Calculus of Variations

Fractional Sobolev metrics on spaces of immersions

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Abstract

We prove that the geodesic equations of all Sobolev metrics of fractional order one and higher on spaces of diffeomorphisms and, more generally, immersions are locally well posed. This result builds on the recently established real analytic dependence of fractional Laplacians on the underlying Riemannian metric. It extends several previous results and applies to a wide range of variational partial differential equations, including the well-known Euler–Arnold equations on diffeomorphism groups as well as the geodesic equations on spaces of manifold-valued curves and surfaces.

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1 Introduction

Background

Many prominent partial differential equations (PDEs) in hydrodynamics admit variational formulations as geodesic equations on an infinite-dimensional manifold of mappings. These include the incompressible Euler [2], Burgers [35], modified Constantin–Lax–Majda [14,19, 61], Camassa–Holm [17,39,48], Hunter–Saxton [30,43], surface quasi-geostrophic [20,60] and Korteweg–de Vries [51] equations of fluid dynamics as well as the governing equation of ideal magneto-hydrodynamics [44,59]. This serves as a strong motivation for the study of Riemannian geometry on mapping space. An additional motivation stems from the field of mathematical shape analysis, which is intimately connected to diffeomorphisms groups and other infinite-dimensional mapping spaces via Grenander's pattern theory [27,63] and elasticity theory [9,53].

The variational formulations allow one to study analytical properties of the PDEs in relation to geometric properties of the underlying infinite-dimensional Riemannian manifold [4,5,13,34,49,52]. Most importantly, local well-posedness of the PDE, including smooth dependence on initial conditions, is closely related to smoothness of the geodesic spray on Sobolev completions of the configuration space [23]. This has been used to show local well-posedness of PDEs in many specific examples, cf. the recent overview article [38]. An extension of this successful methodology to wider classes of PDEs requires an in-depth study of smoothness properties of partial and pseudo differential operators with non-smooth coefficients such as those appearing in the geodesic spray or, more generally, in the Euler–Lagrange equations. This is the topic of the present paper.

Contribution

This article establishes local well-posedness of the geodesic equation for fractional order Sobolev metrics on spaces of diffeomorphisms and, more generally, immersions. A simplified version of our main result reads as follows:

Theorem *On the space of immersions of a closed manifold M into a Riemannian manifold $(N, \bar{g})$, the geodesic equation of the fractional-order Sobolev metric*

$$G_f(h, k) = \int_M \bar{g}\big((1 + \Delta^{f^*\bar{g}})^p h, k\big) \operatorname{vol}^{f^*\bar{g}}, \qquad h, k \in T_f \operatorname{Imm}(M, N),$$

is locally well-posed in the sense of Hadamard for any $p \in [1, \infty)$.

This follows from Theorems 4.4 and 4.6. The result unifies and extends several previously known results:

- For integer-order metrics, local well-posedness on the space of immersions from M to N has been shown in [11]. However, the proof contained a gap, which was closed in [50] for $N = \mathbb{R}^n$, and which is closed in the present article for general N. The strategy of proof, which goes back to Ebin and Marsden [23], is to show that the geodesic spray extends smoothly to certain Sobolev completions of the space. Our generalization to fractional-order metrics builds on recent results about the smoothness of the functional calculus of sectorial operators [6].
- For $N = \mathbb{R}^n$, the set of N-valued immersions becomes a vector space, which simplifies the formulation of the geodesic equation; see Corollary 5.3. The treatment of general manifolds N requires a theory of Sobolev mappings between manifolds, which is developed in Sect. 2.2. Moreover, in the absence of global coordinate systems for these mapping spaces, we recast the geodesic equation using an auxiliary covariant derivative following [11]; see Lemma 2.6 and Theorem 4.3.
- For $M = N$ our result specializes to the diffeomorphism group $\mathrm{Diff}(M)$, which is an open subset of $\mathrm{Imm}(M, M)$. On $\mathrm{Diff}(M)$ we obtain local well-posedness of the geodesic equation for Sobolev metrics of order $p \in [1/2, \infty)$; see Corollary 5.1. Analogous results have been obtained by different methods (smoothness of right-trivializations) for inertia operators that are defined as abstract pseudo-differential operators [3,10,24].
- For $M = S^1$, our result specializes to the space of immersed loops in N. For loops in $N = \mathbb{R}^d$, local well-posedness has been shown by different methods (reparameterization to arc length) in [8]. Our analysis extends this result to manifold-valued loops and also to higher-dimensional and more general base manifolds M.

2 Sobolev mappings

2.1 Setting

We use the notation of [11] and write $\mathbb{N}$ for the natural numbers including zero. Smooth will mean C^∞ and real analytic C^ω. Sobolev regularity is denoted by H^r, and Sobolev spaces $H^s_{H^r}$ of mixed order r in the foot point and s in the fiber are introduced in Theorem 2.4.

Throughout this paper, without any further mention, we fix a real analytic connected closed manifold M of dimension $\dim(M)$ and a real analytic manifold N of dimension $\dim(N) \geq \dim(M)$.

2.2 Sobolev sections of vector bundles

[6, Section 2.3] We write $H^s(\mathbb{R}^m, \mathbb{R}^n)$ for the Sobolev space of order $s \in \mathbb{R}$ of $\mathbb{R}^n$-valued functions on $\mathbb{R}^m$. We will now generalize these spaces to sections of vector bundles. Let E be a vector bundle of rank $n \in \mathbb{N}_{>0}$ over M. We choose a finite vector bundle atlas and a subordinate partition of unity in the following way. Let $(u_i \colon U_i \to u_i(U_i) \subseteq \mathbb{R}^m)_{i \in I}$ be a finite atlas for M, let $(\varphi_i)_{i \in I}$ be a smooth partition of unity subordinated to $(U_i)_{i \in I}$, and let $\psi_i \colon E|U_i \to U_i \times \mathbb{R}^n$ be vector bundle charts. Note that we can choose open sets U_i° such that $\mathrm{supp}(\psi_i) \subset U_i^\circ \subset \overline{U_i^\circ} \subset U_i$ and each $u_i(U_i^\circ)$ is an open set in $\mathbb{R}^m$ with Lipschitz boundary (cf. [15, Appendix H3]). Then we define for each $s \in \mathbb{R}$ and $f \in \Gamma(E)$

$$\|f\|^2_{\Gamma_{H^s}(E)} := \sum_{i \in I} \| \operatorname{pr}_c f \mathbb{R}^n \circ \psi_i \circ (\varphi_i \cdot f) \circ u_i^{-1} \|^2_{H^s(\mathbb{R}^m, \mathbb{R}^n)}.$$

Then $\| \cdot \|_{\Gamma_{H^s}(E)}$ is a norm, which comes from a scalar product, and we write $\Gamma_{H^s}(E)$ for the Hilbert completion of $\Gamma(E)$ under the norm. It turns out that $\Gamma_{H^s}(E)$ is independent of the choice of atlas and partition of unity, up to equivalence of norms. We refer to [58, Section 7] and [28, Section 6.2] for further details.

The following theorem describes module properties of Sobolev sections of vector bundles, which will be used repeatedly throughout the paper.

2.3 Theorem Module properties. [6, Theorem 2.4] *Let E_1, E_2 be vector bundles over M and let $s_1, s_2, s \in \mathbb{R}$ satisfy*

$$s_1 + s_2 \geq 0, \quad \min(s_1, s_2) \geq s, \quad \text{and } s_1 + s_2 - s > \dim(M)/2.$$

Then the tensor product of smooth sections extends to a bounded bilinear mapping

$$\Gamma_{H^{s_1}}(E_1) \times \Gamma_{H^{s_2}}(E_2) \to \Gamma_{H^s}(E_1 \otimes E_2).$$

The following theorem describes the manifold structure of Sobolev mappings between finite-dimensional manifolds. It is an elaboration of [46, 5.2 and 5.4] and an extension to the Sobolev case of parts of [41, Section 42].

2.4 Theorem Sobolev mappings between manifolds. *The following statements hold for any $r \in (\dim(M)/2, \infty)$ and $s, s_1, s_2 \in [-r, r]$:*

(a) *The space $H^r(M, N)$ is a C^∞ and a real analytic manifold. Its tangent space satisfies in a natural (i.e., functorial) way*

$$T H^r(M, N) = H^r(M, TN) \xrightarrow[\pi_{H^r(M,N)}]{(\pi_N)_*} H^r(M, N)$$

   with foot point projection given by $\pi_{H^r}(M, N) = (\pi_N)_* \colon h \mapsto \pi_N \circ h$.

(b) *The space $H^s_{H^r}(M, TN)$ of 'H^s mappings $M \to TN$ with foot point in $H^r(M, N)$' is a real analytic manifold and a real analytic vector bundle over $H^r(M, N)$. Similarly, spaces such as $L(H^{s_1}_{H^r}(M, TN), H^{s_2}_{H^r}(M, TN))$ are real analytic vector bundles over $H^r(M, N)$.*

(c) *The space $\operatorname{Met}_{H^r}(M)$ of all Riemannian metrics of Sobolev regularity H^r is an open subset of the Hilbert space $\Gamma_{H^r}(S^2 T^* M)$, and thus a real analytic manifold.*

Proof (a) Let us recall the chart construction: we use an auxiliary real analytic Riemannian metric $\hat{g}$ on N and its exponential mapping $\exp^{\hat{g}}$; some of its properties are summarized in the following diagram:
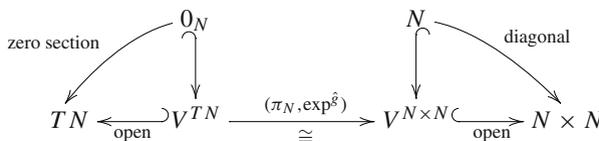
Without loss we may assume that $V^{N \times N}$ is symmetric:

$$(y_1, y_2) \in V^{N \times N} \iff (y_2, y_1) \in V^{N \times N}.$$

A chart, centered at a real analytic $f \in C^\omega(M, N)$, is:

$$H^r(M, N) \supset U_f = \{g : (f, g)(M) \subset V^{N \times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma_{H^r}(f^*TN)$$

$$u_f(g) = (\pi_N, \exp^{\hat{g}})^{-1} \circ (f, g), \quad u_f(g)(x) = (\exp^{\hat{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\hat{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\hat{g}}_{f(x)}(s(x))$$

Note that $\tilde{U}_f$ is open in $\Gamma(f^*TN)$. The charts U_f for $f \in C^\omega(M, N)$ cover $H^r(M, N)$: since $C^\omega(M, N)$ is dense in $H^r(M, N)$ by [41, 42.7] and since $H^r(M, N)$ is continuously embedded in $C^0(M, N)$, a suitable C^0-norm neighborhood of $g \in H^r(M, N)$ contains a real analytic $f \in C^\omega(M, N)$, thus $f \in U_g$, and by symmetry of $V^{N \times N}$ we have $g \in U_f$. The chart changes,

$$\Gamma_{H^r}(f_1^*TN) \supset \tilde{U}_{f_1} \ni s \mapsto (u_{f_2, f_1})_*(s) := (\exp^{\hat{g}}_{f_2})^{-1} \circ \exp^{\hat{g}}_{f_1} \circ s \in \tilde{U}_{f_2} \subset \Gamma_{H^r}(f_2^*TN),$$

for charts centered on real analytic $f_1, f_2 \in C^\omega(M, N)$ are real analytic by Lemma A.5 since $r > \dim(M)/2$.

The tangent bundle $TH^r(M, N)$ is canonically glued from the following vector bundle chart changes, which are real analytic by Lemma A.5 again:

$$\tilde{U}_{f_1} \times \Gamma_{H^r}(f_1^*TN) \ni (s, h) \mapsto (T(u_{f_2, f_1})_*)(s, h) =$$
$$= \left((u_{f_2, f_1})_*(s), (d_{\text{fiber}}u_{f_2, f_1})_*(s, h)\right) \in \tilde{U}_{f_2} \times \Gamma_{H^r}(f_2^*TN) \qquad (1)$$

It has the canonical charts

$$TH^r(M, N) \supset T\tilde{U}_f \xrightarrow[\substack{(T(\exp^{\hat{g}}_f)^{-1}_*)}]{Tu_f} \tilde{U}_f \times \Gamma_{H^r}(f^*TN).$$

These identify $TH^r(M, N)$ canonically with $H^r(M, TN)$ since

$$Tu_f^{-1}(s, s') = T(\exp^{\hat{g}}_f) \circ \text{vl} \circ (s, s') : M \to TN,$$

where we used the vertical lift $\text{vl} : TN \times_N TN \to TTN$ which is given by $\text{vl}(u_x, v_x) = \partial_t|_{t=0}(u_x + t.v_x)$; see [45, 8.12 or 8.13]. The corresponding foot-point projection is then

$$\pi_{H^s(M,N)}(T(\exp^{\hat{g}}_f) \circ \text{vl} \circ (s, s')) = \exp^{\hat{g}}_f \circ s = \pi_N \circ T(\exp^{\hat{g}}_f) \circ (s, s').$$

(b) The canonical chart changes (1) for $TH^r(M, N)$ extend to

$$\tilde{U}_{f_1} \times \Gamma_{H^s}(f_1^*TN) \ni (s, h) \mapsto (Tu_{f_2, f_1})_*(s, h) =$$
$$= \left((u_{f_2, f_1})_*(s), (d_{\text{fiber}}u_{f_2, f_1} \circ s)_*(h)\right) \in \tilde{U}_{f_2} \times \Gamma_{H^s}(f_2^*TN),$$

since $d_{\text{fiber}}u_{f_2, f_1} : f_1^*TN \times_M f_1^*TN = f_1^*(TN \times_N TN) \to f_2^*TN$ is fiber respecting real analytic by the module properties 2.3. Note that $d_{\text{fiber}}u_{f_2, f_1} \circ s$ is then an H^r-section of the bundle $L(f_1^*TN, f_1^*TN) \to M$, which may be applied to the H^s-section h by the module properties 2.3. These extended chart changes then glue the vector bundle

$$H^s_{H^r}(M, TN) \xrightarrow{(\pi_N)_*} H^r(M, TN).$$

(c) The space $\Gamma_{H^r}(S^2T^*M)$ is continuously embedded in $\Gamma_{C^1}(S^2T^*M)$ because $r > \dim(M)/2 + 1$. Thus, the space of metrics is open.

