Perfectness and Simplicity of Certain Groups of Diffeomorphisms

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Zusammenfassung

Die vorliegende Arbeit behandelt die Frage ob gewisse Gruppen von Diffeomorphismen einfach, oder wenigstens perfekt, sind. M. R. Herman und W. Thurston haben gezeigt, daß die Zusammenhangskomponente der Identität der Gruppe aller Diffeomorphismen mit kompakten Träger einer Mannigfaltigkeit einfach ist. Für volumserhaltende Diffeomorphismen hat W. Thurston ein analoges Resultat bewiesen. A. Banyaga hat ähnliche Methoden auf die Automorphismengruppe einer symplektischen Mannigfaltigkeit angewandt. Auch auf die infinitesimale Version davon, nämlich die Perfektheit von Lie Algebren von Vektorfeldern, wird eingegangen. Diese ist um einiges einfacher zu behandeln, es scheint aber keine Möglichkeit zu geben aus ihr die Perfektheit der Gruppe zu erhalten. Trotzdem ist in allen bekannten Fällen die Gruppe genau dann perfekt, wenn es die entsprechende Lie Algebra ist. Für die Einfachheit gilt dies bei weitem nicht. Zum Beispiel hat die Lie Algebra aller Vektorfelder mit kompakten Träger sehr viel Ideale, obwohl die Gruppe dazu einfach ist.

Im ersten Kapitel wird gezeigt, unter welchen Voraussetzungen man von der lokalen Perfektheit einer Diffeomorphismengruppe zu deren Einfachheit gelangt. Außerdem wird ein simplizialer Komplex konstruiert, dessen erste Homologiegruppe gleich der Abelisierung der universellen Überlagerung der Diffeomorphismengruppe ist. Verschwindet diese Homologiegruppe, ist die Diffeomorphismengruppe also perfekt. Danach wird ein Resultat von M. R. Herman und F. Sergeraert diskutiert, welches besagt, daß die Diffeomorphismengruppe des Torus perfekt ist. Daraus folgt auch leicht, daß die Gruppe der Diffeomorphismen des Torus, die die Blätter der Standardblätterung erhalten, perfekt ist. Dies wurde erstmals von T. Rybicki gezeigt, sein Beweis benötigt allerdings eine geblätterte Version des Theorems von Herman.

Im zweiten Kapitel wird zuerst eine modifizierte Unterteilung des Standard Simplex konstruiert. Diese ist etwas umständlicher zu handhaben als die baryzentrische Unterteilung, aber sie erlaubt es eine Fragmentierungsabbildung für modulare Diffeomorphismengruppen zu definieren, welche kettenhomotop zur Identität ist. Dies vereinheitlicht zwei Methoden, nämlich das Fragmentierungs- und das Deformierungslemma, die üblicherweise verwendet werden um Perfektheit der Diffeomorphismengruppe einer Mannigfaltigkeit auf die Perfektheit der entsprechenden Diffeomorphismengruppe des Torus zurückzuführen. Dies wird dann auf den Fall der vollen Diffeomorphismengruppe angewandt, und liefert deren Perfektheit, und in weiterer Folge auch deren Einfachheit. Auch die Gruppe aller Diffeomorphismen, die die Blätter einer Blätterung invariant lassen ist modular, und so liefert diese Methode auch die Perfektheit letzterer Gruppe.

Im dritten Kapitel werden sogenannte lokal konform symplektische Mannigfaltigkeiten behandelt. Das sind Mannigfaltigkeiten mit einer Struktur, die lokal bis auf konforme Äquivalenz wie eine symplektische Struktur aussieht. Ihre Bedeutung rührt einerseits daher, daß jedes gerade dimensionale Blatt einer Jacobi-Mannigfaltigkeit eine lokal konform symplektische Struktur besitzt, und andererseits daher, daß sie als Phasenräume in der Hamiltonschen Mechanik auftreten, siehe [Vai85]. Es wird auch ein Beispiel einer solchen Mannigfaltigkeit gegeben, die keine symplektische Struktur besitzt. Im Weiteren wird die

Automorphismengruppe einer lokal konform symplektischen Struktur betrachtet. Es stellt sich heraus, daß dies eine unendlich dimensionale Lie Gruppe im Sinn von [KM97] ist. Im allgemeinen ist diese Gruppe, wie im symplektischen Fall auch, weder einfach noch perfekt. Der Flux-Homomorphismus und die Calabi-Invariante lassen sich vom symplektischen Fall auf den lokal konform symplektischen Fall verallgemeinern, sie haben jetzt allerdings Werte in getwisteten de Rham Cohomologiegruppen. Außerdem tritt eine neue solche Invariante auf, die im symplektischen Fall immer verschwindet. Das erste Hauptresultat besagt, daß der Kern der Calabi-Invariante eine einfache, also auch perfekte Gruppe ist. Dies verallgemeinert ein bekanntes Resultat von A. Banyaga für symplektische Mannigfaltgkeiten, siehe [Ban78]. Genauer wird die derivierte Reihe der Automorphismengruppe sowie die infinitesimale Version davon, d.h. die derivierte Reihe der entsprechenden Lie Algebra, berechnet. Schlußendlich wird noch gezeigt, daß die Gruppe der Automorphismen schon die Mannigfaltigkeit und die lokal konform symplektische Struktur bestimmt. Im symplektischen Fall wurde dies von A. Banyaga bewiesen.

Im letzten Kapitel wird gezeigt, daß zwei der Invarianten aus dem dritten Kapitel auf die Fundamentalgruppe größerer Diffeomorphismengruppen ausgedehnt werden können, und wie diese Ausdehnungen mit gewissen Erweiterungen von Diffeomorphismengruppen zusammenhängen.

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1. Basic Setup

1.1 Introduction

This work deals with the question, whether certain groups of diffeomorphisms are simple, or at least perfect. M. R. Herman and W. Thurston have shown that the connected component of the group of all compactly supported diffeomorphisms of a manifold is simple, and hence perfect. For the group of volume preserving diffeomorphisms W. Thurston has shown an analogous statement, but it involves the concept of the flux homomorphism. In fact, he showed that the kernel of the flux homomorphism is a simple group. A. Banyaga adapted his methods to the symplectic case and obtained similar simplicity results, although another invariant, the Calabi invariant, appears in the non-compact symplectic case. There are also lots of perfectness results for Lie algebras of vector fields. These are much more easier to prove, but there does not seem to exist a method to obtain the perfectness of the group from the perfectness of the Lie algebra, although in all known cases the group is perfect if and only if the Lie algebra is. This is far from being true for simplicity. For example the Lie algebra of compactly supported vector fields has many ideals, but the corresponding group is simple.

In the first chapter we fix some notation and discuss under which assumptions one obtains simplicity of the group from so called local perfectness. In addition a simplicial complex is introduced. Its first homology group equals the abelianization of the universal covering of the diffeomorphism group in question. Consequently, if this homology group vanishes, the diffeomorphism group is perfect. Then we discuss a well known theorem of Herman, which immediately implies that the group of diffeomorphisms of the torus is simple. Finally we show how this yields the perfectness of the group of leaf preserving diffeomorphisms of the torus with the standard foliation. This is originally due to T. Rybicki, but his proof uses a foliated version of Hermans theorem.

In the second chapter we define a modified subdivision of the standard simplex. Combinatorically it is more difficult to handle than the barycentric subdivision, but it allows to define a fragmentation mapping for all modular groups of diffeomorphisms which is chain homotopic to the identity. This method unifies two main tools, usually used when one tries to reduce the problem of perfectness of a diffeomorphism group of a manifold to the perfectness of the corresponding diffeomorphism group of the torus. We apply this method and obtain simplicity of the full diffeomorphism group and perfectness of the group of leaf preserving diffeomorphisms.

In the third chapter we discuss locally conformally symplectic manifolds. These are manifolds together with a structure which locally, up to conformal equivalence, looks like a symplectic structure. There are two reasons why this structures may be interesting. First, every even dimensional leaf of a Jacobi manifold possesses a locally conformally symplectic structure and second, they occur as phase spaces in Hamiltonian mechanics, cf. [Vai85]. We also give an example of a locally conformally symplectic manifold, which does not admit a symplectic structure. Furthermore we show that the group of automorphisms of a locally conformally symplectic manifold is a Lie group in the sense of [KM97]. In general this group is neither simple nor perfect. The flux homomorphism and the Calabi invariant generalize to the locally conformal case, although they now have values in twisted de Rham cohomology groups. In addition another invariant appears, which is always zero in the symplectic case. The main result of this chapter is that the kernel of the Calabi invariant is simple. This generalizes a well known theorem of A. Banyaga. More precisely we compute the derived series of the automorphism group of a locally conformally symplectic manifold. We also compute the derived series of the corresponding Lie algebra. Finally we show that the automorphism group determines the manifold and the locally conformally symplectic structure, up to conformal equivalence.

In the last chapter we show how one can extend two of the invariants of the third chapter to the fundamental group of larger groups of diffeomorphisms, and how these are related to certain extensions of diffeomorphism groups.

1.2 The Lie Group $\mathrm{Diff}_c^\infty(M)$

Let M be a smooth, paracompact, boundaryless manifold and denote by $\mathfrak{X}_c(M)$ the Lie algebra of compactly supported vector fields. We equipped it with the inductive limit topology $\mathfrak{X}_c(M) = \varinjlim_K \mathfrak{X}_K(M)$, where the limit is over all compact subsets $K \subseteq M$ and $\mathfrak{X}_K(M) := \{X \in \mathfrak{X}(M) : \mathrm{supp}(X) \subseteq K\}$. By $\mathrm{Diff}_c^\infty(M)$ we denote the group of all compactly supported diffeomorphisms of M equipped with the inductive limit topology $\mathrm{Diff}_c^\infty(M) = \varinjlim_K \mathrm{Diff}_K^\infty(M)$, where $\mathrm{Diff}_K^\infty(M) := \{f \in \mathrm{Diff}^\infty(M) : \mathrm{supp}(f) \subseteq K\}$. Recall that $\mathfrak{X}_c(M)$ is a strict inductive limit of Fréchet spaces and that $\mathrm{Diff}_c^\infty(M)$ is a Lie group modeled on $\mathfrak{X}_c(M)$. See [KM97] for this. By $\mathrm{Diff}_c^\infty(M)_\circ$ we denote the connected component of $\mathrm{Diff}_c^\infty(M)$ containing id_M. It consists exactly of those diffeomorphisms f that are compactly diffeotopic to id_M, i.e. there exists a compact set $K \subseteq M$ and a smooth mapping $H : M \times I \to M$, such that $\check{H} : I \to \mathrm{Diff}_K^\infty(M)$, $\check{H}(0) = \mathrm{id}_M$ and $\check{H}(1) = f$. The kinematic tangent space of $\mathrm{Diff}_c^\infty(M)$ at f is

$$T_f \mathrm{Diff}_c^\infty(M) = \Gamma_c(f^*\pi_M : f^*TM \to M) \cong \{X \in C_c^\infty(M, TM) : \pi_M \circ X = f\},$$

i.e. $T_f \mathrm{Diff}_c^\infty(M)$ consist of all compactly supported vector fields along f. In particular we have $T_{\mathrm{id}_M} \mathrm{Diff}_c^\infty(M) = \mathfrak{X}_c(M)$. $\mathrm{Diff}_c^\infty(M)$ admits a smooth exponential mapping $\exp : \mathfrak{X}_c(M) \to \mathrm{Diff}_c^\infty(M)$ namely $\exp(X) = \mathrm{Fl}_1^X$, but it is not even locally surjective around id_M, although $T_0 \exp = \mathrm{id}$. This can be found for example in [Gra88] or [KM97]. The adjoint representation $\mathrm{Ad} : \mathrm{Diff}_c^\infty(M) \to \mathrm{GL}(\mathfrak{X}_c(M))$ is given by $g \mapsto (g^{-1})^*$ since we have

$$\begin{aligned}\mathrm{Ad}(g) \cdot X = T_{\mathrm{id}_M} \mathrm{conj}_g \cdot X &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 (g \circ \exp(tX) \circ g^{-1}) \\ &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 (g \circ \mathrm{Fl}_t^X \circ g^{-1}) = Tg \circ X \circ g^{-1} = (g^{-1})^* X.\end{aligned}$$

Thus for $\mathrm{ad} : \mathfrak{X}_c(M) \to \mathrm{L}\big(\mathfrak{X}_c(M), \mathfrak{X}_c(M)\big)$ we have

$$\begin{aligned}\mathrm{ad}(X)(Y) = (T_{\mathrm{id}_M} \mathrm{Ad} \cdot X)(Y) &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \mathrm{Ad}(\exp(tX))(Y) \\ &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \exp(-tX)^* Y = \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 (\mathrm{Fl}_{-t}^X)^* Y = -[X, Y]\end{aligned}$$

and hence the Lie bracket $[\cdot,\cdot]_{\mathfrak{X}_c(M)} : \mathfrak{X}_c(M) \times \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ is the negative of the usual Lie bracket on vector fields.

The space of $\mathfrak{X}_c(M)$-valued k-forms on $\mathrm{Diff}_c^\infty(M)$ is

$$\Omega^k\big(\mathrm{Diff}_c^\infty(M);\mathfrak{X}_c(M)\big) := \Gamma\big(\mathrm{L}^k_{\mathrm{alt}}\big(T\,\mathrm{Diff}_c^\infty(M), \mathrm{Diff}_c^\infty(M)\times\mathfrak{X}_c(M)\big)\big)$$

the space of smooth sections of the smooth vector bundle

$$\mathrm{L}^k_{\mathrm{alt}}\big(T\,\mathrm{Diff}_c^\infty(M), \mathrm{Diff}_c^\infty(M)\times\mathfrak{X}_c(M)\big) \to \mathrm{Diff}_c^\infty(M).$$

Notice that the right translation $\mu^g : \mathrm{Diff}_c^\infty(M) \to \mathrm{Diff}_c^\infty(M)$, $h \mapsto h\circ g$ has the following derivative $T_h\mu^g : T_h\,\mathrm{Diff}_c^\infty(M) \to T_{h\circ g}\,\mathrm{Diff}_c^\infty(M)$, $X_h \mapsto X_h\circ g$. Thus the right Maurer Cartan form $\kappa^r \in \Omega^1\big(\mathrm{Diff}_c^\infty(M);\mathfrak{X}_c(M)\big)$ is

$$\kappa^r(X_g) := T_g\mu^{g^{-1}}\cdot X_g = X_g\circ g^{-1}.$$

Recall that for any manifold N the space $\Omega^*(N;\mathfrak{X}_c(M))$ is a graded Lie algebra with Lie bracket

$$[\Psi,\Theta]_{\mathfrak{X}_c(M)}(X_1,\dots,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}(p+q)}\mathrm{sign}(\sigma)[\Psi(X_{\sigma(1)},\dots),\Theta(X_{\sigma(p+1)},\dots)]_{\mathfrak{X}_c(M)}$$

where $\Psi\in\Omega^p(N;\mathfrak{X}_c(M))$, $\Theta\in\Omega^q(N;\mathfrak{X}_c(M))$ and $X_i\in\mathfrak{X}(N)$. Moreover the exterior derivative $d:\Omega^k(N;\mathfrak{X}_c(M))\to\Omega^{k+1}(N;\mathfrak{X}_c(M))$ is defined by:

$$\begin{aligned} d\Psi(X_0,\dots,X_k) = &\sum_{i=0}^k(-1)^iX_i\cdot\Psi(X_0,\dots,\hat{\imath},\dots,X_k)\\ &+\sum_{i<j}(-1)^{i+j}\Psi([X_i,X_j],X_0,\dots,\hat{\imath},\dots,\hat{\jmath},\dots,X_k).\end{aligned}$$

In the first term $X_i\cdot\Psi(X_0,\dots,\hat{\imath},\dots,X_k)$ denotes the derivative of $\Psi(X_0,\dots,\hat{\imath},\dots,X_k)\in C^\infty(N;\mathfrak{X}_c(M))$ in direction X_i. κ^r satisfies the left Maurer-Cartan equation:

$$d\kappa^r - \tfrac{1}{2}[\kappa^r,\kappa^r]_{\mathfrak{X}_c(M)} = 0 \in \Omega^2\big(\mathrm{Diff}_c^\infty(M);\mathfrak{X}_c(M)\big) \tag{1.1}$$

For a mapping $f : N \to \mathrm{Diff}_c^\infty(M)$ the right logarithmic derivative $\delta^r f := f^*\kappa^r \in \Omega^1(N;\mathfrak{X}_c(M))$ looks like

$$\delta^r f(X_x) = (f^*\kappa^r)(X_x) = \kappa^r(T_xf\cdot X_x) = T_xf\cdot X_x\circ f(x)^{-1}$$

where $X_x\in T_xN$. For a curve $c:\mathbb{R}\to\mathrm{Diff}_c^\infty(M)$ this yields $\dot c := i_{\partial_t}\delta^r c:\mathbb{R}\to\mathfrak{X}_c(M)$ and

$$\dot c(t) = \delta^r c(\partial_t)(t) = T_tc\cdot\partial_t\circ c(t)^{-1} = \tfrac{\mathrm{d}}{\mathrm{ds}}|_t c(s)\circ c(t)^{-1}.$$

Pulling back (1.1) via $f : N\to\mathrm{Diff}_c^\infty(M)$ we obtain

$$d\delta^r f - \tfrac{1}{2}[\delta^r f,\delta^r f]_{\mathfrak{X}_c(M)} = 0 \in \Omega^2(N;\mathfrak{X}_c(M)). \tag{1.2}$$

The special case we will need later is $f:\mathbb{R}^2\to\mathrm{Diff}_c^\infty(M)$. For $\alpha\in\Omega^1(\mathbb{R}^2;\mathfrak{X}_c(M))$ we have

$$\begin{aligned}\big(d\alpha - \tfrac{1}{2}[\alpha,\alpha]_{\mathfrak{X}_c(M)}\big)(\partial_s,\partial_t) &= \partial_s\cdot\alpha(\partial_t) - \partial_t\cdot\alpha(\partial_s) - \alpha([\partial_s,\partial_t])\\ &\quad - \tfrac{1}{2}\big([\alpha(\partial_s),\alpha(\partial_t)]_{\mathfrak{X}_c(M)} - [\alpha(\partial_t),\alpha(\partial_s)]_{\mathfrak{X}_c(M)}\big)\\ &= \partial_s\cdot\alpha(\partial_t) - \partial_t\cdot\alpha(\partial_s) + [\alpha(\partial_s),\alpha(\partial_t)]\end{aligned} \tag{1.3}$$

and so (1.2) yields

$$\partial_s \cdot \delta^r f(\partial_t) - \partial_t \cdot \delta^r f(\partial_s) + [\delta^r f(\partial_s), \delta^r f(\partial_t)] = 0. \tag{1.4}$$

For $f, g : N \to \operatorname{Diff}_c^\infty(M)$ and $fg := \mu \circ (f, g) : N \to \operatorname{Diff}_c^\infty(M)$ that is $(fg)(x) = f(x) \circ g(x)$ we have the Leibniz rule

$$\delta^r(fg)(x) = \delta^r f(x) + \operatorname{Ad}(f(x)) \cdot \delta^r g(x). \tag{1.5}$$

Finally recall that $\operatorname{Diff}_c^\infty(M)$ is a regular Lie group, i.e. there exists a smooth right evolution mapping

$$\operatorname{Evol}^r : \Omega^1(\mathbb{R}; \mathfrak{X}_c(M)) \to C^\infty\big((\mathbb{R}, 0), (\operatorname{Diff}_c^\infty(M), \operatorname{id}_M)\big)$$

inverse to $\delta^r : C^\infty\big((\mathbb{R}, 0), (\operatorname{Diff}_c^\infty(M), \operatorname{id}_M)\big) \to \Omega^1(\mathbb{R}; \mathfrak{X}_c(M))$. $f := \operatorname{Evol}^r(\alpha)$ is simply the integral curve of the time dependent vector field $X := i_{\partial_t}\alpha \in C^\infty(\mathbb{R}, \mathfrak{X}_c(M))$ on M, i.e. $\frac{\partial}{\partial t} f_t = X_t \circ f$.

1.2.1. Lemma. *Let $X, Y : \mathbb{R} \to \mathfrak{X}_c(M)$ be smooth curves. Then the inhomogeneous linear, differential equation $Z'(t) = [X(t), Z(t)]_{\mathfrak{X}_c(M)} + Y(t)$ has a unique solution $Z : \mathbb{R} \to \mathfrak{X}_c(M)$ with initial value $Z(0)$ and it is given by*

$$Z(t) = \operatorname{Ad}(g(t)) \cdot \Big(\int_0^t \operatorname{Ad}(g(s)^{-1}) \cdot Y(s) ds + Z(0) \Big)$$

where $g : \mathbb{R} \to \operatorname{Diff}_c^\infty(M)$ is such that $g(0) = \operatorname{id}_M$ and $\delta^r g(\partial_t) = X$, i.e. $g = \operatorname{Evol}^r(X dt)$.

Proof. Notice first that we have for $g \in \operatorname{Diff}_c^\infty(M)$, $X \in \mathfrak{X}_c(M)$ and $Y_g \in T_g \operatorname{Diff}_c^\infty(M)$:

$$\begin{aligned} T_g\big(\operatorname{Ad}(\cdot)(X)\big)Y_g &= T_g\big(\operatorname{Ad}(\cdot)(X)\big)T_e\mu^g T_g \mu^{g^{-1}} Y_g \\ &= T_e\big(\operatorname{Ad}(\cdot)(\operatorname{Ad}(g)(X))\big)T_g\mu^{g^{-1}}Y_g = [T_g\mu^{g^{-1}}Y_g, \operatorname{Ad}(g)(X)]_{\mathfrak{X}_c(M)} \end{aligned}$$

So we have

$$\begin{aligned} Z'(t) = &\Big[T_{g(t)}\mu^{g(t)^{-1}} g'(t), \operatorname{Ad}(g(t))\Big(\int_0^t \operatorname{Ad}(g(s)^{-1}) Y(s) ds + Z(0)\Big)\Big]_{\mathfrak{X}_c(M)} \\ &+ \operatorname{Ad}(g(t)) \operatorname{Ad}(g(t)^{-1})(Y(t)) \\ = &[X(t), Z(t)]_{\mathfrak{X}_c(M)} + Y(t) \end{aligned}$$

and Z solves the differential equation. Remains to check uniqueness. Suppose $C : \mathbb{R} \to \mathfrak{X}_c(M)$ is another solution with initial value $C(0) = Z(0)$. Then $D : \mathbb{R} \to \mathfrak{X}_c(M)$, $D(t) := Z(t) - C(t)$ solves $D'(t) = [X(t), D(t)]_{\mathfrak{X}_c(M)}$ with $D(0) = 0$. Let us compute

$$\begin{aligned} &\partial_t \cdot \big(\operatorname{Ad}(g(t)^{-1})(D(t)) \big) \\ &\quad = -\big[T_{g(t)^{-1}}\mu^{g(t)} T_e \mu^{g(t)^{-1}} T_{g(t)}\mu_{g(t)^{-1}} g'(t), \operatorname{Ad}(g(t)^{-1})(D(t))\big]_{\mathfrak{X}_c(M)} \\ &\qquad + \operatorname{Ad}(g(t)^{-1})(D'(t)) \\ &\quad = -\operatorname{Ad}(g(t)^{-1})[T_{g(t)}\mu^{g(t)^{-1}} g'(t), D(t)]_{\mathfrak{X}_c(M)} + \operatorname{Ad}(g(t)^{-1})[X(t), D(t)]_{\mathfrak{X}_c(M)} \\ &\quad = -\operatorname{Ad}(g(t)^{-1})[X(t), D(t)]_{\mathfrak{X}_c(M)} + \operatorname{Ad}(g(t)^{-1})[X(t), D(t)]_{\mathfrak{X}_c(M)} = 0 \end{aligned}$$

So $\operatorname{Ad}(g(t)^{-1})(D(t)) = \operatorname{Ad}(g(0)^{-1})(D(0)) = 0$ and therefore $D(t) = 0$ which yields $Z(t) = C(t)$. ∎

The following can be found in [KM97], where it is proved using principal bundle theory.

1.2.2. Lemma. *Let N be a simply connected manifold, let $\alpha \in \Omega^1(N; \mathfrak{X}_c(M))$ satisfying the Maurer-Cartan equation $d\alpha - \frac{1}{2}[\alpha,\alpha]_{\mathfrak{X}_c(M)} = 0$ and let $x \in N$. Then there exists a unique $g \in C^\infty\big((N,x), (\mathrm{Diff}_c^\infty(M), \mathrm{id}_M)\big)$ such that $\delta^r g = \alpha$.*

Proof. Let $y \in N$ and choose a path $c \in C^\infty(I,N)$ from $x = c(0)$ to $y = c(1)$. If such a g exists then we must have $(g \circ c)(0) = \mathrm{id}_M$ and $c^*\alpha = c^*\delta^r g = \delta^r(g \circ c)$, i.e. $g \circ c = \mathrm{Evol}^r(c^*\alpha)$. Especially we get $g(y) = \mathrm{Evol}^r(c^*\alpha)(1)$ and therefore g is unique.

Let $H \in C^\infty(I \times I, N)$ be a homotopy with $H(s,0) = x$ and $H(s,1) = y$. Define $h \in C^\infty(I \times I, \mathrm{Diff}_c^\infty(M))$ by $h(s,0) = \mathrm{id}_M$ and $\delta^r h(\partial_t) = H^*\alpha(\partial_t)$, in other words $h(s,t) = \mathrm{Evol}^r(\mathrm{inc}_s^* H^*\alpha)(t)$. Pulling back $d\alpha - \frac{1}{2}[\alpha,\alpha]_{\mathfrak{X}_c(M)} = 0$ we obtain $dH^*\alpha - \frac{1}{2}[H^*\alpha, H^*\alpha]_{\mathfrak{X}_c(M)} = 0$ and in view of (1.3) this yields:

$$\partial_s \cdot H^*\alpha(\partial_t) - \partial_t \cdot H^*\alpha(\partial_s) + [H^*\alpha(\partial_s), H^*\alpha(\partial_t)] = 0$$

Using this equation and (1.4) for h we obtain

$$\begin{aligned}\partial_t \cdot (\delta^r h(\partial_s) - H^*\alpha(\partial_s)) &= \\ &= \partial_s \cdot \delta^r h(\partial_t) + [\delta^r h(\partial_s), \delta^r h(\partial_t)] - \partial_s \cdot H^*\alpha(\partial_t) - [H^*\alpha(\partial_s), H^*\alpha(\partial_t)] \\ &= [\delta^r h(\partial_s) - H^*\alpha(\partial_s), \delta^r h(\partial_t)] = [\delta^r h(\partial_t), \delta^r h(\partial_s) - H^*\alpha(\partial_s)]_{\mathfrak{X}_c(M)}\end{aligned}$$

that means $\varphi_s \in C^\infty(I, \mathfrak{X}_c(M))$ given by $\varphi_s(t) := \delta^r h(\partial_s)(s,t) - H^*\alpha(\partial_s)(s,t)$ satisfies the linear differential equation $\varphi_s'(t) = [\delta^r h(\partial_t)(s,t), \varphi_s(t)]_{\mathfrak{X}_c(M)}$ with initial condition $\varphi_s(0) = \delta^r h(\partial_s)(s,0) - H^*\alpha(\partial_s)(s,0) = 0 - 0 = 0$. Hence by the uniqueness part of lemma 1.2.1 we get $\varphi_s(t) = 0$ for all $t \in I$, hence $\delta^r h(\partial_s) = H^*\alpha(\partial_s)$ and therefore $\delta^r h = H^*\alpha$. Especially we have $\delta^r h(\partial_s)(s,1) = H^*\alpha(\partial_s)(s,1) = 0$ and thus $h(s,1)$ is constant in s. Further for $c_i(t) := H(i,t)$, $i = 0,1$ we have

$$c_i^*\alpha = (H \circ \mathrm{inc}_i)^*\alpha = \mathrm{inc}_i^* H^*\alpha = \mathrm{inc}_i^* \delta^r h = \delta^r(h \circ \mathrm{inc}_i)$$

and thus

$$\mathrm{Evol}^r(c_i^*\alpha)(1) = \mathrm{Evol}^r(\delta^r(h \circ \mathrm{inc}_i))(1) = h \circ \mathrm{inc}_i(1) = h(i,1)$$

So $\mathrm{Evol}^r(c_0^*\alpha)(1) = \mathrm{Evol}^r(c_1^*\alpha)(1)$ and since N is simply connected we may define g by $g(y) := \mathrm{Evol}^r(c^*\alpha)(1)$ where c is any path from x to y.

Next consider the mapping $m_s : (I,0) \to (I,0)$ defined by $m_s(t) = st$ where $s \in I$. For every $\beta \in \Omega^1(I, \mathfrak{X}_c(M))$ we have $\delta^r(\mathrm{Evol}^r(\beta) \circ m_s) = m_s^*\delta^r \mathrm{Evol}^r(\beta) = m_s^*\beta$ and hence $\mathrm{Evol}^r(m_s^*\beta) = \mathrm{Evol}^r(\beta) \circ m_s$. So

$$\begin{aligned}\mathrm{Evol}^r(c^*\alpha)(s) &= (\mathrm{Evol}^r(c^*\alpha) \circ m_s)(1) = \mathrm{Evol}^r(m_s^* c^*\alpha)(1) \\ &= \mathrm{Evol}^r((c \circ m_s)^*\alpha)(1) = g(c(s))\end{aligned}$$

that is $g \circ c = \mathrm{Evol}^r(c^*\alpha)$. Hence g maps smooth curves to smooth curves and is thus smooth.

Remains to show that $\delta^r g = \alpha$. But for $X_y \in T_yN$ we choose a curve c form x to y such that $c'(1) = X_y$ and obtain:

$$\begin{aligned}\delta^r g(X_y) &= \delta^r g(T_1 c \cdot \partial_t) = (c^*\delta^r g)(\partial_t)(1) = \delta^r(g \circ c)(\partial_t)(1) \\ &= c^*\alpha(\partial_t)(1) = \alpha(T_1 c \cdot \partial_t) = \alpha(X_y)\end{aligned}$$

This shows $\delta^r g = \alpha$. □

1.2.3. Lemma. *Let N be a manifold, $g \in C^\infty\big(N, \operatorname{Diff}_c^\infty(M)\big)$, $\omega \in C^\infty\big(N, \Omega^k(M)\big)$ and $X_x \in T_xN$. Then we have*

$$X_x \cdot \big(y \mapsto g(y)^*\omega(y)\big) = g(x)^*\big(L_{\delta^r g(X_x)}(w(x)) + (X_x \cdot \omega)\big)$$

for all $x \in N$.

