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CONSTRUCTION OF COMPLETELY INTEGRABLE SYSTEMS BY POISSON MAPPINGS

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Abstract. Pulling back sets of functions in involution by Poisson mappings and adding Casimir functions during the process allows to construct completely integrable systems. Some examples are investigated in detail.

1. Introduction

The standard notion of complete integrability is the so called Liouville-Arnold integrability: a Hamiltonian system on a $2n$-dimensional symplectic manifold M is said to be completely integrable if it has n first integrals in involution which are functionally independent on some open and dense subset of M.

It is natural to extend the notion of complete integrability to sytems defined on Poisson manifolds (N, Λ) by requiring that on each symplectic leaf such system defines a completely integrable system in the usual sense. This generalization implies that an integrable system is associated to a maximal abelian Poisson subalgebra of $(C^\infty(N), \{\quad,\quad\}_\Lambda)$. The dynamical system Γ associated to a 1-form α on N via $\Gamma = i_\alpha \Lambda$ will define a Hamiltonian system on a symplectic leaf S with embedding $\varepsilon_S : S \to N$ if we have $\varepsilon_s^* \alpha = dH_S$.

If this is the case for any symplectic leaf we may write $\alpha = \sum_k f_k\, dg^k$ where the f_k are Casimir functions for Λ. When all the g^k's belong to a sufficiently large set of functions in involution which are functionally independent on each leaf, the dynamical system is completely integrable. For some cases, even if $\varepsilon_S^* d\alpha \neq 0$, we get an integrable system in the generalized sense of [1].

Of course, integrable systems are not easy to find. Recently, in the paper [3] we came accross a beautiful idea to construct completely integrable systems by using coproducts in Poisson-Hopf algebras. In this paper we put this construction into a geometric perspective in order to understand better which are the essential ideas

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that make the construction possible. In addition to this, we construct a full family of Poisson-Hopf algebras associated with a parametrized family of Poisson-Lie structures on the group $SB(2,\mathbb{C})$. The standard Lie-Poisson structures on $SB(2,\mathbb{C})$ with $SU(2)$ and $SL(2,\mathbb{R})$ as dual groups are included in this scheme. This Poisson-Hopf algebras can be viewed as geometrical version of the corresponding quantum groups — deformations of the universal enveloping algebra of the associated Lie algebras. We also present symplectic realizations of the corresponding commutation rules in the deformed algebras.

2. Constructing integrable systems by Poisson maps

2.1. Complete integrability on Poisson manifolds. If (M,Λ) is a Poisson manifold, a Hamiltonian system $H \in C^\infty(M)$ is called *completely integrable* if it admits a complete set of first integrals in involution: There are $f_1,\dots,f_k \in C^\infty(M)$ with $\{f_i,f_k\}_\Lambda = 0$ and $\{f_i,H\}_\Lambda = 0$ such that on each symplectic leaf (or an open dense set of symplectic leaves) H together with a suitable subset of $f_1,\dots,f_k$ is Liouville-Arnold integrable.

2.2. Constructing families of functions in involution by Poisson maps. Let $\Phi_i : (M_{i+1},\Lambda_{i+1}) \to (M_i,\Lambda_i)$ be Poisson maps between manifolds, so that $\{f,g\}_i \circ \Phi_i = \{f\circ\Phi_i, g\circ\Phi_i\}_{i+1}$. If we have a family of functions $\mathcal{F}_1 \subset C^\infty(M_1)$ in involution on (M_1,Λ_1), we may consider the family $\mathcal{F}_2 = (\mathcal{F}_1\circ\Phi_1)\cup C_2 \subset C^\infty(M_2)$ where C_2 is a complete set of Casimir functions on (M_2,Λ_2), and so on:

$$\begin{array}{cc}
(M_1,\Lambda_1) & \mathcal{F}_1, \text{ in involution} \\
\Phi_1 \big\uparrow & \\
(M_2,\Lambda_2) & \mathcal{F}_2 = (\mathcal{F}_1\circ\Phi_1)\cup C_2 \\
\Phi_2 \big\uparrow & \\
(M_3,\Lambda_3) & \mathcal{F}_3 = (\mathcal{F}_2\circ\Phi_2)\cup C_3 \\
\Phi_3 \big\uparrow & \\
.. &
\end{array}$$