$\square$

2.5 Connections, connectors, and sprays

This sections reviews some relations between connections, connectors, and sprays. It holds for general convenient manifolds N, including infinite-dimensional manifolds of mappings, and will be used in this generality in the sequel (see e.g. the proofs of Theorems 4.3 and 4.4).

(a) **Connectors.** [45, 22.8–9] Any connection ∇ on TN is given in terms of a connector $K\colon TTN \to TN$ as follows: For any manifold M and function $h\colon M \to TN$, one has $\nabla h = K \circ Th\colon TM \to TN$. In the subsequent points we fix such a connection and connector on N.

(b) **Pull-backs.** [45, (22.9.6)] For any manifold Q, smooth mapping $g\colon Q \to M$ and $Z_y \in T_yQ$, one has $\nabla_{Tg.Z_y}s = \nabla_{Z_y}(s \circ g)$. Thus, for g-related vector fields $Z \in \mathfrak{X}(Q)$ and $X \in \mathfrak{X}(M)$, one has $\nabla_Z(s \circ g) = (\nabla_X s) \circ g$, as summarized in the following diagram:
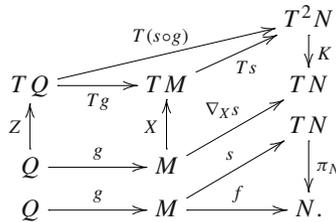
(c) **Torsion.** [45, (22.10.4)] For any smooth mapping $f\colon M \to N$ and vector fields $X, Y \in \mathfrak{X}(M)$ we have

$$\mathrm{Tor}(Tf.X, Tf.Y) = \nabla_X(Tf \circ Y) - \nabla_Y(Tf \circ X) - Tf \circ [X, Y]$$
$$= (K \circ \kappa_M - K) \circ TTf \circ TX \circ Y.$$

(d) **Sprays.** [45, 22.7] Any connection ∇ induces a one-to-one correspondence between fiber-wise quadratic C^α mappings $\Phi\colon TN \to TN$ and C^α sprays $S\colon TN \to TTN$. Here $\nabla_{\partial_t}c_t = \Phi(c_t)$ corresponds to $c_{tt} = S(c_t)$ for curves c in N. Equivalently, in terms of the connector K, the relation between Φ and S is as follows:

The diagram on the left introduces the projections $T(\pi_N)$ and π_{TN}, which define the two vector bundle structures on TTN. The diagram on the right shows that Φ and S are related by $\Phi = K \circ S$.

The following lemma describes how any connection on TN induces via a product-preserving functor from finite to infinite-dimensional manifolds [37,40] a connection on the mapping space $H^s_{H^r}(M, TN)$. The induced connection will be used as an auxiliary tool for expressing the geodesic equation; see Theorem 4.3.

2.6 Lemma Induced connection on mapping spaces. *Let $r \in (\dim(M)/2, \infty)$, $s \in [-r, r]$, and $\alpha \in \{\infty, \omega\}$. Then any C^α connection on TN induces in a natural (i.e., functorial) way a C^α connection on $H^s_{H^r}(M, TN)$.*

Proof Note that $TN \mapsto H^s_{H^r}(M, TN)$ is a product-preserving functor from finite-dimensional manifolds to infinite-dimensional manifolds as described in [40] and [41, Section

31]. Furthermore, note that $T H_{H^r}^s(M, TN) = H^{r,s,r,s}(M, TTN)$, where (r, s, r, s) denotes the Sobolev regularity of the individual components in any local trivialization $TTN \supset TTU \xrightarrow{TTu} u(U) \times (\mathbb{R}^n)^3 \subset (\mathbb{R}^n)^4$ induced by a chart $N \supset U \xrightarrow{u} u(U) \subset \mathbb{R}^n$; cf. the proof of Theorem 2.4. Applying the functor $H_{H^r}^s(M, \cdot)$ to the connector $K : TTN \to TN$ gives the induced connector

$$K_* = H_{H^r}^s(M, K) : T H_{H^r}^s(M, TN) \to H_{H^r}^s(M, TN), \quad h \mapsto K \circ h.$$

The induced connector is C^α by Lemma A.5. □

3 Sobolev immersions

This section collects some results about the differential geometry of immersions with Sobolev regularity. More specifically, it describes the Sobolev regularity of the induced metric, volume form, normal and tangential projections, and fractional Laplacian, as well as variations of these objects with respect to the immersion. Here the term fractional Laplacian is understood as a p-th power of the operator $1 + \Delta$, where Δ is the Bochner Laplacian and $p \in \mathbb{R}$; see [6, Section 3].

3.1 Lemma Geometry of Sobolev immersions. *The following statements hold for any $r \in (\dim(M)/2 + 1, \infty)$ and any smooth Riemannian metric $\bar{g}$ on N:*

(a) *The space $\mathrm{Imm}_{H^r}(M, N)$ of all immersions $f : M \to N$ of Sobolev class H^r is an open subset of the real analytic manifold $H^r(M, N)$.*

(b) *The pull-back metric is well defined and real analytic as a mapping*

$$\mathrm{Imm}^r(M, N) \ni f \mapsto f^* \bar{g} \in \mathrm{Met}_{H^{r-1}}(M) := \Gamma_{H^{r-1}}(S_+^2 T^* M).$$

(c) *The Riemannian volume form is well defined and real analytic as a mapping*

$$\mathrm{Imm}^r(M, N) \ni f \mapsto \mathrm{vol}^{f^*\bar{g}} \in \Gamma_{H^{r-1}}(\mathrm{Vol}\, M).$$

(d) *The tangential projection $\top : T \mathrm{Imm}(M, N) \to \mathfrak{X}(M)$ and the normal projection $\perp : T \mathrm{Imm}(M, N) \to T \mathrm{Imm}(M, N)$ are defined for smooth $h \in T_f \mathrm{Imm}(M, N)$ via the relation $h = Tf.h^\top + h^\perp$, where $\bar{g}(h^\perp(x), T_x f(T_x M)) = 0$ for all $x \in M$. They extend real analytically for any real number $s \in [1 - r, r - 1]$ to*

$$\perp \in \Gamma_{C^\omega}\Big(L(H_{\mathrm{Imm}^r}^s(M, TN), H_{\mathrm{Imm}^r}^s(M, TN))\Big),$$

$$\top \in C^\omega\Big(H_{\mathrm{Imm}^r}^s(M, TN), \mathfrak{X}_{H^s}(M)\Big),$$

where $H_{\mathrm{Imm}^r}^s(M, TN)$ is the space of 'H^s mappings $M \to TN$ with foot point in $\mathrm{Imm}^r(M, N)$' described in Theorem 2.4.

(e) *For any real numbers s, p with $s, s - 2p \in [1 - r, r]$, the fractional Laplacian*

$$f \mapsto (1 + \Delta^{f^*\bar{g}})^p$$

is a real analytic section of the bundle

$$GL(H_{\mathrm{Imm}^r}^s(M, TN), H_{\mathrm{Imm}^r}^{s-2p}(M, TN)).$$

Proof (a) The space $H^r(M, N)$ is continuously embedded in $C^1(M, N)$ because $r > \dim(M)/2 + 1$. Thus, the space of immersions is open.

(b) follows from the formula $f^*\bar{g} = \bar{g}(Tf, Tf)$.

(c) follows from **(b)** and the real analyticity of $g \mapsto \text{vol}^g$; see [6, Lemma 3.3].

(d) Let U be an open subset of M which carries a local frame $X \in \Gamma(GL(\mathbb{R}^m, TU))$. For any $f \in \text{Imm}^r(M, N)$, the Gram-Schmidt algorithm transforms X into an $(f^*\bar{g})$-ortho-normal frame $Y_f \in \Gamma_{H^{r-1}}(GL(\mathbb{R}^m, TU))$, which is given by

$$\forall j \in \{1, \ldots, m\}: \quad Y_f^j = \frac{X^j - \sum_{k=1}^{j-1}(f^*\bar{g})(Y_f^k, X^j)Y_f^k}{\left\| X^j - \sum_{k=1}^{j-1}(f^*\bar{g})(Y_f^k, X^j)Y_f^j \right\|_{f^*\bar{g}}}.$$

This defines a real analytic map

$$Y: \text{Imm}^r(M, N) \to \Gamma_{H^{r-1}}(GL(\mathbb{R}^m, TU)).$$

We write TN as a sub-bundle of a trivial bundle $N \times V$ and denote the corresponding inclusion and projection mappings by

$$i: TN \to N \times V, \quad \pi: N \times V \to TN.$$

This allows one to define a projection from $N \times V$ onto TN and further onto the normal bundle of f, which is real analytic as a map

$$p: \text{Imm}^r(M, N) \to H^{r-1}(U, L(V, V)),$$

$$p_f(x)(v) := v - \sum_{i=1}^{m} \bar{g}\big(\pi(f(x), v), T_x f.Y_f^i(x)\big).$$

This construction can be repeated for any open set $\tilde{U}$ such that $T\tilde{U}$ is parallelizable, and the resulting projections p_f coincide on $U \cap \tilde{U}$. Thus, one obtains a real analytic map

$$p: \text{Imm}^r(M, N) \to H^{r-1}(M, L(V, V)).$$

By the module properties 2.3, this induces a real analytic map

$$\tilde{p}: \text{Imm}^r(M, N) \times H^s(M, V) \to H^s(M, V), \quad \tilde{p}(f, h) := p_f.h.$$

These maps fit into the commutative diagrams

$$
\begin{array}{ccc}
T_{f(x)}N \xrightarrow{\ \perp\ } T_{f(x)}N & & H^s_{\text{Imm}^r}(M, TN) \xrightarrow{\qquad \perp \qquad} H^s_{\text{Imm}^r}(M, TN) \\
\Big\downarrow{\scriptstyle i} \quad \Big\uparrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle i_*} \qquad\qquad\qquad\qquad\quad \Big\uparrow{\scriptstyle \pi_*} \\
V \xrightarrow{\ p_f(x)\ } V & & \text{Imm}^r(M, N) \times H^s(M, V) \xrightarrow{\ \tilde{p}\ } \text{Imm}^r(M, N) \times H^s(M, V)
\end{array}
$$

The maps i_* and π_* are real analytic, as shown in part (b') of the proof of Lemma A.5. Therefore, the map $\perp = \pi_* \circ \tilde{p} \circ i_*$ is real analytic. The tangential projection $h^\top = Tf^{-1}(h - h^\perp)$ is then also real analytic.

(e) There is a bundle E over N such that $TN \oplus E$ is a trivial bundle, i.e., $TN \oplus E \cong N \times V$ for some vector space V. We endow the bundle E with a smooth connection and the bundle $N \times V \cong TN \oplus E$ with the product connection. By construction, the inclusion $i: TN \to N \times V$ and projection $\pi: N \times V \to TN$ respect the connection. At the level of Sobolev sections of these bundles, this means that the natural inclusion and projection

mappings fit into the following commutative diagram with $p = 1$:

$$
\begin{array}{ccc}
H^s_{\mathrm{Imm}^r}(M, TN) & \xrightarrow{\;\;(1+\Delta)^p\;\;} & H^{s-2p}_{\mathrm{Imm}^r}(M, TN) \\
\Big\downarrow{\scriptstyle i_*} & & \Big\uparrow{\scriptstyle \pi_*} \\
\mathrm{Imm}^r(M, N) \times H^s(M, V) & \xrightarrow{(\mathrm{Id},(1+\Delta)^p)} & \mathrm{Imm}^r(M, N) \times H^{s-2p}(M, V)
\end{array}
$$

As the functional calculus preserves commutation relations, this extends to all p. Thus, we have reduced the situation to the bottom row of the diagram, where the fractional Laplacian acts on $H^s(M, V)$. In this case real analytic dependence of the fractional Laplacian on the metric has been shown in [6, Theorem 5.4]. Now the claim follows from the chain rule and (b).