Proof. Notice that $\sigma : \mathbb{R} \to \Omega^k(M)$, $\sigma(t) := w(x) + t(X_x \cdot \omega)$ is a smooth curve satisfying $\sigma(0) = w(x)$ and $\sigma'(0) = (X_x \cdot \omega)$. Moreover $\tau : \mathbb{R} \to \operatorname{Diff}_c^\infty(M)$, $\tau(t) := \exp(t\kappa^r(T_xg \cdot X_x)) \circ g(x)$ is a smooth curve satisfying $\tau(0) = g(x)$ and

$$\begin{aligned}\tau'(0) &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \mu^{g(x)}\big(\exp(t\kappa^r(T_xg \cdot X_x))\big) = T_{\mathrm{id}}\mu^{g(x)} \cdot \kappa^r(T_xg \cdot X_x) \\ &= T_{\mathrm{id}}\mu^{g(x)} T_{g(x)}\mu^{g(x)^{-1}} \cdot T_xg \cdot X_x = T_xg \cdot X_x.\end{aligned}$$

Hence we obtain

$$\begin{aligned}X_x \cdot \big(y \mapsto g(y)^*\omega(y)\big) &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \tau(t)^*\omega(x) + \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 g(x)^*\sigma(t) \\ &= \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \big(\exp(t\kappa^r(T_xg \cdot X_x)) \circ g(x)\big)^*\omega(x) + g(x)^* \tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \sigma(t) \\ &= g(x)^*\big(\tfrac{\mathrm{d}}{\mathrm{dt}}|_0 \exp(t\kappa^r(T_xg \cdot X_x))^*\omega(x) + (X_x \cdot \omega)\big) \\ &= g(x)^*\big(L_{\kappa^r(T_xg \cdot X_x)}\omega(x) + (X_x \cdot \omega)\big) \\ &= g(x)^*\big(L_{(g^*\kappa^r)(X_x)}\omega(x) + (X_x \cdot \omega)\big) \\ &= g(x)^*\big(L_{\delta^r g(X_x)}\omega(x) + (X_x \cdot \omega)\big)\end{aligned}$$

and thus the lemma is proved. □

A well known special case of lemma 1.2.3 is $\frac{\partial}{\partial t}(g_t^*\omega_t) = g_t^*\big(L_{\dot g_t}\omega_t + \frac{\partial}{\partial t}\omega_t\big)$ for smooth curves $g : \mathbb{R} \to \operatorname{Diff}_c^\infty(M)$ and $\omega : \mathbb{R} \to \Omega^k(M)$.

1.2.4. Definition. Let $G \subseteq \operatorname{Diff}_c^\infty(M)$ be a subgroup. Then we set

$$C^\infty(N, G) := \big\{f \in C^\infty\big(N, \operatorname{Diff}_c^\infty(M)\big) : f(x) \in G \quad \forall x \in N\big\}$$

We denote by $G_\circ$ the normal subgroup of all elements g of G that can be joined with the identity by a smooth path in G, i.e. there exists $c \in C^\infty(I, G)$ with $c(0) = \mathrm{id}$ and $c(1) = g$. We call G connected by smooth arcs if $G_\circ = G$. Moreover we denote by $\tilde G$ the group $C^\infty\big((I, 0), (G, \mathrm{id})\big)/\sim$, where two such curves c_0, c_1 are equivalent iff they are smoothly homotopic relative endpoints in G, i.e. there exists $H \in C^\infty(I \times I, G)$ with $H(s, 0) = \mathrm{id}$, $H(s, 1) = c_0(1) = c_1(1)$, $H(0, t) = c_0(t)$ and $H(1, t) = c_1(t)$. Notice that $\pi := \mathrm{ev}_1 : \tilde G \to G$, $c \mapsto c(1)$, is surjective iff G is connected by smooth arcs. Finally set $\boldsymbol{\pi}_1(G) := \ker \pi$. If $G \subseteq \operatorname{Diff}_c^\infty(M)$ is a submanifold then $G_\circ$ is the connected component containing id, $\pi : \tilde G \to G$ is the universal covering of $G_\circ$ and $\boldsymbol{\pi}_1(G)$ is the first homotopy group of G.

1.2.5. Definition. Let $G \subseteq \operatorname{Diff}^\infty(M)$ be a subgroup, and $k \in \mathbb{N}$. We say G acts k-transitive on M if the following holds: For distinct points $x_1, \dots, x_k \in M$ and distinct points $y_1, \dots, y_k \in M$ there exists $g \in G$ such that $g(x_i) = y_i$ for all $1 \le i \le k$. A Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{X}(M)$ is said to act infinitesimal k-transitive if for distinct points $x_1, \dots, x_k \in M$ and $Y_i \in T_{x_i}M$ there exists $X \in \mathfrak{g}$ with $X(x_i) = Y_i$ for all $1 \le i \le k$.

The proof of the following is due to P. W. Michor and C. Vizman, see [KM97].

1.2.6. Proposition. *Let M be a connected manifold of dimension greater than 1. Let $G \subseteq \mathrm{Diff}_c^\infty(M)$ be a subgroup and $\mathfrak{g} \subseteq \mathfrak{X}_c(M)$ a Lie subalgebra such that $\mathrm{Fl}_t^X \in G$ for all $X \in \mathfrak{g}$ and all $t \in \mathbb{R}$. If $\mathfrak{g}$ acts infinitesimal k-transitive on M then $G_\circ$ acts k-transitive on M.*

Proof. Consider

$$\tilde{M} := \big\{(x_1, \dots, x_k) : x_i \neq x_j \text{ for } i \neq j\big\} \subseteq M \times \dots \times M$$

G acts on $\tilde{M}$ by $g \cdot (x_1, \dots, x_k) := \big(g(x_1), \dots, g(x_k)\big)$. It is clear that G acts k-transitive on M iff G acts transitive on $\tilde{M}$. Notice that $\tilde{M}$ is connected, since we have the assumption $\dim(M) > 1$. So it suffices to show that the G-orbits on $\tilde{M}$ are open, for then they are closed too, and connectedness yields that there exists only one orbit. Let $(x_1, \dots, x_k) \in \tilde{M}$ and choose $X_{i,j} \in \mathfrak{g}$, $1 \leq i \leq k$, $1 \leq j \leq n$ such that $\{X_{i,1}(x_i), \dots, X_{i,n}(x_i)\}$ is a basis of $T_{x_i}M$ for all $1 \leq i \leq k$ and $X_{i,j}(x_l) = 0$ for $l \neq i$. Consider the mapping:

$$f : \overbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}^{k \text{ factors}} \to \tilde{M}$$
$$\left(\begin{pmatrix} t_{1,1} \\ \vdots \\ t_{1,n} \end{pmatrix}, \dots, \begin{pmatrix} t_{k,1} \\ \vdots \\ t_{k,n} \end{pmatrix}\right) \mapsto \big(\mathrm{Fl}_{t_{1,1}}^{X_{1,1}} \circ \dots \circ \mathrm{Fl}_{t_{1,n}}^{X_{1,n}} \circ \dots \circ \mathrm{Fl}_{t_{k,1}}^{X_{k,1}} \circ \dots \circ \mathrm{Fl}_{t_{k,n}}^{X_{k,n}}\big) \cdot (x_1, \dots, x_k)$$

We have $f(0, \dots, 0) = (x_1, \dots, x_k)$ and

$$f(0, \dots, t_{i,j} e_j, \dots, 0) = \mathrm{Fl}_{t_{i,j}}^{X_{i,j}} \cdot (x_1, \dots, x_k) = \big(x_1, \dots, \mathrm{Fl}_{t_{i,j}}^{X_{i,j}}(x_i), \dots, x_k\big)$$

where $e_j \in \mathbb{R}^n$ is the j-th unit vector. So $\frac{\partial}{\partial t_{i,j}} f(0) = \big(0, \dots, X_{i,j}(x_i), \dots, 0\big) \in T_{(x_1,\dots,x_k)}\tilde{M}$ and $T_0 f$ is surjective. Using the inverse function theorem we see that f is a local diffeomorphism and so $(x_1, \dots, x_k)$ is in the interior of the G-orbit through $(x_1, \dots, x_k)$. Since $(x_1, \dots, x_k)$ was arbitrary the G-orbits are open. □

1.2.7. Remark. If $\dim(M) = 1$ the statement of proposition 1.2.6 remains true for $k = 1$. The proof is the same, since $\tilde{M}$ is connected in this case, too. Easy examples show that proposition 1.2.6 is false for $\dim(M) = 1$ and $k > 1$.

1.3 From Perfectness to Simplicity

The following result slightly generalizes a result due to W. Thurston (cf. [Ban97]).

1.3.1. Proposition. *Let X be a Hausdorff topological space, $\mathcal{U}$ be a basis of the topology and $G \subseteq \mathrm{Homeo}(X)$ be a subgroup of homeomorphisms on X. Assume we have for all $U \in \mathcal{U}$ a perfect subgroup $G_U \subseteq G \cap \mathrm{Homeo}_U(X)$, satisfying:*

1. *every G-orbit is dense in X (weak transitivity)*
2. *if $\mathcal{V} \subseteq \mathcal{U}$ is a covering of X then $\bigcup_{V \in \mathcal{V}} G_V$ generates G (fragmentation)*
3. *if $U, V \in \mathcal{U}$, $g \in G$, $g(U) \subseteq V$ then $g G_U g^{-1} \subseteq G_V$*

Then G is simple.

Proof. Let $\mathrm{id} \neq g \in G$. We want to show $\mathcal{N}(g) = G$, where $\mathcal{N}(g)$ denotes the normal subgroup in G generated by g. Since $\mathrm{id} \neq g$ we find $x \in X$ with $g(x) \neq x$ and by 1 we also find $h \in G$ with $h(x) \neq x$ and $h(x) \neq g(x)$. Since X is Hausdorff we can separate $x, g(x), h(x)$ by open neighborhoods W_1, W_2, W_3 of $x, g(x), h(x)$ respectively. We let $U := W_1 \cap g^{-1}(W_2) \cap h^{-1}(W_3)$. Then U is an open neighborhood of x and $U, g(U), h(U)$ are pairwise disjoint. We claim

$$[u, v] = [[u, g], [v, h]] \qquad \forall u, v \in \mathrm{Homeo}_U(X) \tag{1.6}$$

Indeed, since $U \cap g(U) = \emptyset$ and $U \cap h(U) = \emptyset$ we have

$$[u, g] = \begin{cases} u & \text{on } U \\ gu^{-1}g^{-1} & \text{on } g(U) \\ \mathrm{id} & \text{elsewhere} \end{cases} \qquad [v, h] = \begin{cases} v & \text{on } U \\ hv^{-1}h^{-1} & \text{on } h(U) \\ \mathrm{id} & \text{elsewhere} \end{cases} \tag{1.7}$$

and so (1.6) holds on $M \setminus (g(U) \cup h(U))$. Remains to check $[[u, g], [v, h]]|_{g(U)\cup h(U)} = \mathrm{id}$ but this follows again from (1.7) and the fact $g(U) \cap h(U) = \emptyset$.

From (1.6) we especially obtain

$$G_U = [G_U, G_U] \subseteq [[G_U, g], [G_U, h]] \subseteq [\mathcal{N}(g), G] \subseteq \mathcal{N}(g)$$

Now let $y \in X$ be arbitrary. From 1 we obtain a neighborhood $U_y \in \mathcal{U}$ of y and $\alpha_y \in G$ with $\alpha_y(U_y) \subseteq U$. Using 3 we get

$$G_{U_y} \subseteq \alpha_y^{-1} G_U \alpha_y \subseteq \alpha_y^{-1} \mathcal{N}(g) \alpha_y \subseteq \mathcal{N}(g)$$

Since $\{U_y : y \in X\}$ covers X, $\bigcup_{y\in X} G_{U_y}$ generates G by 2 and so $G \subseteq \mathcal{N}(g)$. □

The following is another famous result in this direction, due to D. B. A. Epstein, but we will not use it in the sequel. See [Eps70] for the completely elementary proof.

1.3.2. Theorem. *Let X be a paracompact Hausdorff topological space and $G \subseteq \mathrm{Homeo}(X)$ a subgroup of homeomorphisms on X. Assume there exists a basis $\mathcal{U}$ of the topology of X such that the following conditions (Epstein's axioms) are satisfied:*

1. *if $U \in \mathcal{U}$ and $g \in G$ then $g(U) \in \mathcal{U}$*
2. *G acts transitively on $\mathcal{U}$*
3. *if $g \in G$, $U \in \mathcal{U}$ and $\mathcal{V}$ is an open covering of X then there exist $N \in \mathbb{N}$, $g_1, \dots, g_N \in G$ and $V_1, \dots, V_N \in \mathcal{V}$ such that $g = g_1 \cdots g_N$, $\mathrm{supp}(g_i) \subseteq V_i$ and $\mathrm{supp}(g_i) \cup g_{i-1} \cdots g_1(\bar{U}) \neq X$ for all $1 \leq i \leq N$.*

Then every non-trivial subgroup $H \subseteq G$, with $[G, G] \subseteq N_G(H) := \{g \in G : gHg^{-1} \subseteq H\}$, contains $[G, G]$.

1.3.3. Corollary. *In the situation of theorem 1.3.2 the commutator subgroup $[G, G]$ is simple.*

Proof. If H is a non-trivial normal subgroup of $[G, G]$ we have $[G, G] \subseteq N_G(H)$ and hence by theorem 1.3.2 we obtain $[G, G] = H$. □

1.3.4. Corollary. *In the situation of theorem 1.3.2 the commutator subgroup $[G,G]$ is the minimal normal subgroup of G.*

Proof. For a normal subgroup H of G we have $[G,G] \subseteq G = N_G(H)$ and hence theorem 1.3.2 yields $[G,G] \subseteq H$. So any non-trivial normal subgroup contains $[G,G]$. □

1.3.5. Remark. The problem about theorem 1.3.2 is that it is not applicable to groups of diffeomorphisms that preserve e.g. a volume form or a symplectic form since the second axiom does not hold.

1.4 The Simplicial Set $B\bar{G}$

Let $\Delta^n := \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} : 0 \le t_i \le 1, \sum t_i = 1\}$ denote the standard n-simplex and recall that the mappings $\delta_i^n : \Delta^{n-1} \to \Delta^n$, $0 \le i \le n$ and $\sigma_i^{n-1} : \Delta^n \to \Delta^{n-1}$, $0 \le i \le n-1$, given by

$$\begin{aligned}\delta_i^n(t_0,\dots,t_{n-1}) &= (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1})\\ \sigma_i^{n-1}(t_0,\dots,t_n) &= (t_0,\dots,t_i+t_{i+1},\dots,t_n)\end{aligned}$$

satisfy the relations:

$$\begin{aligned}
\delta_j^{n+1}\delta_i^n &= \delta_i^{n+1}\delta_{j-1}^n && 0 \le i < j \le n+1\\
\sigma_j^{n-1}\sigma_i^n &= \sigma_i^{n-1}\sigma_{j+1}^n && 0 \le i \le j \le n-1\\
\sigma_j^n\delta_i^{n+1} &= \delta_i^n\sigma_{j-1}^{n-1} && 0 \le i < j \le n\\
\sigma_j^n\delta_i^{n+1} &= \delta_{i-1}^n\sigma_j^{n-1} && 1 \le j+1 < i \le n+1\\
\sigma_j^n\delta_j^{n+1} &= \sigma_j^n\delta_{j+1}^{n+1} = \mathrm{id}_{\Delta^n} && 0 \le j \le n
\end{aligned}$$

For a subgroup $G \subseteq \mathrm{Diff}_c^\infty(M)$ we let $S_n(G) := C^\infty(\Delta^n, G)$ denote the set of smooth mappings $\Delta^n \to \mathrm{Diff}_c^\infty(M)$ that take values in G. Then $\partial_i^n := (\delta_i^n)^* : S_n(G) \to S_{n-1}(G)$ and $s_i^n := (\sigma_i^n)^* : S_n(G) \to S_{n+1}(G)$ satisfy the well known relations:

$$\begin{aligned}
\partial_i^n\partial_j^{n+1} &= \partial_{j-1}^n\partial_i^{n+1} && 0 \le i < j \le n+1\\
s_i^n s_j^{n-1} &= s_{j+1}^n s_i^{n-1} && 0 \le i \le j \le n-1\\
\partial_i^{n+1} s_j^n &= s_{j-1}^{n-1}\partial_i^n && 0 \le i < j \le n\\
\partial_i^{n+1} s_j^n &= s_j^{n-1}\partial_{i-1}^n && 1 \le j+1 < i \le n+1\\
\partial_j^{n+1} s_j^n &= \partial_{j+1}^{n+1} s_j^n = \mathrm{id}_{S_n(G)} && 0 \le j \le n
\end{aligned}$$

That means $S_n(G)$ together with s_i^n and ∂_i^n form a simplicial complex, which we will denote by $S(G)$. A good reference for simplicial complexes is [May75]. In the sequel we will write ∂_i resp. s_i for ∂_i^n resp. s_i^n if no confusion is possible.

1.4.1. Lemma. *For any subgroup $G \subseteq \mathrm{Diff}_c^\infty(M)$ the simplicial complex $S(G)$ is a Kan complex. That is, it satisfies the following extension condition: For $n+1$ n-simplices $g_0,\dots,g_{k-1},g_{k+1},\dots,g_{n+1}$ which satisfy the compatibility condition $\partial_i g_j = \partial_{j-1} g_i$, $0 \le i < j \le n+1$, $i \ne k$, $j \ne k$ there exists a $(n+1)$-simplex g such that $\partial_i g = g_i$ for $i \ne k$.*

Proof. For $1 \le i \le n$ we consider the mapping

$$r_i : \Delta^n \to \Delta^{n-1} \qquad r_i(t_0,\dots,t_n) := (t_0+t_i,t_1,\dots,t_{i-1},t_{i+1},\dots,t_n)$$

and $\rho_i := r_i^* : S_{*-1}(G) \to S_*(G)$. An easy calculation shows:

$$\begin{aligned} \partial_j \rho_i &= \rho_{i-1}\partial_j \quad 1 \le j < i \\ \partial_i \rho_i &= \mathrm{id} \\ \partial_j \rho_i &= \rho_i \partial_{j-1} \quad i < j \end{aligned} \tag{1.8}$$

Suppose first the case $k = 0$, i.e. we have n-simplices $g_1, \ldots, g_{n+1}$. Define $\lambda_1 := \rho_1 g_1 \in S_{n+1}(G)$ and inductively

$$\lambda_i := \lambda_{i-1}(\rho_i \partial_i \lambda_{i-1}^{-1})(\rho_i g_i) \in S_{n+1}(G)$$

for $2 \le i \le n+1$. Here the multiplication and inverse of simplices is point wise, i.e. $(\alpha\beta)(t_0 \ldots, t_n) := \alpha(t_0, \ldots, t_n)\beta(t_0, \ldots, t_n)$ and $\alpha^{-1}(t_0, \ldots, t_b) = \alpha(t_0, \ldots, t_n)^{-1}$. Clearly this multiplication commutes with the operators s_i, ∂_i and ρ_i.

We claim that $\partial_j \lambda_i = g_j$ for $1 \le j \le i \le n+1$. We will prove the last statement by induction on i. For $i = 1$ this follows immediately from the second equations of (1.8). For the inductive step we calculate

$$\partial_i \lambda_i = (\partial_i \lambda_{i-1})(\partial_i \rho_i \partial_i \lambda_{i-1}^{-1})(\partial_i \rho_i g_i) = (\partial_i \lambda_{i-1})(\partial_i \lambda_{i-1}^{-1}) g_i = g_i$$

where we used again (1.8). For $1 \le j < i$ we have

$$\begin{aligned} \partial_j \lambda_i &= (\partial_j \lambda_{i-1})(\partial_j \rho_i \partial_i \lambda_{i-1}^{-1})(\partial_j \rho_i g_i) = g_j(\rho_{i-1}\partial_j \partial_i \lambda_{i-1}^{-1})(\rho_{i-1}\partial_j g_i) \\ &= g_j(\rho_{i-1}\partial_{i-1}\partial_j \lambda_{i-1}^{-1})(\rho_{i-1}\partial_{i-1} g_j) = g_j(\rho_{i-1}\partial_{i-1} g_j^{-1})(\rho_{i-1}\partial_{i-1} g_j) = g_j \end{aligned}$$

This ends the proof in the case $k = 0$, for $g := \lambda_{n+1}$ has the desired property.

To cope with the case $k \ne 0$ we define mappings

$$p : \Delta^n \to \Delta^n \qquad p(t_0, \ldots, t_n) := (t_n, t_0, \ldots, t_{n-1})$$

and $\pi := p^* : S_*(G) \to S_*(G)$. An easy calculation shows

$$\begin{aligned} \partial_i \pi &= \pi \partial_{i+1} \quad 0 \le i < n \\ \partial_n \pi &= \partial_0 \end{aligned} \tag{1.9}$$

We proceed by induction on k. So we have n-simplices $g_0, \ldots, g_{k-1}, g_{k+1}, \ldots, g_{n+1}$ with $\partial_i g_j = \partial_{j-1} g_i$ for $0 \le i < j \le n+1$, $i \ne k$, $j \ne k$. Define $f_{n+1} := g_0$ and $f_i := \pi g_{i+1}$ for $0 \le i \le n$, $i \ne k-1$, i.e. we have n-simplices $f_0, \ldots, f_{k-2}, f_k, \ldots .f_{n+1}$. Using equations (1.9) we obtain for $0 \le i < n+1$, $i \ne k-1$

$$\partial_i f_{n+1} = \partial_i g_0 = \partial_0 g_{i+1} = \partial_n \pi g_{i+1} = \partial_n f_i$$

and for $0 \le i < j \le n$, $i \ne k-1$, $j \ne k-1$

$$\partial_i f_j = \partial_i \pi g_{j+1} = \pi \partial_{i+1} g_{j+1} = \pi \partial_j g_{i+1} = \partial_{j-1} \pi g_{i+1} = \partial_{j-1} f_i$$

So f_i satisfy the compatibility conditions for $k-1$ and by induction there exists an $n+1$-simplex f with $\partial_i f = f_i$ for $i \ne k-1$. If we define $g := (p^{-1})^* f$ we get

$$\partial_0 g = \partial_0 (p^{-1})^* f = \partial_{n+1} f = f_{n+1} = g_0$$

and for $0 \le i \le n+1$, $i \ne k$

$$\partial_i g = \partial_i (p^{-1})^* f = (p^{-1})^* \partial_{i-1} f = (p^{-1})^* f_{i-1} = g_i$$

hence g is the desired extension. □

1.4.2. Definition. Consider the left action of G on $S(G)$ by simplicial maps $G \times S_n(G) \to S_n(G)$, $(h, g) \mapsto (\mu^{h^{-1}})_* g$, where $\mu^{h^{-1}} : G \to G$ denotes right multiplication with h^{-1}. Since ∂_i and s_i are G-equivariant we can define a new simplicial set $S(B\overline{G}) := S(G)/G$.

1.4.3. Lemma. *Let $G \subseteq \mathrm{Diff}^\infty_c(M)$ be a group with Lie algebra $\mathfrak{g} \subseteq \mathfrak{X}_c(M)$ in the sense that $g \in C^\infty\big((I,0),(G,\mathrm{id})\big) \subseteq C^\infty\big((I,0),(\mathrm{Diff}^\infty_c(M),\mathrm{id})\big)$ if and only if $\delta^r g \in \Omega^1(I;\mathfrak{g}) \subseteq \Omega^1(I;\mathfrak{X}_c(M))$, i.e. $\dot g_t \in \mathfrak{g}$ for all $t \in I$ (cf. page 3). Then there exists a natural one-to-one correspondence between*

1. $S_p(B\overline{G})$
2. $C^\infty\big((\Delta^p, e_0),(G,\mathrm{id})\big)$
3. *$\sigma \in \Omega^1(\Delta^p;\mathfrak{g})$ satisfying $d\sigma - \frac{1}{2}[\sigma,\sigma]_{\mathfrak{g}} = 0$*
4. *$\dim(M)$-codimensional foliations on $\Delta^p \times M$, transversal to $\{t\} \times M$ for all $t \in \Delta$ which have the following property: if $Y_t \in T_t\Delta^p$ and $X \in \mathfrak{X}_c(M)$ such that (Y_t, X) is tangential to this foliation, then $X \in \mathfrak{g}$.*

$[g] \in S_p(B\overline{G})$ corresponds to $g \cdot g(e_0)^{-1} \in C^\infty\big((\Delta^p,e_0),(G,\mathrm{id})\big)$, to $\delta^r g \in \Omega^1(\Delta^p;\mathfrak{g})$ and to the foliation with leaves $\{(t, g(t)(x)) : t \in \Delta^p\}$. If $\sigma \in \Omega^1(\Delta^p;\mathfrak{g})$ then $E_{(t,x)} := \{(Y_t, \sigma(Y_t)(x)) : Y_t \in T_t\Delta^p\} \subseteq T_{(t,x)}(\Delta^p \times M)$ is a distribution, transversal to $\{t\} \times M$ for all $t \in \Delta^p$. It is integrable iff $d\sigma - \frac{1}{2}[\sigma,\sigma]_{\mathfrak{g}} = 0$ and the foliation corresponding to σ is the foliation tangential to this distribution.

Proof. The correspondence between 1 and 2 is obvious. Notice that the right logarithmic derivative $\delta^r g$ does not depend on the representative g of $[g] \in S_p(B\overline{G})$. For $g \in C^\infty\big((\Delta^p,e_0),(\mathrm{Diff}^\infty_c(M),\mathrm{id})\big)$ we have:

$$g \in C^\infty\big((\Delta^p,e_0),(G,\mathrm{id})\big) \qquad \Leftrightarrow \qquad \delta^r g \in \Omega^1(\Delta^p;\mathfrak{g})$$

Indeed if $g \in C^\infty\big((\Delta^p,e_0),(G,\mathrm{id})\big)$ and $Y_t \in T_t\Delta^p$ we choose a path $c : I \to \Delta^p$ connecting e_0 and t with $c'(1) = Y_t$. Then

$$\delta^r g(Y_t) = (c^*\delta^r g)(\partial_t)(1) = \delta^r(g \circ c)(\partial_t)(1) \in \mathfrak{g}$$

for $g \circ c : I \to G$. Suppose conversely $\delta^r g \in \Omega^1(\Delta^p;\mathfrak{g})$ and let $t \in \Delta^p$. We have to show $g(t) \in G$. Let again $c : I \to \Delta^p$ be a path from e_0 to t. Then $\delta^r(g \circ c) = c^*\delta^r g \in \Omega^1(I;\mathfrak{g})$ and thus $g(t) = (g \circ c)(1) \in G$. The correspondence between 2 and 3 now follows from lemma 1.2.2.

If $\sigma \in \Omega^1(\Delta^p;\mathfrak{X}_c(M))$ then $E_{(t,x)} := \{(Y_t, \sigma(Y_t)(x))\}$ is a $\dim(M)$-codimensional distribution on $\Delta^p \times M$ which is transversal to $\{t\} \times M$ for all $t \in \Delta^p$. Conversely every such distribution is of this form. It remains to show that this distribution is integrable iff $d\sigma - \frac{1}{2}[\sigma,\sigma]_{\mathfrak{g}} = 0$. Choose a global frame of vector fields $Y_1, \dots, Y_p \in \mathfrak{X}(\Delta^p)$ and consider $X_i := \big(Y_i, \sigma(Y_i)\big) \in \mathfrak{X}_c(\Delta^p \times M)$. Then X_i span the distribution and it is integrable if and only if $[X_i, X_j]$ is tangential to the distribution $\forall i, j$, i.e.

$$\begin{aligned} 0 &= -\sigma([Y_i,Y_j]) + Tp_M[(Y_i,\sigma(Y_i)),(Y_j,\sigma(Y_j))] \\ &= -\sigma([Y_i,Y_j]) + Y_i \cdot \sigma(Y_j) - Y_j \cdot \sigma(Y_i) + [\sigma(Y_i),\sigma(Y_j)] \\ &= \big(d\sigma - \tfrac{1}{2}[\sigma,\sigma]_{\mathfrak{g}}\big)(Y_i,Y_j) \end{aligned}$$

and since $Y_1, \dots, Y_p$ is a frame this is equivalent to $d\sigma - \frac{1}{2}[\sigma,\sigma]_{\mathfrak{g}} = 0$. $\square$

1.4.4. Corollary. *For any subgroup $G \subseteq \mathrm{Diff}_c^\infty(M)$ the complex $S(B\overline{G})$ is a Kan complex.*

Proof. Let $0 \le k \le n+1$ and given $n+1$ n-simplices $g_0 \dots, g_{k-1}, g_{k+1}, \dots, g_{n+1} \in S_n(B\overline{G})$ such that $\partial_i g_j = \partial_{j-1} g_i \in S_{n-1}(B\overline{G})$ for $0 \le i < j \le n+1$, $i \ne k$, $j \ne k$, i.e. $\partial_i g_j$ and $\partial_{j-1} g_i$ equal up to right multiplication by some element of G. We claim that there exist representatives $f_i \in S_n(G)$ of $g_i \in S_n(B\overline{G})$ with $\partial_i f_j = \partial_{j-1} f_i \in S_{n-1}(G)$ for $0 \le i < j \le n+1$, $i \ne k$, $j \ne k$. To see this suppose first $k \ne 0$. Then we can choose f_j such that $\partial_0 f_j = \partial_{j-1} f_0 \in S_{n-1}(G)$ for $1 \le j \le n+1$, $j \ne k$. Then for $1 \le i < j \le n+1$, $i \ne k$, $j \ne k$ we get

$$\partial_0 \partial_i f_j = \partial_{i-1} \partial_0 f_j = \partial_{i-1} \partial_{j-1} f_0 = \partial_{j-2} \partial_{i-1} f_0 = \partial_{j-2} \partial_0 f_i = \partial_0 \partial_{j-1} f_i$$

and since $\partial_i f_j = \partial_{j-1} f_j \in S_{n-1}(B\overline{G})$ we obtain $\partial_i f_j = \partial_{j-1} f_j \in S_{n-1}(G)$. If $k = 0$ one has to define the representatives f_i of g_i by f_j by $\partial_1 f_j = \partial_{j-1} f_1 \in S_{n-1}(G)$. A similar calculation shows $\partial_1 \partial_i f_j = \partial_1 \partial_{j-1} f_i$ and hence again $\partial_i f_j = \partial_{j-1} f_i \in S_{n-1}(G)$. Since $S(G)$ is a Kan complex by lemma 1.4.1 we find $f \in S_{n+1}(G)$ such that $\partial_i f = f_i \in S_n(G)$ for $i \ne k$ and especially $\partial_i f = f_i = g_i \in S_n(B\overline{G})$. □

If K is a simplicial complex let $C_p(K; \mathbb{Z})$ denote the free abelian group generated by the p-simplices K_p. Moreover consider the differential

$$\partial := \textstyle\sum_{i=0}^p (-1)^i \partial_i : C_p(K; \mathbb{Z}) \to C_{p-1}(K; \mathbb{Z})$$

One easily checks $\partial \circ \partial = 0$ and so one has a complex:

$$\cdots \to C_2(K; \mathbb{Z}) \xrightarrow{\partial} C_1(K; \mathbb{Z}) \xrightarrow{\partial} C_0(K; \mathbb{Z}) \to 0$$

Its homology is, by definition, the homology of K with values in $\mathbb{Z}$. We will write $H_*(K; \mathbb{Z})$ for it.

