negative and satisfies

$$
\begin{aligned}
j_{\pi_1 \oplus \pi_2} &= j_{\pi_1} + j_{\pi_2}, \\
j_{\pi_1 \otimes \pi_2} &= \dim(V_2) j_{\pi_1} + \dim(V_1) j_{\pi_2}, \\
j_\pi &= \frac{\dim(V)}{\dim(\mathfrak{g})} B(\lambda_\pi, \lambda_\pi + \rho),
\end{aligned}
$$

(5)

for an irreducible representation π with highest weight λ_π, where ρ is half the sum of all positive roots. This is due to Dynkin [4] and can be found in [5], p.100. Using this, equation (2) becomes

(2')
$$
\begin{aligned}
\Phi_{\pi_1 \otimes \pi_2}(g) &= \frac{B(\lambda_{\pi_1}, \lambda_{\pi_1} + \rho)}{B(\lambda_{\pi_1}, \lambda_{\pi_1} + \rho) + B(\lambda_{\pi_2}, \lambda_{\pi_2} + \rho)} \frac{\chi_{\pi_2}(g)}{\dim(V_2)} \Phi_{\pi_1}(g) \\
&\quad + \frac{B(\lambda_{\pi_2}, \lambda_{\pi_2} + \rho)}{B(\lambda_{\pi_2}, \lambda_{\pi_2} + \rho) + B(\lambda_{\pi_1}, \lambda_{\pi_1} + \rho)} \frac{\chi_{\pi_1}(g)}{\dim(V_1)} \Phi_{\pi_2}(g)
\end{aligned}
$$

Proof. (1), (2) and (4) are easy computations.

(3) By induction and (5) we check that $j_{\otimes^n \pi} = n \dim(V)^{n-1} j_\pi$ which via (2) leads quickly to the result. $\square$

2.9. Proposition. *For the Cayley mapping of a rational representation of a connected reductive complex algebraic group G, the pullback mapping $\Phi^* : A(\mathfrak{g}) = S^*(\mathfrak{g}^*) \to A(G)$ between the algebras of regular functions is injective, equivariant, and maps the subalgebras of invariant regular functions to each other, $\Phi^* : A(\mathfrak{g})^G \to A(G)^G$. Consequently, $\Phi : G \to \mathfrak{g}$ is a dominant algebraic morphism. By the algebraic Peter-Weyl theorem we have $A(G) = \bigoplus_{\lambda \in D} A_\lambda$ where D is the set of all dominant integral highest weights, and where*

$$A_\lambda = \{f \in A(G) : f(g) = \mathrm{tr}(\pi_\lambda(g)B) \text{ for some } B \in \mathrm{End}(V_\lambda)\}$$

For an irreducible representation π we thus have $\Phi^*(\mathfrak{g}^*) \subset A_\lambda$ where λ is the highest weight of π.

Proof. We prove the last statement. Let λ be the highest weight of π. A_λ is a vector space of dimension $(\dim(V_\lambda))^2$ of functions on G. Take $X \in \mathfrak{g}$ and consider the function $f \in A_\lambda$ given by $f(g) = \mathrm{tr}(\pi(g)\pi'(X))$. Then $f = \Phi^*(B_\pi(\quad, X))$ where $B_\pi(\quad, X) \in \mathfrak{g}^*$. $\square$

3. A SEPARATION OF VARIABLES THEOREM
FOR REDUCTIVE ALGEBRAIC GROUPS

In this section G is a connected reductive complex algebraic group and Φ is the Cayley mapping of a rational representation.

3.1. Rational vector fields on G. Let $x_i : \mathfrak{g} \to \mathbb{C}$ be the coordinate functions for the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, and consider the constant vector fields $\partial_{x_i} = \frac{\partial}{\partial x_i}$.

Theorem. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation.*

Then the pull back vector fields $Y_i := \Phi^* \partial_{x_i}$ for $i = 1, \ldots, n$ are commuting vector fields with rational coefficients on G. The fields $\Psi.Y_i$ are regular vector fields (with algebraic coefficients). The fields Y_i induce an G-equivariant injective algebra homomorphism from the symmetric algebra $S^*\mathfrak{g}$ into the algebra of differential operators on G with rational coefficients with polar divisors contained in the hypersurface $\{h \in G : \Psi(h) = 0\}$.

Proof. For a matrix A we denote by $C(A)$ the classical adjoint or the matrix of algebraic complements which satisfies $A.C(A) = C(A).A = \det(A).\mathrm{Id}$. Applying this to $d\Phi(g)$ we can write the pull back fields as

$$(\Phi^* \partial_{x_i})(g) = d\Phi(g)^{-1}.\partial_{x_i}|_{\Phi(g)} = \frac{1}{\det(d\Phi(g))}.C(d\Phi(g)).\partial_{x_i}$$

$$= \frac{1}{\Psi(g)}.\sum_{j=1}^{n} C(d\Phi(g))_{ij} L_{X_j}(g)$$

These are well defined algebraic vector fields on the Zariski open set $\{\Psi \neq 0\}$ and are Φ-related to the constant fields ∂_{x_i} on $\mathfrak{g}$, thus they commute. $\square$

3.2. Invariants and harmonic functions. For the algebra of regular functions we have $A(\mathfrak{g}) = A(\mathfrak{g})^G \otimes \mathrm{Harm}(\mathfrak{g})$ by [9], theorem 0.2., where the space $\mathrm{Harm}(\mathfrak{g})$ is by definition the space of all regular functions which are killed by all invariant differential operators with constant coefficients. We define $\mathrm{Harm}_\pi(G) := \Phi_\pi^*(\mathrm{Harm}(\mathfrak{g}))$. It is a G-module. Let us denote by $A(G)_\Psi$ the localization at Ψ.

Theorem. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation. Then*

$$A(G)_\Psi = A(G)_\Psi^G \otimes \mathrm{Harm}_\pi(G).$$

Moreover, we have
$$A(G) = A(G)^G \otimes \mathrm{Harm}_\pi(G)$$

if and only if $\Phi : G \to \mathfrak{g}$ maps regular orbits in G to regular orbits in $\mathfrak{g}$.

Note that for the standard representation π of $SL_n(\mathbb{C})$ the Cayley mapping Φ_π carries regular orbits to regular orbits, see (6.2) below. In general this is wrong. If there exists a closed orbit $O = \mathrm{Conj}_G(a)$ in G for a in a Cartan subgroup H, such that $\dim(\Phi(O)) < \dim(O)$ then there exists already a regular orbit with that property, $\mathrm{Ad}(G)(a\,u) \subset \mathfrak{g}$ where u is a principal unipotent element in $(G^a)_o$, see (4.8.3). The orbits through

$$\mathrm{diag}(1,1,1,1,-1,-1,-1,-1) \in SO(8) \cap \mathfrak{so}(8)$$

have this property.

Proof. By [15], theorem A, we have

$$(1) \qquad\qquad A(G) = A(G)^G \otimes H$$

for some G-submodule H of $A(G)$. Then H is by restriction isomorphic to the affine ring $A(\mathrm{Conj}_G(a))$ for any regular orbit $\mathrm{Conj}_G(a)$ in G.

Let λ be any highest weight of G such that the corresponding irreducible G-module V_λ has a non-zero weight space V_λ^0 with weight 0 of dimension $d(\lambda)$. Let H_λ be the primary component of H of type V_λ so that one has the direct sum

$$H = \bigoplus_\lambda H_\lambda.$$

By [9], p 348, proposition 8, the multiplicity of V_λ in H_λ is $d(\lambda)$, since H restricts bijectively to any regular orbit in G; see also [15], theorem A. Thus we may write as a direct sum

$$(2) \qquad\qquad H_\lambda = \bigoplus_{j=1}^{d(\lambda)} H_\lambda^j$$

where H_λ^j is irreducible and hence equivalent to V_λ.

Now let $\mathrm{Harm}(G)_\lambda$ be the primary component of the G-module $\mathrm{Harm}(G) = \Phi^*(\mathrm{Harm}(\mathfrak{g})) \subset A(G)$ of type V_λ. By [9] the multiplicity of V_λ in $\mathrm{Harm}(G)$ is again $d(\lambda)$ so that we have

$$(3) \qquad\qquad \mathrm{Harm}(G)_\lambda = \bigoplus_{j=1}^{d(\lambda)} \mathrm{Harm}(G)_\lambda^j$$

where each $\mathrm{Harm}(G)_\lambda^j$ is an irreducible G-submodule of $A(G) = A(G)^G \otimes H$ of highest weight λ and hence equivalent to V_λ. Thus there exists a $(d(\lambda) \times d(\lambda))$-matrix $S = (s_{ij})$ with entries in $A(G)^G$ so that

$$(4) \qquad\qquad \mathrm{Harm}(G)_\lambda^i = \sum_{j=1}^{d(\lambda)} s_{ij} H_\lambda^j.$$

One would like to replace the abstract G-module H_λ by the equivalent and explicit G-module $\mathrm{Harm}(G)_\lambda$. For that one needs that the matrix S is invertible in $A(G)^G$. The determinant $\det(S)$ is non-zero in $A(G)^G$ by the independence in (2) and (3). One would need that $\det(S)$ is a constant. Let $Z = \mathrm{Zero}(\det(\mathrm{S}))$ be the zero set of $\det(S)$. By Steinberg [18], theorem 1.3, the set of irregular elements in G is of codimension 3. Since $\det(S) \in A(G)^G$ its zero set Z contains full orbits and is of codimension 1, so it is the union of the closures of all regular orbits $\mathcal{O}$ in Z. But $\Phi(\mathcal{O})$ cannot be a regular adjoint orbit in $\mathfrak{g}$ since $\mathrm{Harm}(\mathfrak{g})$ restrict faithfully to $A(\mathrm{Ad}(G).X)$ for every regular orbit in $\mathfrak{g}$. Consequently, if one knew that Φ carried each regular orbit to a regular orbit then Z was empty and hence $\det(S)$ a non-zero constant which proves part of the second assertion.

In the general case, if we localize $A(G)$ and $A(G)^G$ at the function Ψ from (2.4), then we restrict Φ to the Zariski open affine subvariety $G_{\Psi \neq 0}$ where Ψ does not vanish. But there Φ is locally biholomorphic and thus carries regular orbits to regular orbits in $\mathfrak{g}$, see (2.5.2). Hence $\det(S)$ does not vanish on $G_{\Psi \neq 0}$ and is thus invertible in $A(G)^G_\psi$. Consequently we have $A(G)_\Psi = A(G)^G_\Psi . \mathrm{Harm}(G)$.

It remains to prove that $A(G)_\Psi = A(G)^G_\Psi . \mathrm{Harm}(G)$ implies $A(G)_\Psi = A(G)^G_\Psi \otimes \mathrm{Harm}(G)$. Assume for contradiction that this is false. Then there exist linearly independent $b_i \in \mathrm{Harm}(G)$ and linearly independent $a_i \in A(G)^G_\Psi$ for $1 \leq i \leq k$ such that $\sum_i a_i b_i = 0$. Since $G_{\Psi \neq 0}$ is Zariski open in G there exists a regular orbit $\mathcal{O}$ in $G_{\Psi \neq 0}$ such that $a_i(g) \neq 0$ for all i and any $g \in \mathcal{O}$. On the other hand $\sum_i a_i b_i | \mathcal{O} = \sum_i a_i(g) b_i | \mathcal{O}$ vanishes on $\mathcal{O}$. But $\Phi(\mathcal{O})$ is a regular orbit in $\mathfrak{g}$. Let $b_i = \Phi^*(c_i)$ for $c_i \in \mathrm{Harm}(\mathfrak{g})$; then the c_i are linearly independent. Thus the vanishing of $\sum_i a_i(g) c_i$ on the regular orbit $\Phi(\mathcal{O})$ contradicts the fact that $\mathrm{Harm}(\mathfrak{g})$ restricts faithfully onto each regular orbit.