$\square$

The following lemma describes the first variation of the metric and fractional Laplacian. The key point is that the variation in normal directions is more regular than the variation in tangential directions. This will be of importance in Theorem 4.6. The lemma is formulated using an auxiliary connection $\hat{\nabla}$ on N, e.g., the Levi–Civita connection of a Riemannian metric $\bar{g}$ on N.

3.2 Lemma First variation formulas. *Let $\bar{g}$ be a smooth Riemannian metric on N, and let $\hat{\nabla}$ be a C^α connection on N for $\alpha \in \{\infty, \omega\}$.*

(a) *For any $r \in (\dim(M)/2 + 1, \infty)$ and $s \in [2 - r, r]$, the variation of the pull-back metric extends to a real analytic map*

$$
H^s_{\mathrm{Imm}^r}(M, TN) \ni m \mapsto D_{f,m}(f^*\bar{g}) \in \Gamma_{H^{s-1}}(S^2 T^*M).
$$

(b) *For any $r \in (\dim(M)/2 + 2, \infty)$ and $s \in [2 - r, r - 2]$, the variation of the pull-back metric in normal directions extends to a real analytic map*

$$
H^s_{\mathrm{Imm}^r}(M, TN) \ni m \mapsto D_{f,m^\perp}(f^*\bar{g}) \in \Gamma_{H^s}(S^2 T^*M).
$$

(c) *For any $r > \dim(M)/2 + 2$ and $p \in [1, r - 1]$ the variation of the fractional Laplacian in normal directions extends to a C^α map*

$$
H^{2p-r}_{\mathrm{Imm}^r}(M, TN) \ni m \mapsto \hat{\nabla}_{m^\perp}(1 + \Delta^{f^*\bar{g}})^p \in L(H^r_{\mathrm{Imm}^r}(M, TN), H^{1-r}_{\mathrm{Imm}^r}(M, TN)),
$$

where $\hat{\nabla}$ is the induced connection on $GL(H^r_{\mathrm{Imm}^r}(M, TN), H^{1-r}_{\mathrm{Imm}^r}(M, TN))$ described in Lemma 2.6, and $L(H^r_{\mathrm{Imm}^r}(M, TN), H^{1-r}_{\mathrm{Imm}^r}(M, N))$ is the vector bundle over $\mathrm{Imm}^r(M, N)$ described in Theorem 2.4.

Proof We will repeatedly use the module properties 2.3.

(a) follows from the following formula for the first variation of the pull-back metric [11, Lemma 5.5]:

$$
D_{f,m}(f^*\bar{g}) = \bar{g}(\nabla m, Tf) + \bar{g}(Tf, \nabla m)
$$

(b) Splitting the above formula into tangential and normal parts of m yields

$$
D_{f,m}(f^*\bar{g}) = -2\bar{g}(m^\perp, \nabla Tf) + g(\nabla m^\top, \cdot) + g(\cdot, \nabla m^\top).
$$

Now the claim follows from the real analyticity of the projection $\perp$ in Lemma 3.1.

(c') We claim for any bundle E over M with fixed fiber metric and fixed connection (i.e., not depending on g) that the following map is real analytic:

$$\mathrm{Met}_{H^{r-1}}(M) \times \Gamma_{H^s}(S^2 T^* M)) \ni (g, m) \mapsto D_{g,m} \Delta^g \in L(\Gamma_{H^q}(E), \Gamma_{H^{s+q-r-1}}(E)),$$

where $s \in [2 - r, r - 1]$ and $q \in [2 - s, r]$. To prove the claim we proceed similarly to [6, Lemma 3.8]. As the connection on E does not depend on the metric g,

$$D_{g,m} \Delta^g h = -D_{g,m}(\mathrm{Tr}^{g^{-1}} \nabla^g \nabla h) = -(D_{g,m} \mathrm{Tr}^{g^{-1}}) \nabla^g \nabla h - \mathrm{Tr}^{g^{-1}} (D_{g,m} \nabla^g) \nabla h.$$

Here ∇^g is the covariant derivative on $T^* M \otimes E$. The proof of [6, Lemma 3.8] and some multi-linear algebra show that $D_{g,m} \nabla^g$ is tensorial and real analytic as a map

$$\mathrm{Met}_{H^{r-1}}(M) \times \Gamma_{H^s}(S^2 T^* M) \ni (g, m)$$
$$\mapsto D_{g,m} \nabla^g \in \Gamma_{H^{s-1}}(T^* M \otimes L(T^* M \otimes E, T^* M \otimes E)).$$

Moreover, the following maps are real analytic by [6, Lemmas 3.2 and 3.5]:

$$\mathrm{Met}_{H^{r-1}}(M) \ni g \mapsto g^{-1} \in \Gamma_{H^{r-1}}(S^2 T M),$$
$$\mathrm{Met}_{H^{r-1}}(M) \ni g \mapsto \nabla^g \in L(\Gamma_{H^{q-1}}(T^* M \otimes E), \Gamma_{H^{q-2}}(T^* M \otimes T^* M \otimes E)).$$

Together with the module properties 2.3 this establishes (c').

(c") Using (c') we will now study the smooth dependence of fractional Laplacians. In particular we claim for any bundle E over M with fixed fiber metric and fixed connection and any $p \in (1, r - 1)$ that the following map is real analytic:

$$\mathrm{Met}_{H^{r-1}}(M) \times \Gamma_{H^{2p-r}}(S^2 T^* M)) \ni (g, m) \mapsto D_{g,m}(1 + \Delta^g)^p \in L(\Gamma_{H^r}(E), \Gamma_{H^{1-r}}(E)).$$

The claim is a generalization of [6, Lemma 5.5] to perturbations m with even lower Sobolev regularity and uses the fact that the connection on E does not depend on the metric g. Let X, Y, Z be the spaces of operators given by

$$X = L(\Gamma_{H^r}(E), \Gamma_{H^{r-2}}(E)) \cap L(\Gamma_{H^{3-r}}(E), \Gamma_{H^{1-r}}(E)),$$
$$Y = L(\Gamma_{H^r}(E), \Gamma_{H^{-r+2p-1}}(E)) \cap L(\Gamma_{H^{r-2p+2}}(E), \Gamma_{H^{1-r}}(E)),$$
$$Z = L(\Gamma_{H^r}(E), \Gamma_{H^{r-2}}(E)) \cap L(\Gamma_{H^{r-2p+2}}(E), \Gamma_{H^{r-2p}}(E)).$$

Note that the conditions $r > 2$ and $p > 1$ ensure that $X, Y,$ and Z are intersections of operator spaces on distinct Sobolev scales, as required in [6, Theorem 4.5] Moreover, let $U \subseteq X$ be an open neighborhood of $1 + \Delta^g$ with $g \in \mathrm{Met}_{H^{r-1}}(M)$ such that the holomorphic functional calculus is well-defined and holomorphic on U in the sense of [6, Theorem 4.5]. Then the desired map is the composition of the following two maps:

$$\mathrm{Met}_{H^{r-1}}(M) \times \Gamma_{H^{2p-r}}(S^2 T^* M) \in (g, m) \mapsto (1 + \Delta^g, D_{g,m} \Delta^g) \in (X, Y),$$
$$(U, Y) \ni (A, B) \mapsto D_{A,B} A^p \in L(\Gamma_{H^r}(E), \Gamma_{H^{1-r}}(E)).$$

The first map is real analytic by Lemma 3.1.(e) and (c'). The second map has to be interpreted via the following identity, which is shown in the proof of [6, Lemma 5.5] using the resolvent representation of the functional calculus:

$$\forall A \in U, \forall B \in Y \cap Z: \qquad D_{A,B} A^p = A^{r-1-p} D_{A, A^{p-r+1} B} A^p.$$

The right-hand side above is the composition of the following maps, which are again real analytic by [6, Theorem 4.5]:

$$(U, Y) \ni (A, B) \mapsto (A, A^{p-r+1/2}B) \in (U, Z),$$
$$(U, Z) \ni (A, B) \mapsto (A, D_{A,B}A^p) \in U \times L(\Gamma_{H^r}(E), \Gamma_{H^{r-2p}}(E))$$
$$U \times L(\Gamma_{H^r}(E), \Gamma_{H^{r-2p}}(E)) \ni (A, B) \mapsto A^{r-p-1/2}B \in L(\Gamma_{H^r}(E), \Gamma_{H^{1-r}}(E))$$

This proves (c''). Note that (c'') extends to $p = 1$ thanks to (c').

(c) As in the proof of Lemma 3.1.(e), we write i and π for the inclusion and projection mappings of TN, seen as a sub-bundle of a trivial bundle $TN \oplus E \cong N \times V$ with C^α product connection. If we consider i_* and π_* as real analytic sections of operator bundles,

$$i_* \in \Gamma_{C^\omega}(L(H^r_{\mathrm{Imm}^r}(M, TN), H^r_{\mathrm{Imm}^r}(M, N \times V))),$$
$$\pi_* \in \Gamma_{C^\omega}(L(H^{r-2p}_{\mathrm{Imm}^r}(M, TN), H^{r-2p}_{\mathrm{Imm}^r}(M, N \times V))),$$

then the covariant derivative of the fractional Laplacian can be expressed as follows:

$$\hat{\nabla}_{m^\perp}(1 + \Delta^{f^*\bar{g}})^p = (\hat{\nabla}_{m^\perp}\pi_*)(\mathrm{Id}, (1 + \Delta^{f^*\bar{g}})^p)i_*$$
$$+ \pi_*\big(\hat{\nabla}_{m^\perp}(\mathrm{Id}, (1 + \Delta^{f^*\bar{g}})^p)\big)i_* + \pi_*(\mathrm{Id}, (1 + \Delta^{f^*\bar{g}})^p)(\hat{\nabla}_{m^\perp}i_*).$$

The maps i_* and π_* are real analytic, and consequently their covariant derivatives are C^α. According to Lemma 2.6, the canonical connection D on the vector space V induces a real analytic connection on the bundle of bounded linear operators $L(H^r_{\mathrm{Imm}^r}(M, N \times V), H^{1-r}_{\mathrm{Imm}^r}(M, N \times V))$. By general principles, this connection differs from $\hat{\nabla}$ by a C^α tensor field, often called the Christoffel symbol. Thus, it suffices to show that the following map is C^α:

$$H^{2p-r}_{\mathrm{Imm}^r}(M, TN) \ni m \mapsto D_{f,m^\perp}(\mathrm{Id}, (1 + \Delta^{f^*\bar{g}})^p)$$
$$\in L(H^r_{\mathrm{Imm}^r}(M, N \times V), H^{1-r}_{\mathrm{Imm}^r}(M, N \times V)).$$

As D is the canonical connection, this is equivalent to the following map being C^α:

$$H^{2p-r}_{\mathrm{Imm}^r}(M, TN) \ni m \mapsto D_{f,m^\perp}(1 + \Delta^{f^*\bar{g}})^p \in L(H^r(M, V), H^{1-r}(M, V)).$$

By (b) with $s = 2p - r$, the variation of the pull-back metric in normal directions is real analytic as a map

$$H^{2p-r}_{\mathrm{Imm}^r}(M, TN) \ni m \mapsto D_{f,m^\perp}(f^*\bar{g}) \in \Gamma_{H^{2p-r}}(S^2T^*M).$$

Thus, (c) follows from (c'') and the chain rule. $\qquad\square$

4 Weak Riemannian metrics on spaces of immersions

The main result of this section is that the geodesic equation of Sobolev-type metrics is locally well posed under certain conditions on the operator governing the metric. The setting is general and encompasses several examples, including in particular fractional Laplace operators.

4.1 Sobolev-type metrics

Within the setup of Sect. 2.1, we consider Sobolev-type Riemannian metrics on the space of immersions $f : M \to N$ of the form

$$G_f^P(h, k) = \int_M \bar{g}(P_f h, k) \, \mathrm{vol}(f^* \bar{g}), \qquad h, k \in T_f \, \mathrm{Imm}(M, N),$$

where $\bar{g}$ is a C^α Riemannian metric on N for $\alpha \in \{\infty, \omega\}$, and where P is an operator field which satisfies the following conditions for some $p \in [0, \infty)$, some $r_0 \in (\dim(M)/2+1, \infty)$, and all $r \in [r_0, \infty)$:

(a) Assume that P is a C^α section of the bundle

$$GL(H^r_{\mathrm{Imm}^r}(M, TN), H^{r-2p}_{\mathrm{Imm}^r}(M, TN)) \to \mathrm{Imm}^r(M, N),$$

   where GL denotes bounded linear operators with bounded inverse.