2.3. Poisson actions and multiplications. We shall apply the procedure of 2.2 mainly in the following situation: Consider $(M_1\times M_2,\Lambda_1\times\Lambda_2)$. Then for the algebras of smooth functions we have $C^\infty(M_1\times M_2) \cong C^\infty(M_1)\tilde\otimes C^\infty(M_2)$ for some suitable completed tensor product, where $(f_1\otimes f_2)(x,y) = f_1(x)f_2(y)$. Then

$$\{f_1\otimes f_2, g_1\otimes g_2\}_{\Lambda_1\times\Lambda_2} = \{f_1,g_1\}_{\Lambda_1}\otimes g_1g_2 + f_1f_2\otimes\{g_1,g_2\}_{\Lambda_1}.$$

So if c_1 is a Casimir function of (M_1,Λ_1), then $c_1\otimes 1$ is a Casimir function of $(M_1\times M_2,\Lambda_1\times\Lambda_2)$. In this sense The Casimir functions of (M_1,Λ_1) and those of (M_2,Λ_2) extend both to Casimir functions on $(M_1\times M_2,\Lambda_1\times\Lambda_2)$.

If $\Phi : (M\times M,\Lambda\times\Lambda) \to (M,\Lambda)$ is a Poisson map (for example the multiplication of a Lie Poisson group) we may use it for the procedure of 2.2. If Φ is associative then $\Delta_\Phi : f \mapsto f\circ\Phi$ is coassociative. But this is not essential for applying the procedure in which $M_n = \prod^n M$ and Φ_n is a Cartesian product of Φ with identities. For example,

$$M \xleftarrow{\Phi} M \times M \xleftarrow{\Phi\times \mathrm{Id}_M} M \times M \times M \xleftarrow{\mathrm{Id}_M \times\Phi\times \mathrm{Id}_M} M \times M \times M \times M \leftarrow \dots$$

We start with a set of functions $\mathcal{F}_1 \subset C^\infty(M)$ in involution and with a basis $\mathcal{C}$ of all Casimirs. Then $\mathcal{F}_n \subset C^\infty(\prod^n M)$ is given recursively by

$$\mathcal{F}_{n+1} = (\mathcal{F}_n \circ \Phi_n) \cup \{\mathcal{C} \otimes 1 \otimes \cdots \otimes 1, 1 \otimes \mathcal{C} \otimes 1 \otimes \cdots \otimes 1, \dots\}$$

and furnishes a family of functions in involution on $\prod^{n+1} M$. If Φ is associative (so Δ_Φ is coassociative) then the result does not depend on the 'path' chosen to define the Φ_n's.

Another possibility is to consider a Poisson mapping $\Phi : (M \times N, \Lambda_M \times \Lambda_N) \to (N, \Lambda_N)$ (for example a Lie-Poisson action on N of a Lie Poisson group M) and to apply the procedure as follows:

$$N \xleftarrow{\Phi} M \times N \xleftarrow{\mathrm{Id}_M \times \Phi} M \times M \times N \xleftarrow{\mathrm{Id}_{M\times M} \times \Phi} M \times M \times M \times N \leftarrow \dots$$

2.4. We may extend the procedure described in 2.3 as follows. We assume that we have furthermore Poisson manifolds (e.g. symplectic ones) $N_1, \dots N_n$ and Poisson mappings $\varphi_i : N_i \to M$. The product map $\varphi = \varphi_1 \times \dots \times \varphi_n : N_1 \times \dots \times N_n \to \prod^n M$ is a Poisson map. Let $\mathcal{F}_n$ be the set of functions in involution on $\prod^n M$ constructed in 2.3. Then $\mathcal{F}_n \circ \varphi$ is a set of functions in involution on $\prod_i N_i$.