For a Kan complex K one can also define homotopy groups $\boldsymbol{\pi}_i(K)$, see [May75]. For example $\boldsymbol{\pi}_1(K) = \{x \in K_1 : \partial_0 x = \partial_1 x\}/\sim$, where $x \sim y$ iff there exists $z \in K_2$ such that $\partial_1 z = x$, $\partial_2 z = y$ and $\partial_0 z = s_0 \partial_0 x = s_0 \partial_0 y$. In the case $K = B\overline{G}$ there is only one 0-simplex and we obtain $\boldsymbol{\pi}_1(B\overline{G}) = C^\infty\big((\Delta^1, e_0), (G, \mathrm{id}_M)\big)/\sim$, where $g \sim h$ if and only if they are smoothly homotopic relative endpoints, i.e. $\boldsymbol{\pi}_1(B\overline{G}) = \widetilde{G}$, the universal covering of G. By the Huréwitz theorem, which is also valid for Kan complexes (see [May75]), we thus get:

$$H_1(B\overline{G}; \mathbb{Z}) = \frac{\boldsymbol{\pi}_1(B\overline{G})}{[\boldsymbol{\pi}_1(B\overline{G}), \boldsymbol{\pi}_1(B\overline{G})]} = \frac{\widetilde{G}}{[\widetilde{G}, \widetilde{G}]} = H_1(\widetilde{G}; \mathbb{Z})$$

If G is connected by smooth arcs then the projection $\widetilde{G} \to G$ is surjective and so is the induced mapping $\widetilde{G}/[\widetilde{G}, \widetilde{G}] \to G/[G, G]$ too. So perfectness of $\widetilde{G}$ implies perfectness of G and we have shown:

1.4.5. Proposition. *If $G \subseteq \mathrm{Diff}_c^\infty(M)$ is a group of diffeomorphisms which is connected by smooth arcs then $\widetilde{G}$ is perfect if and only if $H_1(B\overline{G}; \mathbb{Z}) = 0$. In this situation G is perfect too.*

1.4.6. Remark. Since there is only one 0-simplex in $B\overline{G}$ and since $\partial = 0 : C_1(B\overline{G}; \mathbb{Z}) \to C_0(B\overline{G}; \mathbb{Z})$ we have $H_0(B\overline{G}; \mathbb{Z}) = \mathbb{Z}$ for every G.

1.4.7. Remark. If $G = \{\mathrm{id}\}$ then $C_*(B\overline{G};\mathbb{Z})$ is

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

and so $H_0(B\overline{G};\mathbb{Z}) = \mathbb{Z}$ and $H_p(B\overline{G};\mathbb{Z}) = 0$ for $p > 0$.

1.4.8. Lemma. *Let $G \subseteq \mathrm{Diff}_c^\infty(M)$ be a subgroup and let $f \in G_\circ$. Then $\mathrm{conj}_f : G \to G$, $\mathrm{conj}_f(g) = fgf^{-1}$ induces a simplicial mapping $(\mathrm{conj}_f)_* : S_*(B\overline{G}) \to S_*(B\overline{G})$ and we have $(\mathrm{conj}_f)_* = \mathrm{id} : H_*(B\overline{G};\mathbb{Z}) \to H_*(B\overline{G};\mathbb{Z})$.*

Proof. If $h \in G$ and $g \in S_p(B\overline{G})$ then

$$\mathrm{conj}_f(gh) = fghf^{-1} = fgf^{-1}fhf^{-1} = (\mathrm{conj}_f\, g)(fhf^{-1}) \tag{1.10}$$

and thus conj_f induces a simplicial mapping $(\mathrm{conj}_f)_* : S_*(B\overline{G}) \to S_*(B\overline{G})$. Moreover left multiplication $\mu_f : G \to G$ is G-equivariant, so it also induces a simplicial mapping $(\mu_f)_* : S_*(B\overline{G}) \to S_*(B\overline{G})$ and we have $(\mu_f)_* = (\mathrm{conj}_f)_* : S_*(B\overline{G}) \to S_*(B\overline{G})$. Since $f \in G_\circ$ we know that μ_f is homotopic to $\mu_{\mathrm{id}} = \mathrm{id}$. Consequently there exists a chain homotopy H from $\mu_f : C_*(G;\mathbb{Z}) \to C_*(G;\mathbb{Z})$ to id in singular homology. Recall that H is constructed by considering $\Delta^p \times I \to G$, $(t,s) \mapsto \mu_{f(s)}g(t)$ together with a simplicial decomposition of $\Delta^p \times I$. Now the latter can be chosen to consist of affine (smooth) simplices, cf. figure 2.5 on page 28. So H maps smooth simplices to smooth simplices. Moreover, since μ_f is G-equivariant, H is G-equivariant too and hence descends to a homotopy $H : C_*(B\overline{G};\mathbb{Z}) \to C_{*+1}(B\overline{G};\mathbb{Z})$ from $(\mathrm{conj}_f)_* = (\mu_f)_*$ to $\mathrm{id} = (\mu_{\mathrm{id}})_*$. □

1.4.9. Lemma. *Let M_1, M_2 be two manifolds and let $G_i \subseteq \mathrm{Diff}_c^\infty(M_i)$, $i = 1, 2$. Consider the disjoint union $M := M_1 \sqcup M_2$ and $G := G_1 \times G_2 \subseteq \mathrm{Diff}_c^\infty(M)$. If $k \geq 1$ and $H_p(B\overline{G_i};\mathbb{Z}) = 0$ for all $1 \leq p < k$ then we have:*

$$H_k(B\overline{G};\mathbb{Z}) \cong H_k(B\overline{G_1};\mathbb{Z}) \oplus H_k(B\overline{G_2};\mathbb{Z})$$

Proof. Arguments similar to the one in the proof of lemma 1.4.8 show that the Eilenberg-Zilber theorem also holds for smooth simplices and descends to $C_*(B\overline{G};\mathbb{Z})$; that is, the complex $C_*(B\overline{G_1};\mathbb{Z}) \otimes C_*(B\overline{G_2};\mathbb{Z})$ computes $H_*(B\overline{G};\mathbb{Z})$. From homological algebra we obtain

$$\begin{aligned} H_k(B\overline{G};\mathbb{Z}) &\cong H_k(B\overline{G_1};\mathbb{Z}) \otimes H_0(B\overline{G_2};\mathbb{Z}) \oplus H_0(B\overline{G_1};\mathbb{Z}) \otimes H_k(B\overline{G_2};\mathbb{Z}) \\ &\cong H_k(B\overline{G_1};\mathbb{Z}) \oplus H_k(B\overline{G_2};\mathbb{Z}) \end{aligned}$$

since we have $H_0(B\overline{G_i};\mathbb{Z}) = \mathbb{Z}$, $H_p(B\overline{G_i};\mathbb{Z}) = 0$ for $1 \leq p < k$, and so there is no torsion involved. □

For a subset $K \subseteq M$ and $G \subseteq \mathrm{Diff}_c^\infty(M)$ we denote by G_K the subgroup of G consisting of the diffeomorphisms having support in K. Moreover if $\mathcal{U}$ is a set of subsets of M then $S^{\mathcal{U}}(B\overline{G})$ denotes the simplicial subcomplex of $S(B\overline{G})$ consisting of simplices which have support in one element of $\mathcal{U}$. Let $C_p^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ denote the p-chains of this complex and $H_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ its homology.

1.4.10. Lemma. *Let $\mathcal{U}$ be a set of sets in M. Then $\varinjlim_K H_*^{\mathcal{U}}(B\overline{G_K};\mathbb{Z}) = H_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$, where the limit is taken over all compact subsets $K \subseteq M$.*

Proof. Obviously we have $\varinjlim_K C_*^{\mathcal{U}}(B\overline{G_K};\mathbb{Z}) = C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ as chain complexes, and since the homology functor commutes with inductive limits we obtain the result. $\square$

Let $\mathrm{sd} : C_*(G;\mathbb{Z}) \to C_*(G;\mathbb{Z})$ be the barycentric subdivision and let $H : C_*(G;\mathbb{Z}) \to C_{*+1}(G;\mathbb{Z})$ be a natural chain homotopy satisfying $\mathrm{sd} - \mathrm{id} = \partial H + H\partial$, such that $H(\mathrm{id}_{\Delta^n})$ consists of affine simplices. Since sd and H are natural they are G-equivariant and support shrinking. Hence they induce mappings $\mathrm{sd} : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$, $H : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_{*+1}^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ for every set $\mathcal{U}$ of sets in M.

We equip the standard simplices $\Delta^p := \{(t_0, \cdots, t_p) : \sum_{i=0}^p t_i = 1\} \subseteq \mathbb{R}^{p+1}$ with the usual Riemannian metric. Further we define the disk bundle of Δ^p to be $D\Delta^p := \{X \in T\Delta^p : \|X\| \leq 1\}$. For a neighborhood of zero $\mathcal{E} \subseteq \mathfrak{X}_c(M)$ we now define

$$S_p^{\mathcal{E},\mathcal{U}}(B\overline{G}) := \{g \in S_p^{\mathcal{U}}(B\overline{G}) : (\delta^r g)(D\Delta^p) \subseteq \mathcal{E}\},$$

where we consider $\delta^r g : T\Delta^p \to \mathfrak{X}_c(M)$. Since every affine mapping $l : \Delta^p \to \Delta^q$ is a contraction, i.e. $\|Tl \cdot X\| \leq \|X\|$ for all $X \in T\Delta^p$, $C_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$ is a sub complex of $C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ and the mappings sd and H preserve this sub complex. Moreover there exist $0 < b_p < 1$ independent of $\mathcal{E}$ and $\mathcal{U}$ such that $\mathrm{sd}\left(C_p^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})\right) \subseteq C_p^{b_p\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$.

1.4.11. Lemma. *Let $\mathcal{U}$ be a set of sets in M and $\mathcal{E} \subseteq \mathfrak{X}_c(M)$ be a neighborhood of zero. Then for every simplex $g \in S_p^{\mathcal{U}}(B\overline{G})$ there exists $m \in \mathbb{N}$ such that $\mathrm{sd}^m(g) \in C_p^{\mathcal{U},\mathcal{E}}(B\overline{G};\mathbb{Z})$.*

Proof. We may assume that $\mathcal{E}$ is convex. Consider $\sigma := \delta^r g$ as smooth mapping $T\Delta^p \to \mathfrak{X}_c(M)$. $D\Delta^p$ is a compact set and so $\sigma(D\Delta^p) \subseteq \mathfrak{X}_c(M)$ is compact and therefore it is absorbed by $\mathcal{E}$, i.e. there exists $N > 0$ such that $\sigma(D\Delta^p) \subseteq N\mathcal{E}$, i.e. $g \in C_p^{N\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$. If we choose m such that $b_p^m N \leq 1$ we obtain $\mathrm{sd}^m(g) \in C_p^{b_p^m N\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z}) \subseteq C_p^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$. $\square$

The next proposition shows that the homology of $S(B\overline{G})$, more generally $S^{\mathcal{U}}(B\overline{G})$, can be computed via small simplices. Therefore it is sometimes called local homology of G.

1.4.12. Proposition. *For every set $\mathcal{U}$ of sets in M and for every neighborhood of zero $\mathcal{E} \subseteq \mathfrak{X}_c(M)$ the inclusion induces an isomorphism in homology $H_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z}) \cong H_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$.*

Proof. For every chain $g \in C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ let $m(g) \in \mathbb{N}_0$ denote the smallest integer such that $\mathrm{sd}^{m(g)}(g) \in C_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$. Notice that such an integer exists by lemma 1.4.11, $m(\partial g) \leq m(g)$ and for $g \in C_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$ we have $m(g) = 0$. Now we define $\bar{H} : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_{*+1}^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ by $\bar{H}(g) := \sum_{j=0}^{m(g)-1} H\,\mathrm{sd}^j(g)$. Using $\mathrm{sd} - \mathrm{id} = \partial H + H\partial$ we obtain

$$\begin{aligned}\partial\bar{H}(g) &= \sum_{0\leq j<m(g)} \mathrm{sd}^{j+1}(g) - \sum_{0\leq j<m(g)} \mathrm{sd}^j(g) - \sum_{0\leq j<m(g)} H\partial\,\mathrm{sd}^j(g)\\ &= \mathrm{sd}^{m(g)}(g) - g - \sum_{0\leq j<m(g)} H\,\mathrm{sd}^j(\partial g)\\ \bar{H}\partial(g) &= \sum_{0\leq j<m(\partial g)} H\,\mathrm{sd}^j(\partial g)\end{aligned}$$

and this yields:

$$g + \partial\bar{H}(g) + \bar{H}\partial(g) = \mathrm{sd}^{m(g)}(g) - \sum_{m(\partial g)\leq j<m(g)} H\,\mathrm{sd}^j(\partial g) \in C_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$$

Let $i : C_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ denote the inclusion and define $r : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_*^{\mathcal{E},\mathcal{U}}(B\overline{G};\mathbb{Z})$ by $r := \mathrm{id} + \partial\bar{H} + \bar{H}\partial$. Both are chain maps, $r \circ i = \mathrm{id}$ and $\bar{H}$ is a chain homotopy $i \circ r \simeq \mathrm{id}$. So r induces an inverse of i in homology. $\square$

1.5 The Torus

1.5.1. Definition. For $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in T^n := \mathbb{R}^n/\mathbb{Z}^n$ let $|k| := \sum_{i=1}^n |k_i| \in \mathbb{Z}$ and $\langle k, \gamma\rangle := \pi\big(\sum_{i=1}^n k_i\tilde{\gamma}_i\big) \in T^1$, where $\pi : \mathbb{R} \to T^1$ denotes the natural projection and $\tilde{\gamma} \in \mathbb{R}^n$ with $\pi(\tilde{\gamma}_i) = \gamma_i$. Obviously this doesn't depend on the choice of $\tilde{\gamma}$. Further for $\delta \in T^1$ we define $||\delta|| := \min_{n\in\mathbb{Z}} |\tilde{\delta} - n|$, where $\pi(\tilde{\delta}) = \delta$. This doesn't depend on the choice of $\tilde{\delta}$.

We say $\gamma \in T^n$ satisfies an diophantine equation iff there exists $\alpha > 0$ and $C > 0$ such that

$$\|\langle k, \gamma\rangle\| \geq \frac{C}{|k|^\alpha} \qquad \forall k \in \mathbb{Z}^n \setminus 0.$$

The existence of $\gamma \in T^n$ satisfying a diophantine equation is guaranteed by the following result due to Kintchine, see [Lan71].

1.5.2. Proposition. *The set $\gamma \in T^n$ satisfying a diophantine equation has measure 1 with respect to the Haar measure on T^n.*

The following theorem is due to M. R. Herman, see [Her73]. His proof uses a deep Nash-Moser-Sergeraert implicit function theorem, see [Ser72].

1.5.3. Theorem. *Let $\gamma \in T^n$ satisfy an diophantine equation, and consider the following smooth mapping:*

$$\begin{aligned} \Phi_\gamma : \mathrm{Diff}^\infty(T^n)_\circ \times T^n &\to \mathrm{Diff}^\infty(T^n)_\circ \\ (f, \lambda) &\mapsto R_\lambda \circ f^{-1} \circ R_\gamma \circ f, \end{aligned}$$

where R_β denotes rotation on T^n by $\beta \in T^n$. Then there exists a C^∞-open neighborhood U of $R_\gamma \in \mathrm{Diff}^\infty(T^n)_\circ$ and a smooth mapping $s : U \to \mathrm{Diff}^\infty(T^n)_\circ \times T^n$ satisfying $\Phi_\gamma \circ s = \mathrm{id}_U$ and $s(R_\gamma) = (\mathrm{id}, 0)$.

In the sequel we consider T^n as a subgroup of $\mathrm{Diff}^\infty(T^n)_\circ$ via $\beta \mapsto R_\beta$. Then we also have $\mathbb{R}^n \cong \widetilde{T^n} \subseteq \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ$. Further for every subset S of a group G we denote by $\mathcal{N}_G(S)$ the normal subgroup S generates in G.

1.5.4. Corollary. *We have $\mathcal{N}(\widetilde{T^n}) := \mathcal{N}_{\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ}(\widetilde{T^n}) = \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ$.*

Proof. By proposition 1.5.2 we can choose $\gamma \in T^n$ satisfying a diophantine equation. Using theorem 1.5.3 we obtain for every $f \in C^\infty\big((I, 0), (\mathrm{Diff}^\infty(T^n)_\circ, \mathrm{id})\big)$, sufficiently close to id, $s_1 = \mathrm{pr}_1 \circ s \circ R_\gamma f \in C^\infty\big((I, 0), (\mathrm{Diff}^\infty(T^n)_\circ, \mathrm{id})\big)$ and $s_2 = pr_2 \circ s \circ R_\gamma f \in C^\infty\big((I, 0), (T^n, 0)\big)$ such that $R_\gamma f = R_{s_2} s_1^{-1} R_\gamma s_1$, i.e. $f = R_{s_2}[R_\gamma^{-1}, s_1^{-1}]$. Let $\alpha : I \to T^n$ be a path connecting 0 with γ. Then $(s, t) \mapsto [R^{-1}_{\alpha(s+(1-s)t)}, s_1(t)^{-1}]$ is a homotopy relative endpoints from $t \mapsto [R^{-1}_{\alpha(t)}, s_1(t)^{-1}]$ to $t \mapsto [R_\gamma^{-1}, s_1(t)^{-1}]$, and we obtain:

$$f = R_{s_2}[t \mapsto R^{-1}_{\alpha(t)}, s_1^{-1}] \in \mathcal{N}(\widetilde{T^n}) \subseteq \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ$$

The f which are close to id generate $C^\infty\big((I, 0), (\mathrm{Diff}^\infty(T^n)_\circ, \mathrm{id})\big)$ as a group, since they contain an open neighborhood of id. Consequently $\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ \subseteq \mathcal{N}(\widetilde{T^n})$, the other inclusion is trivial. ∎

1.5.5. Corollary. $\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ$ *is perfect.*

Proof. We have to show $\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ = [\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ, \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ]$. Consider the action of $\mathrm{PGL}(2,\mathbb{C}) := \mathrm{GL}(2,\mathbb{C})/\mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C} \setminus 0$, on $\mathbb{CP}^1$ given by

$$\begin{aligned} \mathrm{PGL}(2,\mathbb{C}) \times \mathbb{CP}^1 &\to \mathbb{CP}^1 \\ (A,[v]) &\mapsto [Av] \end{aligned}$$

If one thinks of $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$ this becomes the well known action by Möbius transformations

$$\begin{aligned} M : \mathrm{PGL}(2,\mathbb{C}) \times \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} \\ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) &\mapsto \frac{az+b}{cz+d} \end{aligned}$$

Let $H^+ := \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane. It is easy to see that the subgroup of $\mathrm{PGL}(2,\mathbb{C})$ mapping H^+ onto H^+ is $\mathrm{PGL}(2,\mathbb{R}) := \mathrm{GL}(2,\mathbb{R})/\mathbb{R}^*$. Next recall, that the Möbius transformation $z \mapsto \frac{z-i}{z+i} = M_{\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)}(z)$ maps H^+ onto D^2. So $G := \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right) \mathrm{PGL}(2,\mathbb{R}) \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)^{-1} \subseteq \mathrm{PGL}(2,\mathbb{C})$ is the subgroup of Möbius transformations mapping D^2 onto D^2. Certainly G preserves $\partial D^2 = T^1$, and so we have found a subgroup $G \subseteq \mathrm{Diff}^\infty(T^1)_\circ$ which is isomorphic to $\mathrm{PGL}(2,\mathbb{R})$. If $\theta \in T^1 \subseteq \mathbb{C}$ we have $R_\theta = M_{\left(\begin{smallmatrix} \theta & 0 \\ 0 & 1 \end{smallmatrix}\right)}$ and so $T^1 \subseteq G \subseteq \mathrm{Diff}^\infty(T^1)_\circ$. Therefore we have $T^n \subseteq G^n \subseteq \mathrm{Diff}^\infty(T^n)_\circ$, where $G^n := G \times \cdots \times G$. It is well known, see [GOV93] for example, that $\tilde{G}$ is perfect, and therefore $\widetilde{G^n} \cong (\tilde{G})^n$ is perfect too. From corollary 1.5.4 we thus obtain:

$$\begin{aligned} \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ &= \mathcal{N}(\widetilde{T^n}) \subseteq \mathcal{N}(\widetilde{G^n}) = \mathcal{N}([\widetilde{G^n}, \widetilde{G^n}]) \\ &\subseteq \mathcal{N}([\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ, \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ]) = [\widetilde{\mathrm{Diff}}^\infty(T^n)_\circ, \widetilde{\mathrm{Diff}}^\infty(T^n)_\circ] \end{aligned}$$

The other inclusion is trivial. □

1.5.6. Corollary. $H_1\big(B\overline{\mathrm{Diff}}^\infty(T^n); \mathbb{Z}\big) = 0$ *and* $\mathrm{Diff}^\infty(T^n)_\circ$ *is perfect too.*

Proof. This follows immediately from proposition 1.4.5 and corollary 1.5.5. □

1.6 The Foliated Torus

Let $\mathcal{F}$ be a regular foliation of M. Then we denote by $\mathrm{Diff}^\infty_c(M,\mathcal{F})$ the group of compactly supported, leaf preserving diffeomorphisms of M. In this section we consider the torus $T^m \times T^n$ with the foliation $\mathcal{F}$ having $\{\mathrm{pt}\} \times T^n$ as leaves. Then we have

$$\mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F}) \cong C^\infty\big(T^m, \mathrm{Diff}^\infty(T^n)\big)$$

as groups, where the multiplication on the latter group is point wise. Recall that we considered T^n as subgroup of $\mathrm{Diff}^\infty(T^n)$, via $\alpha \mapsto R_\alpha$, the rotation by α, and define a subgroup:

$$H := C^\infty(T^m, T^n)_\circ \subseteq C^\infty\big(T^m, \mathrm{Diff}^\infty(T^n)\big)_\circ \cong \mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F})_\circ$$

In this situation we have the following generalization of corollary 1.5.4.

1.6.1. Corollary. $\mathcal{N}(\widetilde{H}) := \mathcal{N}_{\widetilde{\mathrm{Diff}}^\infty(T^m\times T^n,\mathcal{F})_\circ}(\widetilde{H}) = \widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ$

Proof. Choose $\gamma \in T^n$ satisfying a diophantine equation. Let

$$f \in C^\infty\big((I,0), (\mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F}), \mathrm{id})\big)$$

be sufficiently close to id and consider it as mapping

$$f \in C^\infty\big((I \times T^m, \{0\} \times T^m), (\mathrm{Diff}^\infty(T^n), \mathrm{id})\big).$$

From theorem 1.5.3 we obtain $s_1 = pr_1 \circ s \circ R_\gamma f \in C^\infty\big((I \times T^m, \{0\} \times T^m), (\mathrm{Diff}^\infty(T^n), \mathrm{id})\big)$ and $s_2 = pr_2 \circ s \circ R_\gamma f \in C^\infty\big((I \times T^m, \{0\} \times T^m), (T^n, 0)\big)$ such that $R_\gamma f = R_{s_2} s_1^{-1} R_\gamma s_1$, i.e. $f = R_{s_2}[R_\gamma^{-1}, s_1^{-1}]$. As we did in the proof of corollary 1.5.4 we choose a path α connecting 0 with γ in T^n and show that $[R_\gamma^{-1}, s_1^{-1}] \in C^\infty\big((I \times T^m, \{0\} \times T^m), (\mathrm{Diff}^\infty(T^n), \mathrm{id})\big)$ is homotopic relative $\partial(I \times T^m)$ to $[R_\alpha^{-1}, s_1^{-1}]$. So we get:

$$f = R_{s_2}[R_\alpha^{-1}, s_1^{-1}] \in \mathcal{N}(\widetilde{H}) \subseteq \widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ$$

since $R_{s_2}, R_\alpha \in \widetilde{H}$. This shows $\widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ \subseteq \mathcal{N}(\widetilde{H})$. The other inclusion is trivial. □

1.6.2. Lemma. *Let G be a n-dimensional, perfect Lie group. Then there exists an open neighborhood U of $e \in G$, $h_1, \dots, h_n \in G$ and smooth mappings $s_i : U \to G$, $1 \le i \le n$, such that*

$$g = [s_1(g), h_1][s_2(g), h_2] \cdots [s_n(g), h_n] \qquad \forall g \in U$$

and $s_i(e) = e$.

Proof. Let $h \in G$ and consider the mapping $\kappa_h : G \to G$, $\kappa_h(g) := [g, h]$. An easy calculation shows $T_e\kappa_h = \mathrm{id} - \mathrm{Ad}(h)$. For $h_1, \dots, h_n \in G$ we consider the mapping:

$$\kappa_{(h_1,\dots,h_n)} : G^n \to G \qquad (g_1, \dots, g_n) \mapsto [g_1, h_1] \cdots [g_n, h_n]$$

Its derivative at $(e, \dots, e)$ is:

$$T_{(e,\dots,e)}\kappa_{(h_1,\dots,h_n)}(X_1, \dots, X_n) = (\mathrm{id} - \mathrm{Ad}(h_1))X_1 + \cdots + (\mathrm{id} - \mathrm{Ad}(h_n))X_n$$

It remains to show that we can choose h_i such that $T_{(e,\dots,e)}\kappa_{(h_1,\dots,h_n)} : \mathfrak{g}^n \to \mathfrak{g}$ is onto. Then everything follows from the implicit function theorem. Since $\mathfrak{g}$ is perfect we find $X_1, \dots, X_n \in \mathfrak{g}$ and $Y_1, \dots, Y_n \in \mathfrak{g}$ such that $[X_1, Y_1], \dots, [X_n, Y_n]$ is a basis of $\mathfrak{g}$. We have:

$$\tfrac{\partial}{\partial t_i}|_0\big(\mathrm{id} - \mathrm{Ad}(\exp(t_iY_i))\big)(X_i) = -[Y_i, X_i] = [X_i, Y_i]$$

If we choose $t_i \neq 0$ sufficiently small and let $h_i := \exp(t_iY_i)$ then $(\mathrm{id} - \mathrm{Ad}(h_i))(X_i)$ is a basis of $\mathfrak{g}$, i.e.

$$(\mathrm{id} - \mathrm{Ad}(h_1))(\mathfrak{g}) + \cdots + (\mathrm{id} - \mathrm{Ad}(h_n))(\mathfrak{g}) = \mathfrak{g}$$

and so $T_{(e,\dots,e)}\kappa_{(h_1,\dots,h_n)}$ is onto. □

1.6.3. Corollary. $\widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ$ *is perfect.*

Proof. In the proof of corollary 1.5.5 we constructed a finite dimensional, connected, perfect Lie group G^n with $T^n \subseteq G^n \subseteq \mathrm{Diff}^\infty(T^n)$. Consider now the subgroup:

$$K := C^\infty(T^m, G^n)_\circ \subseteq C^\infty\big(T^m, \mathrm{Diff}^\infty(T^n)\big)_\circ \cong \mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F})_\circ$$

Then we have $H \subseteq K \subseteq \mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F})_\circ$ and thus $\widetilde{H} \subseteq \widetilde{K} \subseteq \widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ$. We claim that $\widetilde{K}$ is perfect. Indeed we have

$$\widetilde{K} \cong C^\infty\big((I,0), C^\infty(T^m, G^n)\big)/\sim_{\partial I} \cong C^\infty\big((I \times T^m, \{0\} \times T^m), (G^n, e)\big)/\sim_{\partial(I \times T^m)}$$

and it follows from lemma 1.6.2 that the latter is perfect. From corollary 1.6.1 we now obtain:

$$\begin{aligned}\widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ &= \mathcal{N}(\widetilde{H}) \subseteq \mathcal{N}(\widetilde{K}) = \mathcal{N}([\widetilde{K}, \widetilde{K}]) \\ &\subseteq \mathcal{N}\big([\widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ, \widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ]\big) \\ &= [\widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ, \widetilde{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F})_\circ]\end{aligned}$$

The other inclusion is trivial. □

Corollary 1.6.3 is due to T. Rybicki, see [Ryb95a]. He proved a slightly stronger (foliated) version of theorem 1.5.3 to show it.

1.6.4. Corollary. *$H_1\big(B\overline{\mathrm{Diff}}^\infty(T^m \times T^n, \mathcal{F}); \mathbb{Z}\big) = 0$ and $\mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F})_\circ$ is perfect too.*

Proof. This is an immediate consequence of proposition 1.4.5 and corollary 1.6.3. □

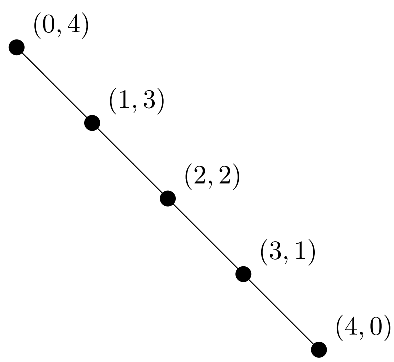
1.6.5. Remark. Notice that $\mathrm{Diff}^\infty(T^m \times T^n, \mathcal{F})$ is not simple. For example the subgroup fixing the points of one distinguished leaf is a proper, normal subgroup.