(b) Assume that P is $\mathrm{Diff}(M)$-equivariant in the sense that one has for all $\varphi \in \mathrm{Diff}(M)$, $f \in \mathrm{Imm}^r(M, N)$, and $h \in T_f \, \mathrm{Imm}^r(M, N)$ that

$$(P_f h) \circ \varphi = P_{f \circ \varphi}(h \circ \varphi).$$

(c) Assume for each $f \in \mathrm{Imm}^r(M, N)$ that the operator P_f is nonnegative and symmetric with respect to the $H^0(g)$ inner product on $T_f \, \mathrm{Imm}^r(M, N)$, i.e., for all $h, k \in T_f \, \mathrm{Imm}^r(M, N)$:

$$\int_M \bar{g}(P_f h, k) \, \mathrm{vol}(g) = \int_M \bar{g}(h, P_f k) \, \mathrm{vol}(g), \qquad \int_M \bar{g}(P_f h, h) \, \mathrm{vol}(g) \geq 0.$$

(d) Assume that the normal part of the adjoint $\mathrm{Adj}(\nabla P)^\perp$, defined by

$$\int_M \bar{g}((\nabla_{m^\perp} P)h, k) \, \mathrm{vol}(g) = \int_M \bar{g}(m, \mathrm{Adj}(\nabla P)^\perp(h, k)) \, \mathrm{vol}(g)$$

   for all $f \in \mathrm{Imm}(M, N)$ and $m, h, k \in T_f \, \mathrm{Imm}$, exists and is a C^α section of the bundle of bilinear maps

$$L^2(H^r_{\mathrm{Imm}^r}(M, TN), H^r_{\mathrm{Imm}^r}(M, TN); H^{r-2p}_{\mathrm{Imm}^r}(M, TN)).$$

   Here ∇ denotes the induced connection (see Lemma 2.6) of the Levi–Civita connection of $\bar{g}$.

4.2 Remark In [11, Section 6.6] we had more complicated conditions, and we implicitly claimed that they imply the conditions in Sect. 4.1 above. There was, however, a significant gap in the argumentation of the main result. Namely, we did not show the smoothness of the extended mappings on Sobolev completions. This article closes this gap and extends the analysis to the larger class of fractional order metrics.

We now derive the geodesic equation of Sobolev-type metrics. Recall that the usual form of the geodesic equation is $f_{tt} = \Gamma_f(f_t, f_t)$, where the time derivatives f_t and f_{tt} as well as the Christoffel symbols Γ are expressed in a chart. This raises the problem that the space $\mathrm{Imm}(M, N)$ lacks canonical charts, unless N admits a global chart. However, $\mathrm{Imm}(M, N)$ carries a canonical connection, namely, the one induced by the metric $\bar{g}$ on N, which has been described in Lemma 2.6. This auxiliary connection, which will be denoted by ∇, allows one to write the geodesic equation as $\nabla_{\partial_t} f_t = \Gamma_f(f_t, f_t)$, where Γ is a difference between two connections and therefore tensorial. In the special case where N is an open subset of Euclidean space, this coincides with the usual derivative $\nabla_{\partial_t} f_t = f_{tt}$; cf. Corollary 5.3.

4.3 Theorem Geodesic equation. [11, Theorem 6.4] *Assume the conditions of Sect.* 4.1. *Then a smooth curve* $f : [0, 1] \to \mathrm{Imm}(M, N)$ *is a critical point of the energy functional*

$$E(f) = \frac{1}{2} \int_0^1 \int_M \bar{g}(P_f f_t, f_t) \, \mathrm{vol}^g \, dt$$

if and only if it satisfies the geodesic equation

$$\nabla_{\partial_t} f_t = \frac{1}{2} P_f^{-1} \Big(\mathrm{Adj}(\nabla P)_f (f_t, f_t)^\perp - 2Tf \, \bar{g}(P_f f_t, \nabla f_t)^\sharp - \bar{g}(P_f f_t, f_t) \, \mathrm{Tr}^g(\nabla Tf) \Big)$$
$$- P_f^{-1} \Big((\nabla_{f_t} P) f_t + \mathrm{Tr}^g \big(\bar{g}(\nabla f_t, Tf) \big) P_f f_t \Big).$$

This also holds for smooth curves in $\mathrm{Imm}^r(M, N)$ *for any* $r \geq r_0$.

Here we used the following notation: $g = f^* \bar{g}$ is the pull-back metric and $\sharp = g^{-1}$ its associated musical isomorphism, the operator P is seen as a map $P \colon \mathrm{Imm} \to GL(T\,\mathrm{Imm}, T\,\mathrm{Imm})$, its composition with f is denoted by $P_f \colon \mathbb{R} \to GL(T\,\mathrm{Imm}, T\,\mathrm{Imm})$, its covariant derivative with respect to the connection on $L(T\,\mathrm{Imm}, T\,\mathrm{Imm})$ induced by ∇ is denoted by $\nabla P \colon T\,\mathrm{Imm} \to L(T\,\mathrm{Imm}, T\,\mathrm{Imm})$, the canonical vector field on $\mathbb{R}$ is denoted by $\partial_t \colon \mathbb{R} \to T\mathbb{R}$, the time derivative $f_t = \partial_t f$ is viewed as a map $f_t \colon \mathbb{R} \times M \to TN$ in the expression $\nabla f_t \colon \mathbb{R} \times M \to T^*M \otimes f^*TN$ and as a map $f_t \colon \mathbb{R} \to T\,\mathrm{Imm}$ elsewhere, the spatial derivative Tf is viewed as a map $Tf \colon \mathbb{R} \times M \to T^*M \otimes f^*TN$, and the map $\nabla Tf \colon \mathbb{R} \times M \to T^*M \otimes T^*M \otimes f^*TN$ is the second fundamental form.

Proof We will consider variations of the curve energy functional along one-parameter families $f \colon (-\varepsilon, \varepsilon) \times [0, 1] \times M \to N$ of curves of immersions with fixed endpoints. The variational parameter will be denoted by $s \in (-\varepsilon, \varepsilon)$, the time-parameter by $t \in [0, 1]$. Then the first variation of the energy $E(f)$ can be calculated as follows:

$$\partial_s E(f) = \partial_s \frac{1}{2} \int_0^1 \int_M \bar{g}(P_f f_t, f_t) \, \mathrm{vol}^g \, dt.$$

As the connection respects $\bar{g}$ and is a derivation of tensor products, and as the operator P_f is symmetric, we have

$$\partial_s E(f) = \frac{1}{2} \int_0^1 \int_M \bar{g} \left((\nabla_{\partial_s} P_f) f_t + 2 P_f \nabla_{\partial_s} f_t + \frac{\partial_s \mathrm{vol}^g}{\mathrm{vol}^g} P_f f_t, f_t \right) \mathrm{vol}^g \, dt.$$

We will treat each of the three summands above separately, making extensive use of properties 2.5 of the (induced) connection ∇:

(a) For the first summand we have by the definition of the adjoint that

$$\frac{1}{2} \int_0^1 \int_M \bar{g}((\nabla_{\partial_s} P_f) f_t, f_t) \, \mathrm{vol}^g \, dt = \frac{1}{2} \int_0^1 \int_M \bar{g}((\nabla_{f_s} P) f_t, f_t) \, \mathrm{vol}^g \, dt$$
$$= \frac{1}{2} \int_0^1 \int_M \bar{g} \Big(f_s, \mathrm{Adj}(\nabla P)(f_t, f_t)^\perp + \mathrm{Adj}(\nabla P)(f_t, f_t)^\top \Big) \mathrm{vol}^g \, dt.$$

To calculate the tangential part of the adjoint, thereby establishing its existence, we need the following formula for the tangential variation of P, which holds for any vector field X on M:

$$(\nabla_{Tf.X} P)(h) = (\nabla_{\partial_t|_0} P_{f \circ \mathrm{Fl}_t^X})(h \circ \mathrm{Fl}_0^X)$$
$$= \nabla_{\partial_t|_0}(P_{f \circ \mathrm{Fl}_t^X}(h \circ \mathrm{Fl}_t^X)) - P_{f \circ \mathrm{Fl}_0^X}(\nabla_{\partial_t|_0}(h \circ \mathrm{Fl}_t^X))$$
$$= \nabla_{\partial_t|_0}(P_f(h) \circ \mathrm{Fl}_t^X) - P_f(\nabla_{\partial_t|_0}(h \circ \mathrm{Fl}_t^X))$$
$$= \nabla_X(P_f(h)) - P_f(\nabla_X h),$$

where Fl_t^X denotes the flow of the vector field X at time t and where we used the equivariance of P in the step from the second to the third line. Using this and the symmetry of P we get

$$\frac{1}{2}\int_0^1 \int_M \bar{g}\left(f_s, \mathrm{Adj}(\nabla P)(f_t, f_t)^\top\right) \mathrm{vol}^g \, dt = \int_M \bar{g}\left((\nabla_{Tf.f_s^\top} P) f_t, f_t\right) \mathrm{vol}(g)$$
$$= \int_M \bar{g}\left(\nabla_{f_s^\top}(P_f f_t) - P_f(\nabla_{f_s^\top} f_t), f_t\right) \mathrm{vol}(g)$$
$$= \int_M \left(\bar{g}(\nabla_{f_s^\top}(P_f f_t), f_t) - \bar{g}(\nabla_{f_s^\top} f_t, P_f f_t)\right) \mathrm{vol}(g)$$
$$= \int_M \bar{g}\left(Tf.f_s^\top, Tf.(\bar{g}(\nabla(P_f f_t), f_t) - \bar{g}(\nabla f_t, P_f f_t))^\sharp\right) \mathrm{vol}(g)$$
$$= \int_M \bar{g}\left(Tf.f_s^\top, Tf.(\nabla \bar{g}(P_f f_t, f_t) - 2\bar{g}(\nabla f_t, P_f f_t))^\sharp\right) \mathrm{vol}(g)$$
$$= \int_M \bar{g}\left(f_s, Tf.(\nabla \bar{g}(P_f f_t, f_t) - 2\bar{g}(\nabla f_t, P_f f_t))^\sharp\right) \mathrm{vol}(g).$$

Thus we obtain the following formula for the first summand of the variation of E:

$$\frac{1}{2}\int_0^1 \int_M \bar{g}((\nabla_{\partial_s} P_f) f_t, f_t) \mathrm{vol}^g \, dt$$
$$= \frac{1}{2}\int_0^1 \int_M \bar{g}\left(f_s, \mathrm{Adj}(\nabla P)(f_t, f_t)^\perp + Tf.(\nabla \bar{g}(P f_t, f_t) - 2\bar{g}(\nabla f_t, P f_t))^\sharp\right) \mathrm{vol}^g \, dt.$$

(b) As P_f is symmetric and the covariant derivative on $\mathrm{Imm}(M, N)$ is torsion-free (see Sect. 2.5), i.e.,

$$\nabla_{\partial_t} f_s - \nabla_{\partial_s} f_t = Tf.[\partial_t, \partial_s] + \mathrm{Tor}(f_t, f_s) = 0,$$

we get for the second summand

$$\int_0^1 \int_M \bar{g}\left(P_f \nabla_{\partial_s} f_t, f_t\right) \mathrm{vol}^g \, dt = \int_0^1 \int_M \bar{g}\left(\nabla_{\partial_t} f_s, P_f f_t\right) \mathrm{vol}^g \, dt.$$

Integration by parts for ∂_t yields

$$\int_0^1 \int_M \bar{g}\left(\nabla_{\partial_t} f_s, P_f f_t\right) \mathrm{vol}^g \, dt$$
$$= \int_0^1 \int_M \left(\bar{g}\left(f_s, -(\nabla_{f_t} P) f_t - P_f(\nabla_{f_t} f_t)\right) - \frac{\partial_t \mathrm{vol}^g}{\mathrm{vol}^g}\right) \mathrm{vol}^g \, dt.$$

To further expand the last term we use the following formula for the variation of the volume form [11, Lemma 5.7]:

$$\frac{\partial_t \, \mathrm{vol}^g}{\mathrm{vol}^g} = \mathrm{Tr}^g \left(\bar{g}(\nabla f_t, Tf) \right) = -\nabla^*(\bar{g}(f_t, Tf)) - \bar{g}(f_t, \mathrm{Tr}^g(\nabla Tf)),$$

where ∇Tf is the second fundamental form and where ∇^* denotes the adjoint of the covariant derivative. Using the first of the above formulas we obtain for the second summand:

$$\int_0^1 \int_M \bar{g} \left(\nabla_{\partial_t} f_s, P_f f_t \right) \mathrm{vol}^g \, dt$$
$$= \int_0^1 \int_M \left(\bar{g} \left(f_s, -(\nabla_{f_t} P) f_t - P_f(\nabla_{f_t} f_t) \right) - \mathrm{Tr}^g \left(\bar{g}(\nabla f_t, Tf) \right) P_f f_t \right) \mathrm{vol}^g \, dt.$$

(c) Using the second version of the variational formula for the volume in the third summand in the variation of the energy yields

$$\frac{1}{2} \int_0^1 \int_M \frac{\partial_s \, \mathrm{vol}^g}{\mathrm{vol}^g} \bar{g} \left(P_f f_t, f_t \right) \mathrm{vol}^g \, dt$$
$$= -\frac{1}{2} \int_0^1 \int_M \left(\nabla^*(\bar{g}(f_s, Tf)) + \bar{g}(f_s, \mathrm{Tr}^g(\nabla Tf)) \right) \bar{g} \left(P_f f_t, f_t \right) \mathrm{vol}^g \, dt$$
$$= -\frac{1}{2} \int_0^1 \int_M \left(\bar{g}(f_s, Tf.(\nabla \bar{g} \left(P_f f_t, f_t \right))^\sharp + \mathrm{Tr}^g(\nabla Tf) \bar{g} \left(P_f f_t, f_t \right) \right) \mathrm{vol}^g \, dt.$$

Taken together, the calculations of (a)–(c) yield

$$\partial_s E(f) = \frac{1}{2} \int_0^1 \int_M \bar{g} \Big(f_s, \mathrm{Adj}(\nabla P)(f_t, f_t)^\perp - 2Tf.\bar{g}(\nabla f_t, P f_t)^\sharp - 2(\nabla_{f_t} P) f_t$$
$$-2P_f(\nabla_{f_t} f_t) - 2 \, \mathrm{Tr}^g \left(\bar{g}(\nabla f_t, Tf) \right) P_f f_t - \mathrm{Tr}^g(\nabla Tf) \bar{g} \left(P_f f_t, f_t \right) \Big) \mathrm{vol}^g \, dt.$$

Setting $\partial_s E(f) = 0$ for arbitrary perturbations f_s yields the geodesic equation on the space $\mathrm{Imm}(M, N)$ of smooth immersions. This statement extends to the space $\mathrm{Imm}^r(M, N)$ of Sobolev immersions because the right-hand side of the geodesic equation is continuous in $f \in C^\infty([0, 1], \mathrm{Imm}^r(M, N))$, as shown in part (a) of the proof of Theorem 4.4. □

We next show well-posedness of the geodesic equation using the Ebin–Marsden approach [23] of extending the geodesic spray to a smooth vector field on $T \, \mathrm{Imm}^r$ for sufficiently high r and showing that the solutions exist on a time interval which is independent of r.