Standard examples of Poisson maps $\varphi_i : N_i \to M$ are the canonical embeddings of symplectic leaves N_i of the Poisson manifold M. In this case, the Casimir functions $1 \otimes 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$ are constants on $N_1 \times \dots \times N_n$, but the coproducts $\Delta_\Phi c$, $(\Delta_\Phi \times \mathrm{Id}_M) \circ \Delta_\Phi c$, etc., are usually no longer Casimirs and hence sometimes give rise to completely integrable systems on $N_1 \times \dots \times N_n$. See example 3.1.

3. Examples

3.1. Example. Let $M = \mathfrak{su}(2)^*$ be the dual space of the Lie algebra $\mathfrak{su}(2)$. It carries a Kostant-Kirillov-Souriau Poisson structure which is given in linear coordinates by

$$\Lambda = z\partial_x \wedge \partial_y + x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x.$$

Since Λ is linear, we have the obvious Poisson map

$$\Phi : M \times M \to M, \qquad \Phi(x_1, y_1, z_1, x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

A Casimir function for Λ is $c = x^2 + y^2 + z^2$. According to our procedure in 2.3 the functions

$$\begin{aligned}
c \circ \Phi &= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2, \\
c \otimes 1 &= x_1^2 + y_1^2 + z_1^2, \\
1 \otimes c &= x_2^2 + y_2^2 + z_2^2, \\
f \circ \Phi &= f(x_1 + x_2, y_1 + y_2, z_1 + z_2),
\end{aligned}$$

are functions in involution on $M\times M$, where f is an arbitrary function on M. If we take for N the symplectic leaf $N=c^{-1}(1)$ which is a 2-dimensional sphere S^2, the Casimir functions $c\otimes 1$ and $1\otimes c$ pull back to constants on $N\times N=S^2\times S^2$. However, $c\circ\Phi$ and $f\circ\Phi$ are in involution and hence $H=\frac{1}{2}(c\circ\Phi)-1=x_1x_2+y_1y_2+z_1z_2$ defines a completely integrable system on the symplectic manifold $N\times N\subset M\times M$. The system defined by the Hamiltonian function H is, in fact, completely integrable on each symplectic leaf of $M\times M$, so that we get a completely integrable system on $M\times M$ whose dynamics is given by the vector field

$$\begin{aligned}\Gamma=&(z_2y_1-y_2z_1)\partial_{x_1}+(z_1y_2-y_1z_2)\partial_{x_2}+\\&(x_2z_1-z_2x_1)\partial_{y_1}+(x_1z_2-z_1x_2)\partial_{y_2}+\\&(y_2x_1-x_2y_1)\partial_{z_1}+(y_1x_2-x_1y_2)\partial_{z_2}.\end{aligned}$$

This vector field is tangent to all products of spheres since $c\otimes 1=x_1^2+y_1^2+z_1^2$ and $1\otimes c=x_2^2+y_2^2+z_2^2$ are first integrals, and on $N\times N$ it induces the motion which can be interpreted as associated with a 'spin-spin'-interaction:

$$\dot{\vec{J_1}}=\vec{J_1}\times\vec{J_2},\quad \dot{\vec{J_2}}=\vec{J_2}\times\vec{J_1};$$

The points on the spheres move in such a way that the velocity of each of them is the vector product of the two position vectors. Stationary solutions occupy the same or opposite points on the sphere.

In sperical coordinates, the same system can be given a different interpretation:

$$z_i=\sin\beta_i,\quad y_i=\cos\beta_i\ \sin\alpha_i,\quad x_i=\cos\beta_i\ \cos\alpha_i.$$

In canonical coordinates $p_i=\sin\beta_i, q_i=\alpha_i$ we get the Hamiltonian function in the form

$$H=p_1p_2+\sqrt{(1-p_1^2)(1-p_2^2)}\cos(q_1-q_2).$$

Since $H_1=\Delta(z^2)-\Delta(c)$ is in involution with $\Delta(z^2)$ we can consider the completely integrable system given by the Hamiltonian

$$H_1=p_1^2+p_2^2-2\sqrt{(1-p_1^2)(1-p_2^2)}\cos(q_1-q_2).$$

Let us remark that our Hamiltonian H is a slight modification of the Hamiltonian

$$H_0=p_1p_2-p_1p_2\cos(q_1-q_2)$$

obtained in [3].