Finally, if we had $A(G) = \mathrm{Harm}_\pi(G)) \otimes A(G)^G$ then $\mathrm{Harm}_\pi(G)) = \Phi^* \mathrm{Harm}(\mathfrak{g}))$ would restrict faithfully to each regular orbit in G. Since the same is true for $\mathrm{Harm}(\mathfrak{g}))$ the mapping Φ had to map regular orbits to regular orbits. $\quad\square$

3.3. Corollary. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation with $\Phi(e) = 0 \in \mathfrak{g}$.*

Then for the G-equivariant extension of the rational function fields $\Phi^* : Q(\mathfrak{g}) \to Q(G)$ the degrees satisfy

$$[Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G].$$

See (6.2) below for the explicit extension in the case of the standard representation of $SL_n(\mathbb{C})$.

Proof. Note that $Q(\mathfrak{g})^G$ is the quotient field of $A(\mathfrak{g})^G$, and $Q(G)^G$ is the quotient field of $A(G)^G$. We have $A(\mathfrak{g}) = A(\mathfrak{g})^G \otimes \mathrm{Harm}(\mathfrak{g})$ by [9], theorem 0.2., and $A(G)_\Psi = A(G)_\Psi^G \otimes \mathrm{Harm}_\pi(G)$ by (3.2), where $\mathrm{Harm}_\pi(G) := \Phi^*(\mathrm{Harm}(\mathfrak{g}))$ is isomorphic to $\mathrm{Harm}(\mathfrak{g})$.

Let $k = [Q(G), Q(\mathfrak{g})]$. Then any $q \in Q(G)$ satisfies a unique monic polynomial of degree $\leq k$ with coefficients in $Q(g)$. Choose $q \in Q(G)^G$. Then the coefficients must be in $Q(g)^G$ since otherwise by conjugating by an element in G we would obtain a new minimal polynomial which contradicts uniqueness. Thus $[Q(G)^G, Q(\mathfrak{g})^G] \leq k$.

On the other hand if $m = [Q(G)^G, Q(\mathfrak{g})^G]$ then there exists q in $Q(G)^G$ which satisfies an equation of degree m over $Q(\mathfrak{g})^G$ and $Q(G)^G = (Q(\mathfrak{g})^G)[q]$. But $A(G)_\Psi^G$ is contained in $Q(G)^G$. Thus $A(G)_\Psi$ is contained in $Q(\mathfrak{g})[q]$ by (3.2). Hence the quotient field $Q(G)$ of $A(G)_\Psi$ is contained in $Q(\mathfrak{g})[q]$. Thus $m = k$. $\square$

4. The behavior of the Jordan decomposition under the Cayley map

In this section G is a connected reductive complex algebraic group and Φ is the generalized Cayley mapping of a rational representation.

4.1. The Jordan decomposition. For references about the Jordan decomposition (additive in the Lie algebra and multiplicative in the algebraic group) see [2] or chapter IV of [6]. In our case given $a \in G$ we write a_s for the semisimple part of a and a_u for the unipotent part of a so that $a = a_s a_u = a_u a_s$. Recall that semisimple means that a_s is G-conjugate to an element in the Cartan subgroup H and a_u unipotent means that a_u is conjugate to an element in the unipotent variety U. We shall use the decomposition $H = T H_\mathbb{R}$ where T is a maximal torus. An element $a \in G$ is called elliptic if a is semisimple and the eigenvalues of $\pi(a)$ are of norm 1 for all π; equivalently, a is conjugated to an element in T. Likewise, an element $a \in G$ is called hyperbolic if a is semisimple and the eigenvalues of $\pi(a)$ are real for all π; or equivalently, if a is conjugated to to an element in $H_\mathbb{R}$. Expanding the multiplicative Jordan decomposition every element $a \in G$ has a unique decomposition

$$a = a_e a_h a_u$$

where a_e and a_h are respectively elliptic and hyperbolic and all three components commute. We say that a is *of positive type* if $a_e = e \in G$.

Analogously for $X \in \mathfrak{g}$ there is the unique (additive) Jordan decomposition $X = X_s + X_n$ where $[X_s, X_n] = 0$ and where X_s is semisimple (conjugate to an element in $\mathfrak{h}$) and X_n is nilpotent (conjugate to an element in the nilcone N). Expanding the additive Jordan decomposition every element $X \in \mathfrak{g}$ has a unique decomposition

$$X = X_e + X_h + X_n$$

where X_e and X_h are respectively elliptic and hyperbolic and all three components commute. We will say that X is *of real type* if $X_e = 0$. Let $\mathfrak{g}_{real}$ be the space of all elements of real type in $\mathfrak{g}$. For information on hyperbolic elements in algebraic groups see [11], especially Section 2 on p. 418.

4.2. Ad-**complete modules.** Let $D \subset h^*$ denote the set of dominant integral weights for G (relative to some fixed Borel subgroup) and for each $\lambda \in D$ let $\pi_\lambda : G \to Aut\, V_\lambda$ be a fixed irreducible representation with highest weight λ.

A completely reducible G-module M will be said to Ad-complete if one has an equivalence

$$M \cong \oplus_{\lambda \in D} \dim(V_\lambda^H)\, V_\lambda,$$

i.e. each irreducible component occurs with multiplicity equal to the dimension of its zero weight space.

Let $\operatorname{Reg}(\mathfrak{g})$ (resp. $\operatorname{Reg}(G)$) be the set of regular elements in $\mathfrak{g}$ (respectively G. We recall the following results.

4.3. Theorem. [9] *For $X \in \operatorname{Reg}(\mathfrak{g})$ one has $A(\operatorname{Ad}_G(X)) = A(\overline{\operatorname{Ad}_G(X)})$ and as a G-module $A(\overline{\operatorname{Ad}_G(X)})$ is* Ad-*complete. Furthermore there exists a section $\operatorname{Reg}_\#(\mathfrak{g})$ of the map $\operatorname{Reg}(\mathfrak{g}) \to \operatorname{Reg}(\mathfrak{g})/G$ which in addition has the property that*

$$(1) \qquad\qquad \operatorname{Reg}_\#(\mathfrak{g}) \to \mathbb{C}^\ell, \qquad X \mapsto (I_1(X), ..., I_\ell(X))$$

is an algebraic isomorphism, where the I_k form a basis of $A(\mathfrak{g})^G$. Finally,

$$(2) \qquad\qquad \mathfrak{g} = \bigcup_{X \in \operatorname{Reg}_\#(\mathfrak{g})} \overline{\operatorname{Ad}_G(X)}$$

is a disjoint union.

Subsequently Steinberg proved the following group-theoretic analogue:

4.4. Theorem. [18] *Assume G is simply-connected semisimple. For $a \in \operatorname{Reg}(G)$ one has*

$$A(\operatorname{Conj}_G(a)) = A(\overline{\operatorname{Conj}_G(a)})$$

and as a G-module $A(\overline{\operatorname{Conj}_G(a)})$ is Ad-*complete. Furthermore there exists a section $\operatorname{Reg}_\#(G)$ of the map $\operatorname{Reg}(G) \to \operatorname{Reg}(G)/G$ which in addition has the property that*

$$(1) \qquad\qquad \operatorname{Reg}_\#(G) \to \mathbb{C}^\ell, \qquad a \mapsto (\chi_1(a), ..., \chi_\ell(a))$$

is an algebraic isomorphism. Here $\{\chi_j\}$ are the characters of the fundamental representations. Finally

$$(2) \qquad\qquad G = \bigcup_{a \in \operatorname{Reg}_\#(G)} \overline{\operatorname{Conj}_G(a)}$$

is a disjoint union.

4.5. Theorem. *Let G be a reductive complex algebraic group and let $\pi : G \to \text{Aut}(V)$ be a locally faithful rational representation of G. Let $a \in G$ be regular. Assume that a is nonsingular with respect to the Cayley map $\Phi = \Phi_\pi$ so that $\Phi(a)$ is regular in $\mathfrak{g}$ by (2.5.2). Then Φ restricts to an isomorphism*

$$(1) \qquad\qquad \Phi : \overline{\text{Conj}_G(a)} \to \overline{\text{Ad}_G(\Phi(a))}$$

of affine varieties.

Proof. Let G^s be the simply-connected covering group of the commutator (semisimple) subgroup G' of G. Let $\gamma : G^s \to G'$ be the covering map. We may write $a = bc$ where $b \in \text{Cent}(G)$ and $c \in G'$. Let $g \in G^s$ be such that $\gamma(g) = c$; note that g is regular. Clearly the mapping $G^s \to G$, given by $h \mapsto b\gamma(h)$, restricts to a surjective G^s-equivariant morphism $\beta : \text{Conj}_{G^s}(g) \to \text{Conj}_G(a)$; thus by continuity β also restricts to a dominant morphism

$$(2) \qquad\qquad \beta : \overline{\text{Conj}_{G^s}(g)} \to \overline{\text{Conj}_G(a)}.$$

But also by continuity one has a dominant morphism

$$(3) \qquad\qquad \Phi : \overline{\text{Conj}_G(a)} \to \overline{\text{Ad}_G(\Phi(a))}$$

and hence if α is the composite of (2) and (3) it defines a dominant morphism

$$(4) \qquad\qquad \alpha : \overline{\text{Conj}_{G^s}(g)} \to \overline{\text{Ad}_G(\Phi(a))}$$

But then the cohomomorphism of (4) is injective. However since the affine algebras in question are Ad-complete by Theorems (4.3) and (4.4), it follows that (4) must be an isomorphism. But then obviously (2) and (3) are isomorphisms. □

4.6. Richardson proved that for semisimple groups the generalized Cayley map defines an isomorphism of the unipotent variety in G with the nilcone in $\mathfrak{g}$. His theorem is very general and includes the case of G defined over fields of finite characteristic as long as the prime is good. An application of Theorem (4.5) yields Richardson's theorem for the complex case.