4.4 Theorem Local well-posedness of the geodesic equation. *Assume the conditions of Sect.* 4.1 *with* $p \geq 1$. *Then the following statements hold for all* $r \in [r_0, \infty)$:

(a) *The initial value problem for geodesics has unique local solutions in* $\mathrm{Imm}^r(M, N)$. *The solutions depend* C^α *on t and on the initial condition* $f_t(0) \in T \, \mathrm{Imm}^r(M, N)$.

(b) *The Riemannian exponential map* $\exp^P$ *exists and is* C^α *on a neighborhood of the zero section in* $T \, \mathrm{Imm}_{H^r}$, *and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Imm}^r(M, N) \times \mathrm{Imm}^r(M, N)$.

(c) *The neighborhoods in (a)–(b) are uniform in r and can be chosen open in the* H^{r_0} *topology. Thus, (a)–(b) continue to hold for* $r = \infty$, *i.e., on the Fréchet manifold* $\mathrm{Imm}(M, N)$ *of smooth immersions.*

Proof (a) This can be shown as in [11, Theorem 6.6]. Let $\Phi(f_t)$ denote the right-hand side of the geodesic equation, i.e.,

$$\Phi(f_t) = \frac{1}{2}P^{-1}\Big(\mathrm{Adj}(\nabla P)(f_t, f_t)^{\perp} - 2Tf\,\bar{g}(Pf_t, \nabla f_t)^{\sharp} - \bar{g}(Pf_t, f_t)\,\mathrm{Tr}^g(\nabla Tf) \Big)$$
$$- P^{-1}\Big((\nabla_{f_t} P)f_t + \mathrm{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big)Pf_t \Big).$$

A term-by-term investigation using the conditions 4.1 and the module properties 2.3 shows that Φ is a fiber-wise quadratic C^{α} map

$$\Phi\colon T\,\mathrm{Imm}^r(M, N) \to T\,\mathrm{Imm}^r(M, N).$$

Here the condition $p \geq 1$ is needed to ensure that the term $P^{-1}\big(\bar{g}(Ph, h)\,\mathrm{Tr}^g(\nabla Tf)\big)$ is again of regularity H^r. The map Φ corresponds uniquely to a C^{α} spray S via the induced connection described in Lemma 2.6. In more detail: The right-hand side diagram in the proof of Sect. 2.5.(d) holds for any manifold N with connector K. Thus, replacing (N, K) by $(\mathrm{Imm}^r(M, N), K_*)$, one obtains the diagram

The spray S is C^{α} because the connection K_* and the map Φ are C^{α}. Therefore, by the theorem of Picard-Lindelöf, S admits a C^{α} flow

$$\mathrm{Fl}^S\colon U \to T\,\mathrm{Imm}^r(M, N)$$

for a maximal open neighborhood U of $\{0\} \times T\,\mathrm{Imm}^r(M, N)$ in $\mathbb{R} \times T\,\mathrm{Imm}^r(M, N)$. The neighborhood U is $\mathrm{Diff}(M)$-invariant thanks to the $\mathrm{Diff}(M)$-equivariance of S.

(b) follows from (a) as in [11, Theorem 6.6], and (c) follows from Lemma B.1 by writing $\mathrm{Imm}(M, N)$ as the intersection of all $\mathrm{Imm}^{r_0+k}(M, N)$ with $k \in \mathbb{N}_{\geq 0}$.

$\square$

4.5 Corollary *Theorem 4.4 with $\alpha = \omega$ remains valid if the assumptions in Sect. 4.1 are modified as follows: the metric $\bar{g}$ is only C^{∞}, and the connection ∇ in condition (d) is replaced by an auxiliary connection $\hat{\nabla}$, which is induced by a torsion-free C^{ω} connection on N, as described in Lemma 2.6.*

Proof In the proof of Theorem 4.3, the geodesic equation is derived by expressing the first variation $\partial_s E$ of the energy functional using the Levi–Civita connection of $\bar{g}$. If the auxiliary connection $\hat{\nabla}$ is used instead, then the following additional terms appear in the formula for $\partial_s E$:

$$\int_0^1 \int_M \Big(-\frac{1}{2}(\hat{\nabla}_{Tf.f_s^{\top}}\bar{g})(Pf_t, f_t) - (\hat{\nabla}_{f_t}\bar{g})(f_s, Pf_t) + \frac{1}{2}(\hat{\nabla}_{f_s}\bar{g})(Pf_t, f_t) \Big)\,\mathrm{vol}^g\,dt$$

Accordingly, letting Ψ denote the right-hand side of the original geodesic equation with ∇P replaced by $\hat{\nabla}P$, i.e.,

$$\Psi(f_t) = \frac{1}{2}P^{-1}\Big(\mathrm{Adj}(\hat{\nabla}P)(f_t, f_t)^{\perp} - 2Tf\,\bar{g}(Pf_t, \nabla f_t)^{\sharp} - \bar{g}(Pf_t, f_t)\,\mathrm{Tr}^g(\nabla Tf) \Big)$$
$$- P^{-1}\Big((\hat{\nabla}_{f_t}P)f_t + \mathrm{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big)Pf_t \Big),$$

the geodesic equation becomes

$$\hat{\nabla}_{\partial_t} f_t = \Psi(f_t) - \frac{1}{2} Tf.\left(\bar{g}^{-1}(\hat{\nabla}\bar{g})(P_f f_t, f_t)\right)^\top - \bar{g}^{-1}(\hat{\nabla}_{f_t}\bar{g})(\cdot, P_f f_t)$$
$$+ \frac{1}{2}\bar{g}^{-1}(\hat{\nabla}\bar{g})(P_f f_t, f_t).$$

One verifies as in the proof of Theorem 4.4 that the right-hand side, seen as a function of f_t, is a fiber-wise quadratic real analytic map $T\,\text{Imm}^r(M, N) \to T\,\text{Imm}^r(M, N)$. As the auxiliary connection $\hat{\nabla}$ is real analytic, this implies that the corresponding spray is real analytic, as well; see Sect. 2.5. Since the spray is independent of the auxiliary connection $\hat{\nabla}$, one may proceed as in the proof of Theorem 4.4.   □

The following theorem shows that (scale-invariant) fractional-order Sobolev metrics satisfy the conditions in Sect. 4.1. This implies local well-posedness of their geodesic equations by Theorem 4.4. Further metrics considered in the literature include curvature weighted metrics and the so-called general elastic metric [33], which can also be formulated in the present framework [12]. The proof takes advantage of the fact that the adjoint in the geodesic equation 4.3 has been split into normal and tangential parts. The normal part has the correct Sobolev regularity thanks to Lemma 3.2. The tangential part incurs a loss of derivatives, but the bad terms cancel out with some other terms in the geodesic equation as shown in part (a) of the proof of Theorem 4.3.

4.6 Theorem *The following operators satisfy the conditions in Sect.* 4.1 *with* $\alpha = \omega$ *for any* $p \in [1, \infty)$ *and* $r_0 \in (\dim(M)/2 + 2, \infty) \cap [p + 1, \infty)$:

$$P_f := \left(1 + \Delta^{f^*\bar{g}}\right)^p, \quad and \quad P_f := \left(\text{Vol}^{-1-\frac{2}{\dim M}} + \text{Vol}^{-1}\Delta^{f^*\bar{g}}\right)^p.$$

Thus, the geodesic equations of these metrics are well posed in the sense of Theorem 4.4.

Proof We will prove this result only for the first field of operators because the proof for the second one is analogous. We shall check conditions (a)–(d) of Sect. 4.1.

(a) follows from Lemma 3.1.
(b) Diff(M)-equivariance of $(1 + \Delta^{f^*\bar{g}})$ is well-known for smooth f and follows in the general case by approximation, noting that the pull-back along a smooth diffeomorphism is a bounded linear map between Sobolev spaces of the same order of regularity [31, Theorem B.2]. As the functional calculus preserves commutation relations, this implies the Diff(M)-equivariance of $(1 + \Delta^g)^p$.
(c) is well-known for smooth f, h, k and follows in the general case by approximation using the continuity of $f \mapsto \langle \cdot, \cdot \rangle_{H^0(f^*\bar{g})}$ established in [6, Lemma 3.3] and the continuity of $f \mapsto P_f$.
(d) Recall from Lemma 3.2 that the derivative of P_f in normal directions extends to a real analytic map

$$H_{\text{Imm}^r}^{2p-r}(M, TN) \ni m \mapsto \left(h \mapsto \hat{\nabla}_{m^\perp} P_f h\right) \in L(H_{\text{Imm}^r}^r(M, TN), H_{\text{Imm}^r}^{1-r}(M, TN)).$$

Equivalently, the following map is real analytic:

$$H_{\text{Imm}^r}^r(M, TN) = T\,\text{Imm}^r(M, N) \ni h \mapsto \left(m \mapsto \hat{\nabla}_{m^\perp} P_f h\right)$$
$$\in L(H_{\text{Imm}^r}^{2p-r}(M, TN), H_{\text{Imm}^r}^{1-r}(M, N)).$$

Dualization using the $H^0(g)$ duality shows that the adjoint is real analytic

$$T\,\text{Imm}^r(M, N) \ni h \mapsto \text{Adj}(\hat{\nabla}P)(h, \cdot)^\perp \in L(H^{r-1}_{\text{Imm}^r}(M, TN), H^{r-2p}_{\text{Imm}^r}(M, TN)).$$

In particular, the adjoint is real analytic

$$T\,\text{Imm}^r(M, N) \ni h \mapsto \text{Adj}(\hat{\nabla}P)(h, \cdot)^\perp \in L(H^r_{\text{Imm}^r}(M, TN), H^{r-2p}_{\text{Imm}^r}(M, TN)).$$

□

4.7 Remark For Sobolev metrics of integer order $p \in \mathbb{N}_{>0}$, condition (d) of Sect. 4.1 can be verified directly by a term-by-term investigation of the following explicit formula for the normal part of the adjoint [11, Section 8.2], assuming that $\hat{\nabla} = \nabla$ is the Levi–Civita connection of $\bar{g}$:

$$\begin{aligned}
\text{Adj}(\nabla P)(h, k)^\perp = {} &2\sum_{i=0}^{p-1} \text{Tr}\left(g^{-1}\nabla Tf g^{-1}\bar{g}(\nabla(1+\Delta)^{p-i-1}h, \nabla(1+\Delta)^i k)\right) \\
&+ \sum_{i=0}^{p-1}\left(\nabla^*\bar{g}(\nabla(1+\Delta)^{p-i-1}h, (1+\Delta)^i k)\right)\text{Tr}^g(\nabla Tf) \\
&+ \sum_{i=0}^{p-1}\text{Tr}^g\left(R^{\bar{g}}((1+\Delta)^{p-i-1}h, \nabla(1+\Delta)^i k)Tf\right) \\
&- \sum_{i=0}^{p-1}\text{Tr}^g\left(R^{\bar{g}}(\nabla(1+\Delta)^{p-i-1}h, (1+\Delta)^i k)Tf\right).
\end{aligned}$$

Here $g = f^*\bar{g}$, $\Delta = \Delta^g$, $\nabla = \nabla^g$, and $R^{\bar{g}}$ denotes the curvature on $(N, \bar{g})$. This direct calculation is consistent with the more general argument of Theorem 4.6.