We can inductively apply our procedure to get a completely integrable system on $\prod^k M$ with Hamiltonian

$$H^{(k)}=\sum_{i<j}^{k}(x_ix_j+y_iy_j+z_iz_j)$$

which reduces in canonical coordinates on $\prod^k N$ to

$$H^{(k)}=\sum_{i<j}^{k}\left(p_ip_j+\sqrt{(1-p_i^2)(1-p_j^2)}\cos(q_i-q_j)\right).$$

3.2. Example. We consider the following symplectic realization [8] of the Lie algebra $\mathfrak{su}(2)$ in $T^*\mathbb{R}^2$:

$$x = \tfrac{1}{2}(q_1q_2 + p_1p_2), \quad y = \tfrac{1}{2}(p_1q_2 - q_1p_2), \quad z = \tfrac{1}{4}(p_1^2 + q_1^2 - p_2^2 - q_2^2).$$

This defines a Poisson morphism

$$\psi : T^*\mathbb{R}^2 \to \mathfrak{su}(2)^*, \quad (q_1, q_2, p_1, p_2) \mapsto (x, y, z),$$

which is the momentum map of the corresponding Hamiltonian action of the group $SU(2)$.

As before, we consider the Casimir function $c = x^2 + y^2 + z^2$ on $\mathfrak{su}(2)$. This time, however,

$$F = c \circ \psi = \tfrac{1}{16}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right)^2$$

is not a Casimir function on the symplectic manifold $T^*\mathbb{R}^2$. The functions in involution on $\mathfrak{su}(2)^* \times \mathfrak{su}(2)^*$ from example 3.1 give rise to functions in involution on $T^*\mathbb{R}^2 \times T^*\mathbb{R}^2 = T^*\mathbb{R}^4$ as in 2.4, where $\tilde{q}_1$ etc. denote the functions on the second copy of $T^*\mathbb{R}^2$:

$$F_1 = F(q_1, q_2, p_1, p_2), \quad F_2 = F(\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2), \quad (\Delta c) \circ \psi = F_1 + F_2 + H,$$

where

$$H = \tfrac{1}{4}\Big((q_1q_2 + p_1p_2)(\tilde{q}_1\tilde{q}_2 + \tilde{p}_1\tilde{p}_2) + (p_1q_2 - q_1p_2)(\tilde{p}_1\tilde{q}_2 - \tilde{q}_1\tilde{p}_2)\Big) + \tfrac{1}{16}(p_1^2 + q_1^2 - p_2^2 - q_2^2)(\tilde{p}_1^2 + \tilde{q}_1^2 - \tilde{p}_2^2 - \tilde{q}_2^2)$$

and $(\Delta f) \circ \psi = f(G_1, G_2, G_3)$, where

$$\begin{aligned} G_1 &= \tfrac{1}{2}(q_1q_2 + p_1p_2 + \tilde{q}_1\tilde{q}_2 + \tilde{p}_1\tilde{p}_2), \\ G_2 &= \tfrac{1}{2}(p_1q_2 - q_1p_2 + \tilde{p}_1\tilde{q}_2 - \tilde{q}_1\tilde{p}_2), \\ G_3 &= \tfrac{1}{4}(p_1^2 + q_1^2 - p_2^2 - q_2^2 + \tilde{p}_1^2 + \tilde{q}_1^2 - \tilde{p}_2^2 - \tilde{q}_2^2). \end{aligned}$$