Theorem. *Let $a \in G$ be a principal unipotent element. Then a is non-singular. Let $U \subset G$ be the unipotent variety $U = \overline{\text{Conj}_G(a)}$. Then*

$$(1) \qquad\qquad \Phi : U \to \overline{\text{Ad}_G(\Phi(a))}$$

is an isomorphism of affine varieties. Furthermore if $\Phi(e) = 0$ (e.g. G is semisimple) then $\Phi(a)$ is principal nilpotent so that $\overline{\text{Ad}_G(\Phi(a))}$ is the nilcone $N \subset \mathfrak{g}$ and hence (1) is an isomorphism

$$(2) \qquad\qquad \Phi : U \to N$$

Proof. Since $e \in U$ and $d\Phi(e)$ is invertible it is immediate that $d\Phi(a)$ is invertible. Thus (1) follows from Theorem (4.5). If $\Phi(e) = 0$ then $0 \in \overline{\text{Ad}_G(\Phi(a))}$. But this implies that $\Phi(a)$ is principal nilpotent and hence one has (2). □

4.7. Corollary. *If G is semisimple and X is a principal nilpotent element in $\mathfrak{g}$ and if π is irreducible, then $\Phi^{-1}(X) \subset G$ consists of $|Z(G)|$ many elements.*

Proof. Let $b \in \Phi^{-1}(X)$. Since $d\Phi(b)$ is invertible we have $b \in G^b = G^X = Z(G) \times (G^X)_0$. Also by (4.6) there is a unique $a \in \Phi^{-1}(X) \cap U$. Since π is irreducible and $Z(G)$ is a finite group, for $c \in Z(G)$ we have $\pi(c) = k_c.1_V$ where $k_c \in S^1 \subset \mathbb{C}$, thus $\operatorname{tr}(\pi'\Phi(c.g)\pi'(Y)) = \operatorname{tr}(\pi(c)\pi(g)\pi'(Y)) = k_c.\operatorname{tr}(\pi(g)\pi'(Y)) = \operatorname{tr}(\pi'\Phi(g)\pi'(k_c.Y))$ for all $Y \in \mathfrak{g}$. This implies $\Phi(c.g) = k_c.\Phi(g)$ for each $g \in G$. Choose the unique elements $a_c \in \Phi^{-1}(\frac{1}{k_c}X) \cap U$ for all $c \in Z(G)$. Then $c.a_c \in G^X$ is in the coset $c.(G^X)_0$ and $\Phi(c.a_c) = X$. So $\Phi^{-1}(X)$ consists of $|Z(G)|$ many elements. $\square$

4.8. For any $a \in G$ (resp. $x \in \mathfrak{g}$) let G^a (resp. G^x) denote the centralizer of a (resp. x) in G. Let $\mathfrak{g}^a = \operatorname{Lie} G^a$ (resp. $\mathfrak{g}^x = \operatorname{Lie} G^x$). If a is semisimple and G^a_o is the identity component of G^a then G^a_o is a reductive subgroup of G (see Proposition 13.19 in [2] on p. 321). If x is semisimple then G^x is connected (see e.g. Theorem 22.3, p. 140 in [6]) and reductive (since $\mathfrak{g}^x$ is clearly reductive). For any semisimple element $b \in G$ let U_b be the unipotent variety in G^b_o and N_b the nilpotent cone in $\mathfrak{g}^b$. Moreover, let $l = \operatorname{rank} G$ and recall that an element $a \in G$ (resp. $x \in \mathfrak{g}$) is called regular if $\dim G^a = l$ or equivalently $\dim \mathfrak{g}^a = l$ (resp. $\dim G^x = l$ or equivalently $\dim \mathfrak{g}^x = l$).

For $a \in G$ with Jordan decomposition $a = a_s a_u$ let $\mathfrak{c} = cent \, \mathfrak{g}^{a_s}$ and let $\mathfrak{s} = [\mathfrak{g}^{a_s}, \mathfrak{g}^{a_s}]$ so that $\mathfrak{s}$ is semisimple and

$$\text{(1)} \qquad\qquad\qquad \mathfrak{g}^{a_s} = \mathfrak{c} \oplus \mathfrak{s}.$$

Proposition. *Let G be a connected reductive algebraic group.*

(2) *Let $a \in G$ so that $a = a_s a_u$. Then a_u is in the unipotent variety U_{a_s} of the reductive subgroup $G^{a_s}_o$ of G. Conversely if $b \in U_{a_s}$ then $g = a_s b$ is the Jordan decomposition of g so that*

$$a_s U_{a_s} = \{b \in G \mid b_s = a_s\}$$

(3) *If $a \in G$ then a is regular if and only if a_u is principal unipotent in $G^{a_s}_o$.*

(4) *If $X \in \mathfrak{g}$ then X is regular if and only if X_n is principal nilpotent in $\mathfrak{g}^{X_s}$.*

(5) *If $a \in G$ let H be a Cartan subgroup of G which contains a_s and let $\mathfrak{h} = \operatorname{Lie} H$. Let C be the center of $G^{a_s}_o$. Then $G^{a_s}_o = CS$ and $a_s \in C$ so that a_s and hence a are in $G^{a_s}_o$. Furthermore $C \subset H$ and $\mathfrak{c} \subset \mathfrak{h}$ and in addition $\operatorname{Lie} C = \mathfrak{c}$.*

In spite of (5) it is not true in general that C is connected. In fact if a_s corresponds to a vertex of the fundamental simplex then $\mathfrak{c} = 0$ and C is finite.

Proof. (2) It clearly suffices to show that a_u is in the identity component of G^{a_s}. But this is immediate from the bijection between U and N defined by the exponential map. The latter implies that if $x = \log(a_u)$ then $x \in \mathfrak{g}^{a_s}$. Hence the one parameter subgroup defined by x is in $G^{a_s}_o$ so that $a_u \in G^{a_s}_o$.

(3) and (4) It suffices by an identical argument to consider only the group case. Clearly by the uniqueness of the Jordan decomposition and (2) we have

$$\text{(6)} \qquad\qquad\qquad \mathfrak{g}^a = \mathfrak{g}^{a_s} \cap \mathfrak{g}^{a_u} = (\mathfrak{g}^{a_s})^{a_u}$$

Of course rank $\mathfrak{g}^{a_s} = l$ since a_s is conjugate to an element in H. Thus

$$(7) \qquad\qquad \text{rank } \mathfrak{c} + \text{rank } \mathfrak{s} = l = \text{rank } G$$

But if S is the semisimple subgroup corresponding to $\mathfrak{s}$ then clearly $U_{a_s} \subset S$ so that $a_u \in S$ and hence by (6) one has

$$(8) \qquad\qquad \mathfrak{g}^a = \mathfrak{c} \oplus \mathfrak{s}^{a_u}$$

so that $\dim \mathfrak{g}^a = \text{rank } \mathfrak{c} + \dim \mathfrak{s}^{a_u}$. But $\dim \mathfrak{s}^{a_u} \geq \text{rank } \mathfrak{s}$ and by definition of principal unipotent one has $\dim \mathfrak{s}^{a_u} = \text{rank } \mathfrak{s}$ if and only if a_u is principal unipotent in S or equivalently in $G_o^{a_s}$. Now the result follows from (7).

(5) That $G_o^{a_s} = C\,S$ follows from (1). Obviously $H \subset G_o^{a_s}$ and hence H is a Cartan subgroup of $G_o^{a_s}$. But since $H \subset G_o^{a_s}$ it follows that $a_s \in G_o^{a_s}$. But then obviously $a_s \in C$. But since the center of a connected reductive group lies in every Cartan subgroup one has $C \subset H$. We get $\text{Lie } C = \mathfrak{c}$ since the centers correspond to each other under the Lie subgroup – Lie subalgebra correspondence. $\square$

4.9. Let $z_a = \log(a_u)$ so that $z_a \in N_{a_s}$. By the theorem of Jacobson-Morosov, see [7], p.983, there exists $h_a \in \mathfrak{s}$ so that $[h_a, z_a] = 2z_a$. But then if $r_a(t) = \exp(t\,h_a)$ one has clearly has that

$$\lim_{t \to -\infty} r_a(t)\, a_u\, r_a(t)^{-1} = 1$$

But of course $r_a(t)$ commutes with a_s since $a_s \in C$. Thus

$$(1) \qquad\qquad \lim_{t \to -\infty} r_a(t)\, a\, r_a(t)^{-1} = a_s$$

4.10. Corollary. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation.*

If for some $a \in G$ the differential $d\Phi(a_s) : T_{a_s}G \to \mathfrak{g}$ is invertible then also $d\Phi(a) : T_aG \to \mathfrak{g}$. This is the case if a_s is hyperbolic, by (5.4) below.

Proof. This is an immediate consequence of (4.9.1). $\square$

4.11. We will begin to establish results leading to the main theorem on the commutativity of the generalized Cayley mapping and the operation of taking the semisimple part for Jordan decompositions. We will use the notation of (4.8).

Theorem. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation. Let $a \in G$.*

Then for the semisimple parts we have $\Phi(a_s) = \Phi(a)_s$ and the Jordan decomposition is the decomposition into components with respect to (4.8.8)

$$\Phi(a) = \Phi(a_s) + \Phi(a)_n \in \mathfrak{g}^a = \mathfrak{c} \oplus \mathfrak{s}^{a_u}.$$

Proof. Let $\Phi(a) = Z + F \in \mathfrak{g}^a = \mathfrak{c} \oplus \mathfrak{s}^{a_u}$ be the decomposition into components with respect to (4.8.8). Recall from (4.9) the curve $r_a : \mathbb{R} \to S$ satisfying $\lim_{t \to -\infty} r_a(t)\, a\, r_a(t)^{-1} = a_s$ by (4.9.1). Hence by the continuity of Φ one has

$$\Phi(a_s) = \lim_{t \to -\infty} \text{Ad}_{r_a(t)}(\Phi(a)) = \lim_{t \to -\infty} \text{Ad}_{r_a(t)}(Z + F) = Z + \lim_{t \to -\infty} \text{Ad}_{r_a(t)} F.$$

By (2.4.2) and (4.8.1) we have $\Phi(a_s) \in \text{Cent}(\mathfrak{g}^{a_s}) = \mathfrak{c}$. But also $\text{Ad}_{r_a(t)} F \in \mathfrak{s}$ by (4.9) so that $\lim_{t \to -\infty} \text{Ad}_{r_a(t)} F = 0$ which implies that F is nilpotent, and $Z = \Phi(a_s)$ which is semisimple since $\mathfrak{c} \subseteq \mathfrak{h}$ by (4.8.5). Finally note that $[Z, F] = 0$ since $\mathfrak{c} = \text{Cent}(g^{a_s})$ so that the result follows. $\square$

4.12. We now consider the nilpotent part of $\Phi(a)$. The situation is more complicated. Let $b \in G$ be semisimple. If $w \in U_b$ then of course

$$(1) \qquad\qquad (bw)_s = b \quad \text{and} \quad (bw)_u = w.$$

For any $w \in U_b$ one has $\Phi(bw)_n \in N_b$ since $\Phi(bw)_n$ is, by the uniqueness of the Jordan decomposition, clearly invariant under $Ad\, b$. One thus obtains a map

$$(2) \qquad\qquad \Phi_b : U_b \to N_b, \quad \Phi_b(w) = \Phi(bw)_n.$$

Proposition. *Let $b \in G$ be semisimple. The map $\Phi_b : U_b \to N_b$ is a regular morphism which commutes with the adjoint actions of G^b.*

Proof. The commutation of Φ_b with the adjoint action of G^b is obvious. The rationality is an immediate consequence of Theorem (4.11) since it clearly imply that

$$(3) \qquad\qquad \Phi_b(w) = \Phi(bw) - \Phi(b). \quad \square$$

4.13. Theorem. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation with $\Phi(e) = 0 \in \mathfrak{g}$. Let $b \in G$ be semisimple and suppose that $d\Phi(b) : T_bG \to \mathfrak{g}$ is invertible. This holds for b hyperbolic, see (5.4).*

   Then $\Phi_b : U_b \to N_b$ is an isomorphism of algebraic varieties.

   This is generalization of theorem (4.6). It also follows directly from the 'lemme fondamental' in [14], as in Richardsons proof of (4.6).