2. Fragmentation and Deformation

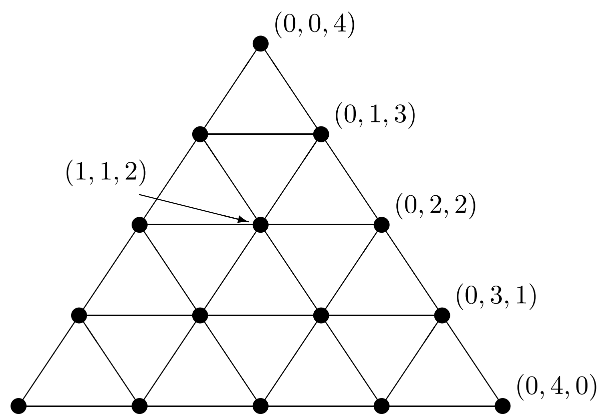
2.1 Modified Subdivision

For $N \in \mathbb{N}$ let $D_N^n := \{(m_0, \dots, m_n) \in \mathbb{N}_0^{n+1} : \sum_{i=0}^n m_i = N\}$. See figure 2.1 on page 19 for low dimensional special cases. If $\lambda \in \Delta^{N-1}$ and $m \in D_N^n$ we define $\tau_m^\lambda \in \Delta^n$ by

Figure 2.1: The index set D_N^n



The bold dots represent the elements of D_4^1. The line is just for convenience, it has no interpretation.



The bold dots represent the elements of D_4^2. The lines are just for convenience, they have no interpretation.

$$\tau_m^\lambda := (\underbrace{\lambda_0 + \dots + \lambda_{m_0-1}}_{m_0}, \underbrace{\lambda_{m_0} + \dots + \lambda_{m_0+m_1-1}}_{m_1}, \dots, \underbrace{\lambda_{m_0+\dots+m_{n-1}} + \dots + \lambda_{m_0+\dots+m_n-1}}_{m_n})$$

See figure 2.2 on page 20 for low dimensional special cases. Moreover we let

$$A_N^n := \{(m, \pi) \in D_N^n \times \mathfrak{S}_n : m + f_{\pi(1)} + \dots + f_{\pi(j)} \in D_N^n \quad \forall 0 \le j \le n\}$$

where $f_i := e_i - e_{i-1}$ and $e_i \in \mathbb{R}^{n+1}$ denote the unit vectors, i.e. the vertexes of Δ^n (cf. figure 2.3 on page 20). If $(m, \pi) \in A_N^n$ we define $\tau_{(m,\pi)}^\lambda : \Delta^n \to \Delta^n$ by $\tau_{(m,\pi)}^\lambda(e_j) := \tau_{m+f_{\pi(1)}+\dots+f_{\pi(j)}}^\lambda$ for $0 \le j \le n$ and extend it affinely. This is shown in two special cases in figure 2.4 on page 21.

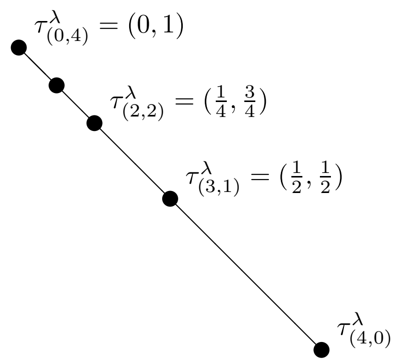
2.1.1. Lemma. *Let $B_N^n := A_N^n \times \{0, \dots, n\}$, $C_N^n := A_N^n \times \{0, \dots, n+1\}$ and for $\alpha = (m, \pi, i) \in C_N^n$ we define $\delta_\alpha := \delta_i$, $\operatorname{sgn}(\alpha) := (-1)^i \operatorname{sgn}(\pi)$, $\tau_\alpha^\lambda := \tau_{(m,\pi)}^\lambda$.*

Then there exist injective mappings $b_N^n : C_N^n \to B_N^{n+1}$ and $c_N^n : B_N^n \setminus b_N^{n-1}(C_N^{n-1}) \to B_N^n \setminus b_N^{n-1}(C_N^{n-1})$ such that

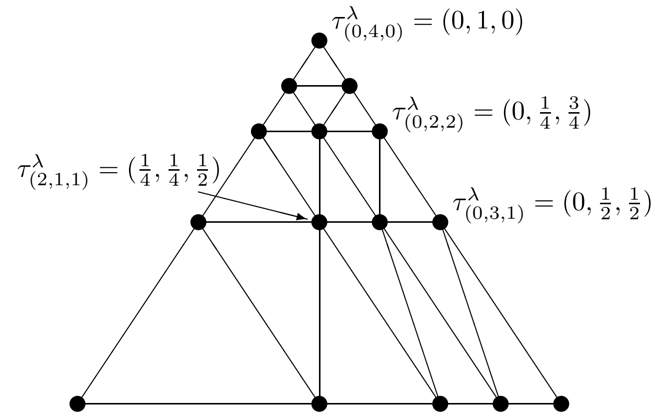
$$\delta_\alpha \circ \tau_\alpha^\lambda = \tau_{b_N^n(\alpha)}^\lambda \circ \delta_{b_N^n(\alpha)} : \Delta^n \to \Delta^{n+1} \quad \forall \alpha \in C_N^n \tag{2.1}$$

$$\tau_\alpha^\lambda \circ \delta_\alpha = \tau_{c_N^n(\alpha)}^\lambda \circ \delta_{c_N^n(\alpha)} : \Delta^{n-1} \to \Delta^n \quad \forall \alpha \in B_N^n \setminus b_N^{n-1}(C_N^{n-1}) \tag{2.2}$$

Figure 2.2: The points $\tau^{\lambda}_{(m,\pi)} \in \Delta^n$

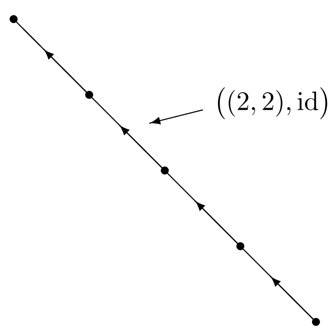


The bold dots represent the elements $\tau^{\lambda}_{(4,0)}, \tau^{\lambda}_{(3,1)}, \ldots \in \Delta^1$ as indicated for $\lambda = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}) \in \Delta^{4-1} = \Delta^3$.

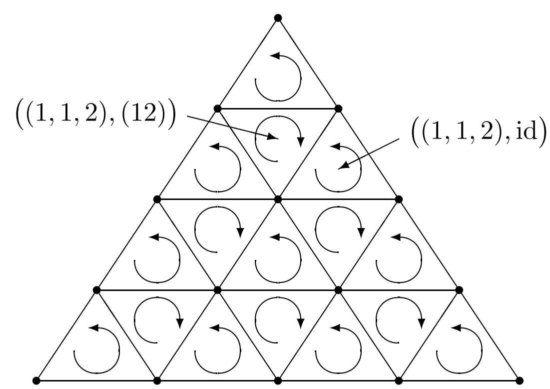


The bold dots represent the elements $\tau^{\lambda}_{(4,0,0)}, \tau^{\lambda}_{(3,1,0)}, \ldots \in \Delta^2$ as indicated for $\lambda = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}) \in \Delta^{4-1} = \Delta^3$.

Figure 2.3: The index set A^n_N



The oriented lines represent the elements of A^1_4. Note that $\big((0,4), \mathrm{id}\big)$ is not contained in A^1_4 as the picture shows.



The oriented 2-simplices represent the elements of A^2_4. Note that $\big((4,0,0), \mathrm{id}\big) \in A^2_4$, but $\big((4,0,0),(12)\big) \notin A^2_4$.

holds for every $\lambda \in \Delta^{N-1}$. *Moreover we have* $c^n_N \circ c^n_N = \mathrm{id}$, $\mathrm{sgn} \circ b^n_N = \mathrm{sgn}$ *and* $\mathrm{sgn} \circ c^n_N = -\,\mathrm{sgn}$. *Especially* c^n_N *has no fixed points.*

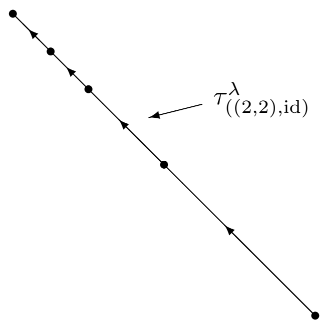
Proof. We regard $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$, i.e. $\pi \in \mathfrak{S}_n$ iff $\pi(n+1) = n+1$. Next we define

$$\begin{aligned} b^n_N : \mathbb{Z}^{n+1} \times \mathfrak{S}_n \times \{0, \ldots, n+1\} &\mapsto \mathbb{Z}^{n+2} \times \mathfrak{S}_{n+1} \times \{0, \ldots, n+1\} \\ (m, \pi, 0) &\mapsto (\delta_0 m - f_1, (1 \cdots n+1) \circ \pi \circ (1 \cdots n+1)^{-1}, 0) \\ (m, \pi, i) &\mapsto (\delta_i m, (i \cdots n+1) \circ \pi \circ (\pi^{-1}(i) \cdots n+1)^{-1}, \pi^{-1}(i)) \end{aligned}$$

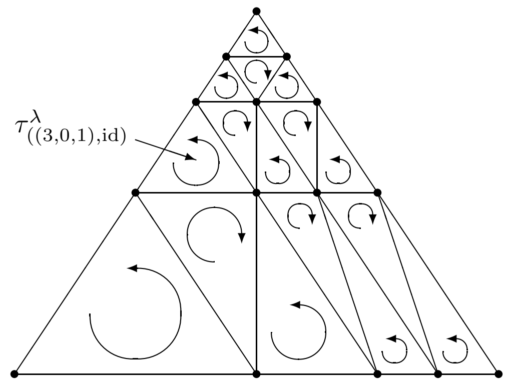
and

$$\begin{aligned} c^n_N : \mathbb{Z}^{n+1} \times \mathfrak{S}_n \times \{0, \ldots, n\} &\to \mathbb{Z}^{n+1} \times \mathfrak{S}_n \times \{0, \ldots, n\} \\ (m, \pi, 0) &\mapsto (m + f_{\pi(1)}, \pi \circ (1 \cdots n), n) \\ (m, \pi, i) &\mapsto (m, \pi \circ (i\, i+1), i) \\ (m, \pi, n) &\mapsto (m - f_{\pi(n)}, \pi \circ (1 \cdots n)^{-1}, 0) \end{aligned}$$

Figure 2.4: The simplices $\tau^{\lambda}_{(m,\pi)}$



The oriented lines represent the image of the singular simplices $\tau^{\lambda}_{((4,0),\mathrm{id})}$, $\tau^{\lambda}_{((3,1),\mathrm{id})}$, ... in Δ^1, again for $\lambda = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}) \in \Delta^{4-1} = \Delta^3$.



The oriented triangles represent the image of the simplices $\tau^{\lambda}_{((4,0,0),\mathrm{id})}$, $\tau^{\lambda}_{((3,1,0),\mathrm{id})}$, ... in Δ^2, again for $\lambda = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}) \in \Delta^{4-1} = \Delta^3$.

It is clear that $c^n_N \circ c^n_N = \mathrm{id}$, especially c^n_N is injective. It is easy to see that b^n_N is injective. Moreover one readily checks that $\mathrm{sgn} \circ b^n_N = \mathrm{sgn}$ and $\mathrm{sgn} \circ c^n_N = -\mathrm{sgn}$.

Next we check $b^n_N(C^n_N) \subseteq B^{n+1}_N$. If $(m, \pi, 0) \in C^n_N$ then $m_0 \geq 1$ and hence $\delta_0 m - f_1 \in D^{n+1}_N$. Moreover if $\pi' := (1 \cdots n+1) \circ \pi \circ (1 \cdots n+1)^{-1}$ we obtain for $1 \leq j \leq n+1$

$$\begin{aligned} \delta_0 m - f_1 + f_{\pi'(1)} + \cdots + f_{\pi'(j)} &= \delta_0 m - f_1 + f_1 + f_{\pi(1)+1} + \cdots + f_{\pi(j-1)+1} \\ &= \delta_0(m + f_{\pi(1)} + \cdots f_{\pi(j-1)}) \in D^{n+1}_N \end{aligned}$$

and so $b^n_N(m, \pi, 0) \in B^{n+1}_N$. Now let $(m, \pi, i) \in C^n_N$ with $1 \leq i \leq n+1$. One checks

$$\begin{aligned} &\delta_i f_{\pi(j)} = f_{\pi'(j)} && 1 \leq j < \pi^{-1}(i) \\ &\delta_i f_{\pi(j)} = f_{\pi'(j+1)} + f_{\pi'(j)} && j = \pi^{-1}(i) \\ &\delta_i f_{\pi(j)} = f_{\pi'(j+1)} && \pi^{-1}(i) < j \leq n \end{aligned}$$

where $\pi' := (i \cdots n+1) \circ \pi \circ (\pi^{-1}(i) \cdots n+1)^{-1}$. It follows immediately from these equations that $\delta_i m + f_{\pi'(1)} + \cdots + f_{\pi'(j)} \in D^{n+1}_N$ for $j \neq \pi^{-1}(i)$. Moreover

$$\delta_i m + f_{\pi'(1)} + \cdots f_{\pi'(\pi^{-1}(i))} = \delta_i(m + f_{\pi(1)} + \cdots + f_{\pi(\pi^{-1}(i)-1)}) + f_i \in D^{n+1}_N$$

for $(m + f_{\pi(1)} + \cdots + f_{\pi(\pi^{-1}(i)-1)})_{i-1} \geq 1$ since $m + f_{\pi(1)} + \cdots + f_{\pi(\pi^{-1}(i)-1)} + f_i \in D^n_N$. So we have shown that $b^n_N(m, \pi, i) \in B^{n+1}_N$.

The most difficult part is to show that $c^n_N(B^n_N \setminus b^{n-1}_N(C^{n-1}_N)) \subseteq B^n_N \setminus b^{n-1}_N(C^{n-1}_N)$. We do this by showing

$$c^n_N(B^n_N) \cap b^{n-1}_N(C^{n-1}_N) = \emptyset \quad \text{and} \quad B^n_N \setminus (c^n_N)^{-1}(B^n_N) \subseteq b^{n-1}_N(C^{n-1}_N).$$

We first show $c^n_N(B^n_N) \cap b^{n-1}_N(C^{n-1}_N) = \emptyset$: Let $(m, \pi, 0) \in B^n_N$ and suppose that there exists $(m', \pi', i') \in C^{n-1}_N$ such that

$$b^{n-1}_N(m', \pi', i') = c^n_N(m, \pi, 0) = (m + f_{\pi(1)}, \pi \circ (1 \cdots n), n)$$

Then $i' = n$, $\pi(1) = n$, $\delta_n m' = m + f_n$ but the latter is a contradiction. Next let $(m, \pi, n) \in B_N^n$ and suppose there exists $(m', \pi', i') \in C_N^{n-1}$ such that

$$b_N^{n-1}(m', \pi', i') = c_N^n(m, \pi, n) = (m - f_{\pi(n)}, \pi \circ (1 \cdots n)^{-1}, 0)$$

Then $i' = 0$, $\pi(n) = 1$, $m - f_1 = \delta_0 m' - f_1$ hence $m = \delta_0 m'$ but this contradicts the fact that $(m, \pi) \in A_N^n$. At last let $(m, \pi, i) \in B_N^n$ and suppose there exists $(m', \pi', i') \in C_N^{n-1}$ such that

$$b_N^{n-1}(m', \pi', i') = c_N^n(m, \pi, i) = (m, \pi \circ (i, i+1), i)$$

Then $\pi'^{-1}(i') = i$, $\pi(i+1) = i'$ and $\pi(i) = i' + 1$. So

$$0 \le (m + f_{\pi(1)} + \cdots + f_{\pi(i-1)} + f_{i'+1})_{i'} = m_{i'} - 1$$

and we obtain $m_{i'} \ge 1$, but this contradicts $m = \delta_{i'} m'$.

Next we show $B_N^n \setminus (c_N^n)^{-1}(B_N^n) \subseteq b_N^{n-1}(C_N^{n-1})$: Let $(m, \pi, 0) \in B_N^n$ and suppose $c_N^n(m, \pi, 0) = (m + f_{\pi(1)}, \pi \circ (1 \cdots n), n) \notin B_N^n$. We show $\pi(1) = 1$. Suppose conversely $\pi(1) \ne 1$. For $1 \le j \le n$ we have $m + f_{\pi(1)} + \cdots f_{\pi(j)} \in D_N^n$ hence

$$m + e_n - e_0 + e_{\pi(1)} - e_{\pi(1)-1} = m + f_{\pi(1)} + \cdots + f_{\pi(n)} + f_{\pi(1)} \notin D_N^n$$

and thus $m_{\pi(1)-1} = 0$. On the other hand we have $0 \le (m + f_{\pi(1)})_{\pi(1)-1} = m_{\pi(1)-1} - 1$ a contradiction. Since $m + e_n - e_0 \in D_N^n$ and $m + e_n - e_0 + e_1 - e_0 \notin D_N^n$ we obtain $m_0 = 1$. So we define $m' := \mathrm{pr}_0(m + f_1) \in D_N^{n-1}$ and $\pi' := (1 \cdots n)^{-1} \circ \pi \circ (1 \cdots n) \in \mathfrak{S}_{n-1}$, where $\mathrm{pr}_i : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ forgets about the i-th component. Obviously we have $b_N^{n-1}(m', \pi', 0) = (m, \pi, 0)$ and it remains to show that $(m', \pi', 0) \in C_N^{n-1}$, i.e. $(m', \pi') \in A_N^{n-1}$. But for $0 \le j \le n-1$ we have

$$\begin{aligned} m' + f_{\pi'(1)} + \cdots + f_{\pi'(j)} &= m' + f_{\pi(2)-1} + \cdots + f_{\pi(j+1)-1} \\ &= \mathrm{pr}_0(m + f_1 + f_{\pi(2)} + \cdots + f_{\pi(j+1)}) \in D_N^{n-1} \end{aligned}$$

since $(m + f_1 + f_{\pi(2)} + \cdots + f_{\pi(j+1)})_0 = 0$.

Next consider the case $(m, \pi, n) \in B_N^n$ and suppose $c_N^n(m, \pi, n) = (m - f_{\pi(n)}, \pi \circ (1 \cdots n)^{-1}, 0) \notin B_N^n$. For $1 \le j \le n-1$ we have $m - f_{\pi(n)} + f_{\pi(n)} + f_{\pi(1)} + \cdots + f_{\pi(j)} \in D_N^n$ and so $m - f_{\pi(n)} \notin D_N^n$ hence $m_{\pi(n)} = 0$. We have $m - f_{\pi(n)} + e_n - e_0 \in D_N^n$ and by looking at the $\pi(n)$-th coordinate we get $\pi(n) = n$ and $m_n = 0$. We define $\pi' := \pi \in \mathfrak{S}_{n-1}$ and $m' := \mathrm{pr}_n(m) \in D_N^{n-1}$. Obviously we have $b_N^{n-1}(m', \pi', n) = (m, \pi, n)$ and so it remains to show $(m', \pi', n) \in C_N^{n-1}$, i.e. $(m', \pi') \in A_N^{n-1}$. But for $0 \le j \le n-1$ we have

$$m' + f_{\pi'(1)} + \cdots + f_{\pi'(j)} = \mathrm{pr}_n(m + f_{\pi(1)} + \cdots + f_{\pi(j)}) \in D_N^{n-1}$$

since $(m + f_{\pi(1)} + \cdots + f_{\pi(j)})_n = 0$.

Consider now the last case $(m, \pi, i) \in B_N^n$ and suppose $c_N^n(m, \pi, i) = (m, \pi \circ (i, i+1), i) \notin B_N^n$. One easily sees $m + f_{\pi(1)} + \cdots + f_{\pi(i-1)} + f_{\pi(i+1)} \notin D_N^n$, $m + f_{\pi(1)} + \cdots + f_{\pi(i-1)} + f_{\pi(i+1)} + f_{\pi(i)} \in D_N^n$ and therefore we obtain $\pi(i+1) - 1 = \pi(i)$ and $m_{\pi(i)} = 0$. We let $\pi' := (\pi(i) \cdots n)^{-1} \circ \pi \circ (i \cdots n) \in \mathfrak{S}_{n-1}$, $m' := \mathrm{pr}_{\pi(i)}(m) \in D_N^{n-1}$ and $i' := \pi(i)$. Then we obviously have $b_N^{n-1}(m', \pi', i') = (m, \pi, i)$ and it remains to show $(m', \pi', i') \in C_N^{n-1}$, i.e. $(m', \pi') \in A_N^{n-1}$, but this follows easily from the equations

$$\begin{aligned} \mathrm{pr}_{\pi(i)}(f_{\pi(j)}) &= f_{\pi'(j)} & 1 \le j < i \\ \mathrm{pr}_{\pi(i)}(f_{\pi(j)}) &= f_{\pi'(j-1)} & i+1 < j \le n \\ \mathrm{pr}_{\pi(i)}(f_{\pi(i)} + f_{\pi(i+1)}) &= f_{\pi'(i)} \end{aligned}$$

and the fact that the $\pi(i)$-th coordinates of $f_{\pi(j)}$ for $j \neq i, i+1$ and $f_{\pi(i)} + f_{\pi(i+1)}$ are 0.

Since the mappings in equations (2.1) and (2.2) are affine it suffices to check them on the vertexes $e_j \in \Delta^n$, resp. $e_j \in \Delta^{n-1}$, but these are now easy calculations. □

2.1.2. Definition. For $N \in \mathbb{N}$ we let $\lambda := (\frac{1}{N}, \dots, \frac{1}{N}) \in \Delta^{N-1}$ and

$$d_N^n := \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)\tau_{(m,\pi)}^\lambda \in C_n(\Delta^n; \mathbb{Z})$$

For any topological space we define the modified subdivision

$$\operatorname{sd}_N : C_*(X; \mathbb{Z}) \to C_*(X; \mathbb{Z})$$

on a n-simplex $\sigma : \Delta^n \to X$ by $\operatorname{sd}_N(\sigma) := \sigma_*(d_N^n)$ and extend it $\mathbb{Z}$-linear.

2.1.3. Theorem. *For any topological space X the modified subdivision $\operatorname{sd}_N : C_*(X; \mathbb{Z}) \to C_*(X; \mathbb{Z})$ is a chain map natural in X and it is natural chain homotopic to the identity.*

Proof. Let $f : X \to Y$ be continuous and let $\sigma : \Delta^n \to X$ be a simplex. Then we have

$$(\operatorname{sd}_N \circ f_*)(\sigma) = \operatorname{sd}_N(f \circ \sigma) = (f \circ \sigma)_* d_N^n = f_* \sigma_* d_N^n = (f_* \circ \operatorname{sd}_N)(\sigma)$$

and so sd_N is natural.

The most difficult part is to show that sd_N is a chain map. So let $\sigma : \Delta^n \to X$ be a simplex. Then $(\partial \circ \operatorname{sd}_N)(\sigma) = \partial \sigma_* d_N^n = \sigma_* \partial d_N^n$ and

$$(\operatorname{sd}_N \circ \partial)(\sigma) = \sum_{i=0}^n (-1)^i \operatorname{sd}_N(\sigma \circ \delta_i) = \sigma_* \Big(\sum_{i=0}^n (-1)^i \operatorname{sd}_N(\delta_i) \Big) = \sigma_* \Big(\sum_{i=0}^n (-1)^i (\delta_i)_* d_N^{n-1} \Big)$$

so it remains to show $\partial d_N^n = \sum_{i=0}^n (-1)^i (\delta_i)_* d_N^{n-1} \in C_{n-1}(\Delta^n; \mathbb{Z})$. Using lemma 2.1.1 we obtain

$$\partial d_N^n = \sum_{i=0}^n (-1)^i \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)\tau_{(m,\pi)}^\lambda \circ \delta_i = \sum_{\alpha\in B_N^n} \operatorname{sgn}(\alpha)\tau_\alpha^\lambda \circ \delta_\alpha$$

where $\lambda = (\frac{1}{N}, \dots, \frac{1}{N}) \in \Delta^{N-1}$, and

$$\begin{aligned} \sum_{i=0}^n (-1)^i (\delta_i)_* d_N^{n-1} &= \sum_{i=0}^n (-1)^i \sum_{(m,\pi)\in A_N^{n-1}} \operatorname{sgn}(\pi)\delta_i \circ \tau_{(m,\pi)}^\lambda = \sum_{\alpha\in C_N^{n-1}} \operatorname{sgn}(\alpha)\delta_\alpha \circ \tau_\alpha^\lambda \\ &= \sum_{\alpha\in C_N^{n-1}} \operatorname{sgn}(b_N^{n-1}(\alpha))\tau_{b_N^{n-1}(\alpha)}^\lambda \circ \delta_{b_N^{n-1}(\alpha)} \\ &= \sum_{\alpha\in b_N^{n-1}(C_N^{n-1})} \operatorname{sgn}(\alpha)\tau_\alpha^\lambda \circ \delta_\alpha \end{aligned}$$

For the last equation we used the fact that b_N^{n-1} is injective. It now remains to show that

$$\sum_{\alpha\in B_N^n \setminus b_N^{n-1}(C_N^{n-1})} \operatorname{sgn}(\alpha)\tau_\alpha^\lambda \circ \delta_\alpha = 0$$

but this follows from the fact that c_N^n is an involution on the set $B_N^n \setminus b_N^{n-1}(C_N^{n-1})$ without fixed points, equation (2.2) and $\operatorname{sgn} \circ c_N^n = -\operatorname{sgn}$.

The proof that sd_N is natural chain homotopic to the identity is standard and uses the method of acyclic models. Notice first that $\mathrm{sd}_N = \mathrm{id} : C_0(\Delta^0;\mathbb{Z}) \to C_0(\Delta^0;\mathbb{Z})$. So we define $H_j^N := 0 : C_j(X;\mathbb{Z}) \to C_{j+1}(X;\mathbb{Z})$ for $j \leq 0$. Then H_j^N is clearly natural and we have $\partial H_j^N + H_{j-1}\partial = \mathrm{sd}_N - \mathrm{id}$ for $j \leq 0$.

As usual we proceed by induction on j. Suppose we already have $H_j^N : C_j(X;\mathbb{Z}) \to C_{j+1}(X;\mathbb{Z})$, natural in X satisfying $\partial H_j^N + H_{j-1}^N\partial = \mathrm{sd}_N - \mathrm{id}$. Using the last equation we obtain

$$\partial(\mathrm{sd}_N - \mathrm{id} - H_j^N\partial) = \partial\,\mathrm{sd}_N - \partial - (\mathrm{sd}_N - \mathrm{id} - H_{j-1}^N\partial)\partial = 0$$

for sd_N is a chain map. Especially for $\mathrm{id}_{\Delta^{j+1}} \in C_{j+1}(\Delta^{j+1};\mathbb{Z})$ we have $\partial(\mathrm{sd}_N - \mathrm{id} - H_j^N\partial)(\mathrm{id}_{\Delta^{j+1}}) = 0$ and since $H_{j+1}(\Delta^{j+1};\mathbb{Z}) = 0$ there exists $s_{j+1}^N \in C_{j+2}(\Delta^{j+1};\mathbb{Z})$ such that $\partial s_{j+1}^N = (\mathrm{sd}_N - \mathrm{id} - H_j^N\partial)(\mathrm{id}_{\Delta^{j+1}})$. Notice that we can choose s_{j+1}^N such that it consists of affine simplices. We now define $H_{j+1}^N : C_{j+1}(X;\mathbb{Z}) \to C_{j+2}(X;\mathbb{Z})$ by $H_{j+1}^N(\sigma) := \sigma_* s_{j+1}^N$. The latter is natural, for given a continuous map $f : X \to Y$ we get

$$(H_{j+1}^N \circ f_*)(\sigma) = (f\circ\sigma)_* s_{j+1}^N = f_*\sigma_* s_{j+1}^N = (f_* \circ H_{j+1}^N)(\sigma).$$

At last we compute

$$\begin{aligned}(\partial H_{j+1}^N + H_j^N\partial)(\sigma) &= \partial\sigma_* s_{j+1}^N + H_j^N\partial\sigma_*\mathrm{id}_{\Delta^{j+1}}\\ &= \sigma_*(\mathrm{sd}_N - \mathrm{id} - H_j^N\partial)(\mathrm{id}_{\Delta^{j+1}}) + \sigma_* H_j^N\partial\mathrm{id}_{\Delta^{j+1}}\\ &= (\mathrm{sd}_N - \mathrm{id})\sigma_*\mathrm{id}_{\Delta^{j+1}} = (\mathrm{sd}_N - \mathrm{id})(\sigma)\end{aligned}$$

Notice that for the last computation it was essential that sd_N is a natural chain map. □

2.1.4. Corollary. *Let $G \subseteq \mathrm{Diff}_c^\infty(M)$ be a subgroup with Lie algebra $\mathfrak{g}$, in the sense that $g \in C^\infty\big((I,0),(G,\mathrm{id})\big) \subseteq C^\infty\big((I,0),(\mathrm{Diff}_c^\infty(M),\mathrm{id})\big)$ if and only if $\delta^r g \in \Omega^1(I;\mathfrak{g}) \subseteq \Omega^1(I;\mathfrak{X}_c(M))$ For any set $\mathcal{U}$ of sets in M the modified subdivision induces a mapping $\mathrm{sd}_N : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ which is homotopic to the identity. Moreover if a simplex in $S_n(B\overline{G})$ is considered as 1-form $\sigma \in \Omega^1(\Delta^n;\mathfrak{g})$ then*

$$\mathrm{sd}_N(\sigma) = \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda)^*\sigma$$

where $\lambda = (\frac{1}{N},\dots,\frac{1}{N})$, and if the simplex is considered as foliation $\mathcal{F}$ on $\Delta^n \times M$ we have

$$\mathrm{sd}_N(\mathcal{F}) = \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda \times \mathrm{id}_M)^*\mathcal{F}$$

Proof. Since $\mathrm{sd}_N : C_*(G;\mathbb{Z}) \to C_*(G;\mathbb{Z})$ is natural it is G-equivariant and hence induces $\mathrm{sd}_N : C_*(B\overline{G};\mathbb{Z}) \to C_*(B\overline{G};\mathbb{Z})$. Again by the naturality sd_N is support shrinking and hence we obtain $\mathrm{sd}_N : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$. Similarly the homotopy induces a mapping $H^N : C_*^{\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_{*+1}^{\mathcal{U}}(B\overline{G};\mathbb{Z})$.