Hence, we have 4 independent functions in involution on $T^*\mathbb{R}^4$ which define completely integrable systems. As Hamiltonian functions we can take the pure interaction term H. The trajectories of the corresponding dynamics Γ_H lie on the intersections of the level sets of F_1 and F_2 (which are, topologically, products of 3-dimensional spheres) and the level sets of H, and, say, G_1 (which are, generically, 4-dimensional tori). Note that G_2 and G_3 are additional constants of the motion. The whole set $\{F_1, F_2, H, G_1, G_2, G_3\}$ is, however, not independent, since $G_1^2 + G_2^2 + G_3^2 = F_1 + F_2 + H$. The dynamics Γ_H on $T^*\mathbb{R}^4 \cong \mathbb{R}^8$ is described by a rather complicated vector field whose coefficients are polynomials of degree 3.

The functions G_1, G_2, G_3 define the diagonal action of $SU(2)$ on $T^*\mathbb{R}^2 \times T^*\mathbb{R}^2$ which preserves Γ_H. The dynamics on $S^2 \times S^2$ from example 3.1 can be obtained via symplectic reduction with respect to this action.

3.3. Example. Let us now consider the Lie group $M = SB(2,\mathbb{C})$ of all matrices of the form

$$A = \begin{pmatrix} e^{-kz/2} & x+iy \\ 0 & e^{kz/2} \end{pmatrix},$$

where $k \neq 0$ is fixed and $z, x, y \in \mathbb{R}$ may be viewed as global coordinates on $SB(2,\mathbb{C})$. The coproduct corresponding to the group multiplication $\Phi : SB(2,\mathbb{C}) \times SB(2,\mathbb{C}) \to SB(2,\mathbb{C})$ is given by

$$\begin{aligned} \Delta(z) &= z \otimes 1 + 1 \otimes z, \\ \Delta(x) &= x \otimes e^{kz/2} + e^{-kz/2} \otimes x, \\ \Delta(y) &= y \otimes e^{kz/2} + e^{-kz/2} \otimes y. \end{aligned}$$

Φ is a Poisson map for each of the following Poisson structures (parameterized by $\alpha, \beta, \gamma \in \mathbb{R}$):

$$(*) \qquad \{z, x\} = \beta y, \quad \{y, z\} = \alpha x, \quad \{x, y\} = \gamma \frac{\sin(kz)}{k}.$$

For $\alpha = \beta = \gamma = 1$ we get the 'classsical realization' of the quantum $SU(2)$ group, and for $\alpha = \beta = -\gamma = 1$ we get the 'classsical realization' of the quantum $SL(2,\mathbb{R})$ of Drinfeld and Jimbo, [6].

For $k \to 0$ we get the Lie algebras

$$\{z, x\} = \beta y, \quad \{y, z\} = \alpha x, \quad \{x, y\} = \gamma z,$$

with the standard cobrackets $\Delta(u) = u \otimes 1 + 1 \otimes u$, corresponding to the addition in $\mathfrak{g}^*$.

A Casimir function for the Poisson bracket (*) is $c = \alpha x^2 + \beta y^2 + \frac{4\gamma}{k^2} \sinh^2(kz/2)$, which for $k \to 0$ goes to $c_0 = \alpha x^2 + \beta y^2 + \gamma z^2$, see [7].

The generic symplectic leaves of the Poisson structure are 2-dimensional (except for the trivial case $\alpha = \beta = \gamma = 0$). As in example 3.1 the Hamiltonian $H = \frac{1}{2}\Delta(c)$ defines a completely integrable system on $SB(2,\mathbb{C}) \times SB(2,\mathbb{C})$. In the coordinates x, y, z the Hamiltonian H has the form

$$\begin{aligned} H = \tfrac{1}{2} &\left(\alpha x_1^2 + \beta y_1^2 + \tfrac{4\gamma}{k^2} \sinh^2(\tfrac{kz_1}{2})\right) e^{kz_2} \\ &+ \tfrac{1}{2} \left(\alpha x_2^2 + \beta y_2^2 + \tfrac{4\gamma}{k^2} \sinh^2(\tfrac{kz_2}{2})\right) e^{-kz_1} \\ &+ \left(\alpha x_1 x_2 + \beta y_1 y_2 + \tfrac{4\gamma}{k^2} \sinh(\tfrac{kz_1}{2}) \sinh(\tfrac{kz_2}{2})\right) e^{k(z_1 - z_2)/2}. \end{aligned}$$

In the limit for $k \to 0$ we get

$$H_0 = c_0 \otimes 1 + 1 \otimes c_0 + (\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)$$

and for $\alpha = \beta = \gamma = 1$ we are in the situation of example 3.1.