5 Special cases

This section describes several applications of the general well-posedness result, Theorem 4.4. First, we consider the geodesic equation of right-invariant Sobolev metrics on the diffeomorphism group $\text{Diff}(M)$. In Eulerian coordinates, this equation is called Euler–Arnold [2] or EPDiff [29] equation and reads as

$$m_t + \nabla_u m + \bar{g}(\nabla u, m) + (\text{div}\,u)m = 0, \qquad m := P_{\text{Id}}u, \qquad u := \varphi_t \circ \varphi^{-1}.$$

In Lagrangian coordinates, the equation takes the form shown in the following corollary. The conditions for local well-posedness in this corollary agree with the ones in [3], where metrics governed by a general class of pseudo-differential operators are investigated. The proof is an application of Theorem 4.4 to $\text{Diff}(M)$, seen as an open subset of $\text{Imm}(M, M)$. Moreover, the proof extends Theorem 4.4 to lower Sobolev regularity using some cancellations which are due to the vanishing normal bundle. The notation is as in Theorem 4.4.

5.1 Corollary Diffeomorphisms. *A smooth curve* $\varphi\colon [0, 1] \to \text{Diff}(M)$ *is a critical point of the energy functional*

$$E(\varphi) = \frac{1}{2}\int_0^1\int_M \bar{g}(P_\varphi\varphi_t, \varphi_t)\,\text{vol}^g\,dt$$

if and only if it satisfies the geodesic equation

$$\nabla_{\partial_t} \varphi_t = P_\varphi^{-1}\Big(-T\varphi\, \bar{g}(P_\varphi \varphi_t, \nabla \varphi_t)^\sharp - (\nabla_{\varphi_t} P)\varphi_t - \operatorname{Tr}^g\big(\bar{g}(\nabla \varphi_t, T\varphi)\big) P_\varphi \varphi_t \Big).$$

The geodesic equation is well-posed in the sense of Theorem 4.4 if P satisfies conditions (a)–(c) of Sect. 4.1 for some $p \in [1/2, \infty)$ and all $r \in [r_0, \infty)$ with $r_0 \in (\dim(M)/2 + 1, \infty)$. In particular, this is the case if $P = (1 + \Delta)^p$ with

- *$p \in [1, \infty)$ and $r \in (\dim(M)/2 + 1, \infty) \cap [p + 1, \infty)$; or*
- *$p \in [1/2, 1)$ and $r \in (\dim(M)/2 + 1, \infty) \cap [p + 3/2, \infty)$.*

Proof The formula for the geodesic equation follows from Theorem 4.3 because the terms $\operatorname{Adj}(\nabla P)^\perp$ and $\nabla Tf = (\nabla Tf)^\perp$ vanish. To show well-posedness of the geodesic equation, note that condition (d) of Sect. 4.1 is trivially satisfied because $\operatorname{Adj}(\nabla P)^\perp$ vanishes. Moreover, note that the condition $p \in [1, \infty)$ in Theorem 4.4 can be replaced by the weaker condition $p \in [1/2, \infty)$ because the term ∇Tf, which is of second order in f, vanishes. This can be seen by a term-by-term investigation of the right-hand side of the geodesic equation as in the proof of Theorem 4.4. Therefore, the geodesic equation is well-posed for any operator field P satisfying conditions (a)–(c) of Sect. 4.1 for some $p \in [1/2, \infty)$ and all $r \in [r_0, \infty)$ with $r_0 \in (\dim(M)/2 + 1, \infty)$, as claimed.

It remains to verify these conditions for the specific operator $P = (1 + \Delta)^p$. Condition (a) for $p \geq 1$ follows from Lemma 3.1, and condition (a) for $p \in [1/2, 1)$ is verified as follows. We split the operator P_φ in two components,

$$P_\varphi = (1 + \Delta^{\varphi^* \bar{g}})^{-1}(1 + \Delta^{\varphi^* \bar{g}})^{1+p}.$$

As $1 + p \geq 1$, Lemma 3.1 shows that the operator $(1 + \Delta^{\varphi^* \bar{g}})^{1+p}$ is a real analytic section of the bundle

$$GL(H^r_{\operatorname{Diff}^r}(M, TM), H^{r-2p-2}_{\operatorname{Diff}^r}(M, TM)) \to \operatorname{Diff}^r(M)$$

for any r such that $r - 2p - 2 \geq 1 - r$, i.e., $r \geq p + 3/2$. Similarly, under even weaker conditions, the operator $(1 + \Delta^{\varphi^* \bar{g}})^{-1}$ is a real analytic section of the bundle

$$GL(H^{r-2p-2}_{\operatorname{Diff}^r}(M, TM), H^{r-2p}_{\operatorname{Diff}^r}(M, TM)) \to \operatorname{Diff}^r(M).$$

By the chain rule, the operator P_φ is real analytic as required in condition (a). Conditions (b) and (c) can be verified as in the proof of Theorem 4.6. $\qquad\square$

Next we consider reparametrization-invariant Sobolev metrics on spaces of immersed curves, i.e., we consider the special case $M = S^1$. Our interest in these spaces stems from their fundamental role in the field of mathematical shape analysis; see e.g. [7,9,36,56,62] for $\mathbb{R}^n$-valued curves and [18,42,54,55] for manifold-valued curves. For curves in $\mathbb{R}^n$ local well-posedness of the geodesic equation for integer-order metrics has been shown in [47]. This has recently been extended to fractional-order metrics in [8]. The following corollary of our main result further generalizes this to fractional-order metrics on spaces of manifold-valued curves:

5.2 Corollary Curves. *A smooth curve $c \colon [0, 1] \to \operatorname{Imm}(S^1, N)$ is a critical point of the energy functional*

$$E(c) = \frac{1}{2} \int_0^1 \int_M \bar{g}(P_c c_t, c_t) |\partial_\theta c| d\theta dt$$

if and only if it satisfies the geodesic equation

$$\nabla_{\partial_t} c_t = \frac{1}{2} P_c^{-1} \Big(\mathrm{Adj}(\nabla P)_c(c_t, c_t)^\perp - 2\,\bar{g}(P_c c_t, \nabla_{\partial_s} c_t) v_c - \bar{g}(P_c c_t, c_t)\, H_c \Big)$$
$$- P_c^{-1} \Big((\nabla_{c_t} P) c_t + \big(\bar{g}(\nabla_{\partial_s} c_t, v_c)\big) P_c c_t \Big),$$

where $\partial_s = |c_\theta|^{-1} \partial_\theta$ *denotes the normalization of the coordinate vector field* ∂_θ, $v_c = \partial_s c$ *the unit-length tangent vector, and* $H_c = (\nabla_{\partial_s} v_c)^\perp$ *the vector-valued curvature of c.*

If the operator P satisfies the conditions of Sect. 4.1 for some $p \in [1, \infty)$ and all $r \in [r_0, \infty)$ with $r_0 \in (\dim(M)/2 + 1, \infty)$, then the geodesic equation is well-posed in the sense of Theorem 4.4. This is in particular the case for the operator $P = (1 - \nabla_{\partial_s} \nabla_{\partial_s})^p$ if $p \in [1, \infty)$ and $r \in (\dim(M)/2 + 1, \infty) \cap [p + 1, \infty)$.

Proof This follows directly from Theorems 4.3, 4.4 and 4.6. □

The last special case to be discussed in this section is $N = \mathbb{R}^n$, which includes in particular the space of surfaces in $\mathbb{R}^3$. In the article [11] we proved a local well-posedness result for integer-order metrics. The proof given there had a gap, which has been corrected in the article [50]. The following corollary of our main result extends this to fractional order metrics:

5.3 Corollary Flat ambient space. *A smooth curve* $f : [0, 1] \to \mathrm{Imm}(M, \mathbb{R}^n)$ *is a critical point of the energy functional*

$$E(f) = \frac{1}{2} \int_0^1 \int_M \langle P_f f_t, f_t \rangle \, \mathrm{vol}^g \, dt$$

if and only if it satisfies the geodesic equation

$$f_{tt} = \frac{1}{2} P_f^{-1} \Big(\mathrm{Adj}(dP)_f(f_t, f_t)^\perp - 2 df \, \langle P_f f_t, df_t \rangle^\sharp - \langle P_f f_t, f_t \rangle \, H_f \Big)$$
$$- P_f^{-1} \Big((\nabla_{f_t} P) f_t + \mathrm{Tr}^g \big(\langle df_t, df \rangle \big) P_f f_t \Big),$$

where $\langle \cdot, \cdot \rangle$ *denotes the Euclidean scalar product on* $\mathbb{R}^n$, $g = f^* \langle \cdot, \cdot \rangle$ *the induced pullback metric on* M, *and* $H_f = \mathrm{Tr}^g (d^2 f)^\perp$ *the vector-valued mean curvature of f.*

If the operator P satisfies the conditions of Sect. 4.1 for some $p \in [1, \infty)$ and all $r \in [r_0, \infty)$ with $r_0 \in (\dim(M)/2 + 1, \infty)$, then the geodesic equation is well-posed in the sense of Theorem 4.4. This is in particular the case for the operator $P = (1 + \Delta)^p$ with $p \in [1, \infty)$ and $r \in (\dim(M)/2 + 1, \infty) \cap [p + 1, \infty)$.

Proof This follows from Theorems 4.3, 4.4, and 4.6 with $N = \mathbb{R}^n$, noting that the covariant derivative on $\mathbb{R}^n$ and the induced covariant derivative on $\mathrm{Imm}^r(M, \mathbb{R}^n)$ coincide with ordinary derivatives. □

Appendix A: The push-forward operator on Sobolev spaces

A.1 Theorem Smooth curves in convenient vector spaces. [26, 4.1.19] *Let* $c \colon \mathbb{R} \to E$ *be a curve in a convenient vector space* E. *Let* $\mathcal{V} \subset E'$ *be a subset of bounded linear functionals such that the bornology of* E *has a basis of* $\sigma(E, \mathcal{V})$-*closed sets. Then the following are equivalent:*

(a) *c is smooth*
(b) *For each $k \in \mathbb{N}$ there exists a locally bounded curve $c^k \colon \mathbb{R} \to E$ such that for each $\ell \in \mathcal{V}$ the function $\ell \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\ell \circ c)^{(k)} = \ell \circ c^k$.*

If E is reflexive, then for any point separating subset $\mathcal{V} \subset E'$ the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed subsets, by [26, 4.1.23].

This theorem is surprisingly strong: Note that $\mathcal{V}$ does not need to recognize bounded sets. We shall use the theorem in situations where $\mathcal{V}$ is just the set of all point evaluations on suitable Sobolev spaces.

A.2 Lemma Smooth curves in Sobolev spaces of sections. *Let E be a vector bundle over M, and let ∇ be a connection on E. Then it holds for each $r \in (\dim(M)/2, \infty)$ that the space $C^\infty(\mathbb{R}, \Gamma_{H^r}(E))$ of smooth curves in $\Gamma_{H^r}(E)$ consists of all continuous mappings $c \colon \mathbb{R} \times M \to E$ with $p \circ c = \mathrm{pr}_2 \colon \mathbb{R} \times M \to M$ such that:*

- *For each $x \in M$ the curve $t \mapsto c(t, x) \in E_x$ is smooth; let $(\partial_t^p c)(t, x) = \partial_t^p(c(t, x))$, and*
- *For each $p \in \mathbb{N}_{\geq 0}$, the curve $\partial_t^p c$ has values in $\Gamma_{H^r}(E)$ so that $\partial_t^p c \colon \mathbb{R} \to \Gamma_{H^r}(E)$, and $t \mapsto \|\partial_t c(t, \ \ )\|_{H^r}$ is bounded, locally in t.*

Proof To see this we first choose a second vector bundle $F \to M$ such that $E \oplus_M F$ is a trivial bundle, i.e., isomorphic to $M \times \mathbb{R}^n$ for some $n \in \mathbb{N}$. Then $\Gamma_{H^r}(E)$ is a direct summand in $H^r(M, \mathbb{R}^n)$, so that we may assume without loss that E is a trivial bundle, and then, that it is 1-dimensional. So we have to identify $C^\infty(\mathbb{R}, H^r(M, \mathbb{R}))$. But in this situation we can just apply Theorem A.1 for the set $\mathcal{V} \subset H^s(M, \mathbb{R})'$ consisting just of all point evaluations $\mathrm{ev}_x \colon H^r(M, \mathbb{R}) \to \mathbb{R}$. □

A.3 Lemma Function spaces of mixed smoothness. *Let U be an open subset of a finite-dimensional vector space, let $r \in (\dim(M)/2, \infty)$, let $\alpha \in \{\infty, \omega\}$, and let $C^\alpha(U) = \varprojlim_p E_p$ be the representation of the complete locally convex space $C^\alpha(U)$ as a projective limit of Banach spaces E_p. Then*

$$H^r C^\alpha(M \times U) := C^\alpha(U, H^r(M)) = H^r(M) \hat{\otimes} C^\alpha(U) = H^r(M, C^\alpha(U)),$$

where $\hat{\otimes}$ is the injective, projective, or bornological tensor product, or any tensor product in-between, and where $H^r(M, C^\alpha(U))$ is defined as the projective limit $\varprojlim_p H^r(M, E_p)$.