Proof. The proof of (4.6) that $\Phi : U \to N$ is an algebraic isomorphism is a consequence of the fact that Φ carries a principal unipotent orbit to a principal nilpotent orbit. Replacing G by the semisimple algebraic group S (using the notation of (4.8) where $b = a_s$) the same argument yields the isomorphism of Φ_b as soon as we demonstrate that Φ_b carries a principal unipotent orbit in U_b to a principal nilpotent orbit in N_b. Let $w \in U_b$ be a principal unipotent element in S. But bw is a regular element in G by (4.8.3). Using the notation of the proof of Proposition (4.8) where $a = bw$, $a_s = b$, $a_u = w$ one has $\dim \mathfrak{g}^{bw} = l$ and $\mathfrak{g}^{bw} = \mathfrak{c} \oplus \mathfrak{s}^w$ where if $c = \dim \mathfrak{c}$ and $s = \mathfrak{s}^w$ then $l = c + s$ and $s = \operatorname{rank} \mathfrak{s}$. One the other hand if $\mathfrak{v} = \mathfrak{s}^{\Phi_b(w)}$ and $v = \dim \mathfrak{v}$ then $v \geq s$ and $v = s$ if and only if $\Phi_b(w)$ is principal nilpotent in S. But, by Theorem (4.11) clearly

$$(1) \qquad\qquad \mathfrak{c} \oplus \mathfrak{v} \subset \mathfrak{g}^{\Phi(bw)}$$

But now by assumption $d\Phi(b)$ is invertible. Thus $d\Phi(bw)$ is invertible by Proposition (4.10). But then $\mathfrak{g}^{bw} = \mathfrak{g}^{\Phi(bw)}$ by Theorem 1.8. In particular $\dim \mathfrak{g}^{\Phi(bw)} = l$. But the left side of (1) has dimension $c + v$. Thus one must have $v = s$ (and equality in (1)). Hence $\Phi_b(w)$ is a principal nilpotent in S. $\quad \square$

5. The mapping degree of the Cayley mapping

5.1. The mapping degree of Φ for compact G. Let now G be a compact group and the ground field be the reals. Then $\Phi(G)$ is compact in $\mathfrak{g}$, so $\Phi : G \to \mathfrak{g}$ is not surjective. Let us embed $\mathfrak{g}$ into the one-point-compactification $\mathfrak{g} \cup \infty = S^n$, then the topological mapping degree of $\Phi : G \to \mathfrak{g} \cup \infty$ is 0 since Φ is not surjective. Thus over a regular value of Φ which exists by the theorem of Sard, there is an even number of sheets: on one half of these Φ is orientation preserving, on the other half it is orientation reversing.

5.2. The mapping degree in the complex case. Since in general there are conjugacy orbits on G which map to points in $\mathfrak{g}$, the mapping Φ is not proper in the sense of (Hausdorff) topology (which means that compact sets have compact inverse images; in the usual topology on $\mathfrak{g}$). But by (2.4.1) the mapping Φ induces a mapping between the algebraic orbit spaces $\bar{\Phi} : G // \operatorname{Conj}_G \to \mathfrak{g} // \operatorname{Ad}_G$, i.e. the affine varieties with coordinate rings $A(G)^G$ and $A(\mathfrak{g})^G$, respectively.

Theorem. *Let G be a connected reductive complex Lie group and let Φ be the Cayley mapping of a representation. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra with Cartan subgroup H. If the Cayley mapping $\Phi : H \to \mathfrak{h}$ is proper then we have:*

(1) *The mapping degree of $\Phi : H \to \mathfrak{h}$ is positive and consequently the mapping is surjective.*

(2) *$\bar{\Phi} : G // \operatorname{Conj}_G \to \mathfrak{g} // \operatorname{Ad}_G$ is a proper mapping for the Hausdorff topologies on the affine varieties. Thus $\bar{\Phi}$ has positive mapping degree and is surjective.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra with Cartan subgroup H. By (2.4.5) we have $\Phi : H \to \mathfrak{h}$, and we assume that this mapping is proper.

(1) $\Phi : H \to \mathfrak{h}$ is a proper smooth mapping, thus its mapping degree is defined as the value of the induced mapping in the top De Rham cohomology with compact supports which is isomorphic to $\mathbb{R}$ via integration:

$$\deg(\Phi) = \Phi^* : H_c^l(\mathfrak{h}; \mathbb{R}) = \mathbb{R} \to H_c^l(H, \mathbb{R}) = \mathbb{R}, \quad l = \dim(\mathfrak{h}).$$

Choose a regular value Y of Φ. Then for each $g \in \Phi^{-1}(Y)$ the tangent mapping $T_g\Phi$ is invertible and orientation preserving, since it is complex holomorphic. Thus the mapping degree is the number of sheets over a regular point, which is positive. But then $\Phi : H \to \mathfrak{h}$ has to be surjective: if not, its image is closed, and a n-form with support in the complement is pulled backed to 0 on H, in contradiction to the positivity of the mapping degree.

(2) Since $G // \operatorname{Conj}_G \cong H/W$ and $\mathfrak{g} // \operatorname{Ad}_G \cong \mathfrak{h}/W$ where W is the Weyl group, the result follows directly from (1). $\square$

5.3. Lemma. *Suppose that for reals r_i we have $\sum_{i=1}^N r_i = 0$. Then*

$$\sum_{i=1}^N r_i e^{r_i} \geq \frac{1}{2N} \sum_{i=1}^N r_i^2.$$

Proof. We separate negative and positive summands and consider for $s_i > 0$ and $t_j > 0$ the expression

$$-\sum_{i=1}^n s_i e^{-s_i} + \sum_{j=1}^m t_j e^{t_j}, \qquad \sum_{i=1}^n s_i = A = \sum_{j=1}^m t_j.$$

The function $f(x) = x\,e^x$ is strictly convex for $x > 0$, thus

$$\frac{1}{m}\sum_{j=1}^{m} f(t_j) \geq f\Big(\frac{1}{m}\sum_{j=1}^{m} t_j\Big) = f(A/m)$$

with equality only if all t_j are equal. Moreover, $-\sum_{i=1}^{n} s_i e^{-s_i} \geq -\sum_{i=1}^{n} s_i = -A$. For the right hand side we have $A^2 = \Big(\sum_{j=1}^{m} t_j\Big)^2 \geq \sum_{j=1}^{m} t_j^2$ and similarly $A^2 \geq \sum_{i=1}^{n} s_i^2$. So finally

$$-\sum_{i=1}^{n} s_i e^{-s_i} + \sum_{j=1}^{m} t_j e^{t_j} \geq -A + m\frac{A}{m} e^{A/m} \geq -A + A\Big(1 + \frac{A}{m}\Big) = \frac{A^2}{m} = \frac{1}{2m} 2A^2$$

$$\geq \frac{1}{2m}\Big(\sum_{i=1}^{n} s_i^2 + \sum_{j=1}^{m} t_j^2\Big). \quad \square$$

5.4. Theorem. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation π. Let $G_{\mathrm{hyp}} \subset G$ be the subset of all semisimple hyperbolic elements in G, and let $\mathfrak{g}_{\mathrm{hyp}}$ be the set of all semisimple hyperbolic elements in the Lie algebra $\mathfrak{g}$.*

Then for each $g \in G_{\mathrm{hyp}}$ the differential $d\Phi(g) : T_g G \to \mathfrak{g}$ is invertible.

Moreover, the mapping $\Phi_\pi : G_{\mathrm{hyp}} \to \mathfrak{g}_{\mathrm{hyp}}$ is bijective and a diffeomorphism between real subvarieties if $\Phi_\pi(e) = 0$, or also if the representation π is self-contragredient.

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra with Cartan subgroup H. Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} + i\mathfrak{h}_{\mathbb{R}}$. Let $g \in G$ be a hyperbolic element which we assume to lie in H. Then $g = \exp X$ where $X \in \mathfrak{h}_{\mathbb{R}}$, so $g \in H_{\mathbb{R}}$. We have $\Phi(g) \in \mathfrak{h}_{\mathbb{R}}$ since

$$\mathrm{tr}(\pi'(\Phi(g))\pi'(Y)) = \mathrm{tr}(\pi(g)\pi'(Y)) \in \mathbb{R}$$

for all $Y \in \mathfrak{h}_{\mathbb{R}}$. We choose a maximal compact subgroup $G_u \subset G$ such that $i\mathfrak{h}_{\mathbb{R}} \subset \mathrm{Lie}(G_u) =: \mathfrak{g}_u$. Then there exists a Hermitian inner product on V (a Hilbert space structure) such that all elements of $\pi(G_u)$ are unitary. Then $\pi'(\mathfrak{g}_u)$ consists of skew Hermitian operators, and $\pi'(i\mathfrak{g}_u)$ consists of Hermitian operators. Thus $\pi'(\mathfrak{h}_{\mathbb{R}})$ and $\pi(g)$ are Hermitian operators. We use $\mathrm{tr}(AB)$ as real positive inner product on the the space of Hermitian operators on V.

We have $g = h^2$ for $h = \exp(\frac{1}{2}\log(g)) \in H_{\mathbb{R}}$. We claim that $\pi(g) : V \to V$ is a positive definite Hermitian operator, i.e. $(X, Y) \mapsto \mathrm{tr}(\pi(g)\pi'(X)\pi'(Y)^*)$ is a positive definite Hermitian form on $\mathfrak{g}$. Namely, for $X \in \mathfrak{g}$ arbitrary we have $X = X_1 + X_2$ for unique $X_1 \in \mathfrak{g}_u$ and $X_2 \in i\mathfrak{g}_u$. Let $X^* := X_1 - X_2$, then $\pi'(X^*)$ equals the adjoint operator $\pi'(X)^*$. We have

$$\mathrm{tr}(\pi(g)\pi'(X)\pi'(X)^*) = \mathrm{tr}(\pi(h)^2 \pi'(X)\pi(X)^*) =$$
$$= \mathrm{tr}(\pi(h)\pi'(X)(\pi(h)\pi'(X))^*) = \|\pi(h)\pi'(X)\|^2 > 0.$$

By (2.4.7) this implies that $d\Phi(g) : T_g G \to \mathfrak{g}$ is invertible.

For the second assertion, we claim that in both cases we have for $C > 0$

(1) $$\|\pi'(\Phi(g))\| \geq C\|\pi'(X)\|.$$

If $\Phi(e) = 0$ we have

$$0 = \mathrm{tr}(\pi'(\Phi(e))\pi'(X)) = \mathrm{tr}(\pi(e)\pi'(X)) = \mathrm{tr}(\pi'(X))$$
$$= \sum_{\lambda \in \mathrm{weight}(\pi)} \mathrm{tr}(\pi'(X)|V_\lambda) = \sum_{\lambda \in \mathrm{weight}(\pi)} \dim(V_\lambda)\lambda(X).$$

Thus by lemma (5.3) we have

$$\sum_{\lambda \in \mathrm{weight}(\pi)} \dim(V_\lambda)\lambda(X)e^{\lambda(X)} \geq C \sum_{\lambda \in \mathrm{weight}(\pi)} \dim(V_\lambda)\lambda(X)^2$$

for a positive constant C. Then we have by Cauchy Schwarz,

$$\|\pi'\Phi(g)\|\|\pi'(X)\| \geq \mathrm{tr}(\pi(g)\pi'(X)) = \sum_{\lambda \in \mathrm{weight}(\pi)} \mathrm{tr}(\pi(\exp(X))\pi'(X)|V_\lambda)$$
$$= \sum_{\lambda \in \mathrm{weight}(\pi)} \dim(V_\lambda)\lambda(X)e^{\lambda(X)} \geq C \sum_\lambda \lambda(X)^2 \dim(V_\lambda)$$
$$= C \sum_\lambda \mathrm{tr}(\pi'(X)^2|V_\lambda) = C \, \mathrm{tr}(\pi'(X)^2) = C\|\pi'(X)\|^2.$$

If the representation π is self-contragredient, a similar argument works using $re^r - re^{-r} \geq 2r$ for $r > 0$ instead of the inequality (5.3).