In all generality, however, it is difficult to express the dynamics explicitly since we deal simultaneously with a parametrized family of structures for which even the topology of the symplectic leaves changes.

Even in the case $\alpha = \beta = \gamma = 1$, where it is known [2] that $(SB(2,\mathbb{C}),\Lambda)$ is equivalent, as a Poisson manifold, to $\mathfrak{su}(2)^*$ with the Kostant-Kirillov-Souriau structure Λ_0 described in example 3.1, the dynamics described by H in the deformed case may differ from that of example 3.1. The reason is that $(SB(2,\mathbb{C}),\Lambda)$ is not equivalent to $(\mathfrak{su}(2),\Lambda_0)$ as a Lie Poisson group, since $SU(2,\mathbb{C})$ is not commutative. In particular, the deformed coproduct is not cocommutative and the interaction we obtain is not symmetric.

In order to work in canonical coordinates let us introduce a symplectic realization of the commutation rules (*) with $\alpha = \delta^2 > 0$ and $\beta = 1$:

$$X = \sqrt{a^2 - \tfrac{4\gamma}{k^2}\sinh^2(\tfrac{kp}{2})}\,\frac{\sin(\delta q)}{\delta}, \quad Y = \sqrt{a^2 - \tfrac{4\gamma}{k^2}\sinh^2(\tfrac{kp}{2})}\,\cos(\delta q), \quad Z = p,$$

where $a \geq 0$ and $a > 0$ if $\gamma = 0$. In particular, if $\gamma > 0$ we get the deformed $SU(2)$, and if $\gamma < 0$ we get the deformed $SL(2,\mathbb{R})$. In this realization the Casimir function is $c = a^2$ and the Hamiltonian reads

$$\begin{aligned} H = e^{k(p_2-p_1)/2}\Bigg(\sqrt{\Big(a^2 - \tfrac{4\gamma}{k^2}\sinh^2(\tfrac{kp_1}{2})\Big)\Big(a^2 - \tfrac{4\gamma}{k^2}\sinh^2(\tfrac{kp_2}{2})\Big)}\,\cos(\delta(q_1-q_2)) + \\ + a^2\cosh(k\tfrac{p_1+p_2}{2}) + \tfrac{4\gamma}{k^2}\sinh(\tfrac{kp_1}{2})\sinh(\tfrac{kp_2}{2})\Bigg). \end{aligned}$$

This Hamiltonian is quite complicated. But if we put $a = 0$, $\gamma = -1$, and $\delta = 1$ we get

$$H_1 = 4e^{k(p_2-p_1)/2}\tfrac{1}{k^2}\sinh(\tfrac{kp_1}{2})\sinh(\tfrac{kp_2}{2})(\cos(q_1-q_2)-1)$$

which is the Hamiltonian obtained in [3] for the deformed $SL(2,\mathbb{R})$.