The lemma justifies the following notation, which shall be used in Lemma A.5 below. If E_1 and E_2 are vector bundles over M, and $U \subseteq E_1$ is an open neighborhood of the image of an H^r section, then we write $\Gamma_{H^r}(C^\alpha(U, E_2))$ for the set of all fiber-preserving functions $F \colon U \to E_2$ which have regularity $H^r C^\alpha$ in every C^α vector bundle chart of E_1. Loosely speaking, these are sections of regularity H^r in the foot point and regularity C^α in the fibers.

Proof The space $C^\infty(U)$ is nuclear by [57, Corollary to Theorem 51.4], and the space $C^\omega(U)$ is nuclear as a countable inductive limit of nuclear spaces of holomorphic functions [41,

Theorem 30.11]. Let $\otimes_\varepsilon, \otimes_\pi$, and $\otimes_\beta$ be the injective, projective, and bornological completed tensor products, respectively. Then

$$C^\alpha(U) \otimes_\varepsilon H^r(M) = C^\alpha(U) \otimes_\pi H^r(M) = C^\alpha(U) \otimes_\beta H^r(M),$$

where the first equality holds because $C^\alpha(U)$ is nuclear, and the second equality holds by [41, Proposition 5.8] using that $H^r(M)$ is a normed space, and $C^\omega(V)$ is an (LF)-space and therefore bornological. Thus, all tensor spaces $C^\alpha(U)\hat{\otimes}H^r(M)$ are equal. Moreover,

$$C^\infty(U, H^r(M)) = C^\infty(U) \otimes_\varepsilon H^r(M)$$

by [57, Theorem 44.1], and

$$C^\omega(U, H^r(M)) = \varprojlim_{\tilde{U}} \mathcal{H}(\tilde{U}, H^r(M)) = \varprojlim_{\tilde{U}} \mathcal{H}(\tilde{U})\hat{\otimes}H^r(M) = C^\omega(U)\hat{\otimes}H^r(M)$$

by [32, Corollary 16.7.5], where $\mathcal{H}$ denotes holomorphic functions and $\tilde{U}$ are open neighborhoods of U in the complexification of the underlying vector space. Let Δ_2 be the natural norm on L^2 functions [22, 7.1]. Then

$$H^r(M)\hat{\otimes}C^\alpha(U) = H^r(M)\hat{\otimes}_{\Delta_2}C^\alpha(U) = \varprojlim_p H^r(M)\hat{\otimes}_{\Delta_2} E_p = \varprojlim_p H^r(M, E_p),$$

where the first equality holds because $\varepsilon \leq \Delta_2 \leq \pi$ [22, 7.1], the second one by the definition of tensor products of locally convex spaces [22, 35.2], and the third one because the fractional Laplacian $(1 + \Delta^g) \colon H^r(M) \to L^2(M)$ with respect to any auxiliary Riemannian metric $g \in \mathrm{Met}(M)$ is an isometry and because $L^2(M, E_p) = L^2(M) \otimes_{\Delta_2} E_p$ by the definition of Δ_2 [22, 7.2]. □

A.4 Lemma Push-forward of functions. *Let U be an open subset of $\mathbb{R}$, and let $r \in (\dim(M)/2, \infty)$. Then $H^r(M, U)$ is open in $H^r(M, \mathbb{R})$, and the following statements hold.*

(a) *The following map is smooth:*

$$H^r C^\infty(M \times U) \times H^r(M, U) \ni (F, h) \mapsto F \circ (\mathrm{Id}_M, h) \in H^r(M).$$

(b) *The following map is real analytic:*

$$H^r C^\omega(M \times U) \times H^r(M, U) \ni (F, h) \mapsto F \circ (\mathrm{Id}_M, h) \in H^r(M).$$

Proof The set $\Gamma_{H^r}(U)$ is open in $\Gamma_{H^r}(E_1)$ because $\Gamma_{H^r}(E_1)$ is continuously included in $\Gamma_C(E_1)$ thanks to the Sobolev embedding theorem.

(a) follows from the more general statement Lemma A.5.(a).

(b') As an intermediate step, we claim that the following map is real analytic:

$$C^\omega(U) \times H^r(M, U) \ni (f, h) \mapsto f \circ h \in H^r(M).$$

For any $f \in C^\omega(U)$ and $h \in H^r(M, U)$, the composition $f \circ h$ coincides with the Riesz functional calculus $f(h)$, which is defined as follows [21, Theorem 4.7]. As the spectrum $\sigma(h)$ equals the range of h, which is a compact subset of U, there is a set of positively oriented curves $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ in $U \setminus \sigma(h)$ such that $\sigma(h)$ is inside of Γ, and $\mathbb{C} \setminus U$ is outside of Γ [21, Proposition 4.4]. Then one defines $f(h)$ as the following Bochner integral over the resolvent of h:

$$f(h) = \frac{-1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda)(h - \lambda)^{-1} d\lambda$$

For any fixed Γ, this integral is well-defined and real analytic as claimed.

(b) The following map is real analytic thanks to (b') and the boundedness of multiplication $H^r(M) \times H^r(M) \to H^r(M)$:

$$H^r(M) \times C^\omega(U) \times H^r(M, U) \ni (a, f, h) \mapsto (a \otimes f) \circ (\mathrm{Id}_M, h) \in H^r(M),$$

where $(a \otimes f) \circ (\mathrm{Id}_M, h)$ denotes the map $x \mapsto a(x) f(h(x))$. Equivalently, by the real analytic exponential law [41, 11.18], the following map is real analytic:

$$H^r(M) \times C^\omega(U) \ni (a, f) \mapsto \big(h \mapsto (a \otimes f) \circ (\mathrm{Id}_M, h)\big) \in C^\omega(H^r(M, U), H^r(M)).$$

This map is bilinear and real analytic, and therefore bounded. By the universal property of the bornological tensor product $\otimes_\beta$ [41, 5.7], it descends to a bounded linear map

$$H^r(M) \otimes_\beta C^\omega(U) \ni F \mapsto \big(h \mapsto F \circ (\mathrm{Id}_M, h)\big) \in C^\omega(H^r(M, U), H^r(M)).$$

The domain of this map equals $H^r C^\omega(M \times U)$ by Lemma A.3.

$\square$

A.5 Lemma Push-forward of sections. *Let E_1, E_2 be vector bundles over M, let $U \subset E_1$ be an open neighborhood of the image of a smooth section, let $F: U \to E_2$ be a fiber preserving function, and let $r \in (\dim(M)/2, \infty)$. Then $\Gamma_{H^r}(U)$ is open in $\Gamma_{H^r}(E_1)$, and the following statements hold:*

(a) *If F is smooth or belongs to $\Gamma_{H^r}(C^\infty(U, E_2))$, then the push-forward F_* is smooth:*

$$F_*: \Gamma_{H^r}(U) \to \Gamma_{H^r}(E_2), \quad h \mapsto F \circ h.$$

(b) *If F is real analytic or belongs to $\Gamma_{H^r}(C^\omega(U, E_2))$, then the pushforward F_* is real analytic.*

The notation $\Gamma_{H^r}(C^\infty(U, E_2))$ and $\Gamma_{H^r}(C^\omega(U, E_2))$ is explained in "Section A.3".

Proof (a) Let $c: \mathbb{R} \ni t \mapsto c(t, \cdot) \in \Gamma_{H^r}(U)$ be a smooth curve. As $r > \dim(M)/2$, it holds for each $x \in M$ that the mapping $\mathbb{R} \ni t \mapsto F_x(c(t, x)) \in (E_2)_x$ is smooth. By the Faà di Bruno formula (see [25] for the 1-dimensional version, preceded in [1] by 55 years), we have for each $p \in \mathbb{N}_{>0}, t \in \mathbb{R}$, and $x \in M$ that

$$\partial_t^p F_x(c(t, x)) = \sum_{j \in \mathbb{N}_{>0}} \sum_{\substack{\alpha \in \mathbb{N}_{>0}^j \\ \alpha_1 + \cdots + \alpha_j = p}} \frac{1}{j!} d^j (F_x)(c(t, x)) \Big(\frac{\partial_t^{(\alpha_1)} c(t, x)}{\alpha_1!}, \ldots, \frac{\partial_t^{(\alpha_j)} c(t, x)}{\alpha_j!} \Big).$$

For each $x \in M$ and $\alpha_x \in (E_2)_x^*$ the mapping $s \mapsto \langle s(x), \alpha_x \rangle$ is a continuous linear functional on the Hilbert space $\Gamma_{H^r}(E_2)$. The set $\mathcal{V}_2$ of all of these functionals separates points and therefore satisfies the condition of Theorem A.1. We also have for each $p \in \mathbb{N}_{>0}, t \in \mathbb{R}$, and $x \in M$ that

$$\partial_t^p \langle F_x(c(t, x)), \alpha_x \rangle = \langle \partial_t^p F_x(c(t, x)), \alpha_x \rangle.$$

Using the explicit expressions for $\partial_t^p F_x(c(t, x))$ from above we may apply Lemma A.2 to conclude that $t \mapsto F(c(t, \ ))$ is a smooth curve $\mathbb{R} \to \Gamma_{H^r}(E_2)$. Thus, F_* is a smooth mapping, and we have shown (a).

(b') We claim that (b) holds when F is fiber-wise linear. Then F can be identified with a map in $\check{F} \in \Gamma_{H^r}(L(E_1, E_2))$. For any $h \in \Gamma_{H^r}(E_1)$, the composition $F \circ h$ equals the trace $\check{F}.h$, which is real analytic in h by the module properties 2.3.

(b) To prove the general case, we write E_1 and E_2 as sub-bundles of a trivial bundle $M \times V$. The corresponding inclusion and projection mappings are real analytic mappings of vector bundles and are denoted by

$$i_1 \colon E_1 \to M \times V, \quad i_2 \colon E_2 \to M \times V, \quad \pi_1 \colon M \times V \to E_1, \quad \pi_2 \colon M \times V \to E_2.$$

Then the set $\tilde{U} := \pi_1^{-1}(U) \subseteq M \times V$ and the map $\tilde{F} := i_2 \circ F \circ \pi_1$ fit into the following commutative diagrams:

$$
\begin{array}{ccc}
U \xrightarrow{\ F\ } E_2 & \qquad & \Gamma_{H^r}(U) \xrightarrow{\ F_*\ } \Gamma_{H^r}(E_2) \\
\downarrow{\scriptstyle i_1} \quad \uparrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle (i_1)_*} \quad \uparrow{\scriptstyle (\pi_2)_*} \\
\tilde{U} \xrightarrow{\ \tilde{F}\ } M \times V & & \Gamma_{H^r}(\tilde{U}) \xrightarrow{\ \tilde{F}_*\ } \Gamma_{H^r}(M \times V)
\end{array}
$$

All maps in the diagram on the left are real analytic by definition. The map $(\tilde{F})_*$ is real analytic by Lemma A.4.(b) applied component-wise to the trivial bundle $M \times V$, and the maps $(i_1)_*$ and $(\pi_2)_*$ are real analytic by (b'). Therefore, $F_* = (\pi_2)_* \circ (\tilde{F})_* \circ (i_1)_*$ is real analytic, which proves (b).

□

Appendix B: A real analytic no-loss no-gain result

The following lemma is a variant of the no-loss-no-gain theorem of Ebin and Marsden [23], adapted to the real analytic sprays on spaces of immersions as in the setting of Theorem 4.4. The proof is a minor adaptation of the proof in [23]; see also [16].

B.1 Lemma Real analytic no-loss no-gain. *Let $r_0 > \dim(M)/2 + 1$ and let $\alpha \in \{\infty, \omega\}$. For each $r \geq r_0$, let S^r be a $\mathrm{Diff}(M)$-invariant C^α vector field on $T\,\mathrm{Imm}^r(M, N)$ such that $T i_{r,s} \circ S^r = S^s \circ i_{r,s}$ where $i_{r,s} \colon T\,\mathrm{Imm}^r(M, N) \to T\,\mathrm{Imm}^s(M, N)$ is the C^α-embedding for $r_0 \leq s < r$. By the theorem of Picard-Lindelöf each S^r has a maximal C^α-flow $\mathrm{Fl}^{S^r} \colon U^r \to T\,\mathrm{Imm}^r(M, N)$ for an open neighborhood U^r of $\{0\} \times T\,\mathrm{Imm}^r(M, N)$ in $\mathbb{R} \times T\,\mathrm{Imm}^r(M, N)$.*
*   Then $U^r = U^s \cap (\mathbb{R} \times T\,\mathrm{Imm}^r(M, N))$ for all $r_0 + 1 \leq r$ and $r_0 \leq s \leq r$. Thus, there is no loss or gain in regularity during the evolution along any S^r for $r \geq r_0 + 1$.*

Proof (a) We shall use the following result [23, Lemma 12.2]: *Any $h \in H^r(M, TN)$ such that $Th \circ X \in H^r(M, TTN)$ for all $X \in \mathfrak{X}(M)$ satisfies $h \in H^{r+1}(M, TN)$.*

(b) For $h \in T\,\mathrm{Imm}^r(M, N)$ let J_h^r be the open interval such that $U^r \cap (\mathbb{R} \times \{h\}) = J_h^r \times \{h\}$, i.e., J_h^r is the maximal domain of the integral curve of S^r through h in $T\,\mathrm{Imm}^r(M, N)$; see [41, 32.14]. Since $i_{r,s} \circ \mathrm{Fl}_t^{S^r} = \mathrm{Fl}_t^{S^s} \circ$ (see [41, 32.16]), for $h \in T\,\mathrm{Imm}^r(M, N)$ we have $J_h^r \subseteq J_h^s$ for $r_0 \leq s < r$.