If $\tau : \Delta^n \to \Delta^n$ and $g \in C^\infty(\Delta^n, G)$ then $\delta^r(g_*\tau) = \delta^r(\tau^* g) = \tau^*\delta^r g$ and thus

$$\delta^r(\mathrm{sd}_N(g)) = \delta^r(g_* d_N^n) = \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)\delta^r(g_*\tau_{(m,\pi)}^\lambda) = \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda)^*(\delta^r g)$$

which yields the first description. Denote the foliation corresponding to g by $\mathcal{F}(g)$. It has leaves $\{(t, g(t)(x)) : t \in \Delta^n\}$. Now the mapping $\tau \times \mathrm{id}_M$ maps the leaves of $\mathcal{F}(g \circ \tau)$ to the leaves of $\mathcal{F}(g)$ and so $(\tau \times \mathrm{id}_M)^* \mathcal{F}(g) = \mathcal{F}(g_* \tau)$. Hence we have

$$\mathcal{F}(\mathrm{sd}_N(g)) = \mathcal{F}(g_* d_N^n) = \sum_{(m,\pi) \in A_N^n} \mathrm{sgn}(\pi) \mathcal{F}(g_* \tau^\lambda_{(m,\pi)}) = \sum_{(m,\pi) \in A_N^n} \mathrm{sgn}(\pi) (\tau^\lambda_{(m,\pi)} \times \mathrm{id}_M)^* \mathcal{F}(g)$$

which yields the second description. □

2.1.5. Remark. We could have proven lemma 1.4.11 and proposition 1.4.12 as well by using sd_N for some fixed $N \geq 2$ instead of sd. In this case the constants b_p used in the proof of lemma 1.4.11 would be $\frac{\sqrt{2}}{N}$, independent of p.

2.2 Fragmentation and Deformation for Modular Groups

2.2.1. Definition. A Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{X}_c(M)$ is called *modular* if it is a C^0-closed $C^\infty(M;\mathbb{R})$-submodule of $\mathfrak{X}_c(M)$. A subgroup $G \subseteq \mathrm{Diff}_c^\infty(M)$ is called modular with Lie algebra $\mathfrak{g}$, if $\mathfrak{g}$ is modular and $\mathfrak{g}$ is the Lie algebra of G in the following sense:

$$g \in C^\infty(I, G) \qquad \Leftrightarrow \qquad \delta^r g \in \Omega^1(I; \mathfrak{g})$$

for $g \in C^\infty\big((I, 0), (\mathrm{Diff}_c^\infty(M); \mathrm{id})\big)$.

Examples for modular Lie algebras are $\mathfrak{X}_c(M)$ and $\mathfrak{X}_c(M, \mathcal{F})$, where $\mathcal{F}$ is a foliation on M (possibly with non-constant rank, see [Ste74] and [Ste80]) and $\mathfrak{X}_c(M, \mathcal{F})$ denotes the compactly supported vector fields that are tangential to $\mathcal{F}$. Moreover if $\mathfrak{g}$ is modular and $K \subseteq M$ is a fixed compact set then $\mathfrak{g}_K := \{X \in \mathfrak{g} : \mathrm{supp}(X) \subseteq K\}$ is again modular and so further examples are $\mathfrak{X}_K(M)$, $\mathfrak{X}_K(M, \mathcal{F})$. The corresponding modular groups are $\mathrm{Diff}_c^\infty(M)$, $\mathrm{Diff}_c^\infty(M, \mathcal{F})$, where the latter denotes the group of leave preserving diffeomorphisms.

2.2.2. Lemma. *Let $V \subseteq \mathfrak{X}_c(M)$ be a C^0-closed $C^\infty(M;\mathbb{R})$-submodule. For $x \in M$ we let $E_x := \{X(x) : X \in V\} \subseteq T_x M$. Then $V = \{X \in \mathfrak{X}_c(M) : X(x) \in E_x \quad \forall x \in M\}$.*

Proof. One inclusion ($\subseteq$) is trivial, we show the other one. So let $X \in \mathfrak{X}_c(M)$ such that $X(x) \in E_x$ for all $x \in M$ and suppose conversely $X \notin V$. Since V is C^0-closed there exists $\varepsilon \in C^\infty(M; \mathbb{R}^+)$ with:

$$Y \in \mathfrak{X}_c(M) : \|Y(y) - X(y)\| \leq \varepsilon(y) \quad \forall y \in M \quad \Rightarrow \quad Y \notin V$$

For all $x \in M$ we choose $Y_x \in V$ with $X(x) = Y_x(x)$ and a neighborhood U_x of x such that $\|Y_x(y) - X(y)\| \leq \varepsilon(y)$ for all $y \in U_x$. Since the support of X is compact we find $x_1, \ldots, x_n$ with $U_{x_1} \cup \cdots \cup U_{x_n} \supseteq \mathrm{supp}(X)$. Finally we choose a partition of unity $\lambda_0, \lambda_1, \ldots, \lambda_n$ subordinated to $\{M \setminus \mathrm{supp}(X), U_{x_1}, \ldots, U_{x_n}\}$ (i.e. $\mathrm{supp}(\lambda_0) \subseteq M \setminus \mathrm{supp}(X)$, $\mathrm{supp}(\lambda_i) \subseteq U_{x_i}$) and define $Y := \sum_{i=1}^n \lambda_i Y_{x_i} \in V$. For all $y \in M$ we then obtain

$$\|Y(y) - X(y)\| = \Big\| \sum_{i=1}^n \lambda_i(y) \big(Y_{x_i}(y) - X(y)\big) \Big\| \leq \sum_{i=1}^n \underbrace{\lambda_i(y) \|Y_{x_i}(y) - X(y)\|}_{\leq \lambda_i(y)\varepsilon(y)} \leq \varepsilon(y)$$

and therefore $Y \notin V$, a contradiction. □

2.2.3. Lemma. *Let $\tau : \Delta^p \times M \to \Delta^q \times M$ be smooth with $\mathrm{pr}_M \circ \tau = \mathrm{pr}_M$ and let G be modular with Lie algebra $\mathfrak{g}$. If $\sigma \in S_q(B\overline{G})$ such that the foliation corresponding to σ is transversal to $\tau(t, \cdot) : M \to \Delta^q \times M$ for all $t \in \Delta^p$ then $\tau^*\sigma \in S_p(B\overline{G})$. Moreover we have* $\mathrm{supp}(\tau^*\sigma) \subseteq \mathrm{supp}(\sigma)$.

Proof. Obviously $\tau^*\sigma$ is a foliation on $\Delta^p \times M$ with $\mathrm{codim}(\tau^*\sigma) = \dim(M)$ which is transversal to the horizontal foliation with leaves $\{(t,x) : x \in M\}$ and so we obtain at least $\tau^*\sigma \in S_p(B\overline{\mathrm{Diff}}_c^\infty(M)_\circ)$. If $Y \in T_t\Delta^p$, the defining equation for $(\tau^*\sigma)(Y)$ is

$$\sigma\big(T_{(t,x)}(\mathrm{pr}_{\Delta^q} \circ \tau) \cdot (Y, (\tau^*\sigma)(Y)(x))\big)(x) = T_{(t,x)}(\mathrm{pr}_M \circ \tau) \cdot (Y, (\tau^*\sigma)(Y)(x)) = (\tau^*\sigma)(Y)(x)$$

So we see that $(\tau^*\sigma)(Y)(x) \in E_x := \{X(x) : X \in \mathfrak{g}\}$ for all $x \in M$ hence by lemma 2.2.2 we obtain $(\tau^*\sigma)(Y) \in \mathfrak{g}$ and thus $\tau^*\sigma \in S_p(B\overline{G})$. □

2.2.4. Lemma. *Let G be modular with Lie algebra $\mathfrak{g}$, $\tau : M \to \Delta^n$ be smooth and define*

$$\mathcal{E}_\tau := \{X \in \mathfrak{g} : \|T_x\tau \cdot X_x\| < 1 \quad \forall x \in M\} \subseteq \mathfrak{g}.$$

Then $\mathcal{E}_\tau$ is a zero neighborhood in $\mathfrak{g}$, and for $\sigma \in S_n^{\mathcal{E}_\tau}(B\overline{G})$ the foliation on $\Delta^n \times M$ corresponding to σ is transversal to $(\tau, \mathrm{id}_M) : M \to \Delta^n \times M$.

Proof. Define $p : TM \to \mathbb{R}_0^+$ by $p(X_x) = \|T_x\tau \cdot X_x\|$. This is a continuous function and hence $U := p^{-1}([0,1)) \subseteq TM$ is an open neighborhood of the image of the zero section $0 \in \mathfrak{g}$. Now $X \in \mathcal{E}_\tau$ iff $X \in \mathfrak{g}$ and $\mathrm{Im}(X) \subseteq U$, i.e. $\mathcal{E}_\tau$ consists of vector fields that are C^0-near zero, and therefore $\mathcal{E}_\tau$ is a neighborhood of $0 \in \mathfrak{g}$.

Since (τ, id_M) is immersive and by dimensional reasons we only have to show:

$$T_x(\tau, \mathrm{id}_M) \cdot T_xM \cap \{(Y, \sigma(Y)(x)) : Y \in T_{\tau(x)}\Delta^n\} = 0$$

Suppose conversely there exist $x \in M$, $0 \neq Y \in T_{\tau(x)}\Delta^n$ and $X \in T_xM$ such that

$$(T_x\tau \cdot X, X) = T_x(\tau, \mathrm{id}_M) \cdot X = (Y, \sigma(Y)(x))$$

We may of course assume that $\|Y\| = 1$, but then we obtain

$$1 = \|Y\| = \|T_x\tau \cdot X\| = \|T_x\tau \cdot \sigma(Y)(x)\| < 1$$

since $\sigma(Y) \in \mathcal{E}_\tau$. □

2.2.5. Corollary. *Let G be modular with Lie algebra $\mathfrak{g}$, $\tau_i : M \to \Delta^n$ for $i = 1, \dots, N$ and define $\mathcal{E} := \bigcap_{i=1}^N \mathcal{E}_{\tau_i}$. Then $\mathcal{E}$ is a zero neighborhood in $\mathfrak{g}$ and for $\sigma \in S_n^{\mathcal{E}}(B\overline{G})$ the foliation on $\Delta^n \times M$ corresponding to σ is transversal to (μ, id_M), where $\mu := \sum_{i=1}^N t_i\tau_i$ is any convex combination of the τ_i, i.e. $0 \leq t_i \leq 1$ and $\sum_{i=1}^N t_i = 1$.*

Proof. First we show $\mathcal{E} \subseteq \mathcal{E}_\mu$. So given $X \in \mathcal{E}$ and $x \in M$ we have

$$\|T_x\mu \cdot X_x\| = \|T_x \textstyle\sum_{i=1}^N t_i\tau_i \cdot X_x\| \leq \sum_{i=1}^N t_i\|T_x\tau_i \cdot X_x\| < \sum_{i=1}^N t_i = 1$$

and hence $X \in \mathcal{E}_\mu$. So by lemma 2.2.4 the foliation corresponding to $\sigma \in S_n^{\mathcal{E}}(B\overline{G}) \subseteq S_n^{\mathcal{E}_\mu}(B\overline{G})$ is transversal to (μ, id_M). □

If $N \in \mathbb{N}$, $\lambda \in C^\infty(M, \Delta^{N-1})$ and $m \in D_N^n$ we define $\tau_m^\lambda : M \to \Delta^n$ by $\tau_m^\lambda(x) := \tau_m^{\lambda(x)}$ and we set $\mathcal{E}_n^\lambda := \bigcap_{m \in D_N^n} \mathcal{E}_{\tau_m^\lambda}$. An easy calculation shows that $\delta_i \circ \tau_m^\lambda = \tau_{\delta_i m}^\lambda$ and so we obtain $\mathcal{E}_n^\lambda \subseteq \mathcal{E}_{n-1}^\lambda$. Indeed for $x \in M$, $X \in \mathcal{E}_n^\lambda$ and $m \in D_N^{n-1}$ we have

$$\|T_x \tau_m^\lambda \cdot X_x\| = \|T_{\tau_m^\lambda(x)} \delta_0 T_x \tau_m^\lambda \cdot X_x\| = \|T_x \tau_{\delta_0 m}^\lambda \cdot X_x\| < 1$$

since $\delta_0 : \Delta^{n-1} \to \Delta^n$ is an isometry and $\delta_0 m \in D_N^n$. So $C_*^{\mathcal{E}_*^\lambda}(B\overline{G}; \mathbb{Z})$ is a chain complex. For $(m, \pi) \in A_N^n$ we can define $\tau_{(m,\pi)}^\lambda : \Delta^n \times M \to \Delta^n \times M$ by $\tau_{(m,\pi)}^\lambda(t, x) = (\tau_{(m,\pi)}^{\lambda(x)}(t), x)$. Since $\mathrm{pr}_{\Delta^n} \circ \tau_{(m,\pi)}^\lambda(t, \cdot)$ is a convex combination of $\tau_{m'}^\lambda$, $m' \in D_N^n$ we obtain from corollary 2.2.5 $\tau_{(m,\pi)}^\lambda(t, \cdot) = (\mathrm{pr}_{\Delta^n} \circ \tau_{(m,\pi)}^\lambda(t, \cdot), \mathrm{id}_M)$ is transversal to the foliation corresponding to $\sigma \in S_n^{\mathcal{E}_n^\lambda}(B\overline{G})$. If we assume $\mathfrak{g}$ to be modular lemma 2.2.3 yields $(\tau_{(m,\pi)}^\lambda)^* \sigma \in S_n(B\overline{G})$.

2.2.6. Definition. Let G be modular with Lie algebra $\mathfrak{g}$, $\mathcal{U}$ a set of sets in M, $N \in \mathbb{N}$ and $\lambda \in C^\infty(M, \Delta^{N-1})$. Then we define

$$\begin{array}{rcl} \varphi_n^\lambda : C_n^{\mathcal{E}_n^\lambda, \mathcal{U}}(B\overline{G}; \mathbb{Z}) & \to & C_n^{\mathcal{U}}(B\overline{G}; \mathbb{Z}) \\ \sigma & \mapsto & \sum\limits_{(m,\pi) \in A_N^n} \mathrm{sgn}(\pi) (\tau_{(m,\pi)}^\lambda)^* \sigma \end{array}$$

where the simplex σ is considered as foliation on $\Delta^n \times M$.

2.2.7. Remark. If we choose $\lambda(x) = (\frac{1}{N}, \dots, \frac{1}{N})$ then $\tau_{(m,\pi)}^\lambda = \tau_{(m,\pi)}^{(\frac{1}{N}, \dots, \frac{1}{N})} \times \mathrm{id}_M$, $\mathcal{E}_n^\lambda = \mathfrak{g}$, and hence by corollary 2.1.4 $\varphi^\lambda = \mathrm{sd}_N$.

2.2.8. Theorem. *Let G be a modular with Lie algebra $\mathfrak{g}$, $\mathcal{U}$ a set of sets in M, $N \in \mathbb{N}$ and $\lambda \in C^\infty(M, \Delta^{N-1})$. Then*

$$\varphi_*^\lambda : C_*^{\mathcal{E}_*^\lambda, \mathcal{U}}(B\overline{G}; \mathbb{Z}) \to C_*^{\mathcal{U}}(B\overline{G}; \mathbb{Z})$$

is a chain map which is homotopic to the inclusion.

Proof. It follows immediately from lemma 2.1.1 that we have

$$(\delta_\alpha \times \mathrm{id}_M) \circ \tau_\alpha^\lambda = \tau_{b_N^n(\alpha)}^\lambda \circ (\delta_{b_N^n(\alpha)} \times \mathrm{id}_M) \quad \forall \alpha \in C_N^n \tag{2.3}$$

$$\tau_\alpha^\lambda \circ (\delta_\alpha \times \mathrm{id}_M) = \tau_{c_N^n(\alpha)}^\lambda \circ (\delta_{c_N^n(\alpha)} \times \mathrm{id}_M) \quad \forall \alpha \in B_N^n \setminus b_N^{n-1}(C_N^{n-1}) \tag{2.4}$$

Now the calculation that φ_*^λ is a chain map is very similar to the calculation that sd_N is a chain map. First we have

$$\begin{aligned} (\partial \circ \varphi_n^\lambda)(\sigma) &= \sum_{i=0}^n (-1)^i \sum_{(m,\pi) \in A_N^n} \mathrm{sgn}(\pi) (\delta_i \times \mathrm{id}_M)^* (\tau_{(m,\pi)}^\lambda)^* \sigma \\ &= \sum_{\alpha \in B_N^n} \mathrm{sgn}(\alpha) (\delta_\alpha \times \mathrm{id}_M)^* (\tau_\alpha^\lambda)^* \sigma \end{aligned}$$

and

$$
\begin{aligned}
(\varphi_n^\lambda \circ \partial)(\sigma) &= \sum_{i=0}^{n}(-1)^i \sum_{(m,\pi)\in A_N^{n-1}} \operatorname{sgn}(\pi)(\tau^\lambda_{(m,\pi)})^*(\delta_i \times \mathrm{id}_M)^*\sigma \\
&= \sum_{\alpha\in C_N^{n-1}} \operatorname{sgn}(\alpha)(\tau^\lambda_\alpha)^*(\delta_\alpha \times \mathrm{id}_M)^*\sigma \\
&= \sum_{\alpha\in C_N^{n-1}} \operatorname{sgn}(b_N^{n-1}(\alpha))(\delta_{b_N^{n-1}(\alpha)} \times \mathrm{id}_M)^*(\tau^\lambda_{b_N^{n-1}(\alpha)})^*\sigma \\
&= \sum_{\alpha\in b_N^{n-1}(C_N^{n-1})} \operatorname{sgn}(\alpha)(\delta_\alpha \times \mathrm{id}_M)^*(\tau^\lambda_\alpha)^*\sigma
\end{aligned}
$$

So it remains to show that

$$
\sum_{\alpha\in B_N^n\setminus b_N^{n-1}(C_N^{n-1})} \operatorname{sgn}(\alpha)(\delta_\alpha \times \mathrm{id}_M)^*(\tau^\lambda_\alpha)^*\sigma = 0
$$

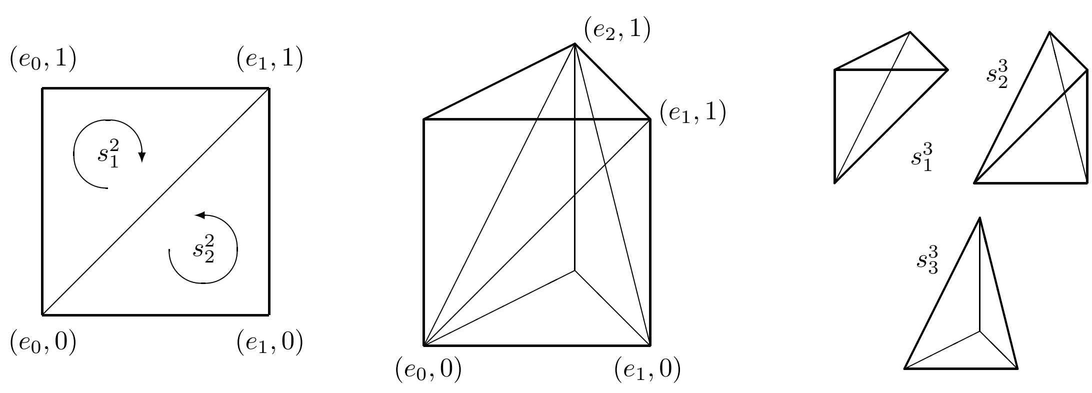
but this follows again from the fact that c_N^n is an involution on the set $B_N^n \setminus b_N^{n-1}(C_N^{n-1})$ without fixed points, equation (2.4) and $\operatorname{sgn}\circ c_N^n = -\operatorname{sgn}$.

Next we show that φ^λ is homotopic to $\mathrm{sd}_N|_{C_*^{\mathcal{E}^\lambda_*,\mathcal{U}}(B\overline{G};\mathbb{Z})}$. We are then finished since by theorem 2.1.3 the latter is homotopic to the inclusion. First we subdivide $\Delta^n \times I$ into $n+1$ simplices in the following way. For $1 \le i \le n+1$ we define $s_i^{n+1} : \Delta^{n+1} \to \Delta^n \times I$ by

$$
s_i^{n+1}(e_j) := \begin{cases} (e_j, 0) & 0 \le j < i \\ (e_{j-1}, 1) & i \le j \le n+1 \end{cases}
$$

and extend it affinely (see figure 2.5). An easy calculation shows

Figure 2.5: Subdivision of $\Delta^n \times I$



$$
\begin{aligned}
s_1^{n+1} \circ \delta_0 &= \mathrm{inc}_1 \\
s_{n+1}^{n+1} \circ \delta_{n+1} &= \mathrm{inc}_0 \\
s_k^{n+1} \circ \delta_k &= s_{k+1}^{n+1} \circ \delta_k && 1 \le k \le n \\
s_i^{n+1} \circ \delta_k &= (\delta_k \times \mathrm{id}_I) \circ s_{i-1}^n && 1 \le k+1 < i \le n+1 \\
s_i^{n+1} \circ \delta_k &= (\delta_{k-1} \times \mathrm{id}_I) \circ s_i^n && 1 \le i < k \le n+1
\end{aligned}
$$

For $(m,\pi) \in A_N^n$ we define $T^\lambda_{(m,\pi)} : \Delta^n \times I \times M \to \Delta^n \times M$ by

$$T^\lambda_{(m,\pi)}(t,0,x) = \tau^\lambda_{(m,\pi)}(t,x) = (\tau^{\lambda(x)}_{(m,\pi)}(t),x)$$
$$T^\lambda_{(m,\pi)}(t,1,x) = \tau^{\lambda_1}_{(m,\pi)}(t,x) = (\tau^{(\frac{1}{N},\dots,\frac{1}{N})}_{(m,\pi)}(t),x)$$

and extend it affinely, where $\lambda_1 \in C^\infty(M,\Delta^{N-1})$ is constant $\lambda_1(x) = (\frac{1}{N},\dots,\frac{1}{N})$. Next we define a homotopy $H : C_*^{\mathcal{E}_*^\lambda,\mathcal{U}}(B\overline{G};\mathbb{Z}) \to C_{*+1}^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ on an n-simplex σ by the following formula:

$$H(\sigma) := \sum_{i=1}^{n+1}(-1)^i \sum_{(m,\pi)\in A_N^n} \operatorname{sgn}(\pi)(s_i^{n+1}\times \mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma \in C^{\mathcal{U}}_{n+1}(B\overline{G};\mathbb{Z})$$

Notice that $\operatorname{supp}((s_i^{n+1}\times\mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma) \subseteq \operatorname{supp}(\sigma)$. We claim that this is the desired homotopy from φ^λ to $\mathrm{sd}_N|_{C_*^{\mathcal{E}_*^\lambda,\mathcal{U}}}(B\overline{G};\mathbb{Z})$. From the equations (2.3) and (2.4) we obtain immediately

$$(\delta_\alpha\times\mathrm{id}_M)\circ T^\lambda_\alpha = T^\lambda_{b_N^n(\alpha)}\circ(\delta_{b_N^n(\alpha)}\times\mathrm{id}_{I\times M}) \quad \forall\alpha\in C_N^n \tag{2.5}$$
$$T^\lambda_\alpha\circ(\delta_\alpha\times\mathrm{id}_{I\times M}) = T^\lambda_{c_N^n(\alpha)}\circ(\delta_{c_N^n(\alpha)}\times\mathrm{id}_{I\times M}) \quad \forall\alpha\in B_N^n\setminus b_N^{n-1}(C_N^{n-1}) \tag{2.6}$$

Moreover we have $T^\lambda_{(m,\pi)}\circ(\mathrm{inc}_1\times\mathrm{id}_M) = \tau^{\lambda_1}_{(m,\pi)} = \tau^{(\frac{1}{N},\dots,\frac{1}{N})}_{(m,\pi)}\times\mathrm{id}_M$ and $T^\lambda_{(m,\pi)}\circ(\mathrm{inc}_0\times\mathrm{id}_M) = \tau^\lambda_{(m,\pi)}$ and hence we get

$$\begin{aligned}
(\partial H)(\sigma) &= \sum_{k=0}^{n+1}(-1)^k\sum_{i=1}^{n+1}(-1)^i\sum_{(m,\pi)\in A_N^n}\operatorname{sgn}(\pi)((s_i^{n+1}\circ\delta_k)\times\mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma\\
&= \sum_{(m,\pi)\in A_N^n}\operatorname{sgn}(\pi)(\mathrm{inc}_0\times\mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma - \sum_{(m,\pi)\in A_N^n}\operatorname{sgn}(\pi)(\mathrm{inc}_1\times\mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma\\
&\quad+\sum_{1\le k+1<i\le n+1}(-1)^i\sum_{(m,\pi)\in A_N^n}(-1)^k\operatorname{sgn}(\pi)(((\delta_k\times\mathrm{id}_I)\circ s_{i-1}^n)\times\mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma\\
&\quad+\sum_{1\le i<k\le n+1}(-1)^i\sum_{(m,\pi)\in A_N^n}(-1)^k\operatorname{sgn}(\pi)(((\delta_{k-1}\times\mathrm{id}_I)\circ s_i^n)\times\mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*\sigma\\
&= \sum_{(m,\pi)\in A_N^n}\operatorname{sgn}(\pi)(\tau^\lambda_{(m,\pi)})^*\sigma - \sum_{(m,\pi)\in A_N^n}\operatorname{sgn}(\pi)(\tau^{(\frac{1}{N},\dots,\frac{1}{N})}_{(m,\pi)}\times\mathrm{id}_M)^*\sigma\\
&\quad-\sum_{i=1}^{n}(-1)^i\sum_{k=0}^{n}(-1)^k\sum_{(m,\pi)\in A_N^n}\operatorname{sgn}(\pi)(s_i^n\times\mathrm{id}_M)^*(\delta_k\times\mathrm{id}_{I\times M})^*(T^\lambda_{(m,\pi)})^*\sigma\\
&= \varphi^\lambda(\sigma)-\mathrm{sd}_N(\sigma)-\sum_{i=1}^{n}(-1)^i\sum_{\alpha\in B_N^n}\operatorname{sgn}(\alpha)(s_i^n\times\mathrm{id}_M)^*(\delta_\alpha\times\mathrm{id}_{I\times M})^*(T^\lambda_\alpha)^*\sigma
\end{aligned}$$

where we used the description of sd_N of corollary 2.1.4. On the other hand we have

$$\begin{aligned}(H\partial)(\sigma) &= \sum_{k=0}^{n}(-1)^k\sum_{i=1}^{n}(-1)^i \sum_{(m,\pi)\in A_N^{n-1}} \mathrm{sgn}(\pi)(s_i^n\times \mathrm{id}_M)^*(T^\lambda_{(m,\pi)})^*(\delta_k\times\mathrm{id}_M)^*\sigma \\ &= \sum_{i=1}^{n}(-1)^i\sum_{\alpha\in C_N^{n-1}}\mathrm{sgn}(\alpha)(s_i^n\times\mathrm{id}_M)^*(T^\lambda_\alpha)^*(\delta_\alpha\times\mathrm{id}_M)^*\sigma \\ &= \sum_{i=1}^{n}(-1)^i\sum_{\alpha\in C_N^{n-1}}\mathrm{sgn}(b_N^{n-1}(\alpha))(s_i^n\times\mathrm{id}_M)^*(\delta_{b_N^{n-1}(\alpha)}\times\mathrm{id}_{I\times M})^*(T^\lambda_{b_N^{n-1}(\alpha)})^*\sigma \\ &= \sum_{i=1}^{n}(-1)^i\sum_{\alpha\in b_N^{n-1}(C_N^{n-1})}\mathrm{sgn}(\alpha)(s_i^n\times\mathrm{id}_M)^*(\delta_\alpha\times\mathrm{id}_{I\times M})^*(T^\lambda_\alpha)^*\sigma\end{aligned}$$

So it remains to show

$$0=\sum_{i=1}^{n}(-1)^i\sum_{\alpha\in B_N^n\setminus b_N^{n-1}(C_N^{n-1})}\mathrm{sgn}(\alpha)(s_i^n\times\mathrm{id}_M)^*(\delta_\alpha\times\mathrm{id}_{I\times M})^*(T^\lambda_\alpha)^*\sigma$$

but for $1\le i\le n$ we even have

$$0=\sum_{\alpha\in B_N^n\setminus b_N^{n-1}(C_N^{n-1})}\mathrm{sgn}(\alpha)(s_i^n\times\mathrm{id}_M)^*(\delta_\alpha\times\mathrm{id}_{I\times M})^*(T^\lambda_\alpha)^*\sigma$$

since c_N^n is an involution without fixed points on $B_N^n\setminus b_N^{n-1}(C_N^{n-1})$, we have equation (2.6) and $\mathrm{sgn}\circ c_N^n=-\mathrm{sgn}$. □

2.2.9. Remark. Notice first that for $1\le j\le n$ and $(m,\pi)\in A_N^n$

$$\tau^\lambda_{(m,\pi)}(e_{j-1},x)=\tau^\lambda_{(m,\pi)}(e_j,x)\qquad \forall x\notin W_m^{\pi(j)}:=\mathrm{supp}(\lambda_{m_0+\dots+m_{\pi(j)-1}-1})$$

So $\tau^\lambda_{(m,\pi)}(\cdot,x)$ is constant for $x\notin\bigcup_{i=1}^n W_m^i$ and $\mathrm{supp}((\tau^\lambda_{(m,\pi)})^*\sigma)\subseteq\bigcup_{i=1}^n W_m^i$. If λ is subordinated to an open cover $\mathcal{U}$, and $\mathcal{U}^{(n)}:=\{U_1\cup\dots\cup U_n : U_i\in\mathcal{U}\}$ we obtain

$$\varphi_n^\lambda : C_n^{\mathcal{E}_n^\lambda}(B\overline{G};\mathbb{Z})\to C_n^{\mathcal{U}^{(n)}}(B\overline{G};\mathbb{Z})$$

and therefore the name *fragmentation mapping*.

Moreover if $m_i=0$ for some $0\le i<n$ then $W_m^j=W_m^k$ for some $1\le j\ne k\le n$ and we even get $(\tau^\lambda_{(m,\pi)})^*\sigma\in S_n^{\mathcal{U}^{(n-1)}}(B\overline{G})$.