3.4. Example. A slight modification of the previous example which, at least formally, is not dealing with Lie-Poisson groups, is the the following. Let M be the space of all upper triangular matrices of the form

$$A = \begin{pmatrix} a & x+iy \\ 0 & b \end{pmatrix},$$

where $a, b, x, y \in \mathbb{R}$. We use these as coordinates on M. We consider the Poisson structure Λ on M with Poisson bracket

$$(**) \qquad \begin{aligned} &\{x,a\} = \beta ya, \quad \{x,b\} = -\beta yb, \quad \{x,y\} = -\gamma(b^2-a^2), \\ &\{y,a\} = \alpha xa, \quad \{y,b\} = \alpha xb, \qquad \{a,b\} = 0, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ paramerterize a family of Poisson brackets. Matrix multiplication on M leads to the coproduct:

$$\Delta(a) = a\otimes a, \quad \Delta(b) = b\otimes b, \quad \Delta(x) = a\otimes x + x\otimes b, \quad \Delta(y) = a\otimes y + y\otimes b.$$

The matrix multiplication turns out to be a Poisson mapping with respect to all brackets (**). But M is not a Lie-Poisson group since it contains elements which are not invertible. On the other hand, all Λ are tangent to $SB(2,\mathbb{C}) \subset M$ and give there the brackets (*) with slightly modified coeficients α, β, γ, if we parameterize $a = e^{-kz/2}$, $b = e^{kz/2}$.

The Poisson tensor has 2 independent Casimirs: $c_1 = ab$ and $c_2 = \alpha x^2 + \beta y^2 + \gamma(a^2 + b^2)$. The symplectic leaves are, generically, 2-dimensional (we assume that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$), and as before,

$$\begin{aligned} H = \tfrac{1}{2}\Delta(c_2) = {} & \tfrac{1}{2}a_1^2(\alpha x_2^2 + \beta y_2^2 + \gamma(a_2^2 + b_2^2)) + \\ & \tfrac{1}{2}b_2^2(\alpha x_1^2 + \beta y_1^2 + \gamma(a_1^2 + b_1^2)) + \\ & a_1 b_2(\alpha x_1 x_2 + \beta y_1 y_2 - \gamma a_1 b_2) \end{aligned}$$

describes a completely integrable system on $M \times M$ which, on $SB(2, \mathbb{C}) = c_1^{-1}(1)$, coincides with a system from example 3.3.

3.5. Example. This is of different type. Let $D = G.G^* = G^*.G$ be a complete Drinfeld double group. The decompositions give us two Poisson projections $\pi_1, \pi_2 : D \to G^*$ onto the Lie-Poisson group G^*. Let $\mathcal{F}, \mathcal{F}'$ be two families of functions in involution on G^*. It is known that pull backs by π_1 and by π_2 commute with respect to the symplectic structure on D. So we can take $\mathcal{F}_2 = (\mathcal{F}_1 \circ \pi_1) \cup (\mathcal{F}_2 \circ \pi_2)$ as a set of functions in involution on D.

For example,

$$SL(2, \mathbb{C}) = SU(2).SB(2, \mathbb{C}) = SB(2, \mathbb{C}).SU(2),$$

where π_1 denotes the projection onto the left factor $SB(2, \mathbb{C})$, and π_2 onto the the right one. Let $\mathcal{F}$ consist of the Casimir C and some function f. Then the functions $c \circ \pi_1, f \circ \pi_1, g \circ \pi_2$ are in involution on D, where f and g are arbitrary in $C^\infty(SB(2, \mathbb{C}))$, and they are generically independent, so we have:

> $H = f \circ \pi_1$ *is a completely integrable system on the symplectic* $SL(2, \mathbb{C})$ *for any* $f \in C^\infty(SB(2, \mathbb{C}))$.

3.6. Concluding remarks.

1. We have shown that by using Poisson-compatible coproducts it is possible to generate interacting systems while preserving the complete integrability. We have given some examples. These can be extended to arbitrary Lie Poisson pairs.

2. The interaction we get is a 2-body interaction, one may wonder if it would not be possible to obtain non-factorizable n-body interactions by using n-ary brackets or n-ary operations ([7], [8], [9]).

3. The composition procedure does not use the existence of an inverse for each element in the product, therefore the procedure may be extended to any algebra. Is it possible to obtain interacting systems with fermionic degrees of freedom by using graded algebras?

4. The procedure may clearly be extended to infinite dimensions. Can one use it to obtain interacting fields?

We shall come back to some of these questions in the near future.

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