Now we may finish the proof. Note that $\exp : \mathfrak{h}_\mathbb{R} \to H_\mathbb{R}$ is a diffeomorphism with inverse $\log : H_\mathbb{R} \to \mathfrak{h}_\mathbb{R}$. Thus estimate (1) implies that $\Phi : H_\mathbb{R} \to \mathfrak{h}_\mathbb{R}$ is a proper mapping. It is also a local diffeomorphism, thus a covering mapping and a diffeomorphism since $\mathfrak{h}_\mathbb{R}$ is vector space.

Finally, each hyperbolic element $g \in G$ is contained in $H_\mathbb{R}$ for a suitable Cartan subgroup H, and the above arguments show that $\Phi : G_{\mathrm{hyp}} \to \mathfrak{g}_{\mathrm{hyp}}$ is locally a diffeomorphism and is surjective. It is also injective: Let $\Phi(g_1) = Y = \Phi(g_2)$. Then g_1 and some conjugate $h\,g_2\,h^{-1}$ lie in the same Cartan subgroup $H_\mathbb{R}$ on which Φ is a diffeomorphism, thus $g_1 = h.g_2.h^{-1}$. By equivariancy we have $\mathrm{Ad}(h)Y = Y$. Since Φ is a local diffeomorphism near g_1, the orbits have the same dimension, thus g_1 and Y have the same connected component of the isotropy group. But isotropy groups of semisimple elements of the Lie algebra are connected, as mentioned in (4.8). Thus $g_1 = g_2$. $\square$

5.5. Theorem. *Let G be a connected reductive complex algebraic group and let Φ be the Cayley mapping of a rational representation with $\Phi(e) = 0 \in \mathfrak{g}$.*

Then $\Phi : G_{\mathrm{pos}} \to \mathfrak{g}_{\mathrm{real}}$ is bijective and a fiber respecting isomorphism of real algebraic varieties, where G_{pos} and $\mathfrak{g}_{\mathrm{real}}$ are defined in (4.1).

Proof. Let $g \in G_{\mathrm{pos}}$, then $g_s = g_h$, thus by (5.4) $d\Phi(g_s) : T_{g_s}G \to \mathfrak{g}$ is invertible and thus by theorem (4.13) $\Phi_{g_s} : U_{g_s} \to N_{g_s}$ is an automorphism of algebraic varieties,

where U_{g_s} is the unipotent variety in the reductive group $G_o^{g_s}$, see (4.8). The set $G_{\text{pos}} = \bigsqcup_{h \in G_{\text{hyp}}} h.U_h \to G_{\text{hyp}}$ is a fibration with complex algebraic varieties as fibers and the real algebraic variety G_{hyp} as base. Likewise $\mathfrak{g}_{\text{real}} = \bigsqcup_{h \in G_{\text{hyp}}} (\Phi(h) + N_h) \to \mathfrak{g}_{\text{hyp}}$ is a is a fibration with the nilpotent cones N_h as fibers. $\Phi : G_{\text{hyp}} \to \mathfrak{g}_{\text{hyp}}$ is given by $\Phi(g) = \Phi(g_s) + \Phi_{g_s}(g_u) = \Phi(g)_s + \Phi(g)_n$ and is a fiber respecting isomorphism by theorem (4.13) and (5.4). $\square$

6. EXAMPLES

6.1. The Cayley mappings for the representations of $SL_2(\mathbb{C})$.

The standard representation $SL_2(\mathbb{C}) \subset \text{End}(\mathbb{C}^2)$. Here $\Phi(A) = A - \frac{1}{2} \operatorname{tr}(A) 1_{\mathbb{C}^2}$ and $\Psi(A) = \frac{1}{2} \operatorname{tr}(A)$ for $A \in SL_2$. Φ is surjective and proper and has mapping degree 2. On the Cartan subgroup we get

$$\Phi_{\pi_1} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1-a^2}{2a} & 0 \\ 0 & \frac{a^2-1}{2a} \end{pmatrix}, \qquad \Psi_{\pi_1} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{a^2+1}{2a}.$$

For the $(n+1)$-dimensional representation we computed

$$\Phi_{\pi_n} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{3}{n^3 + 3n^2 + 2n} \sum_{p=0}^{n} (n - 2p) a^{n-2p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Psi_{\pi_n} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{-108}{(n^3 + 3n^2 + 2n)^3} \sum_{p_1,p_2,p_3=0}^{n} (n - 2p_1)^2 (n - p_2) n a^{3n - 2(p_1+p_2+p_3)}$$

6.2. Example: The standard representation of $SL_n(\mathbb{C})$. *The standard representation $SL_n(\mathbb{C}) \subset \text{End}(\mathbb{C}^n)$. Here $\Phi(A) = A - \frac{1}{n} \operatorname{tr}(A) 1_{\mathbb{C}^n}$, and Φ has mapping degree n, which coincides with the degree of the smooth hypersurface $SL_n \subset \text{End}(\mathbb{C}^n)$. Thus Φ maps regular orbits in SL_n to regular orbits in $\mathfrak{sl}_n$, and by (3.2) we have*

$$A(SL_n) = A(SL_n)^{SL_n} \otimes \text{Harm}(SL_n)).$$

The set G_{sing} where the differential of the Cayley transform is singular is given by $\Psi^{-1}(0) = (SL_n \cap \mathfrak{sl}_n)^{-1} = \{A \in SL_n : \operatorname{tr}(A^{-1}) = 0\}$.
Let $\chi \in A(SL_n)$ be given by $\chi(A) = \frac{1}{n} \operatorname{tr}(A)$. Then

$$A(SL_n)^{SL_n} = A(\mathfrak{sl}_n)^{SL_n}[\chi],$$
$$Q(SL_n)^{SL_n} = Q(\mathfrak{sl}_n)^{SL_n}[\chi],$$
$$Q(SL_n) = Q(\mathfrak{sl}_n)[\chi],$$

in terms of (3.3).

This can be shown as follows: $\text{End}(\mathbb{C}^n) = \mathfrak{sl}_n \oplus \mathbb{C}.1_n$ is an orthogonal direct sum, thus the orthogonal projection is given by $\operatorname{pr}(A) = A - \frac{1}{n} \operatorname{tr}(A)$. Of course $\Phi = \operatorname{pr} |SL_n$. For $Y \in \mathfrak{sl}_n$ we have $A \in \Phi^{-1}(Y)$ if and only if $A - Y = t.1_n$ for some $t \in \mathbb{C}$. The set of all these t is given by the equation $\det(Y + t.1_n) = 1$. Generically there are n solution, thus the degree of Φ is n, and Φ is surjective.

For $A \in SL_n$ and $X \in \mathfrak{sl}_n$ then $A.X$ is a typical tangent vector in $T_A(SL_n)$. Now $d\Phi(A) = \Phi : T_A(SL_n) \to \mathfrak{sl}_n$ is not invertible if its kernel $\mathbb{C}.1_n \cap T_A(SL_n)$ is non trivial, so there exists an $X \in \mathfrak{sl}_n$ with $A.X = 1_n$ (by scaling X appropriately). But then $X = A^{-1} \in \mathfrak{sl}_n$ and thus $\mathrm{tr}(A^{-1}) = 0$.

Let us discuss the adjunction now. The fiber over any $X \in \mathfrak{sl}_n$ consists of all $A = X + t.1_n$ where t runs through the roots of the polynomial $f_X(t) = \mathrm{tr}(t.1_n + X) - 1$, and moreover $t = \chi(A)$, by the formula for Φ. Let us consider the expansion

$$f_X(t) = \mathrm{tr}(t.1_n + X) - 1 = \sum_{j=0}^{n} p_j(X) \, t^j,$$

where $p_j \in A(\mathfrak{sl}_n)^{SL_n}$ form a system of generators with $p_0(X) = \det(X) - 1$, $p_{n-1}(X) = \mathrm{tr}(X) = 0$, and $p_n(X) = 1$. We have

$$f_X(\chi) = \mathrm{tr}(t.1_n + X) - 1 = \sum_{j=0}^{n} p_j(X) \, \chi^j = 0,$$

and this is the minimal polynomial for χ over $A(\mathfrak{sl}_n)^{SL_n}$. Finally note that χ is $1/n$ times the character of the standard representation and Φ^* pulls back the elements $p_j \in A(\mathfrak{sl}_n)^{SL_n}$ to the characters of the remaining fundamental representations of SL_n. Thus $A(SL_n)^{SL_n} = A(\mathfrak{sl}_n)^{SL_n}[\chi]$. The other assertions follow from (3.3).

6.3. The standard representation of $O_n(\mathbb{C})$. In the standard representation $O_n(\mathbb{C}) \to \mathrm{End}(\mathbb{C}^n)$ we have

$$O_n(\mathbb{C}) = \{A \in \mathrm{End}(\mathbb{C}^n) : AA^\top = 1_n\}$$
$$\mathfrak{so}_n(\mathbb{C}) = \{X \in \mathrm{End}(\mathbb{C}^n) : X + X^\top = 0\}$$
$$\Phi(A) = \tfrac{1}{2}(A - A^\top).$$

7. Spin representations and Cayley transforms

In this section we treat only the complex group $\mathrm{Spin}(n, \mathbb{C})$. Some of the results here were inspired by [13]. We first recall notations and results from [12].

7.1. Clifford multiplication in terms of exterior algebra operations. Let V be a complex vector space of finite dimension $n > 0$. Assume that $B_V = (\ , \ )$ is a fixed nonsingular symmetric bilinear form on V. Its extends naturally to a symmetric nonsingular bilinear form $B_{\wedge V}$ on the exterior algebra $\wedge V$. The graded structure on $\wedge V$ induces a $\mathbb{Z}$-graded structure on the operator algebra $\mathrm{End}(\wedge V)$ and also on the $\mathbb{Z}$-graded super Lie algebra $\mathrm{Der}(\wedge V)$ of graded super derivations. For $\beta \in \mathrm{End}(\wedge V)$ we denote its transpose by $\beta^t \in \mathrm{End}(\wedge V)$ so that $(\beta u, v) = (u, \beta^t v)$ for any $u, v \in \wedge V$.

For any $u \in \wedge V$ let $\epsilon(u) \in \mathrm{End}(\wedge V)$ (left exterior multiplication) be defined so that $\epsilon(u)w = u \wedge w$ for any w. For any $u \in \wedge V$ let $\iota(u) = \epsilon(u)^t$. If $u \in \wedge^k V$ then $\epsilon(u) \in \mathrm{End}^k(\wedge V)$ and $\iota(u) \in \mathrm{End}^{-k}(\wedge V)$. Furthermore if $x \in V$ then $\iota(x) \in \mathrm{Der}^{-1}\wedge V$. In fact $\iota(x)$ is the unique element of $\mathrm{Der}^{-1}(\wedge V)$ such that $\iota(x)y = (x, y)$ where we have identified $\mathbb{C} = \wedge^0 V$. For a basis $\{z_j\}$ of V we have

$\mathrm{Der}(\wedge V) = \bigoplus_j \epsilon(\wedge V)\iota(z_j)$. Let $\kappa \in \mathrm{End}^0(\wedge V)$ be defined so that $\kappa = (-1)^k$ on $\wedge^k V$. Let $\wedge^{\mathrm{even}} V$ and $\wedge^{\mathrm{odd}} V$ be the eigenspaces for κ corresponding, respectively, to the eigenvalues 1 and -1.