2.2.10. Theorem. *Let G be modular with Lie algebra $\mathfrak{g}$, $\mathcal{U}$ be an open covering of M and define $\mathcal{U}^{(n)}:=\{U_1\cup\dots\cup U_n : U_i\in\mathcal{U}\}$. Then the inclusion induces isomorphisms*

$$H_k^{\mathcal{U}^{(n)}}(B\overline{G};\mathbb{Z})\to H_k(B\overline{G};\mathbb{Z})\qquad\forall k\le n$$

Proof. It suffices to show that for a fixed compact set $K\subseteq M$ the inclusions induce isomorphisms $H_k^{\mathcal{U}^{(n)}}(B\overline{G_K};\mathbb{Z})\cong H_k(B\overline{G_K};\mathbb{Z})$ for all $k\le n$. For then by lemma 1.4.10 we obtain

$$H_k^{\mathcal{U}^{(n)}}(B\overline{G};\mathbb{Z})\cong\varinjlim H_k^{\mathcal{U}^{(n)}}(B\overline{G_K};\mathbb{Z})\cong\varinjlim H_k(B\overline{G_K};\mathbb{Z})\cong H_k(B\overline{G};\mathbb{Z})$$

First we show surjectivity: Let $\{\lambda_1, \lambda_2, \dots\}$ be a partition of unity subordinated to $\mathcal{U}$, assume that $\sum_{i=1}^{N-1} \lambda_i|_K = 1$ and let λ_0 be the sum of the remaining λ_i. Then $\lambda := (\lambda_0, \dots, \lambda_{N-1}) \in C^\infty(M, \Delta^{N-1})$. Given $[\sigma] \in H_k(B\overline{G_K}; \mathbb{Z})$. By proposition 1.4.12 we may assume that $\sigma \in C_k^{\mathcal{E}_n^\lambda}(B\overline{G_K}; \mathbb{Z})$. By theorem 2.2.8 we get $[\sigma] = [\varphi_k^\lambda(\sigma)] \in H_k(B\overline{G_K}; \mathbb{Z})$, but by remark 2.2.9 $\varphi_k^\lambda(\sigma) \in C_k^{\mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z})$ and we have proven surjectivity.

Next we prove injectivity: It suffices to show that the inclusion induce injective mappings $i'_* : H_k^{\mathcal{E}_{n+1}^\lambda, \mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z}) \to H_k^{\mathcal{E}_{n+1}^\lambda}(B\overline{G_K}; \mathbb{Z})$ for all $k \leq n$, since by proposition 1.4.12 we have a commutative square

$$\begin{array}{ccc} H_*^{\mathcal{E}_{n+1}^\lambda, \mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z}) & \xrightarrow{i'_*} & H_*^{\mathcal{E}_{n+1}^\lambda}(B\overline{G_K}; \mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ H_*^{\mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z}) & \xrightarrow{i^*} & H_*(B\overline{G_K}; \mathbb{Z}) \end{array}$$

Next choose an open covering $\mathcal{V}$ with the property

$$V_1, V_2 \in \mathcal{V}, V_1 \cap V_2 \neq \emptyset \Rightarrow V_1 \cup V_2 \subseteq U \in \mathcal{U}$$

As above we let $\{\lambda_1, \lambda_2, \dots\}$ be a partition of unity subordinated to $\mathcal{V}$ with $\sum_{i=1}^{N-1} \lambda_i|_K = 1$ and we let λ_0 be the sum of the remaining λ_i. Again we obtain $\lambda := (\lambda_0, \dots, \lambda_{N-1}) \in C^\infty(M; \Delta^{N-1})$. If $m \in D_N^{n+1}$ we let $W_m^j := \operatorname{supp}(\lambda_{m_0+\dots+m_{j-1}-1})$ for $1 \leq j \leq n+1$.

If $0 = i'_*[\sigma]$ then there exists $\rho \in C_{k+1}^{\mathcal{E}_{n+1}^\lambda}(B\overline{G_K}; \mathbb{Z})$ such that $\partial\rho = \sigma$ and so $\varphi_k^\lambda(\sigma) = \partial\varphi_{k+1}^\lambda\rho$, but $\varphi_{k+1}^\lambda\rho \in C_{k+1}^{\mathcal{V}^{(n+1)}}(B\overline{G_K}; \mathbb{Z})$. If we define

$$\kappa := \sum_{\substack{(m,\pi)\in A_N^{n+1} \\ m_i=0 \text{ for some } 0\leq i<n+1}} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda)^*\rho + \sum_{\substack{(m,\pi)\in A_N^{n+1} \\ m_i\geq 1 \text{ for all } 0\leq i<n+1 \\ W_m^j \text{ not pairwise disjoint}}} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda)^*\rho$$

then $\kappa \in C_{k+1}^{\mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z})$, because of remark 2.2.9 and the construction of $\mathcal{V}$. We claim that $\partial\kappa = \partial\varphi_{k+1}^\lambda\rho$. Then we are done since from theorem 2.2.8 we would obtain $[\sigma] = [\varphi_k^\lambda\sigma] = 0 \in H_k^{\mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z})$ and hence $[\sigma] = 0 \in H_k^{\mathcal{E}_{n+1}^\lambda, \mathcal{U}^{(n)}}(B\overline{G_K}; \mathbb{Z})$.

So it remains to show $\partial(\varphi_{k+1}^\lambda\rho - \kappa) = 0$, but

$$\varphi_{k+1}^\lambda\rho - \kappa = \sum_{\substack{(m,\pi)\in A_N^{n+1} \\ m_i\geq 1 \text{ for all } 0\leq i<n+1 \\ W_m^j \text{ pairwise disjoint}}} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda)^*\rho = \sum_{\substack{m\in D_N^{n+1} \\ m_i\geq 1 \text{ for all } 0\leq i<n+1 \\ W_m^j \text{ pairwise disjoint}}} \sum_{\pi\in\mathfrak{S}_{n+1}} \operatorname{sgn}(\pi)(\tau_{(m,\pi)}^\lambda)^*\rho$$

Let $m \in D_N^{n+1}$ with $m_i \geq 1$ for $0 \leq i < n+1$ and W_m^j pairwise disjoint. If $x \notin W_m^{\pi(1)}$ we have

$$\begin{aligned} \tau_{(m,\pi)}^\lambda \circ (\delta_0 \times \mathrm{id}_M)(e_j, x) &= (\tau_{m+f_{\pi(1)}+\dots+f_{\pi(j+1)}}^{\lambda(x)}, x) \\ &= (\tau_{m+f_{\pi(2)}+\dots+f_{\pi(j+1)}}^{\lambda(x)}, x) = \tau_{(m,\pi\circ(1\cdots n+1))}^\lambda \circ (\delta_{n+1} \times \mathrm{id}_M)(e_j, x) \end{aligned}$$

and if $x \in W_m^{\pi(1)}$ (hence $x \notin W_m^{\pi(j)}$ for all $1 < j \leq n+1$) we obtain

$$\tau_{(m,\pi)}^\lambda \circ (\delta_0 \times \mathrm{id}_M)(e_j, x) = (\tau_{m+f_{\pi(1)}}^{\lambda(x)}, x) \text{ and } \tau_{(m,\pi\circ(1\cdots n+1))}^\lambda \circ (\delta_{n+1} \times \mathrm{id}_M)(e_j, x) = (\tau_m^{\lambda(x)}, x)$$

So we get $(\tau^\lambda_{(m,\pi)} \circ (\delta_0 \times \mathrm{id}_M))^*\rho = (\tau^\lambda_{(m,\pi\circ(1\cdots n+1))} \circ (\delta_{n+1} \times \mathrm{id}_M))^*\rho$. Looking a bit closer to the mapping c_N^{n+1} of lemma 2.1.1 we see that

$$\begin{aligned}
\partial\Big(\sum_{\pi\in\mathfrak{S}_{n+1}} \mathrm{sgn}(\pi)(\tau^\lambda_{(m,\pi)})^*\rho\Big) &= \sum_{i=0}^{n+1}(-1)^i \sum_{\pi\in\mathfrak{S}_{n+1}} \mathrm{sgn}(\pi)(\tau^\lambda_{(m,\pi)} \circ (\delta_i \times \mathrm{id}_M))^*\rho \\
&= \sum_{\pi\in\mathfrak{S}_{n+1}} \mathrm{sgn}(\pi)(\tau^\lambda_{(m,\pi)} \circ (\delta_0 \times \mathrm{id}_M))^*\rho \\
&\quad + (-1)^{n+1} \sum_{\pi\in\mathfrak{S}_{n+1}} \mathrm{sgn}(\pi)(\tau^\lambda_{(m,\pi)} \circ (\delta_{n+1} \times \mathrm{id}_M))^*\rho \\
&= \sum_{\pi\in\mathfrak{S}_{n+1}} \mathrm{sgn}(\pi)(\tau^\lambda_{(m,\pi\circ(1\cdots n+1))} \circ (\delta_{n+1} \times \mathrm{id}_M))^*\rho \\
&\quad + (-1)^{n+1} \sum_{\pi\in\mathfrak{S}_{n+1}} \mathrm{sgn}(\pi)(\tau^\lambda_{(m,\pi)} \circ (\delta_{n+1} \times \mathrm{id}_M))^*\rho \\
&= 0
\end{aligned}$$

and therefore $\partial(\varphi^\lambda_{k+1}\sigma - \rho) = 0$. □

We will also make use of the following fragmentation lemma. Its proof is completely independent of the preceding material in this section.

2.2.11. Lemma. *Let G be modular with Lie algebra $\mathfrak{g}$ and let $\mathcal{U}$ be an open covering of M. Then every $g \in C^\infty\big((I,0),(G,\mathrm{id})\big)$ has a decomposition $g = g_1 \cdots g_n$, where each g_i is supported in some $U_i \in \mathcal{U}$ and $g_i \in C^\infty\big((I,0),(G_{U_i},\mathrm{id})\big)$.*

Proof. Fix a compact set $K \subseteq M$ and recall that we have a continuous mapping

$$H_K : C^\infty(I,\mathfrak{g}_K) \to C^\infty\big((I,0),(G,\mathrm{id})\big) \qquad \alpha \mapsto \mathrm{Evol}(\alpha)$$

where $\mathfrak{g}_K := \mathfrak{g} \cap \mathfrak{X}_K(M)$. It follows immediately from the Leibniz rule (1.5) that H_K is a homomorphism of topological groups if we set:

$$(\alpha\beta)(t) := \alpha_t + (H_K(\alpha)(t)^{-1})^*\beta_t$$

It suffices to show that every $g \in \mathrm{Im}(H_K)$ has the desired decomposition, for $\bigcup_K \mathrm{Im}(H_K) = C^\infty\big((I,0),(G,\mathrm{id})\big)$.

Now choose $U_1,\dots,U_n \in \mathcal{U}$ covering K, open sets V_i, W_i with $\bar{W}_i \subseteq V_i \subseteq \bar{V}_i \subseteq U_i$ such that W_i still cover K and a partition of unity $\{\lambda_0,\lambda_1,\dots,\lambda_n\}$ subordinated to $\{M \setminus K, W_1,\dots,W_n\}$. Consider the open neighborhoods $\mathcal{W}_i$ of the identity:

$$\mathcal{W}_i := \big\{g \in C^\infty\big((I,0),(G,\mathrm{id})\big) : g_t(M \setminus \bar{V}_i) \subseteq M \setminus \bar{W}_i \quad \forall t \in I\big\}$$

and define an open neighborhood of $0 \in C^\infty(I,\mathfrak{g}_K)$

$$\mathcal{W}_K := \big\{\alpha \in C^\infty(I,\mathfrak{g}_K) : H_K(\textstyle\sum_{j=0}^{i-1}\lambda_j\alpha) \in \mathcal{W}_i \quad \forall 1 \le i \le n\big\}$$

Since $\mathcal{W}_K$ is open it generates $C^\infty(I,\mathfrak{g}_K)$ as a group and so $H_K(\mathcal{W}_K)$ generates $\mathrm{Im}(H_K)$. Consequently it suffices to show that every $g \in H_K(\mathcal{W}_K)$ has the desired decomposition. For $\alpha \in \mathcal{W}_K$ we set $f_i := H_K(\sum_{j=0}^{i}\lambda_j\alpha)$, $i = 0,\dots,n$. Then we have $f_0 = \mathrm{id}$, $f_n = H_K(\alpha)$

and if we let $g_i := f_{i-1}^{-1} f_i$, $i = 1, \dots, n$ we obtain $H_K(\alpha) = g_1 \cdots g_n$. It remains to show that g_i is supported in U_i, but this follows from

$$\begin{aligned} g_i &= f_{i-1}^{-1} f_i \\ &= H_K\big(t \mapsto -f_{i-1}(t)^*(\textstyle\sum_{j=0}^{i-1} \lambda_j \alpha_t) + f_{i-1}(t)^* \sum_{j=0}^{i} \lambda_j \alpha_t\big) \\ &= H_K\big(t \mapsto f_{i-1}(t)^*(\lambda_i \alpha_t)\big) \end{aligned}$$

for we have $\alpha \in \mathcal{W}_K$, therefore $f_{i-1} = H_K\big(\sum_{j=0}^{i-1} \lambda_j \alpha\big) \in \mathcal{W}_i$ and consequently the support of the mapping $t \mapsto f_{i-1}(t)^*(\lambda_i \alpha_t)$ is contained in $\bar{V}_i \subseteq U_i$. □

2.2.12. Corollary. *Let G be modular with Lie algebra $\mathfrak{g}$, let $\mathcal{U}$ be an open covering of M and assume that G is connected by smooth arcs. Then every $g \in G$ has a decomposition $g = g_1 \cdots g_n$, with $g_i \in G_{U_i}$ for some $U_i \in \mathcal{U}$.*

Proof. This is an immediate consequence of lemma 2.2.11 and the fact that the projection $\mathrm{ev}_1 : C^\infty\big((I, 0), (G, \mathrm{id})\big) \to G$ is onto, in this situation. □

2.3 Local versus Global

2.3.1. Lemma. *Let $G \subseteq \mathrm{Diff}_c^\infty(M)$ be a subgroup and let $\mathcal{U}$ be an open covering of M. Then*

$$0 \leftarrow C_*^{\mathcal{U}}(B\overline{G}; \mathbb{Z}) \xleftarrow{\varepsilon} \bigoplus_{U \in \mathcal{U}} C_*(B\overline{G_U}; \mathbb{Z}) \xleftarrow{\delta} \bigoplus_{U < V \in \mathcal{U}} C_*(B\overline{G_{U \cap V}}; \mathbb{Z}) \leftarrow \cdots$$

is an exact sequence of chain complexes, the Čech complex with respect to the covering $\mathcal{U}$ of the pre-cosheaf $U \mapsto C_*(B\overline{G_U}; \mathbb{Z})$.

Proof. It suffices to show this for finite $\mathcal{U}$. For $|\mathcal{U}| = 1$ this is trivial. For $|\mathcal{U}| = 2$ the sequence looks like:

$$0 \leftarrow C_*^{\{U,V\}}(B\overline{G}; \mathbb{Z}) \leftarrow C_*(B\overline{G_U}; \mathbb{Z}) \oplus C_*(B\overline{G_V}; \mathbb{Z}) \leftarrow C_*(B\overline{G_{U \cap V}}; \mathbb{Z}) \leftarrow 0 \tag{2.7}$$

which is easily seen to be exact. One proceeds by induction on $|\mathcal{U}|$, but we only consider the case $|\mathcal{U}| = 3$; for $|\mathcal{U}| > 3$ the proof is similar. Let $\mathcal{U} = \{U, V, W\}$. Then we have a commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \hookrightarrow & C_*(B\overline{G_{U\cap V\cap W}}) & \twoheadrightarrow & C_*(B\overline{G_{U\cap V\cap W}}) \\
\downarrow & & \downarrow & & \downarrow \\
C_*(B\overline{G_{V\cap W}}) & \hookrightarrow & C_*(B\overline{G_{U\cap V}})\oplus C_*(B\overline{G_{U\cap W}})\oplus C_*(B\overline{G_{V\cap W}}) & \twoheadrightarrow & C_*(B\overline{G_{U\cap V}})\oplus C_*(B\overline{G_{U\cap W}}) \\
\downarrow & & \downarrow & & \downarrow \\
C_*(B\overline{G_V})\oplus C_*(B\overline{G_W}) & \hookrightarrow & C_*(B\overline{G_U})\oplus C_*(B\overline{G_V})\oplus C_*(B\overline{G_W}) & \twoheadrightarrow & C_*(B\overline{G_U}) \\
\downarrow & & \downarrow & & \downarrow \\
C_*^{\{V,W\}}(B\overline{G}) & \hookrightarrow & C_*^{\mathcal{U}}(B\overline{G}) & \twoheadrightarrow & C_*^{\mathcal{U}}(B\overline{G})/C_*^{\{V,W\}}(B\overline{G}) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

All rows are exact, hence we consider this diagram as short exact sequence of chain complexes. The first column is exact by the induction hypothesis. Moreover the third column is exact since it is the composition of the two exact sequences

$$0 \to C_*(B\overline{G_{U\cap V\cap W}}) \to C_*(B\overline{G_{U\cap V}}) \oplus C_*(B\overline{G_{U\cap W}}) \to C_*^{\{V\cap U, W\cap U\}}(B\overline{G}) \to 0$$

and

$$0 \to C_*^{\{V\cap U, W\cap U\}}(B\overline{G}) \to C_*(B\overline{G_U}) \to C_*^{\mathcal{U}}(B\overline{G})/C_*^{\{V,W\}}(B\overline{G}) \to 0.$$

The second is obviously exact, and the first sequence is also exact by the induction hypothesis, since $|\{V \cap U, W \cap U\}| = 2$. Summing up, we have a short exact sequence of chain complexes and two of them have zero homology. So the third one, that is the middle column, has zero homology too. □

2.3.2. Corollary. *Let $G \subseteq \mathrm{Diff}_c^\infty(M)$ be a subgroup and let $\mathcal{U}$ be an open covering of M. Then there exists a spectral sequence with E^1-term*

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\bigoplus_{U\in\mathcal{U}} H_2(B\overline{G_U};\mathbb{Z}) & \longleftarrow & \bigoplus_{U<V\in\mathcal{U}} H_2(B\overline{G_{U\cap V}};\mathbb{Z}) & \longleftarrow & \bigoplus_{U<V<W\in\mathcal{U}} H_2(B\overline{G_{U\cap V\cap W}};\mathbb{Z}) & & \cdots \\
\bigoplus_{U\in\mathcal{U}} H_1(B\overline{G_U};\mathbb{Z}) & \longleftarrow & \bigoplus_{U<V\in\mathcal{U}} H_1(B\overline{G_{U\cap V}};\mathbb{Z}) & \longleftarrow & \bigoplus_{U<V<W\in\mathcal{U}} H_1(B\overline{G_{U\cap V\cap W}};\mathbb{Z}) & & \cdots \\
\bigoplus_{U\in\mathcal{U}} \mathbb{Z} & \longleftarrow & \bigoplus_{U<V\in\mathcal{U}} \mathbb{Z} & \longleftarrow & \bigoplus_{U<V<W\in\mathcal{U}} \mathbb{Z} & & \cdots
\end{array}$$

converging to $H_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$. Moreover the bottom row of the E^2-term is: $\mathbb{Z} \quad 0 \quad 0 \quad \cdots$

Proof. Consider the double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\bigoplus_{U\in\mathcal{U}} C_2(B\overline{G_U};\mathbb{Z}) & \xleftarrow{\delta} & \bigoplus_{U<V\in\mathcal{U}} C_2(B\overline{G_{U\cap V}};\mathbb{Z}) & \xleftarrow{\delta} & \bigoplus_{U<V<W\in\mathcal{U}} C_2(B\overline{G_{U\cap V\cap W}};\mathbb{Z}) & & \cdots \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
\bigoplus_{U\in\mathcal{U}} C_1(B\overline{G_U};\mathbb{Z}) & \xleftarrow{\delta} & \bigoplus_{U<V\in\mathcal{U}} C_1(B\overline{G_{U\cap V}};\mathbb{Z}) & \xleftarrow{\delta} & \bigoplus_{U<V<W\in\mathcal{U}} C_1(B\overline{G_{U\cap V\cap W}};\mathbb{Z}) & & \cdots \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
\bigoplus_{U\in\mathcal{U}} C_0(B\overline{G_U};\mathbb{Z}) & \xleftarrow{\delta} & \bigoplus_{U<V\in\mathcal{U}} C_0(B\overline{G_{U\cap V}};\mathbb{Z}) & \xleftarrow{\delta} & \bigoplus_{U<V<W\in\mathcal{U}} C_0(B\overline{G_{U\cap V\cap W}};\mathbb{Z}) & & \cdots
\end{array}$$

By lemma 2.3.1 it computes $H_*^{\mathcal{U}}(B\overline{G};\mathbb{Z})$ and the E^1-term is as claimed, since we have $H_0(B\overline{G};\mathbb{Z}) = \mathbb{Z}$ for every G, cf. remark 1.4.6. Notice that $H_0(B\overline{G_U};\mathbb{Z}) = \mathbb{Z}$ even if U is not connected, e.g. $H_0(B\overline{G_\emptyset};\mathbb{Z}) = \mathbb{Z}$ too. So the bottom row of the E^1-term is acyclic, i.e. the bottom row of the E^2-term is as claimed. □

2.4 Simplicity of $\mathrm{Diff}_c^\infty(M)_\circ$

2.4.1. Definition. An admissible covering of M is an open covering $\mathcal{U}$ such that for every $k \in \mathbb{N}$ and $U_1, \ldots, U_k \in \mathcal{U}$ the intersection $U_1 \cap \cdots \cap U_k$ is a disjoint union of open balls or empty.

2.4.2. Lemma. *Let A be an Abelian group, X a topological space and $\mathcal{U}$ an admissible covering of X. Then the Čech complex of the pre-cosheaf $U \mapsto H_0(U;A)$*

$$0 \leftarrow \bigoplus_{U\in\mathcal{U}} H_0(U;A) \xleftarrow{\delta} \bigoplus_{U<V\in\mathcal{U}} H_0(U\cap V;A) \xleftarrow{\delta} \bigoplus_{U<V<W\in\mathcal{U}} H_0(U\cap V\cap W;A) \leftarrow \cdots$$

computes $H_*(X;A)$.

Proof. Let $C_*(X;A)$ denote the singular chains with coefficients in A and let $C_*^{\mathcal{U}}(X;A)$ denote the chains made up from simplices each of which lies in some set of the cover $\mathcal{U}$. Then

$$0 \leftarrow C_*^{\mathcal{U}}(X;A) \xleftarrow{\varepsilon} \bigoplus_{U\in\mathcal{U}} C_*(U;A) \xleftarrow{\delta} \bigoplus_{U<V\in\mathcal{U}} C_*(U\cap V;A) \leftarrow \cdots$$

is an exact sequence of chain complexes, see [BT82] for example. Since $H_*^{\mathcal{U}}(X;A) = H_*(X;A)$ the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\bigoplus_{U\in\mathcal{U}} C_2(U;A) & \xleftarrow{\delta} & \bigoplus_{U<V\in\mathcal{U}} C_2(U\cap V;A) & \xleftarrow{\delta} & \bigoplus_{U<V<W\in\mathcal{U}} C_2(U\cap V\cap W;A) & & \cdots \\
\downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
\bigoplus_{U\in\mathcal{U}} C_1(U;A) & \xleftarrow{\delta} & \bigoplus_{U<V\in\mathcal{U}} C_1(U\cap V;A) & \xleftarrow{\delta} & \bigoplus_{U<V<W\in\mathcal{U}} C_1(U\cap V\cap W;A) & & \cdots \\
\downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
\bigoplus_{U\in\mathcal{U}} C_0(U;A) & \xleftarrow{\delta} & \bigoplus_{U<V\in\mathcal{U}} C_0(U\cap V;A) & \xleftarrow{\delta} & \bigoplus_{U<V<W\in\mathcal{U}} C_0(U\cap V\cap W;A) & & \cdots
\end{array}$$

computes $H_*(X;A)$. But the E^1-term of the corresponding spectral sequence degenerate to

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
0 & & 0 & & 0 & & \cdots \\
\bigoplus_{U \in \mathcal{U}} H_0(U;A) & \longleftarrow & \bigoplus_{U<V \in \mathcal{U}} H_0(U \cap V;A) & \longleftarrow & \bigoplus_{U<V<W \in \mathcal{U}} H_0(U \cap V \cap W;A) & & \cdots
\end{array}$$

since we have $H_i(U_1 \cap \cdots \cap U_k; A) = 0$ for all $i > 0$. ☐

2.4.3. Theorem. *Let $A_p^n := H_p\big(B\overline{\mathrm{Diff}}_c^\infty(\mathbb{R}^n)_\circ;\mathbb{Z}\big)$ and let M be an n-dimensional orientable manifold. If $A_p^n = 0$ for all $1 \le p < k$ then we have*

$$H_k\big(B\overline{\mathrm{Diff}}_c^\infty(M)_\circ;\mathbb{Z}\big) \cong H_0(M;A_k^n)$$

where $H_*(M;A_k^n)$ denotes ordinary singular homology of M with coefficients in A_k^n.

Proof. Suppose U is an open ball in M and choose an orientation preserving diffeomorphism $u : U \to \mathbb{R}^n$. Then

$$S_p^{\{U\}}\big(B\overline{\mathrm{Diff}}_c^\infty(M)\big) \xrightarrow{(\mathrm{conj}_u)_*} S_p\big(B\overline{\mathrm{Diff}}_c^\infty(\mathbb{R}^n)\big)$$

is a simplicial isomorphism and we have an induced isomorphism:

$$\varphi_u : H_p^{\{U\}}\big(B\overline{\mathrm{Diff}}_c^\infty(M)_\circ;\mathbb{Z}\big) \to H_p\big(B\overline{\mathrm{Diff}}_c^\infty(\mathbb{R}^n);\mathbb{Z}\big) = A_p^n \cong H_0(U;A_p^n)$$

We claim that it does not depend on the choice of the chart u. Indeed, suppose $v : U \to \mathbb{R}^n$ is another chart and $c = \sum \lambda_i g_i \in C_p^{\{U\}}\big(B\overline{\mathrm{Diff}}_c^\infty(M);\mathbb{Z}\big)$. This is a finite sum and every g_i has compact support, so there exists a closed ball $B \subseteq U$ with $\mathrm{supp}(g_i) \subseteq B$ for all i. Since orientation preserving embeddings of closed balls are diffeotopic (see [Hir76] for example) we find $h \in \mathrm{Diff}_c^\infty(\mathbb{R}^n)_\circ$ with $v = h \circ u$ on B. So $(\mathrm{conj}_v)_*(c) = (\mathrm{conj}_h)_* \circ (\mathrm{conj}_u)_*(c)$ and since $(\mathrm{conj}_h)_* = \mathrm{id}$ in homology (see lemma 1.4.8) we have $\varphi_u([c]) = \varphi_v([c])$. So φ_u does not depend on u and we will write φ_U in the sequel. If W is a disjoint union of open balls $W = \bigsqcup U_i$ we have $\mathrm{Diff}_c^\infty(W)_\circ \cong \mathrm{Diff}_c^\infty(U_1)_\circ \times \cdots$, so lemma 1.4.9 yields

$$H_k^{\{W\}}\big(\mathrm{Diff}_c^\infty(M);\mathbb{Z}\big) \cong \bigoplus H_k^{\{U_i\}}\big(\mathrm{Diff}_c^\infty(M);\mathbb{Z}\big)$$

and we obtain an isomorphism:

$$\varphi_W := \bigoplus \varphi_{U_i} : H_k^{\{W\}}\big(\mathrm{Diff}_c^\infty(M);\mathbb{Z}\big) \to \bigoplus H_0(U_i;A_k^n) \cong H_0(W;A_k^n)$$

Now choose an open covering $\mathcal{U}$ of M such that $\mathcal{U}^k$ is admissible, cf. theorem 2.2.10. Because of the isomorphism φ and lemma 2.4.2, the E^2-term of the spectral sequence in corollary 2.3.2 with respect to the covering $\mathcal{U}^k$ looks like:

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & & \\
H_0(M;A_k^n) & H_1(M;A_k^n) & H_2(M;A_k^n) & \cdots & k\text{-th row} \\
0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & & \\
\mathbb{Z} & 0 & 0 & \cdots &
\end{array}$$

Since the spectral sequence converges to $H_*^{\mathcal{U}^k}\big(B\overline{\mathrm{Diff}}_c^\infty(M);\mathbb{Z}\big)$ we obtain

$$H_k^{\mathcal{U}^k}\big(B\overline{\mathrm{Diff}}_c^\infty(M);\mathbb{Z}\big) \cong H_0(M;A_k^n)$$

and from theorem 2.2.10 $H_k^{\mathcal{U}^k}\big(B\overline{\mathrm{Diff}}_c^\infty(M);\mathbb{Z}\big) \cong H_k\big(B\overline{\mathrm{Diff}}_c^\infty(M);\mathbb{Z}\big)$. □

2.4.4. Remark. It is an immediate consequence of theorem 2.4.3 that for a n-dimensional, connected and orientable M the following groups are isomorphic

1. $H_k\big(B\overline{\mathrm{Diff}}_c^\infty(\mathbb{R}^n);\mathbb{Z}\big)$
2. $H_k\big(B\overline{\mathrm{Diff}}_c^\infty(M);\mathbb{Z}\big)$

where k is the first positive integer such that one of them is non-zero. This is a special case of a theorem due to W. Thurston [Thu74], which states that both are isomorphic to a homology group of a certain classifying space, see [Mat79] for proofs.

2.4.5. Corollary. *$H_1\big(B\overline{\mathrm{Diff}}_c^\infty(M)_\circ;\mathbb{Z}\big) = 0$ for every orientable manifold M. Moreover $\widetilde{\mathrm{Diff}}_c^\infty(M)_\circ$ and $\mathrm{Diff}_c^\infty(M)_\circ$ are perfect.*

Proof. Recall from corollary 1.5.6 that we have $H_1\big(B\overline{\mathrm{Diff}}^\infty(T^n);\mathbb{Z}\big) = 0$. From theorem 2.4.3 we thus get

$$0 = H_1\big(B\overline{\mathrm{Diff}}^\infty(T^n);\mathbb{Z}\big) = H_0(T^n;A_1^n) = A_1^n$$

and, using again theorem 2.4.3, we obtain

$$H_1\big(B\overline{\mathrm{Diff}}_c^\infty(M)_\circ;\mathbb{Z}\big) = H_0(M;A_1^n) = 0$$

where $n = \dim(M)$. The perfectness statement now follows from proposition 1.4.5. □

2.4.6. Corollary. *$\mathrm{Diff}_c^\infty(M)_\circ$ is simple for every manifold M.*

Proof. We want to apply proposition 1.3.1. Let $\mathcal{U}$ be the set of all orientable open subsets of M, $G := \mathrm{Diff}_c^\infty(M)_\circ$ and for $U \in \mathcal{U}$ we set $G_U := \mathrm{Diff}_c^\infty(U)_\circ$. G_U is perfect by corollary 2.4.5. G acts transitively on M since we have proposition 1.2.6, see also remark 1.2.7. G has the fragmentation property since we have corollary 2.2.12. The third assumption of proposition 1.3.1 is obviously satisfied, and consequently G is simple. □

2.5 Perfectness of $\mathrm{Diff}_c^\infty(M,\mathcal{F})_\circ$

In [Ryb95a] T. Rybicki showed that the component containing the identity of the group of leave preserving diffeomorphisms is perfect. In this chapter we will give a different proof of this theorem.