(c) **Claim.** *For $h \in T\,\mathrm{Imm}^{r+1}(M, N)$ we have $J_h^r = J_h^{r+1}$.*

Since S^r is invariant under the pullback action of $\mathrm{Diff}(M)$, we have for $h \in T\,\mathrm{Imm}^{r+1}(M, N)$ and any $X \in \mathfrak{X}(M)$ that

$$\mathrm{Fl}_t^{S^r}(h \circ \mathrm{Fl}_u^X) = \mathrm{Fl}_t^{S^r}(h) \circ \mathrm{Fl}_u^X .$$

Differentiating both side we get

$$
\begin{aligned}
T(\mathrm{Fl}_t^{S^r}(h)) \circ X &= \partial_u|_0 (\mathrm{Fl}_t^{S^r}(h) \circ \mathrm{Fl}_u^X) = \partial_u|_0 (\mathrm{Fl}_t^{S^r}(h \circ \mathrm{Fl}_u^X)) \\
&= T(\mathrm{Fl}_t^{S^r})(\partial_u|_0 (h \circ \mathrm{Fl}_u^X)) = T(\mathrm{Fl}_t^{S^r})(Th \circ X)
\end{aligned}
$$

Since $Th \circ X \in H^r(M, TTN)$ we see that $T(\mathrm{Fl}_t^{sr}(h)) \circ X \in H^r(M, TTN)$. By result (a) we get $\mathrm{Fl}_t^{sr}(h) \in T\,\mathrm{Imm}^{r+1}(M, N)$, and thus $J_h^r \supseteq J_h^{r+1}$. The converse inclusion is (b).

(d)   Let $r_0 + 1 \leq s < r < s + 1$ and let $h \in T\,\mathrm{Imm}^r(M, N)$. Then

$$J_h^r \subseteq J_h^s \subseteq J_h^{r-1} = J_h^r,$$

where the inclusions follow from (b), (b), and (c) , respectively. Thus we have $J_h^r = J_h^s = J_h^{r-1}$. $\square$

References

1. Arbogast, L.F.A.: Du calcul des dérivations. Levrault, Strasbourg (1800)
2. Arnold, V.I.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier (Grenoble), **16**(fasc. 1), 319–361 (1966)
3. Bauer, M., Bruveris, M., Cismas, E., Escher, J., Kolev, B.: Well-posedness of the EPDiff equation with a pseudo-differential inertia operator. To appear in J. Differ. Equ. (2020). https://doi.org/10.1016/j.jde.2019.12.008
4. Bauer, M., Bruveris, M., Harms, P., Michor, P.W.: Vanishing geodesic distance for the Riemannian metric with geodesic equation the KdV-equation. Ann. Glob. Anal. Geom. **41**(4), 461–472 (2012)
5. Bauer, M., Bruveris, M., Harms, P., Michor, P.W.: Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. Ann. Glob. Anal. Geom. **44**(1), 5–21 (2013)
6. Bauer, M., Bruveris, M., Harms, P., Michor, P.W.: Smooth perturbations of the functional calculus and applications to Riemannian geometry on spaces of metrics (2018). arXiv:1810.03169
7. Bauer, M., Bruveris, M., Harms, P., Møller-Andersen, J.: A numerical framework for Sobolev metrics on the space of curves. SIAM J. Imaging Sci. **10**(1), 47–73 (2017)
8. Bauer, M., Bruveris, M., Kolev, B.: Fractional Sobolev metrics on spaces of immersed curves. Calc. Var. Partial. Differ. Equ. **57**(1), 27 (2018)
9. Bauer, M., Bruveris, M., Michor, P.W.: Overview of the geometries of shape spaces and diffeomorphism groups. J. Math. Imaging Vis. **50**(1–2), 60–97 (2014)
10. Bauer, M., Escher, J., Kolev, B.: Local and global well-posedness of the fractional order EPDiff equation on $\mathbb{R}^d$. J. Differ. Equ. **258**(6), 2010–2053 (2015)
11. Bauer, M., Harms, P., Michor, P.W.: Sobolev metrics on shape space of surfaces. J. Geom. Mech. **3**(4), 389–438 (2011)
12. Bauer, M., Harms, P., Michor, P.W.: Sobolev metrics on shape space, II: weighted Sobolev metrics and almost local metrics. J. Geom. Mech. **4**(4), 365–383 (2012)
13. Bauer, M., Harms, P., Preston, S.C.: Vanishing distance phenomena and the geometric approach to SQG. Arch. Ration. Mech. Anal. **235**, 1445–1466 (2020). https://doi.org/10.1007/s00205-019-01449-7
14. Bauer, M., Kolev, B., Preston, S.C.: Geometric investigations of a vorticity model equation. J. Differ. Equ. **260**(1), 478–516 (2016)
15. Behzadan, A., Holst, M.: On certain geometric operators between Sobolev spaces of sections of tensor bundles on compact manifolds equipped with rough metrics (2017). arXiv:1704.07930
16. Bruveris, M.: Regularity of maps between Sobolev spaces. Ann. Glob. Anal. Geom. **52**(1), 11–24 (2017)
17. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**(11), 1661 (1993)
18. Celledoni, E., Eidnes, S., Schmeding, A.: Shape analysis on homogeneous spaces: a generalised SRVT framework. In: The Abel Symposium. Springer, pp. 187–220 (2016)
19. Constantin, P., Lax, P.D., Majda, A.: A simple one-dimensional model for the three-dimensional vorticity equation. Commun. Pure Appl. Math. **38**(6), 715–724 (1985)
20. Constantin, P., Majda, A.J., Tabak, E.: Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. Nonlinearity **7**(6), 1495 (1994)
21. Conway, J.B.: A Course in Functional Analysis, vol. 96. Springer, Berlin (2013)
22. Defant, A., Floret, K.: Tensor Norms and Operator Ideals, vol. 176. Elsevier, Amsterdam (1992)
23. Ebin, D.G., Marsden, J.E.: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. **92**, 102–163 (1970)
24. Escher, J., Kolev, B.: Right-invariant Sobolev metrics of fractional order on the diffeomorphism group of the circle. J. Geom. Mech. **6**(3), 335–372 (2014)
25. Faà di Bruno, C.F.: Note sur une nouvelle formule du calcul différentielle. Q. J. Math. **1**, 359–360 (1855)

26. Frölicher, A., Kriegl, A.: Linear Spaces and Differentiation Theory. Pure and Applied Mathematics (New York). Wiley, Chichester (1988)
27. Grenander, U., Miller, M.I.: Computational anatomy: an emerging discipline. Q. Appl. Math. **56**(4), 617–694 (1998)
28. Große, N., Schneider, C.: Sobolev spaces on Riemannian manifolds with bounded geometry: general coordinates and traces. Math. Nachr. **286**(16), 1586–1613 (2013)
29. Holm, D.D., Marsden, J.E.: Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation. In: The Breadth of Symplectic and Poisson Geometry, volume 232 of Progress in Mathematics. Birkhäuser Boston, Boston, MA, pp. 203–235 (2005)
30. Hunter, J.K., Saxton, R.: Dynamics of director fields. SIAM J. Appl. Math. **51**(6), 1498–1521 (1991)
31. Inci, H., Kappeler, T., Topalov, P.: On the regularity of the composition of diffeomorphisms. Mem. Am. Math. Soc. **226**(1062):vi+60 (2013)
32. Jarchow, H.: Locally Convex Spaces. Springer, Berlin (2012)
33. Jermyn, I., Kurtek, S., Laga, H., Srivastava, A.: Elastic shape analysis of three-dimensional objects. Synth. Lect. Comput. Vis. **12**, 1–185 (2017)
34. Jerrard, R.L., Maor, C.: Vanishing geodesic distance for right-invariant Sobolev metrics on diffeomorphism groups. Ann. Glob. Anal. Geom. **55**(4), 631–656 (2019)
35. Khesin, B., Wendt, R.: The geometry of infinite-dimensional groups, vol. 51. Springer, Berlin (2008)
36. Klassen, E., Srivastava, A., Mio, M., Joshi, S.H.: Analysis of planar shapes using geodesic paths on shape spaces. IEEE Trans. Pattern Anal. Mach. Intell. **26**(3), 372–383 (2004)
37. Kolář, I., Michor, P.W., Slovák, J.: Natural Operations in Differential Geometry. Springer, Berlin (1993)
38. Kolev, B.: Local well-posedness of the EPDiff equation: a survey. J. Geom. Mech. **9**(2), 167–189 (2017)
39. Kouranbaeva, S.: The Camassa–Holm equation as a geodesic flow on the diffeomorphism group. J. Math. Phys. **40**(2), 857–868 (1999)
40. Kriegl, A., Michor, P.W.: Product preserving functors of infinite-dimensional manifolds. Arch. Math. (Brno) **32**(4), 289–306 (1996)
41. Kriegl, A., Michor, P.W.: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, vol. 53. American Mathematical Society, Providence (1997)
42. Le Brigant, A.: Computing distances and geodesics between manifold-valued curves in the SRV framework. J. Geom. Mech. **9**(2), 131–156 (2017)
43. Lenells, J.: The Hunter–Saxton equation describes the geodesic flow on a sphere. J. Geom. Phys. **57**(10), 2049–2064 (2007)
44. Marsden, J.E., Ratiu, T., Weinstein, A.: Semidirect products and reduction in mechanics. Trans. Am. Math. Soc. **281**(1), 147–177 (1984)
45. Michor, P.W.: Topics in Differential Geometry. Graduate Studies in Mathematics, vol. 93. American Mathematical Society, Providence (2008)
46. Michor, P.W.: Manifolds of mappings for continuum mechanics. In Geometric Continuum Mechanics—An Overview. Birkhauser, pp. 1–60 (2020)
47. Michor, P.W., Mumford, D.: An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach. Appl. Comput. Harmon. Anal. **23**(1), 74–113 (2007)
48. Misiołek, G.: A shallow water equation as a geodesic flow on the Bott–Virasoro group. J. Geom. Phys. **24**(3), 203–208 (1998)
49. Misiołek, G., Preston, S.C.: Fredholm properties of Riemannian exponential maps on diffeomorphism groups. Invent. Math. **179**(1), 191 (2010)
50. Müller, O.: Applying the index theorem to non-smooth operators. J. Geom. Phys. **116**, 140–145 (2017)
51. Ovsienko, V.Y., Khesin, B.A.: Korteweg–de Vries superequation as an Euler equation. Funct. Anal. Appl. **21**(4), 329–331 (1987)
52. Shnirel'man, A.I.: On the geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid. Math. USSR Sbornik **56**(1), 79 (1987)
53. Srivastava, A., Klassen, E.: Functional and Shape Data Analysis. Springer, New York (2016). https://doi.org/10.1007/978-1-4939-4020-2
54. Su, J., Kurtek, S., Klassen, E., Srivastava, A., et al.: Statistical analysis of trajectories on Riemannian manifolds: bird migration, hurricane tracking and video surveillance. Ann. Appl. Stat. **8**(1), 530–552 (2014)
55. Su, Z., Klassen, E., Bauer, M.: Comparing curves in homogeneous spaces. Differ. Geom. Appl. **60**, 9–32 (2018)
56. Sundaramoorthi, G., Mennucci, A., Soatto, S., Yezzi, A.: A new geometric metric in the space of curves, and applications to tracking deforming objects by prediction and filtering. SIAM J. Imaging Sci. **4**(1), 109–145 (2011)

57. Treves, F.: Topological Vector Spaces, Distributions and Kernels: Pure and Applied Mathematics. Academic Press, New York (1967)
58. Triebel, H.: Theory of Function Spaces II. Monographs in Mathematics, vol. 84. Birkhäuser, Boston (1992)
59. Vishik, S., Dolzhanskii, F.: Analogs of the Euler–Lagrange equations and magnetohydrodynamics equations related to Lie groups. Sov. Math. Dokl. **19**, 149–153 (1978)
60. Washabaugh, P.: The SQG equation as a geodesic equation. Arch. Ration. Mech. Anal. **222**(3), 1269–1284 (2016)
61. Wunsch, M.: On the geodesic flow on the group of diffeomorphisms of the circle with a fractional Sobolev right-invariant metric. J. Nonlinear Math. Phys. **17**(1), 7–11 (2010)
62. Younes, L.: Computable elastic distances between shapes. SIAM J. Appl. Math. **58**(2), 565–586 (1998)
63. Younes, L.: Shapes and Diffeomorphisms, vol. 171. Springer, Berlin (2010)