We recall that the Clifford algebra $C(V)$ over V with respect to $B_V|V$ is the tensor algebra $T(V)$ over V modulo the ideal generated by all elements in $T(V)$ of the form $x \otimes x - (x, x)$ where $x \in V$ and we regard $\mathbb{C} = T^0(V)$. We note that $\epsilon(x)\iota(y) + \iota(y)\epsilon(x) = (x, y) \in \mathrm{End}(\wedge V)$. For any $x \in V$ let $\gamma(x) = \epsilon(x) + \iota(x)$ then $\gamma(x)^2 = (x, x)$. Thus the correspondence $x \mapsto \gamma(x)$ extends to a homomorphism $C(V) \to \mathrm{End}(\wedge V)$, denoted by $u \mapsto \gamma(u)$, defining the structure of a $C(V)$-module on $\wedge V$. This leads to the map $u \mapsto \gamma(u)1$, denoted by $\psi : C(V) \to \wedge V$. By Theorem II.1.6, p. 41 in [3] this map ψ is bijective and we will identify $C(V)$ with $\wedge V$, using ψ. We consequently then recognize that the linear space $\wedge V = C(V)$ has 2 multiplicative structures. If $u, v \in \wedge V$ there is exterior multiplication $u \wedge v$ and Clifford multiplication $uv = \gamma(u)v$. Also we will write $C^0(V) = \wedge^{\mathrm{even}} V$ and $C^1(V) = \wedge^{\mathrm{odd}} V$ so that $C^i(V)C^j(V) \subset C^{i+j}(V)$ for $i, j \in \mathbb{Z}/(2)$. That is, the parity automorphism κ of the exterior algebra is also an automorphism of the Clifford algebra. In addition if $\alpha \in \mathrm{End}\,\wedge V$ is the unique involutory antiautomorphism of $\wedge V$, as an exterior algebra, such that $\alpha(x) = x$ for all $x \in V$, then α is also an antiautomorphism of the Clifford algebra structure; explicitly, $\alpha = (-1)^{\frac{k(k-1)}{2}}$ on $\wedge^k V$. Using α and κ one can show easily that for $x \in V$ and $u \in C(V)$ we have $ux = (\varepsilon(x) - \iota(x))\kappa(u)$.

Let $\mathrm{Spin}(V)$ be the set of all $g \in C^0(V)$ such that $g\alpha(g) = 1$ and $gx\alpha(g) \in V$ for all $x \in V$. This is an algebraic group under Clifford multiplication. For any $g \in \mathrm{Spin}\,V$ let $T(g) \in GL(V)$ be the mapping given by $T(g)x = gx\alpha(g)$ for $x \in V$.

The Lie algebra $\mathrm{Lie}\,\mathrm{Spin}\,V$ is a Lie subalgebra of $C^0(V)$, in fact $\mathrm{Lie}\,\mathrm{Spin}\,V = \wedge^2 V$ as explained on p. 68 in [3]. We need this in detail: Let $u \in \wedge^2 V$. For any $x \in V$ one has $\iota(x)u \in V$ and one can define $\tau(u) \in \mathfrak{so}(V) \subset \mathrm{End}(V)$ by $\tau(u)x = -2\iota(x)u$.

For any $z \in \mathrm{End}\,V$ there exists a unique operator $\xi(z) \in \mathrm{Der}^0(\wedge V)$ which extends the action of z on V. Clearly $\xi : \mathrm{End}\,V \to \mathrm{Der}^0(\wedge V)$ is a Lie algebra isomorphism.

7.2. Theorem. $\mathrm{Spin}\,V \subset C^0(V)$ *is a connected Lie group. If* $g \in \mathrm{Spin}\,V$ *then* $T(g) \in SO(V)$ *and* $T : \mathrm{Spin}\,V \to SO(V)$ *is an epimorphism with kernel* $\{\pm 1\}$. *In particular* T *defines* $\mathrm{Spin}\,V$ *as a double cover of* $SO(V)$.

The subspace $\wedge^2 V$ *is a Lie subalgebra of* $C^0(V)$. *In fact* $\wedge^2 V = \mathrm{Lie}\,\mathrm{Spin}\,V$ *so that* $\mathrm{Spin}\,V$ *is the group generated by all* $\exp(u)$ *for* $u \in \wedge^2 V$ *where exponentiation is with respect to Clifford multiplication.*

Furthermore the map $\tau : \wedge^2 V \to \mathrm{Lie}\,SO(V)$ *is a Lie algebra isomorphism and is the differential of the double cover* $T : \mathrm{Spin}\,V \to SO(V)$. *Finally if* $u \in \wedge^2 V$ *then* $\mathrm{ad}(u) = \xi(\tau(u))$ *so that not only is* $\mathrm{ad}(u)$ *a derivation of both exterior algebra and Clifford algebra structures on* $C(V)$ *but also* $\mathrm{ad} : \wedge^2 V \to \mathrm{Der}^0 \wedge V$ *is a faithful representation of the Lie algebra* $\wedge^2 V$ *on* $\wedge V$.

7.3. Let $\{z_i\}$ be a B_V-orthonormal basis of V. On these elements the Clifford product equals the exterior product, $z_i z_j = z_i \wedge z_j$. Let $N = \{1, \ldots, n\}$ and let $\mathcal{I}$ be the set of all subsets of N. We regard any subset I as ordered: $I = \{i_1 < \ldots < i_k\}$. Let

$$z_I = z_{i_1} \cdots z_{i_k} = z_{i_1} \wedge \cdots \wedge z_{i_k}.$$

The set of elements $\{z_I\}$ with $|I| = k$, is a $B_{\wedge V}$-orthonormal basis of $\wedge^k V$ so that the set $\{z_I\}, I \in \mathcal{I}, |I|$ is even (resp. odd) is a $B_{\wedge V}$-orthonormal basis of $C^0(V)$ (resp. $C^1(V)$) and the full set $\{z_I\}, I \in \mathcal{I}$ is a $B_{\wedge V}$-orthonormal basis of $C(V)$. Moreover for $I, J \in \mathcal{I}$ let $I \diamond J \in \mathcal{I}$ be the symmetric difference $(I \setminus J) \cup (J \setminus I)$. Then

$$z_I z_J = c_{I,J} z_{I \diamond J}, \qquad \text{where } c_{I,J} \in \{1, -1\},$$
$$(z_I)^{-1} = \alpha(z_I) = (-1)^{|I|(|I|-1)/2} z_I.$$

7.4. The spin module. The following description of the spin module S is uniform in n and the spin representation will always be a direct sum of 2 (possibly equivalent) irreducible representations.

Let $Z = \wedge^0 V + \wedge^n V$ so that Z is a 2-dimensional subspace of $C(V)$ and let $C^Z(V)$ be the centralizer of Z in $C(V)$. There exists (uniquely up to sign) an element $\mu \in \wedge^n V$ such that $\mu^2 = 1$. Hence if we put $e_+ = \frac{1}{2}(1 + \mu)$ and $e_- = \frac{1}{2}(1 - \mu)$ then $\{e_+, e_-\} \subset Z$ are orthogonal idempotents in the sense of ring theory and $1 = e_+ + e_-$. In particular, Z is a subalgebra of $C(V)$ and is isomorphic to $e_+ Z \oplus e_- Z = \mathbb{C} \oplus \mathbb{C} \subset C^Z(V)$. We have $x\mu = (-1)^{n+1} \mu v$ for any $v \in V$ so that $C^0(V) \subset C^Z(V)$ and in fact $C^Z(V) = ZC^0(V)$. Let $C^Z_+(V) = e_+ C^Z(V)$ and $C^Z_-(V) = e_- C^Z(V)$. Let $n_1 = [\frac{n+1}{2}]$. It is well known (see e.g. II.2.4 and II.2.6 in [3]) that both $C^Z_+(V)$ and $C^Z_-(V)$ are each isomorphic to a $2^{n_1-1} \times 2^{n_1-1}$ matrix algebra so that $C^Z(V)$ is a semisimple associative algebra of dimension 2^{2n_1-1} and the unique decomposition of $C^Z(V)$ into simple ideals is given by $C^Z(V) = C^Z_+(V) \oplus C^Z_-(V)$. In particular $Z = \text{Cent}(C^Z(V))$ and e_+ and e_- are, respectively, the identity elements of $C^Z_+(V)$ and $C^Z_-(V)$. Thus there exists a $C^Z(V)$-module S (the spin module) S of dimension 2^{n_1}, defined by a representation $\sigma : C^Z(V) \to \text{End } S$ characterized uniquely, up to equivalence, by the condition that if $S_+ = e_+ S$ and $S_- = e_- S$ then $S = S_+ \oplus S_-$ is the unique decomposition of S into irreducible $C^Z(V)$-modules. Let $\sigma_+ : C^Z(V) \to \text{End } S_+$ and $\sigma_- : C^Z(V) \to \text{End } S_-$ be the corresponding irreducible representations.

Recall $\text{Spin}(V) \subset C^0(V) \subset C^Z(Z)$. The restriction of σ to $\text{Spin}(V)$ will be denoted by Spin so that Spin is a representation of $\text{Spin}(V)$ on S. Its differential, also denoted by Spin, is the restriction σ to $\wedge^2 V$. Thus one has a Lie algebra representation Spin $: \wedge^2 V \to \text{End } S$. Replacing σ by $\sigma_\pm$ and S by $S_\pm$ one similarly has representations $\text{Spin}_\pm$ of $\text{Spin } V$ and $\wedge^2 V$ on $S_\pm$. Since $\wedge^2 V$ generates $C^0(V)$, both $\text{Spin}_\pm$ are irreducible representations.

Finally, let $n_0 = [\frac{n}{2}]$ so that one has $n = n_0 + n_1$. Note that if n is even then $n_1 = n_0 = \frac{1}{2}n$, whereas if n is odd then $n_1 = n_0 + 1$ so that $n_1 = \frac{1}{2}(n + 1)$ and $n_0 = \frac{1}{2}(n - 1)$.

7.5. Let $\text{pr}_k : C(V) \to \wedge^k V$ be the projection defined by the graded structure of $\wedge V$. We identify $\wedge^0 V$ with $\mathbb{C}$ so that the image of pr_0 is scalar-valued.

The Clifford algebra $C(V)$ as a module over itself by left multiplication (γ) is equivalent to 2^{n_0} copies of the $C(V)$ spin module S with respect to σ (see (7.4)). One has that $2^n = 2^{n_0} 2^{n_1}$

Proposition. *For any $w \in C(V)$ one has*

$$\mathrm{pr}_0(w) = \frac{1}{2^n} \operatorname{tr} \gamma(w) = \frac{1}{2^{n_1}} \operatorname{tr} \sigma(w).$$

Proof. Of course for $c \in \wedge^0 V$ one obviously has that $c = \frac{1}{2^n} \operatorname{tr} \gamma(c)$. With the notation of (7.3), it suffices to prove that $\operatorname{tr} \gamma(z_I) = 0$ if $k > 0$. But this follows from (7.3). $\square$

7.6. We now note that the bilinear form given by $\operatorname{tr} \sigma$ is essentially given by $B_{\wedge V}$.

Proposition. *Let $u, w \in C(V)$ Then*

$$\operatorname{tr} \sigma(u)\sigma(w) = 2^{n_1}(u, \alpha(w))$$

Proof. The left side is just $\operatorname{tr} \sigma(uw)$. By proposition (7.5) this is just $2^{n_1} \mathrm{pr}_0(uw)$. Using the basis z_I of (7.3) and writing $u = \sum_I u_I z_I$ and $w = \sum_I w_I z_I$ it is immediate from (7.3) that $\mathrm{pr}_0(uw) = \sum_I u_I w_I \mathrm{pr}_0(z_I^2)$. But clearly by (7.3) one has $(z_I, \alpha(z_J)) = 0$ for $I \neq J$ and $z_I^2 = \mathrm{pr}_0(z_I^2) = (z_I, \alpha(z_I))$. $\square$

7.7. The Cayley mapping. The generalized Cayley mapping $\Phi_T : \mathrm{Spin}(V) \to \mathrm{Lie}\,\mathrm{Spin}(V)$ for the representation $T : \mathrm{Spin}(V) \to SO(V)$ is given by $\operatorname{tr}(\sigma(g)\sigma(y)) = \operatorname{tr}(\sigma(\Phi_T(g))\sigma(y))$. By (7.2) one has $\mathrm{Lie}\,\mathrm{Spin}(V) = \wedge^2 V$.