2.5.1. Proposition. *$H_1\big(B\overline{\mathrm{Diff}}^\infty(\mathbb{R}^m\times\mathbb{R}^n,\mathcal{F});\mathbb{Z}\big)=0$, where $\mathcal{F}$ is the foliation with leaves $\{\mathrm{pt}\}\times\mathbb{R}^n$.*

Proof. Consider the torus $T^m\times T^n$ with the foliation $\mathcal{G}$ having $\{\mathrm{pt}\}\times T^n$ as leaves and choose a good covering $\mathcal{U}$ of $T^n\times T^m$ such that $\mathcal{G}|_U\cong\mathcal{F}$ for every $U\in\mathcal{U}$. From theorem 2.2.10 and corollary 1.6.4 we obtain:

$$H_1^{\mathcal{U}}\big(B\overline{\mathrm{Diff}}^\infty(T^m\times T^n,\mathcal{G});\mathbb{Z}\big)\cong H_1\big(B\overline{\mathrm{Diff}}^\infty(T^m\times T^n,\mathcal{G});\mathbb{Z}\big)=0$$

In view of corollary 2.3.2 this can only be the case if $H_1\big(B\overline{\mathrm{Diff}}_c^\infty(U,\mathcal{G}|_U);\mathbb{Z}\big)=0$ for every $U\in\mathcal{U}$, and since $(U,\mathcal{G}|_U)\cong(\mathbb{R}^m\times\mathbb{R}^n,\mathcal{F})$ we get $H_1\big(B\overline{\mathrm{Diff}}_c^\infty(\mathbb{R}^m\times\mathbb{R}^n,\mathcal{F});\mathbb{Z}\big)=0$. □

2.5.2. Theorem. *Let $\mathcal{F}$ be a regular foliation of M. Then $H_1\big(B\overline{\mathrm{Diff}}_c^\infty(M,\mathcal{F});\mathbb{Z}\big)=0$.*

Proof. Choose an open covering $\mathcal{U}$ of M such that $(U,\mathcal{F}|_U)\cong(\mathbb{R}^m\times\mathbb{R}^n,\mathcal{G})$ where $\mathcal{G}$ is the product foliation we considered in proposition 2.5.1. Since $H_1\big(\overline{\mathrm{Diff}}_c^\infty(U,\mathcal{F}|_U);\mathbb{Z}\big)=0$ by proposition 2.5.1 it follows from corollary 2.3.2 that $H_1^{\mathcal{U}}\big(B\overline{\mathrm{Diff}}_c^\infty(M,\mathcal{F});\mathbb{Z}\big)=0$ and so

$$H_1\big(B\overline{\mathrm{Diff}}_c^\infty(M,\mathcal{F});\mathbb{Z}\big)\cong H_1^{\mathcal{U}}\big(B\overline{\mathrm{Diff}}_c^\infty(M,\mathcal{F});\mathbb{Z}\big)=0$$

for we have theorem 2.2.10. □

2.5.3. Corollary. *Let $\mathcal{F}$ be a regular foliation of a manifold M. Then $\widetilde{\mathrm{Diff}}_c^\infty(M,\mathcal{F})_\circ$ and $\mathrm{Diff}_c^\infty(M,\mathcal{F})_\circ$ are perfect.*

Proof. This is an immediate consequence of theorem 2.5.2 and proposition 1.4.5. □

3. Locally Conformally Symplectic Manifolds

3.1 d^ω-Cohomology

Let ω be a closed 1-form on a manifold M and define

$$d^\omega : \Omega^*(M) \to \Omega^{*+1}(M) \qquad d^\omega(\alpha) := d\alpha + \omega \wedge \alpha$$

Obviously we have $d^\omega \circ d^\omega = 0$ and we may define the d^ω-cohomology $H^*_{d^\omega}(M)$ and similarly d^ω_c-cohomology with compact supports $H^*_{d^\omega_c}(M)$. Suppose $[\omega'] = [\omega] \in H^1(M)$ and choose $a \in \Omega^0(M)$ with $\omega' = \omega + \frac{da}{a} = \omega + d(\ln|a|)$. Then there are isomorphisms $\frac{1}{a} : H^*_{d^\omega} \cong H^*_{d^{\omega'}}(M)$ and $\frac{1}{a} : H^*_{d^\omega_c} \cong H^*_{d^{\omega'}_c}(M)$ given by multiplication with $\frac{1}{a}$. So for an exact ω the d^ω-cohomology is isomorphic to the ordinary de Rham cohomology.

For closed 1-forms ω_1, ω_2 an easy calculation shows

$$d^{\omega_1+\omega_2}(\sigma \wedge \tau) = d^{\omega_1}\sigma \wedge \tau + (-1)^{|\sigma|}\sigma \wedge d^{\omega_2}\tau$$

and hence the wedge product induces a bilinear mapping

$$\wedge : H^k_{d^{\omega_1}}(M) \times H^l_{d^{\omega_2}}(M) \to H^{k+l}_{d^{\omega_1+\omega_2}}(M)$$

and similar for compact supports.

For a smooth $g : M \to N$ we have an induced mapping $g^* : H^*_{d^\omega}(N) \to H^*_{d^{g^*\omega}}(M)$. If g is proper then we also have an induced mapping $g^* : H^*_{d^\omega_c}(N) \to H^*_{d_c^{g^*\omega}}(M)$.

3.1.1. Lemma. *Let ω be a closed 1-form on N and let $g : M \times I \to N$ be a smooth homotopy. Define $a \in C^\infty(M \times I, \mathbb{R})$ by $a_t := \exp\left(\int_0^t \operatorname{inc}_s^* i_{\partial_t} g^*\omega ds\right)$ where $\operatorname{inc}_s : M \to M \times I$, $\operatorname{inc}_s(x) := (x, s)$. Then*

$$a_1 g_1^* = a_0 g_0^* : H^*_{d^\omega}(N) \to H^*_{d^{g_0^*\omega}}(M)$$

If g is proper the same holds with compact supports.

Proof. Notice that the definition of a is such that $g_t^*\omega = g_0^*\omega + d(\ln|a_t|)$. One defines a mapping $H : \Omega^*(N) \to \Omega^{*-1}(M)$ by $H(\sigma) := \int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} g^*\sigma dt$ and checks that it is a chain homotopy, i.e. $d^{g_0^*\omega}H(\sigma) + H(d^\omega\sigma) = a_1 g_1^*\sigma - a_0 g_0^*\sigma$. Indeed we have

$$\begin{aligned}
d^{g_0^*\omega}H(\sigma) + H(d^\omega\sigma) &= d^{g_0^*\omega} \int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} g^*\sigma dt + \int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} g^* d^\omega \sigma dt \\
&= \int_0^1 a_t d^{g_t^*\omega} \operatorname{inc}_t^* i_{\partial_t} g^*\sigma dt + \int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} d^{g^*\omega} g^*\sigma dt \\
&= \int_0^1 a_t \operatorname{inc}_t^* d^{g^*\omega} i_{\partial_t} g^*\sigma dt + \int_0^1 a_t \operatorname{inc}_t^* i_{\partial_t} d^{g^*\omega} g^*\sigma dt \\
&= \int_0^1 a_t \operatorname{inc}_t^* \big(L_{\partial_t} g^*\sigma + i_{\partial t} g^*\omega \wedge g^*\sigma\big) dt \\
&= \int_0^1 a_t \tfrac{\partial}{\partial t}(g_t^*\sigma) + (\tfrac{\partial}{\partial t} a_t) g_t^*\sigma dt \\
&= \int_0^1 \tfrac{\partial}{\partial t}(a_t g_t^*\sigma) dt = a_1 g_1^*\sigma - a_0 g_0^*\sigma
\end{aligned}$$

where we used $d^\omega i_X\alpha + i_X d^\omega\alpha = L_X\alpha + i_X\omega \wedge \alpha$ and $\frac{\partial}{\partial_t} a_t = a_t \operatorname{inc}_t^* i_{\partial_t} g^*\omega$. □

3.1.2. Corollary (Relative Poincaré Lemma). *Let $i : N \to M$ be a closed submanifold and $\omega \in \Omega^1(M)$ closed. If $\alpha \in \Omega^k(M)$ satisfies $d^\omega\alpha = 0$, $i^*\alpha = 0$ then there exists an open neighborhood U of N and $\varphi \in \Omega^{k-1}(U)$, which vanishes on N, such that $d^\omega\varphi = \alpha|_U$.*

Proof. By choosing a tubular neighborhood of N in M we may assume that $\pi : M \to N$ is a vector bundle and $i : N \to M$ is the zero section. Consider the homotopy $g : M \times I \to M$ defined by $g_t(x) := (1-t)x$. Since $g_0 = \mathrm{id}$, $g_1 = i \circ \pi$ lemma 3.1.1 yields

$$d^\omega H(\alpha) + H(d^\omega\alpha) = a_1\pi^* i^*\alpha - \alpha$$

and since $d^\omega\alpha = 0$, $i^*\alpha = 0$ we obtain $\alpha = d^\omega(-H(\alpha))$. It remains to show that $H(\alpha)$ vanishes on $i(N)$. For $t < 1$ we have $\mathrm{inc}_t^* i_{\partial t} g^*\alpha = g_t^* i_{\dot g_t}\alpha$ and $\dot g_t(i(x)) = 0$. Therefore

$$H(\alpha)(i(x)) = \int_0^1 a_t(\mathrm{inc}_t^* i_{\partial_t} g^*\alpha)(i(x))dt = \int_0^1 (a_t g_t^* i_{\dot g_t}\alpha)(i(x))dt = 0$$

and $\varphi := -H(\alpha)$ vanishes on $i(N)$. □

Suppose M is the union of two open subsets U, V. Then the following is a short exact sequence of cochain complexes

$$0 \to \big(\Omega^*(M), d^\omega\big) \xrightarrow{\alpha} \big(\Omega^*(U) \oplus \Omega^*(V), d^{\omega|_U} \oplus d^{\omega|_V}\big) \xrightarrow{\beta} \big(\Omega^*(U \cap V), d^{\omega|_{U\cap V}}\big) \to 0$$

where $\alpha(\sigma) = (\sigma|_U, \sigma|_V)$ and $\beta(\sigma,\tau) = \sigma|_{U\cap V} - \tau|_{U\cap V}$. So we obtain

3.1.3. Lemma. *Let M be the union of two open subsets U and V. Then there exists a long exact sequence*

$$\cdots \to H^k_{d^\omega}(M) \xrightarrow{\alpha_*} H^k_{d^{\omega|_U}}(U) \oplus H^k_{d^{\omega|_V}}(V) \xrightarrow{\beta_*} H^k_{d^{\omega|_{U\cap V}}}(U \cap V) \xrightarrow{\delta} H^{k+1}_{d^\omega}(M) \to \cdots$$

and $\delta([\sigma]) = [d\lambda_V \wedge \sigma] = -[d\lambda_U \wedge \sigma]$, where $\{\lambda_U, \lambda_V\}$ is a partition of unity subordinated to $\{U, V\}$ and the forms under consideration are assumed to be extended by 0 to the whole of M.

Similarly there is an exact sequence of cochain complexes

$$0 \to \big(\Omega_c^*(U \cap V), d_c^{\omega|_{U\cap V}}\big) \xrightarrow{\beta} \big(\Omega_c^*(U) \oplus \Omega_c^*(V), d_c^{\omega|_U} \oplus d_c^{\omega|_V}\big) \xrightarrow{\alpha} \big(\Omega_c^*(M), d_c^\omega\big) \to 0$$

where $\beta(\sigma) = (\sigma, -\sigma)$ and $\alpha(\sigma,\tau) = \sigma + \tau$ and everything is assumed to be extended by 0. So we have

3.1.4. Lemma. *If M is the union of two open subsets U and V then there exists a long exact sequence*

$$\cdots \to H^{k-1}_{d_c^\omega}(M) \xrightarrow{\delta} H^k_{d_c^{\omega|_{U\cap V}}}(U \cap V) \xrightarrow{\beta_*} H^k_{d_c^{\omega|_U}}(U) \oplus H^k_{d_c^{\omega|_V}}(V) \xrightarrow{\alpha_*} H^k_{d_c^\omega}(M) \to \cdots$$

where $\delta[\sigma] = [d\lambda_U \wedge \sigma|_{U\cap V}] = -[d\lambda_V \wedge \sigma|_{U\cap V}]$ and $\{\lambda_U, \lambda_V\}$ is a partition of unity subordinated to $\{U, V\}$.

A covering $\mathcal{U}$ of a manifold M is called good if for all $m \in \mathbb{N}$ and $U_1, \ldots, U_m \in \mathcal{U}$ the intersection $U_1 \cap \cdots \cap U_m$ is either empty or contractible. Using a Riemannian metric and geodesically convex open sets one easily sees that every manifold admits a good covering and these are cofinal in all coverings.

Using the Mayer Vietoris sequence inductively and the fact that for contractible sets the d^ω-cohomology is isomorphic to the de Rham cohomology, and hence finite dimensional, we immediately obtain

3.1.5. Corollary. *Suppose M admits a finite good covering. Then $H^*_{d^\omega}(M)$ and $H^*_{d^\omega_c}(M)$ are finite dimensional. Especially this is true for compact manifolds.*

For an oriented manifold of dimension n we may define a pairing by

$$\langle\cdot,\cdot\rangle_\omega : H^*_{d-\omega}(M) \times H^{n-*}_{d^\omega_c}(M) \xrightarrow{\wedge} H^n_c(M) \xrightarrow{\int} \mathbb{R}$$

If $\omega' = \omega + \frac{da}{a} = \omega + d(\ln|a|)$ then $-\omega' = -\omega + d(\ln|\frac{1}{a}|)$ so $\frac{1}{a} : H^*_{d^\omega_c}(M) \cong H^*_{d^{\omega'}_c}(M)$, $a : H^*_{d-\omega}(M) \cong H^*_{d-\omega'}(M)$ and $\langle a[\sigma], \frac{1}{a}[\tau]\rangle_{\omega'} = \langle[\sigma],[\tau]\rangle_\omega$. Hence if ω is exact this pairing is non-degenerated by ordinary Poincaré duality.

3.1.6. Proposition. *On an oriented manifold of dimension n the mappings defined by*

$$D^*_\omega : H^*_{d-\omega}(M) \to H^{n-*}_{d^\omega_c}(M)^* \qquad D^*_\omega([\sigma])([\tau]) := \langle[\sigma],[\tau]\rangle_\omega$$

are isomorphisms.

Proof. If M is a disjoint union of open balls then we have

$$\big(H^*_{d^\omega_c}(\bigsqcup U_i)\big)^* \cong \big(\bigoplus H^*_{d^\omega_c}(U_i)\big)^* \cong \prod H^*_{d^\omega_c}(U_i)^*$$

and via this isomorphism D^*_ω corresponds to $\prod D^*_{\omega|_{U_i}}$ and is therefore an isomorphism. Using the explicit description of the connecting homomorphisms δ in lemma 3.1.3 and lemma 3.1.4 one easily checks that the following diagram commutes up to sign:

$$\begin{array}{ccccccc}
H^k_{d-\omega}(M) & \xrightarrow{\alpha_*} & H^k_{d-\omega|_U}(U) \oplus H^k_{d-\omega|_V}(V) & \xrightarrow{\beta_*} & H^k_{d-\omega|_{U\cap V}}(U\cap V) & \xrightarrow{\delta} & H^{k+1}_{d-\omega}(M) \\
\downarrow{\scriptstyle D^k_\omega} & & \downarrow{\scriptstyle D^k_{\omega|_U}\oplus D^k_{\omega|_V}} & & \downarrow{\scriptstyle D^k_{\omega|_{U\cap V}}} & & \downarrow{\scriptstyle D^{k+1}_\omega} \\
H^{n-k}_{d^\omega_c}(M)^* & \xrightarrow{(\alpha_*)^*} & H^{n-k}_{d^{\omega|_U}_c}(U)^* \oplus H^{n-k}_{d^{\omega|_V}_c}(V)^* & \xrightarrow{(\beta_*)^*} & H^{n-k}_{d^{\omega|_{U\cap V}}_c}(U\cap V)^* & \xrightarrow{\delta^*} & H^{n-k-1}_{d^\omega_c}(M)^*
\end{array}$$

So if Poincaré duality holds for U, V and $U\cap V$ it also holds for $U\cup V$ by the five lemma. Finally one chooses a good covering $\mathcal{U}$ such that every $U \in \mathcal{U}$ does only intersect finitely many other sets of $\mathcal{U}$. Then we can write $M = W_1 \cup \cdots \cup W_n$ where every W_i is a disjoint union of open balls in $\mathcal{U}$. Since Poincaré duality holds for W_i, W_j and $W_i \cap W_j$ (the latter is also an disjoint union of open balls) it holds also for $W_i \cup W_j$. Proceeding inductively finishes the proof. $\square$

3.1.7. Example. Let $[f] \in H^0_{d^\omega}(M)$, i.e. $f \in C^\infty(M,\mathbb{R})$ and $d^\omega f = 0$. Consider the set $Z := \{x \in M : f(x) = 0\}$. It is of course closed. We show that it is open too. Let $x \in Z$ and choose a contractible neighborhood U of x. Then $\omega|_U = d(\ln|a|)$ for some nowhere vanishing function a on U and $\frac{1}{a} : H^*_{d^{\omega|_U}}(U) \cong H^*(U)$. So $\frac{1}{a}f|_U$ is a constant function on U and since it vanishes in x it vanishes on U, that is $U \subseteq Z$. For connected M this yields that $H^0_{d^\omega}(M)$ and similarly $H^0_{d^\omega_c}(M)$ is at most 1-dimensional.

Let M be connected and oriented. Then $i^* : H^0_{d^\omega}(M) \to H^0_{d^{\omega|_U}}(U)$ is injective and by Poincaré duality $i_* : H^n_{d^{\omega|_U}_c}(U) \to H^n_{d^\omega_c}(M)$ is onto. So generators of $H^n_{d^\omega_c}(M)$ can be chosen to have arbitrary small supports.

Let $M = S^1$ and $\omega = \lambda d\theta$, where $0 \neq \lambda \in \mathbb{R}$, be a generator of its first de Rham cohomology. We claim that $H^0_{d^\omega}(S^1) = 0$. So let $f \in \Omega^0(S^1)$ be d^ω-closed. We consider f

as periodic function on $\mathbb{R}$ then $\omega = \lambda dx$. The condition $d^\omega f = 0$ translates to $f' + \lambda f = 0$, but this has no non-trivial periodic solution, hence $f = 0$. So $H^0_{d^\omega}(S^1) = 0$ for every non-exact ω.

Let M be connected and ω a closed 1-form that is not exact. Then there exists a mapping $i : S^1 \to M$ such that $i^*\omega$ is not exact. Now let $f \in \Omega^0(M)$. By the previous paragraph $i^* f = 0$ and hence by connectedness $f = 0$. So we have shown $H^0_{d^\omega}(M) = 0$ and similarly $H^0_{d^\omega_c}(M) = 0$ for every connected M and any non-exact ω. Using Poincaré duality we also obtain $H^n_{d^\omega}(M) = 0$ and $H^n_{d^\omega_c}(M) = 0$ for every oriented, connected, n-dimensional M and every non-exact ω. Using the orientation covering one sees that the assumption orientable is superfluous. A different proof of $H^n_{d^\omega}(M) = 0$ can be found in [GL84].

3.1.8. Example. Consider $M = \mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$ and let ω resp. η be a generator of $H^1(M)$ supported in $(-\infty,0) \times \{\mathbb{R}\}$ resp. $U := (0,\infty) \times \mathbb{R}$. Then obviously $d^\omega \eta = 0$ and $\eta|_U$ cannot be $d^{\omega|_U} = d$-exact. From the Mayer Vietoris sequence one sees that η generates $H^1_{d^\omega}(M)$.

Suppose we have two manifolds M_1, M_2 and two closed 1-forms ω_1 resp. ω_2 on M_1 resp. M_2. Let $\omega := \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega_2 \in \Omega^1(M_1 \times M_2)$ and define a mapping

$$\Psi : \Omega^k(M_1) \times \Omega^l(M_2) \to \Omega^{k+l}(M_1 \times M_2) \qquad (\alpha, \beta) \mapsto \mathrm{pr}_1^* \alpha \wedge \mathrm{pr}_2^* \beta.$$

One easily checks $d^\omega(\Psi(\alpha,\beta)) = \Psi(d^{\omega_1}\alpha, \beta) + (-1)^{|\alpha|}\Psi(\alpha, d^{\omega_2}\beta)$ and hence we have an induced mapping

$$H^*_{d^{\omega_1}}(M_1) \otimes H^*_{d^{\omega_2}}(M_2) \to H^*_{d^\omega}(M_1 \times M_2)$$

As in ordinary de Rham cohomology one proves that under the assumption that one of the two manifolds has finite dimensional cohomology, Ψ is an isomorphism. Using this and example 3.1.8 one obtains manifolds with arbitrarily complicated d^ω-cohomology and non-exact ω.

3.1.9. Theorem. *Let $\mathcal{F}_\omega(U) := \{f \in C^\infty(U,\mathbb{R}) : d^\omega f = 0\}$. Then $\mathcal{F}_\omega$ is a locally constant sheaf and $H^*(M;\mathcal{F}_\omega) \cong H^*_{d^\omega}(M)$. So the d^ω-cohomology is a kind of twisted de Rham cohomology.*

Proof. For any $x \in M$ we can choose a contractible neighborhood U of x and a function $a \in C^\infty(U, \mathbb{R}^+)$ such that $\omega = d \ln a$. Then multiplication with a defines a isomorphism of the sheaf $\mathcal{F}|_U$ and the constant sheaf $\mathbb{R}$ on U. So $\mathcal{F}_\omega$ is a locally constant sheaf. Moreover we have a fine resolution of $\mathcal{F}_\omega$

$$0 \to \mathcal{F}_\omega \to \Omega^0 \xrightarrow{d^\omega} \Omega^1 \xrightarrow{d^\omega} \Omega^2 \to \cdots$$

and hence $H^*(M;\mathcal{F}_\omega) \cong H^*_{d^\omega}(M)$, by the theorem of de Rham which can be found in [Bre67] for example. □

Let ω be a closed 1-form on M, define $B_\omega := M \times \mathbb{R}$ and let $\pi : B_\omega \to M$ denote the projection. Then $t\pi^*\omega + dt$ is a nowhere vanishing 1-form on B_ω which satisfies $d(t\pi^*\omega + dt) = (t\pi^*\omega + dt) \wedge \pi^*\omega$. Hence it defines a codimension one foliation on B_ω. We provide B_ω with the topology which has as basis the leaves of $\pi^{-1}(U)$ for $U \subseteq M$ open. This is a (locally trivial) bundle of coefficients on M, see [Ste51] for example. A section is simply a smooth function f on M such that $0 = f\omega + df = d^\omega f$. So the sheaf corresponding to B_ω is simply $\mathcal{F}_\omega$ and we have

3.1.10. Corollary. *$H^*_{d^\omega}(M) \cong H^*(M, B_\omega)$, where the latter denotes cohomology with values in the bundle of coefficients B_ω, see [Ste51].*

Consider $\Omega^*_c(M) = \varinjlim_K \Omega^*_K(M)$ with the inductive limit topology, where the limit is over all compact K and $\Omega_K(M)$ denotes the forms with support contained in K. This is a strict inductive limit of Fréchet spaces and hence a complete, separated locally convex vector space. We provide $\operatorname{im} d^\omega_c \subseteq \ker d^\omega_c \subseteq \Omega^*_c(M)$ with the initial topologies and put the quotient topology on $H^*_{d^\omega_c}(M)$.

3.1.11. Theorem. *Let ω be a closed 1-form on a manifold M. Then $H^*_{d^\omega_c}(M)$ is a strict inductive limit of separated, finite dimensional topological vector spaces and hence a complete, separated locally convex vector space.*

Proof. First we assume that M is oriented. $d^\omega_c : \Omega^*_c(M) \to \Omega^{*+1}_c(M)$ is continuous and hence $\ker d^\omega_c \subseteq \Omega^*_c(M)$ is closed. By Poincaré duality $\sigma \in \ker d^\omega_c$ is contained in $\operatorname{im} d^\omega_c$ if and only if

$$\textstyle\int_M \tau \wedge \sigma = 0 \qquad \forall \tau \in \ker\left(d^{-\omega} : \Omega^*(M) \to \Omega^{*+1}(M)\right)$$

but these are continuous conditions and so $\operatorname{im} d^\omega_c \subseteq \ker d^\omega_c$ is closed.

Let $d^\omega_K := d^\omega|_{\Omega^*_K(M)} : \Omega^*_K(M) \to \Omega^{*+1}_K(M)$. It is a general fact that if $E = \varinjlim E_n$ is a strict inductive limit and $F \subseteq E$ is a (not necessarily closed) subspace then $F = \varinjlim(E_n \cap F)$ as strict inductive limit. Applying this twice we obtain

$$\varinjlim_K \ker d^\omega_K = \ker d^\omega_c \quad \text{and} \quad \varinjlim_K\left(\Omega^*_K(M) \cap \operatorname{im} d^\omega_c\right) = \operatorname{im} d^\omega_c.$$

Since $\ker d^\omega_K$ is a Fréchet space and $\operatorname{im} d^\omega_c \subseteq \Omega^*_c(M)$ is closed, $\frac{\ker d^\omega_K}{\Omega^*_K(M) \cap \operatorname{im} d^\omega_c}$ is separated. We claim that it is finite dimensional for nice K.

So assume that K is a compact, $\dim(M)$-dimensional submanifold with boundary. Let $i : \partial K \hookrightarrow K$ denote the inclusion. We let $\Omega^*(K, \partial K) := \{\alpha \in \Omega^*(K) : i^*\alpha = 0\}$ and denote by $H^*_{d^{\omega|_K}}(K, \partial K)$ the corresponding cohomology, i.e. the relative cohomology. As usual we have a long exact sequence

$$\cdots \to H^*_{d^{\omega|_K}}(K, \partial K) \to H^*_{d^{\omega|_K}}(K) \xrightarrow{i^*} H^*_{d^{i^*\omega}}(\partial K) \xrightarrow{\delta} H^{*+1}_{d^{\omega|_K}}(K, \partial K) \to \cdots$$

and so $H^*_{d^{\omega|_K}}(K, \partial K)$ is finite dimensional by corollary 3.1.5. We have a mapping $\Omega^*_K(M) \to \Omega^*(K, \partial K)$ and we claim that the induced mapping $H^*_{d^\omega_K}(M) \to H^*_{d^{\omega|_K}}(K, \partial K)$ is injective. To see this let $\alpha \in \Omega^*_K(M)$ be d^ω-closed and such that $\alpha|_K = d^{\omega|_K}\beta$ for some $\beta \in \Omega^*(K, \partial K)$. Next choose a smooth homotopy $g : K \times I \to K$ with $g_0 = \mathrm{id}_K$, $g_t(\partial K) \subseteq \partial K$ and such that there exists an open neighborhood U of ∂K with $g_1(U) \subseteq \partial K$. From lemma 3.1.1 we get

$$d^{\omega|_K}\left(\textstyle\int_0^1 a_t \operatorname{inc}^*_t i_{\partial_t} g^*\alpha dt\right) = a_1 g_1^*(\alpha|_K) - a_0 g_0^*(\alpha|_K) = d^{\omega|_K}(a_1 g_1^*\beta) - \alpha|_K$$

By the choice of g we see that $g_1^*\beta$ is zero on U an hence can be extended by 0 to the whole of M. Moreover one sees that $\operatorname{inc}^*_t i_{\partial_t} g^*(\alpha|_K)$ is flat along ∂K and so the integral in the equation above can also be extended to M by 0. But this shows that $[\alpha] = 0 \in H^*_{d^\omega_K}(M)$. Since we have an injective mapping from $H^*_{d^\omega_K}(M)$ into the finite dimensional vector space $H^*_{d^{\omega|_K}}(K, \partial K)$ the space $H^*_{d^\omega_K}(M)$ has to be finite dimensional and hence $\frac{\ker d^\omega_K}{\Omega^*_K(M) \cap \operatorname{im} d^\omega_c} \subseteq \frac{\ker d^\omega_K}{\operatorname{Im} d^\omega_K} = H^*_{d^\omega_K}(M)$ too.

Since the inductive limit can be computed via these nice K we obtain

$$H^*_{d^\omega_c}(M) = \ker d^\omega_c / \operatorname{im} d^\omega_c = \frac{\varinjlim_K \ker d^\omega_K}{\varinjlim_K (\Omega^*_K(M) \cap \operatorname{im} d^\omega_c)} = \varinjlim_K \frac{\ker d^\omega_K}{\Omega^*_K(M) \cap \operatorname{im} d^\omega_c}$$

as strict inductive limit and the steps $\frac{\ker d^\omega_K}{\Omega^*_K(M)\cap \operatorname{im} d^\omega_c}$ are separated, finite dimensional topological vector spaces.

If M is non-orientable let $\pi : \tilde{M} \to M$ denote the orientation covering and let $f : \tilde{M} \to \tilde{M}$ be the unique non-trivial deck transformation. Then

$$\Omega^*_c(\tilde{M}) = \Omega^{*,\text{even}}_c(\tilde{M}) \oplus \Omega^{*,\text{odd}}_c(\tilde{M}) =: \{\sigma : f^*\sigma = \sigma\} \oplus \{\sigma : f^*\sigma = -\sigma\}$$

Its easily seen that $\pi^* : \Omega^*_c(M) \cong \Omega^{*,\text{even}}_c(\tilde{M})$ and hence $H^*_{d^\omega_c}(M) \cong H^{*,\text{even}}_{d^{\pi^*\omega}_c}(\tilde{M})$ which is a closed subspace of $H^*_{d^{\pi^*\omega}_c}(\tilde{M})$. Since the latter is a strict inductive limit of separated finite dimensional topological vector spaces $H^*_{d^\omega_c}(M)$ is so too. □

For every manifold N and every complete locally convex vector space E we define $C^\infty_c(N, E) = \varinjlim_K C^\infty_K(N, E)$. The following is a slight generalization of an argument due to A. Banyaga, see [Ban78] and [Ban97].