Proposition. *Let $\mathfrak{g} \in \mathrm{Spin}(V)$. Then $\Phi_T(g) = \mathrm{pr}_2(g)$*

Proof. Let $y \in \wedge^2 V$. Let $x = \Phi_T(g)$ and let $z = \mathrm{pr}_2(g)$. Note that $\alpha(y) = -y$. By (7.6) one has

$$\operatorname{tr}(\sigma(g)\sigma(y)) = -2^{n_1}(g, y) = -2^{n_1}(\mathrm{pr}_2(g), y) = \operatorname{tr}(\sigma(\mathrm{pr}_2(g))\sigma(y)). \square$$

7.8. Let $\theta : SO(V) \to \mathrm{Aut}(\wedge V)$ be the representation so that if $a \in SO(V)$ then $\theta(a)$ is the unique exterior algebra automorphism which extends the action of a on V. Clearly

$$\operatorname{tr} \theta(a) = \det(1 + a)$$

Proposition. *For any $g \in \mathrm{Spin}(V)$ and $w \in C(V)$ then*

$$\theta(T(g))w = g\,w g^{-1}$$

In particular for $\theta(a)$, for $a \in SO(V)$ is an automorphism of both the Clifford and exterior algebra structures on $C(V)$.

Proof. Using the notation of (7.1) and (7.2) it is clear that $\xi | \mathrm{Lie}\, SO(V)$ is the differential of θ. To prove the proposition it suffices to establish its infinitesimal analogue. But this is stated in (7.2). $\square$

7.9. Proposition. *The tensor product representation* $\mathrm{Spin} \otimes \mathrm{Spin}$ *of* $\mathrm{Spin}(V)$ *on* $S \otimes S$ *descends to* $SO(V)$ *and is equivalent to* θ *if* n *is even and 2 copies of* θ *if* n *is odd.*

This is well known; we include a proof in the conventions used above.

Proof. Assume that M is a matrix algebra and β is an antiautomorphism of M. Let L be a minimal left ideal of M so that $(\dim L)^2 = \dim M$. Then clearly $R = \beta(L)$ is a minimal right ideal of M. There exists $v \in M$ such that $LvR \neq 0$ so that $LvR = M$ since LvR is a 2-sided ideal and M is simple. This implies that the map

$$\mu : L \otimes L \to M, \qquad \mu(a \otimes b) = a\,v\,\beta(b)$$

is surjective and thus a linear isomorphism by dimension. If $g \in M$ then clearly $\mu(ga \otimes gb) = g\mu(a \otimes b)\beta(g)$, which by proposition (7.8) proves the assertion in case n is even by choosing $M = C(V)$, $\beta = \alpha$, $L = S$ and $g \in \mathrm{Spin}(V)$.

If n is odd then $C(V) = M_1 \oplus M_2$ where M_i, $i = 1, 2$, are matrix algebras. Then $S = L_1 \oplus L_2$ where L_i, $i = 1, 2$, are respectively minimal left ideals in M_i (and left ideals in $C(V)$). Then

(1) $$S \otimes S = L_1 \otimes L_1 \oplus L_1 \otimes L_2 \oplus L_2 \otimes L_1 \oplus L_2 \otimes L_2$$

We have $\mathrm{Spin}(V) \subset C^0(V)$ which implies that all 4 summands on the right side of (1) define $\mathrm{Spin}(V)$-equivalent submodules of $\mathrm{Spin} \otimes \mathrm{Spin} : \mathrm{Spin}(V) \to \mathrm{Aut}(S \otimes S)$, since the parity automorphism $\kappa \in \mathrm{Aut}\, C(V)$ from (7.1) of course fixes $C^0(V)$ but interchanges M_1 and M_2. This is clear from the definition of $e_\pm$ in (7.4) since $\kappa = -1$ on $\wedge^n V$. Now if $i \in \{1, 2\}$ let $i' \in \{1, 2\}$ be defined so that $\alpha(M_i) = M_{i'}$. But then $\alpha(L_i) = R_{i'}$ is a minimal right ideal (and right ideal in $C(V)$) in $M_{i'}$. But then

$$C(V) = L_1\,M_1\,R_{1'} \oplus L_1\,M_1\,R_{2'} \ \oplus L_2\,M_2\,R_{1'} \oplus L_2\,M_2\,R_{2'},$$

where 2 of the 4 components are identically zero and the remaining 2 are M_1 and M_2. Note that the 2-sided ideal M_i in $C(V)$ is stable under θ by proposition (7.8). The argument in the even case then readily implies that θ is equivalent to 2 of the 4 components in (1) and hence $\mathrm{Spin} \otimes \mathrm{Spin}$ is equivalent to 2 copies of θ. $\square$

7.10. Theorem. *Let* $g \in \mathrm{Spin}(V)$ *then* $\mathrm{pr}_0(g) = 0$ *if* $g \notin \mathrm{Spin}(V)^* := \{g \in \mathrm{Spin}(V) : \det(1 + T(g)) \neq 0\}$. *On the other hand if* $g \in \mathrm{Spin}(V)^*$ *then*

$$\mathrm{pr}_0(g) = \frac{1}{2^{n/2}} \sqrt{\det(1 + T(g))}$$

for one of the two choices of the square root. Both choices are taken for the two elements $g, g' \in \mathrm{Spin}(V)$ *such that* $T(g) = T(g')$.

Proof. Apply Proposition (7.5) to the case where $w = g \in \mathrm{Spin}(V)$. Then

$$\mathrm{pr}_0(g)^2 = \frac{1}{2^{2n_1}} (\mathrm{tr}\, \mathrm{Spin}(g))^2.$$

But $(\operatorname{tr} \operatorname{Spin}(g))^2 = \operatorname{tr}(\operatorname{Spin} \otimes \operatorname{Spin})(g)$. If n is even then $2n_1 = n$ and we have $\operatorname{tr}(\operatorname{Spin} \otimes \operatorname{Spin})(g) = \det(1 + T(g))$ by (7.8) and proposition (7.9). Thus in this case

$$(1) \qquad\qquad \operatorname{pr}_0(g)^2 = \frac{1}{2^n} \det(1 + T(g)).$$

Now assume that n is odd. Then $\operatorname{tr}(\operatorname{Spin} \otimes \operatorname{Spin})(g) = 2 \det(1 + T(g))$ by (7.8) and Proposition (7.9). But since $2n_1 - 1 = n$ equation (1) holds for all n. The theorem now follows from: If $g_1, g_2 \in \operatorname{Spin}(V)$ then $T(g_1) = T(g_2)$ if and only if $g_1 = \pm g_2$. $\quad\square$

7.11. For any $u \in \wedge^2 V$ let $\hat{e}^u$ denote the exponential of u using exterior (and hence commutative - since $\wedge^{\text{even}} V$ is commutative) multiplication. If $w \in \wedge^2 V$ and $x \in V$ then by (7.1) and (7.2) one has, by differentiating an exponential,

$$-\iota(x)\hat{e}^{2w} = \varepsilon([w, x])e^{2w},$$

where the bracket is Clifford commutation. But then also

$$(-\varepsilon([w, x]) - \iota(x))\hat{e}^{2w} = (\iota(x) + \varepsilon([w, x]))\hat{e}^{2w}.$$

Adding $(\varepsilon(x) + \iota([w, x]))\hat{e}^{2w}$ to both sides of the last equation yields

$$(\varepsilon(x - [w, x]) - \iota(x - [w, x]))\hat{e}^{2w} = (\varepsilon(x + [w, x]) + \iota(x + [w, x]))\hat{e}^{2w}.$$

But this is just Clifford multiplication. If $y \in V$ and $u \in C(V)$ then $(\varepsilon(y) + \iota(y))u = yu$ and, if $u \in C^0(V)$, $(\varepsilon(y) - \iota(y))u = uy$, both by (7.1). We have proved:

Proposition. *Let $x \in V$ and $w \in \wedge^2 V$. Then*

$$\hat{e}^{2w}(x - [w, x]) = (x + [w, x])\hat{e}^{2w}.$$

7.12. Let $s \in \operatorname{Lie} SO(V)$ and let V_s^1 and V_s^{-1} be, respectively, the s-eigenspaces for the eigenvalues ± 1. For $x, y \in V$ the equation

$$(sx, y) = -(x, sy)$$

readily implies

Lemma. *The subspaces V_s^1 and V_s^{-1} are isotropic with respect to the symmetric bilinear form B_V and are nonsingularly paired by B_V. In particular*

$$\dim V_s^1 = \dim V_s^{-1}$$

Consequently $1 + s$ is invertible if and only if $1 - s$ is invertible.

7.13. Using lemma (7.12) we define the Zariski open subset of $\operatorname{Lie} SO(V)$ and $SO(V)$ by

$$\operatorname{Lie} SO(V)^{(*)} = \{s \in \operatorname{Lie} SO(V) : \dim(V_s^1) = \dim(V_s^{-1}) = 0\}$$
$$= \{s \in \operatorname{Lie} SO(V) : 1 + s \text{ and } 1 - s \text{ are invertible}\},$$
$$SO(V)^* = \{a \in SO(V) \mid \det(1 + a) \neq 0\},$$

so that in the notation of theorem (7.2) and theorem (7.10) one has

$$\operatorname{Spin}(V)^* = T^{-1}(SO(V)^*).$$

Proposition. *The subsets* $\operatorname{Lie} SO(V)^{(*)}$ *and* $SO(V)^*$ *are algebraically isomorphic via the mapping*

$$(1) \qquad \operatorname{Lie} SO(V)^{(*)} \to SO(V)^*, \qquad s \mapsto a = \frac{1-s}{1+s}, \qquad s = \frac{1-a}{1+a} \leftarrow a$$

for $s \in \operatorname{Lie} SO(V)^{(*)}$ *and* $a \in SO(V)^*$. *In addition one has the relation* $(1+a)(1+s) = 2$.

Proof. If $b \in \operatorname{End} V$ let b^t be the transpose endomorphism with respect to B_V. Since $s^t = -s$ for $s \in \operatorname{Lie} SO(V)$ it is immediate from (1) that $a^t = a^{-1}$ so that a is orthogonal. Since $\operatorname{Lie} SO(V)^{(*)}$ is Zariski open it is connected and $0 \in \operatorname{Lie} SO(V)^{(*)}$. It follows then that $a \in SO(V)$. Furthermore $(1+a)(1+s) = 2$ is immediate from (1) and hence $a \in SO(V)^*$. The map $s \mapsto a$ is injective since one recovers s from a by (1). Conversely if $a \in SO(V)^*$, then $s \in \operatorname{Lie} SO(V)$ since $a^t = a^{-1}$ implies $s^t = -s$. But this formula yields $(1+a)(1+s) = 2$ so that $s \in \operatorname{Lie} SO(V)^{(*)}$. Hence the mapping (1) is bijective. $\square$

7.14. In general if $b \in \operatorname{End} V$ is such that $1 + b$ is invertible we will put

$$\Gamma(b) = \frac{1-b}{1+b}$$

Also let $\wedge^2 V^{(*)}$ be the inverse image of $\operatorname{Lie} SO(V)^{(*)}$ under the isomorphism $\tau : \wedge^2 V \to \operatorname{Lie} SO(V)$.

Theorem. *Let* $g \in \operatorname{Spin}(V)^*$. *Let the sign of square root be chosen so that*

$$\operatorname{pr}_0(g) = \frac{1}{2^{n/2}} \sqrt{\det(1 + T(g))}$$

(see Theorem (7.10)). Then putting $c = \operatorname{pr}_0(g)$ *one has*

$$g = c\, \hat{e}^{-2w}$$

where $w = \tau^{-1}(\Gamma(T(g))) \in \wedge^2 V$. *In particular* $w \in \wedge^2 V^{(*)}$.

Proof. We have $\Gamma(\tau(w)) = T(g)$. But by proposition (7.11), for any $x \in V$, one has $\hat{e}^{-2w}((1 + \tau(w))x) = ((1 - \tau(w))x)\hat{e}^{-2w}$. Putting $y = (1 + \tau(w))x$ one therefore has

$$\hat{e}^{-2w}\, y = T(g)(y)\, \hat{e}^{-2w}$$

On the other hand $g \, y \, g^{-1} = T(g) \, (y)$ so that

$$y \, g^{-1} = g^{-1} T(g)(y)$$

But the last two equations imply that $g^{-1} e^{-2w}$ commutes with y. Since $y \in V$ is arbitrary this implies that $g^{-1} e^{-2w} \in \mathrm{Cent}(C(V)) \cap C^0(V)$. But $\mathrm{Cent}(C(V)) \cap C^0(V) = \mathbb{C}$. Hence there exists $d \in \mathbb{C}$ such that $d \, e^{-2w} = g$. But by applying pr_0 it follows that $d = c$. $\square$

7.15. We determine the generalized Cayley mapping $\Phi : \mathrm{Spin}(V) \to \mathrm{Lie}\,\mathrm{Spin}(V)$, corresponding to to the representation Spin, for elements g in the Zariski open set $\mathrm{Spin}(V)^*$.

Theorem. *Let $g \in \mathrm{Spin}(V)^*$. Let the sign of square root be chosen so that*

$$\mathrm{pr}_0(g) = \frac{1}{2^{n/2}} \, \sqrt{\det(1 + T(g))},$$

according to theorem (7.10). Then

$$\Phi_{\mathrm{Spin}}(g) = -2 \, \mathrm{pr}_0(g) \, \tau^{-1}(\Gamma(T(g))) \in \wedge^2 V.$$

Thus the generalized Cayley mapping $\Phi_{\mathrm{Spin}} : \mathrm{Spin}(V) \to \wedge^2 V$ factors to the classical Cayley transform $\Gamma : SO(V)^* \to \mathrm{Lie}\,\mathrm{Spin}(V)^{(*)}$, up to multiplication by a regular function, via the natural identifications.
For the degree of the spin representation we have

$$\deg(\mathrm{Spin}) = \begin{cases} n & \text{if n is even,} \\ n-1 & \text{if n is odd.} \end{cases}$$

Let $\chi \in A(\mathrm{Spin}(n, \mathbb{C}))$ be given by $\chi(a) = \sqrt{\det(1 + T(a))} = 2^{n/2} \, \mathrm{pr}_0(a)$. Then

$$Q(\mathrm{Spin}(n))^{\mathrm{Spin}(n)} = Q(\mathrm{Lie}\,\mathrm{Spin}(n))^{\mathrm{Spin}(n)}[\chi],$$
$$Q(\mathrm{Spin}(n)) = Q(\mathrm{Lie}\,\mathrm{Spin}(n))[\chi],$$

in terms of (3.3).

Proof. By (7.7) one has $\Phi_{\mathrm{Spin}}(g) = \mathrm{pr}_2(g)$ but since the exponential in theorem (7.14) is an exponential for exterior multiplication the result follows from this theorem.

Let us now determine $\deg(\mathrm{Spin})$. Consider the fiber of Φ over a generic X in $\mathrm{Lie}\,\mathrm{Spin}(n)$. Let $G_o = \{a \in \mathrm{Spin}(n) \mid \det(1 + T(a)) = 0\}$. We may assume that X is in the complement of $\Phi(G_o)$. Then modulo some fixed scalar (which can be ignored in determining the cardinality of the fiber) if $a \in \mathrm{Spin}(n)$ is in the fiber then

$$\sqrt{\det(1 + T(a))}(1 - T(a))(1 + T(a))^{-1} = X.$$

Put $t = \sqrt{\det(1 + T(a))}$ so that $(1 - T(a))(1 + T(a))^{-1} = X/t$. Recall that an equation of matrices $(1 - c)(1 + c)^{-1} = d$ is symmetric in c and d and that $(1 + c)(1 + d) = 2$. Thus

(1) $$(1 + T(a))(1 + X/t) = 2.$$

Taking the determinant of both sides of (1) one has

(2) $$t^2 \det(1 + X/t) = 2^n$$

or

(3) $$f_X(t) = \det(t + X) - 2^n t^{n-2} = 0.$$

Comparing (1) and (2) we get $\det(1 + T(a)) = t^2$ which checks with the definition of t. Given a non-zero root t of (3) the corresponding $T(a)$ is given by $T(a) = (1 - X/t)(1 + X/t)^{-1}$. But t is a specific square root of $\det(1 + T(a))$ and hence a itself is uniquely defined.

Now (3) is a polynomial of degree n in t with leading term t^n and constant term $\det(X)$. If n is even then $\det(X) \neq 0$ for generic X and we also may assume that the polynomial $\det(t + X)$ has pairwise different roots. Replacing X by a large multiple rX and considering

$$\tfrac{1}{r^n} f_{rX}(t) = \tfrac{1}{r^n} \left(\det(t + rX) - 2^n t^{n-2} \right) = \det(\tfrac{t}{r} + X) - \tfrac{2^n}{r^2}(\tfrac{t}{r})^{n-2}$$

which approaches $\det(t/r + X)$ we see that also the polynomial (3) has pairwise distinct roots and they are all non-zero since $\det(X) \neq 0$.

If n is odd then $\det(X) = 0$ but generically the rank of X is $n - 1$; by the argument above (3) has again pairwise distinct roots for generic X, but one of them is 0 so that there are $n - 1$ different non-zero roots.

Let us discuss the adjunction now. We use the above description of the fiber of Φ over a generic X. Let us consider the expansion

(4) $$f_X(t) = \det(t + X) - 2^n t^{n-2} = \sum_{j=0}^{n} p_j(X)\, t^j,$$

where $p_j \in A(\mathrm{Lie\,Spin}(n))^{\mathrm{Spin}(n)}$ form a system of generators with $p_0(X) = \det(X)$ which is 0 for odd n, and $p_n(X) = 1$. We have

$$f_X(\chi) = \det(\chi + X) - 2^n t^{n-2} = \sum_{j=0}^{n} p_j(X)\, \chi^j = 0,$$

since t in the discussion above corresponds to $\chi(a)$, and this is the minimal polynomial for χ over $Q(\mathrm{Lie\,Spin}(n))^{\mathrm{Spin}(n)}$ for n even. For n odd the minimal polynomial is $f_X(\chi)/\chi$. The reason for this is that $f_X(t)$ has exactly $\deg(\mathrm{Spin})$ many pairwise nonzero roots for generic X, in a Zariski dense open set which we may determine as the complement to $\Phi(\{g : \Psi(g) = 0\})$. $\square$

7.16. The Cayley transform for the ρ-representation of any semisimple Lie group. Let G be any simply connected semisimple Lie group. One has a homomorphism

(1) $$G \to \mathrm{Spin}(\mathfrak{g})$$

which lifts the adjoint representation. The Lie algebra of $\mathrm{Spin}(\mathfrak{g})$ is $\wedge^2 \mathfrak{g}$. Thus for any $a \in G$ there exists $u = u(a) \in \wedge^2 \mathfrak{g}$ where

(2) $$u = \sqrt{\det(1 + Ada)}(1 - Ada)/(1 + Ada)$$

On the other hand $\wedge \mathfrak{g}$ is a differential chain complex with respect to a boundary operator c where $c : \wedge^2 \mathfrak{g} \to \mathfrak{g}$ is given by $c(X \wedge Z) = [X, Z]$ for $X, Z \in \mathfrak{g}$.

Theorem. *Let G be any simply connected semisimple Lie group. For the ρ-representation one has up to a fixed scalar multiple*

$$(3) \qquad \Phi(a) = c(u).$$

Proof. Actually the first reference establishing that the restriction (using (1)) of the spin representation s of $\mathrm{Spin}(\mathfrak{g})$ to G is a multiple of the ρ-representation is in reference 9 in [12] (see top paragraph on p. 358 where ρ has been written as g). But our result (2.8.1) on direct sums (here a multiple of the same representation, thus we may use (2.8.1) for G semisimple) we may use the restriction $\pi = s|G$ to compute Φ_ρ. But then $\pi' : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is an injection. Let $\mathfrak{m}$ be the orthocomplement of $\pi'(\mathfrak{g})$ in $\wedge^2 \mathfrak{g}$ so that one has a direct sum $\wedge^2 \mathfrak{g} = \pi'(\mathfrak{g}) + \mathfrak{m}$ with corresponding projection $p : \wedge^2 \mathfrak{g} \to \pi'(\mathfrak{g})$. Let $a \in G$ and put $X = \Phi(a)$. In the notation of (3) we must prove that $X = c(u)$ up to a scalar multiple. But one clearly has

$$(4) \qquad \pi'(X) = p(u)$$

That is for some $v \in \mathfrak{m}$ one has

$$(5) \qquad u = v + \pi'(X)$$

But now besides the boundary operator c on $\wedge \mathfrak{g}$ one has a coboundary operator d on $\wedge \mathfrak{g}$. See section 2 in [10] where ∂ in [10] is written as c here. One has $d : \mathfrak{g} \to \wedge^2 \mathfrak{g}$. The homomorphism π' is δ in (69) of [12]. By (106) in [12] one has $\pi' = d/2$, so that (4) becomes

$$(6) \qquad d(X)/2 = p(u)$$

But by (94) in [12] one has that $dc + cd$ is $1/2$ times the Casimir operator on $\wedge \mathfrak{g}$. But the Casimir is normalized so that it takes the value 1 on $\mathfrak{g}$. However c vanishes on $\mathfrak{g} = \wedge^1 \mathfrak{g}$. Thus $cd = 1/2$ on $\mathfrak{g}$. Thus upon applying c to (6) yields

$$(7) \qquad X/4 = c(pu).$$

On the other hand $\mathfrak{m}$ is the kernel of c on $\wedge^2 \mathfrak{g}$ by (4.4.4) in [10]. Thus c vanishes on v in (5) and by applying c to both sides of (5) one has $c(u) = c(pu)$ and hence $X/4 = c(u)$. But this is just (3) up to a scalar. $\square$

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