3.1.12. Corollary. *Let N, M be manifolds and ω a closed 1-form on M. Then every $f \in C^\infty_c(N, \operatorname{im} d^\omega_c)$ can be lifted, i.e. there exists $\tilde{f} \in C^\infty_c(N, \Omega^*_c(M))$, with $d^\omega_c \circ \tilde{f} = f$.*

Proof. Since $d^\omega_c : \Omega^*_c(M) \to \operatorname{im} d^\omega_c$ is onto and $\operatorname{im} d^\omega_c$ is complete the mapping

$$d^\omega_c \widehat{\otimes}_\pi \operatorname{id}_{C^\infty_c(N,\mathbb{R})} : \Omega^*_c(M) \widehat{\otimes}_\pi C^\infty_c(N, \mathbb{R}) \to \operatorname{im} d^\omega_c \widehat{\otimes}_\pi C^\infty_c(N, \mathbb{R})$$

is surjective. Since $C^\infty_c(N, \mathbb{R})$ is nuclear we obtain

$$\Omega^*_c(M) \widehat{\otimes}_\pi C^\infty_c(N, \mathbb{R}) \cong \Omega^*_c(M) \widehat{\otimes}_\varepsilon C^\infty_c(N, \mathbb{R}) \cong C^\infty_c(N, \Omega^*_c(M))$$

and

$$\operatorname{im} d^\omega_c \widehat{\otimes}_\pi C^\infty_c(N, \mathbb{R}) \cong \operatorname{im} d^\omega_c \widehat{\otimes}_\varepsilon C^\infty_c(N, \mathbb{R}) \cong C^\infty_c(N, \operatorname{im} d^\omega_c)$$

Via these isomorphisms $d^\omega_c \widehat{\otimes}_\pi \operatorname{id}_{C^\infty_c(N,\mathbb{R})}$ corresponds to $(d^\omega_c)_*$ and hence the latter is surjective too. See [Jar81] for the functional analysis involved. □

3.2 Locally Conformally Symplectic Manifolds

3.2.1. Definition. A locally conformally symplectic manifold is a triple (M, Ω, ω) where M is a $2n$-dimensional manifold, ω is a closed 1-form and Ω is a non-degenerated 2-form satisfying $0 = d^\omega\Omega = d\Omega + \omega \wedge \Omega$. Since Ω is non-degenerated we get a canonical vector bundle isomorphism $\flat : TM \cong T^*M$ given by $X \mapsto i_X\Omega$. By $\sharp$ we denote the inverse of $\flat$.

If $\dim M > 2$ then ω is uniquely determined by Ω. Otherwise there would exist a not everywhere vanishing ω' with $\omega' \wedge \Omega = 0$. Let $x \in M$ with $\omega'(x) \neq 0$. But then $\Omega(x) = \omega'(x) \wedge \eta$ for some $\eta \in \bigwedge^1 T^*_x M$. Indeed for any finite dimensional vector space V and $0 \neq w \in V$ one has an exact sequence

$$0 \to \mathbb{R} \xrightarrow{w\wedge\cdot} V \xrightarrow{w\wedge\cdot} \Lambda^2 V \xrightarrow{w\wedge\cdot} \Lambda^3 V \xrightarrow{w\wedge\cdot} \cdots \xrightarrow{w\wedge\cdot} \Lambda^{\dim V} V \to 0$$

But this would yield $\Omega^2(x) = 0$ a contradiction.

If (M, Ω, ω) is a locally conformally symplectic manifold and a is a nowhere vanishing function on M then $(M, \frac{1}{a}\Omega, \omega + \frac{da}{a})$ is again a locally conformally symplectic manifold. Two locally conformally symplectic manifolds (M, Ω, ω) and (M, Ω', ω') are called conformally equivalent iff there exists a nowhere vanishing function a on M with $\Omega' = \frac{1}{a}\Omega$ and $\omega' = \omega + \frac{da}{a} = \omega + d(\ln |a|)$. In this case we will write $(M, \Omega, \omega) \sim (M, \Omega', \omega')$ or $(M, \Omega, \omega) \overset{a}{\sim} (M, \Omega', \omega')$. So (M, Ω, ω) is conformally equivalent to a symplectic manifold iff $[\omega] = 0 \in H^1(M)$. It is obvious that conformal equivalence is an equivalence relation on the set of all locally conformally symplectic structures on M.

Suppose $\dim M = 2n$. A submanifold $i : L \hookrightarrow M$ is called Lagrangian iff $\dim L = n$ and $i^*\Omega = 0$. Notice that the Lagrangian submanifolds remain the same if we change (M, Ω, ω) conformally.

3.2.2. Remark. There is another way to look at locally conformally symplectic manifolds, see [Lee43]. Suppose we have an open covering $\mathcal{U}$ of a manifold M and for every $U \in \mathcal{U}$ a symplectic form $\Omega_U \in \Omega^2(U)$ such that $\Omega_U|_{U\cap V} = c_{UV}\Omega_V|_{U\cap V}$ for some locally constant functions $c_{UV} \in C^\infty(U \cap V, \mathbb{R}^+)$ (if $\dim(M) > 2$ then the c_{UV} are automatically locally constant). We have

$$\Omega_U|_{U\cap V\cap W} = c_{UV}\Omega_V|_{U\cap V\cap W} = c_{UV}c_{VW}\Omega_W|_{U\cap V\cap W}$$

and thus $c_{UV}c_{VW}|_{U\cap V\cap W} = c_{UW}|_{U\cap V\cap W}$. We set $\alpha_{UV} := \ln c_{UV} \in C^\infty(U\cap V, \mathbb{R})$ and obtain $\alpha_{VW} - \alpha_{UW} + \alpha_{UV} = 0$ on $U \cap V \cap W$. Next we choose a partition of unity $\{\lambda_U : U \in \mathcal{U}\}$ subordinated to $\mathcal{U}$ (i.e. $\operatorname{supp} \lambda_U \subseteq U$) and set,

$$\beta_U := \textstyle\sum_{W\in\mathcal{U}} \lambda_W \alpha_{UW}$$

where $\lambda_W\alpha_{UW}$ is extended by 0 to U. Then we get

$$\beta_U - \beta_V = \sum_{W\in\mathcal{U}} \lambda_W\alpha_{UW} - \sum_{W\in\mathcal{U}} \lambda_W\alpha_{VW} = \sum_{W\in\mathcal{U}} \lambda_W\big(\alpha_{UW} - \alpha_{VW}\big) = \sum_{W\in\mathcal{U}} \lambda_W\alpha_{UV} = \alpha_{UV}$$

on $U\cap V$. So we have found $f_U := \exp \beta_U \in C^\infty(U, \mathbb{R}^+)$ with $c_{UV} = \frac{f_U}{f_V}|_{U\cap V}$. We then have $0 = d\ln c_{UV} = d\ln f_U|_{U\cap V} - d\ln f_V|_{U\cap V}$ and we may define a closed 1-form $\omega \in \Omega^1(M)$ by $\omega|_U = d\ln f_U$. Moreover since we have $\Omega_U|_{U\cap V} = \frac{f_U}{f_V}\Omega_V|_{U\cap V}$ we also can define a non-degenerated 2-form $\Omega \in \Omega^2(M)$ by $\Omega|_U := \frac{1}{f_U}\Omega_U$. Finally we have

$$(d^\omega\Omega)|_U = d^{d\ln f_U}\big(\tfrac{1}{f_U}\Omega_U\big) = \tfrac{1}{f_U}d\Omega_U = 0$$

and thus (M, Ω, ω) is a locally conformally symplectic manifold. This locally conformally symplectic structure depends of course on the choice of f_U, but if we choose $f'_U \in C^\infty(U; \mathbb{R}^+)$ with $c_{UV} = \frac{f'_U}{f'_V}|_{U\cap V}$ the corresponding locally conformally symplectic structure (M, Ω', ω') is conformally equivalent to (M, Ω, ω). Indeed we have $\frac{f_U}{f_V}|_{U\cap V} = c_{UV} = \frac{f'_U}{f'_V}|_{U\cap V}$, so $\frac{f_U}{f'_U} = \frac{f_V}{f'_V}$ and we obtain a well defined function $a \in C^\infty(M, \mathbb{R}^+)$ with $a|_U = \frac{f_U}{f'_U}$. Moreover we have $\omega|_U = d\ln f_U = d\ln a|_U + d\ln f'_U = (d\ln a + \omega')|_U$ as well as $\Omega|_U = \frac{1}{f_U}\Omega_U = \frac{1}{a}\frac{1}{f'_U}\Omega_U = \frac{1}{a}\Omega'|_U$, i.e. $(M, \Omega', \omega') \overset{a}{\sim} (M, \Omega, \omega)$.

Conversely, if (M, Ω, ω) is a locally conformally symplectic manifold and $\mathcal{U}$ is an open covering such that $\omega|_U \in \Omega^1(U)$ is exact (if $\mathcal{U}$ is a covering of contractible open sets this is always true), then we find $f_U \in C^\infty(U, \mathbb{R}^+)$ with $\omega|_U = d\ln f_U$ and we let $\Omega_U := f_U\Omega|_U \in \Omega^2(U)$. Ω_U is closed for we have $0 = (d^\omega\Omega)|_U = d^{d\ln f_U}\left(\frac{1}{f_U}\Omega_U\right) = \frac{1}{f_U}d\Omega_U$ and it is obviously non-degenerated, i.e. Ω_U is a symplectic structure on U. Moreover we have

$$\Omega_U|_{U\cap V} = f_U\Omega|_{U\cap V} = \tfrac{f_U}{f_V} f_V\Omega|_{U\cap V} = \tfrac{f_U}{f_V}\Omega_V|_{U\cap V} =: c_{UV}\Omega_V|_{U\cap V}$$

and $d\ln\frac{f_U}{f_V}|_{U\cap V} = d\ln f_U|_{U\cap V} - d\ln f_V|_{U\cap V} = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$, i.e. $\ln\frac{f_U}{f_V}|_{U\cap V}$ is locally constant, and so is $c_{UV} = \frac{f_U}{f_V}|_{U\cap V}$ too.

3.2.3. Remark. To construct the f_U in remark 3.2.2 we actually used the fact that the sheaf $C^\infty(\cdot, \mathbb{R}^+) \cong C^\infty(\cdot, \mathbb{R})$ is a fine sheaf and is thus acyclic. Especially every 1-cocycle $\alpha_{VW} - \alpha_{UW} + \alpha_{UV} = 0$ is a boundary, i.e. $\alpha_{UV} = \beta_U - \beta_V$. For this argument it is essential that c_{UV} has values in $\mathbb{R}^+$ rather than in $\mathbb{R}^* := \mathbb{R}\setminus 0$. If we only have $c_{UV} \in C^\infty(U\cap V, \mathbb{R}^*)$ then the cohomology class $[c] \in H^1(M; C^\infty(\cdot, \mathbb{R}^*)) \cong H^1(M; \mathbb{Z}_2)$ is the obstruction to find f_U with $c_{UV} = \frac{f_U}{f_V}$.

For example the Möbius strip does not admit a locally conformally symplectic structure since it is non-orientable. But one can cover it by three open sets $\{U, V, W\}$ and easily find symplectic forms on them with $\Omega_U = \Omega_V$ on $U\cap V$, $\Omega_V = \Omega_W$ on $V\cap W$ and $\Omega_U = -\Omega_W$ on $U\cap W$.

3.2.4. Example. Let M be a n-dimensional manifold and let ω be a closed 1 form on M. Let Θ denote the canonical 1-form on T^*M. Recall that for $\alpha \in \Omega^1(M)$ considered as mapping $\alpha : M \to T^*M$ one has $\alpha^*\Theta = \alpha$. Define $\omega' := \pi^*\omega$, $\Omega' := d^{\omega'}\Theta$. Then (T^*M, Ω', ω') is a locally conformally symplectic manifold. Indeed let (U, q) be a chart for M and $(U' := \pi^{-1}(U), (q, p))$ be the induced chart for T^*M. It is well known that $\Theta|_{U'} = \sum_{i=1}^n p_i dq^i$. Moreover we have $\omega'|_{U'} = \sum_{j=1}^n w_j dq^j$ for some $w_j \in C^\infty(U', \mathbb{R})$. So we obtain

$$\begin{aligned}(\Omega')^n|_{U'} &= (d^{\omega'}\Theta)^n|_{U'} = (d\Theta + \omega'\wedge\Theta)^n|_{U'}\\ &= (d\Theta)^n|_{U'} + n(d\Theta)^{n-1}\wedge\omega'\wedge\Theta|_{U'}\\ &= \left(\textstyle\sum_{i=1}^n dp_i\wedge dq^i\right)^n + n\left(\textstyle\sum_{i=1}^n dp_i\wedge dq^i\right)^{n-1}\wedge\left(\textstyle\sum_{j=1}^n w_j dq^j\right)\wedge\left(\textstyle\sum_{k=1}^n p_k dq^k\right)\\ &= \left(\textstyle\sum_{i=1}^n dp_i\wedge dq^i\right)^n = n dp_1\wedge dq^1\wedge dp_2\wedge dq^2\wedge\cdots\end{aligned}$$

since every summand in the other term contains $n+1$ dq^i's and is hence zero. This shows that Ω' is non-degenerated.

For $\alpha \in \Omega^1(M)$ we have $\alpha^*\Omega' = \alpha^* d^{\omega'}\Theta = d^{\alpha^*\omega'}\alpha^*\Theta = d^\omega\alpha$. So $\mathrm{im}(\alpha)$ is a Lagrangian submanifold of (T^*M, Ω', ω') if and only if $d^\omega\alpha = 0$. Since $\pi^* : H^*(M) \cong H^*(T^*M)$ we have (T^*M, Ω', ω') is conformally equivalent to a symplectic manifold if and only if $[\omega] = 0 \in H^1(M)$.

3.2.5. Example. On S^3 there exists a global frame of 1-forms $\alpha, \beta, \gamma \in \Omega^1(S^3)$ satisfying

$$d\alpha = \beta\wedge\gamma \qquad d\beta = \gamma\wedge\alpha \qquad d\gamma = \alpha\wedge\beta.$$

This is because S^3 is a Lie group with Lie algebra $\mathfrak{so}(3, \mathbb{R})$ and the latter has a basis $\{A, B, C\}$ satisfying

$$[A, B] = C \qquad [B, C] = A \qquad [C, A] = B.$$

If $\{a,b,c\}$ denotes the dual basis to $\{A,B,C\}$ then α,β,γ are the left invariant 1-forms corresponding to a,b,c respectively. Let $\omega := dt \in \Omega^1(S^1)$ and $\Omega := d^\omega\alpha \in \Omega^2(S^1 \times S^3)$. Since

$$\Omega^2 = (d^\omega\alpha)^2 = (d\alpha + \omega\wedge\alpha)^2 = (\beta\wedge\gamma + dt\wedge\alpha)^2 = 2dt\wedge\alpha\wedge\beta\wedge\gamma$$

Ω is non-degenerated and $(S^1\times S^3,\Omega,\omega)$ is a locally conformally symplectic manifold.

However $S^1\times S^3$ does not admit a symplectic structure, for this would be exact since $H^2(S^1\times S^3) = 0$ and hence would give rise to an exact volume form on $S^1\times S^3$, a contradiction since $S^1\times S^3$ is compact.

If g is a diffeomorphism of M then $(M, g^*\Omega, g^*\omega)$ is again a locally conformally symplectic manifold. We write $\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ for the group of all compactly supported diffeomorphisms that preserve the locally conformally symplectic structure up to conformal equivalence, i.e.

$$\mathrm{Diff}^\infty_c(M,\Omega,\omega) := \big\{g\in \mathrm{Diff}^\infty_c(M) : (M,g^*\Omega,g^*\omega)\sim(M,\Omega,\omega)\big\}$$

More explicitly $g\in\mathrm{Diff}^\infty_c(M,\Omega,\omega)$ iff there exists $a\in C^\infty(M,\mathbb{R}\setminus 0)$ such that $g^*\Omega = \frac{1}{a}\Omega$ and $g^*\omega = \omega + d(\ln|a|)$. If $\dim M > 2$ then the first equation implies the second since ω is unique. Moreover we define

$$\mathfrak{X}_c(M,\Omega,\omega) := \big\{X\in\mathfrak{X}_c(M) : \exists f\in C^\infty(M,\mathbb{R}) \text{ with } L_X\Omega = -f\Omega, L_X\omega = df\big\}$$

Again, if $\dim M > 2$ then the equation $L_X\Omega = -f\Omega$ implies the equation $L_X\omega = df$. Indeed $d\Omega + \omega\wedge\Omega = 0$ and $L_X\Omega = -f\Omega$ give

$$\begin{aligned}0 &= dL_X\Omega + L_X\omega\wedge\Omega + \omega\wedge L_X\Omega = -d(f\Omega) + L_X\omega\wedge\Omega - f\omega\wedge\Omega\\ &= -df\wedge\Omega + f\omega\wedge\Omega + L_X\omega\wedge\Omega - f\omega\wedge\Omega = (L_X\omega - df)\wedge\Omega\end{aligned}$$

and so $L_X\omega = df$.

3.2.6. Lemma. *Let $g\in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}^\infty_c(M),\mathrm{id})\big)$. Then*

$$g\in C^\infty\big(\mathbb{R},\mathrm{Diff}^\infty_c(M,\Omega,\omega)\big) \quad\Leftrightarrow\quad \delta^r g\in\Omega^1\big(\mathbb{R};\mathfrak{X}_c(M,\Omega,\omega)\big) \quad\Leftrightarrow\quad \dot g_t\in\mathfrak{X}_c(M,\Omega,\omega).$$

Especially $\mathrm{Fl}^X\in C^\infty\big(\mathbb{R},\mathrm{Diff}^\infty_c(M,\Omega,\omega)\big)$ *iff* $X\in\mathfrak{X}_c(M,\Omega,\omega)$.

Proof. Suppose we have $g : (\mathbb{R},0)\to\big(\mathrm{Diff}^\infty_c(M,\Omega,\omega),\mathrm{id}\big)$. Then there exists $a\in C^\infty(\mathbb{R}\times M,\mathbb{R})$ with $g_t^*\Omega = \frac{1}{a_t}\Omega$ and $g_t^*\omega = \omega + d(\ln|a_t|)$. Differentiating these equations with respect to t we obtain $L_{\dot g_t}\Omega = -(g_t^{-1})^*(\frac{\dot a_t}{a_t})\Omega$ and $L_{\dot g_t}\omega = d((g_t^{-1})^*\frac{\dot a_t}{a_t})$, where $\dot a_t := \frac{\partial}{\partial t}a_t$. Hence $\dot g_t\in\mathfrak{X}_c(M,\Omega,\omega)$ with $f_{\dot g_t} = (g_t^{-1})^*\frac{\dot a_t}{a_t}$.

Suppose conversely $L_{\dot g_t}\Omega = -f_t\Omega$ and $L_{\dot g_t}\omega = df_t$. Then we define $a_t := \exp(\int_0^t g_s^* f_s ds)$. It satisfies $g_t^* f_t = \frac{\dot a_t}{a_t}$ and $a_0 = 1$. So we obtain the following differential equation for $g_t^*\Omega$

$$\tfrac{\partial}{\partial t}(g_t^*\Omega) = -\tfrac{\dot a_t}{a_t}(g_t^*\Omega) \quad\text{with initial condition}\quad g_0^*\Omega = \Omega.$$

This equation has a solution namely $\frac{1}{a_t}\Omega$ and since the solution is unique (evaluate everything at points $x\in M$ and obtain differential equations in finite dimensional spaces) we obtain $g_t^*\Omega = \frac{1}{a_t}\Omega$. Similarly we check $g_t^*\omega = \omega + d(\ln|a_t|)$. For $t = 0$ this follows from $a_0 = 1$. Moreover we have

$$\tfrac{\partial}{\partial t}(g_t^*\omega) = g_t^* L_{\dot g_t}\omega = g_t^* df_t = d(g_t^* f_t) = d(\tfrac{\dot a_t}{a_t}) = \tfrac{\partial}{\partial t}(\omega + d\ln|a_t|)$$

and so equality holds for all t. □

3.2.7. Lemma. *Let (M,Ω_0,ω_0), (M,Ω_1,ω_1) be two locally conformally symplectic structures on M, $i: N\to M$ a closed submanifold such that $\Omega_0=\Omega_1$ on $TM|_N$ and $i^*\omega_0=i^*\omega_1$. Then there exist open neighborhoods U, V of N and a diffeomorphism $f: U\to V$, which is the identity on N, and such that $(U,f^*\Omega_1,f^*\omega_1)$ is conformally equivalent to $(U,\Omega_0|_U,\omega_0|_U)$.*

Proof. Since $i^*\omega_0=i^*\omega_1$ there exists a neighborhood U of $i(N)$ and a nowhere vanishing function a on U such that $\omega_0=\omega_1+d\ln|a|$ on U and $i^*a=1$. So changing (Ω_0,ω_0) conformally (by a) we may assume $\omega_0=\omega_1=\omega$.

Since we have $\Omega_1=\Omega_0$ on $TM|_N$ there exists a possibly smaller open neighborhood U of N such that $\Omega_t:=t(\Omega_1-\Omega_0)+\Omega_0$ is non-degenerated on U for all $t\in I$. So $(U,\Omega_t|_U,\omega|_U)$ is a locally conformally symplectic manifold for all $t\in I$. Moreover we have $i^*(\Omega_1-\Omega_0)=0$ and hence by corollary 3.1.2 there exists $\alpha\in\Omega^1(U)$, which vanishes on N such that $d^\omega\alpha=\Omega_1-\Omega_0=\frac{\partial}{\partial t}\Omega_t$ on a possibly smaller U. We define a time dependent vector field $X_t\in\mathfrak{X}(U)$, by $i_{X_t}\Omega_t:=-\alpha$ and denote the curve of local diffeomorphisms corresponding to it by f_t. Since X_t vanishes on $i(N)$, f_t is the identity on $i(N)$ and hence it is possible to shrink U, such that f_t is defined on U for all $t\in I$. Next we have

$$\begin{aligned}\tfrac{\partial}{\partial t}(f_t^*\Omega_t)&=f_t^*(L_{X_t}\Omega_t+\tfrac{\partial}{\partial t}\Omega_t)=f_t^*(di_{X_t}\Omega_t+i_{X_t}d\Omega_t+d^\omega\alpha)\\&=f_t^*\big(-d\alpha-i_{X_t}(\omega\wedge\Omega_t)+d^\omega\alpha\big)=f_t^*\big(\omega\wedge\alpha-(i_{X_t}\omega)\Omega_t+\omega\wedge i_{X_t}\Omega_t\big)\\&=-(f_t^*i_{X_t}\omega)(f_t^*\Omega_t)=-(\mathrm{inc}_t^*\,i_{\partial_t}f^*\omega)(f_t^*\Omega_t)=-\tfrac{\dot a_t}{a_t}(f_t^*\Omega_t)\end{aligned}$$

with $a_t:=\exp\big(\int_0^t\mathrm{inc}_s^*\,i_{\partial_t}f^*\omega ds\big)$. Since the solution of such a differential equation is unique we obtain $f_t^*\Omega_t=\frac{1}{a_t}\Omega_0$. Moreover by the definition of a_t we have $f_t^*\omega=\omega+d\ln|a_t|$ and hence especially $(U,f_1^*\Omega_1,f_1^*\omega)\overset{a_1}{\sim}(U,\Omega_0|_U,\omega|_U)$. □

3.2.8. Definition. Let M be a manifold. $J\in T^*M\otimes TM$ is called an almost complex structure if $J^2=-\mathrm{id}$. Let $\Omega\in\Omega^2(M)$ be non-degenerated. Then J is called Ω-compatible if the following two conditions are satisfied:

1. $\Omega(X,JX)>0$ for all $0\neq X\in TM$
2. $\Omega(JX,Y)+\Omega(X,JY)=0$ for all $X,Y\in T_xM$, $x\in M$

3.2.9. Lemma. *Let Ω be a non-degenerated 2-form on M. Then the space of all Ω-compatible almost complex structures is non-empty and contractible.*

Proof. Let $R(M)$ denote the space of Riemannian metrics on M and $J(M,\Omega)$ the space of all Ω-compatible almost complex structures on M. We have a mapping:

$$i: J(M,\Omega)\to R(M)\qquad J\mapsto g_J\quad\text{where}\quad g_J(X,Y):=\Omega(X,JY)$$

Given a Riemannian metric g we define $A\in T^*M\otimes TM$ by $\Omega(X,AY)=g(X,Y)$. Then

$$g(AX,Y)=\Omega(AX,AY)=-\Omega(AY,AX)=-g(AY,X)=-g(X,AY)=-g(A^*X,Y)$$

and hence $A^*=-A$. Moreover A is invertible and so using polar decomposition there exist unique $B,J\in T^*M\otimes TM$ such that $B^*=B$, $B>0$, $JJ^*=\mathrm{id}$ and $A=BJ$. From

$$BJ=A=-A^*=-J^*B^*=-J^*B=(J^*BJ)(-J^*)$$

and the uniqueness of the polar decomposition we obtain $-J^* = J$ and $BJ = JB$. So $J^2 = -\mathrm{id}$. Moreover we have:

$$\Omega(BX, JBX) = \Omega(BX, AX) = g(BX, X) > 0 \qquad \forall X \neq 0$$

and

$$\begin{aligned}\Omega(JBX, BY) + \Omega(BX, JBY) &= \Omega(AX, BY) + \Omega(BX, AY)\\ &= -g(BY, X) + g(BX, Y)\\ &= -g(Y, BX) + g(BX, Y) = 0\end{aligned}$$

Since B is onto this shows that $J_g := J$ is a Ω-compatible almost complex structure. So we have another mapping

$$r : R(M) \to J(M, \Omega) \qquad g \mapsto J_g$$

One readily sees that $r \circ i = \mathrm{id}$ and hence $J(M, \Omega)$ is a retract of $R(M)$. Since the latter is non-empty and contractible $J(M, \Omega)$ is so too. □

3.2.10. Lemma. *Let (M, Ω_0, ω_0) and (M, Ω_1, ω_1) be two locally conformally symplectic structures on M, suppose $i : L \to M$ is a Lagrangian submanifold for both structures and $[i^*\omega_0] = [i^*\omega_1] \in H^1(L)$. Then there exist open neighborhoods U, V of L and a diffeomorphism $f : U \to V$, which is the identity on L, such that $(U, f^*\Omega_1, f^*\omega_1)$ is conformally equivalent to $(U, \Omega_0|_U, \omega_0|_U)$.*

Proof. Choose Ω_i-compatible almost complex structures J_i on M, $i = 0, 1$. Since L is a Lagrangian submanifold we obtain $J_i(TL) \oplus TL = TM|_L$. Moreover Ω_i induces an isomorphism of vector bundles $\flat_i : J_i(TL) \cong T^*L$. We define an isomorphism of vector bundles $A : TM|_L \to TM|_L$ by

$$A|_{TL} = \mathrm{id} \qquad \text{and} \qquad A|_{J_0(TL)} = \flat_1^{-1}\flat_0 : J_0(TL) \to J_1(TL)$$

One easily checks

$$\Omega_1(AX, AY) = \Omega_0(X, Y) \qquad \forall X, Y \in T_xL, x \in L.$$

Denote by $\bar{A} : N(L) \to N(L)$ the isomorphism induced from A, where $N(L)$ denotes the normal bundle of L in M. Next choose a tubular neighborhood $g : N(L) \to U$ of L and define a diffeomorphism $h := g \circ \bar{A} \circ g^{-1} : U \to U$. Then we obtain $h|_L = \mathrm{id}$ and $h^*\Omega_1 = \Omega_0$ along L. Moreover we have $i^*(h^*\omega_1) = i^*\omega_1 = i^*\omega_0$ and we may apply lemma 3.2.7 to finish the proof. □

3.2.11. Corollary. *Let (M, Ω, ω) be a locally conformally symplectic manifold and let $i : L \to M$ be a Lagrangian submanifold. Then there exists an open neighborhood U of the zero section in T^*L, an open neighborhood V of L in M and a diffeomorphism $f : U \to V$ such that $(U, f^*\Omega, f^*\omega)$ is conformally equivalent to $(U, d^{\pi^*i^*\omega}\Theta|_U, \pi^*i^*\omega|_U)$ (cf. example 3.2.4).*

Proof. Choose a compatible almost complex structure J on (M, Ω, ω). Then $J(TL)$ is a Lagrangian complement to TL, i.e. $TM|_L \cong TL \oplus J(TL)$ and hence $N(TL) := TM|_L/TL \cong J(TL)$. Moreover Ω induces an isomorphism $J(TL) \cong T^*L$. Now choose a tubular neighborhood $g : T^*L \cong N(L) \to U \subseteq M$ of L. Then the image of the zero section $s : L \to T^*L$ is a Lagrangian submanifold for $(T^*L, g^*\Omega, g^*\omega)$ and $(T^*L, d^{\pi^*i^*\omega}\Theta, \pi^*i^*\omega)$. Moreover $s^*g^*\omega = i^*\omega = s^*\pi^*i^*\omega$ and we can apply lemma 3.2.10. □

3.2.12. Lemma. *Let M be compact and ω a closed 1-form on M. Suppose (M,Ω_0,ω) and (M,Ω_1,ω) are two locally conformally symplectic structures, which are C^0-close and such that $[\Omega_1]=[\Omega_0]\in H^2_{d^\omega}(M)$. Then there exists a diffeomorphism $f\in \mathrm{Diff}^\infty(M)_\circ$ such that $(M,f^*\Omega_1,f^*\omega)$ is conformally equivalent to (M,Ω_0,ω).*

Proof. $\Omega_t := t(\Omega_1-\Omega_0)+\Omega_0$ is non-degenerated for all $t\in I$, since Ω_1 and Ω_0 are C^0-close. Since $[\Omega_1]=[\Omega_0]\in H^2_{d^\omega}(M)$ there exists $\alpha\in\Omega^1(M)$ such that $d^\omega\alpha=\Omega_1-\Omega_0=\frac{\partial}{\partial t}\Omega_t$. Let X_t be the time dependent vector field on M defined by $i_{X_t}\Omega_t=-\alpha$ and f_t the curve of diffeomorphisms corresponding to it. The same calculation as in the proof of lemma 3.2.7 yields $f_t^*\Omega_t=\frac{1}{a_t}\Omega_0$ and $f_t^*\omega=\omega+d\ln|a_t|$, hence f_1 is the desired diffeomorphism. □

3.3 Jacobi Manifolds

Let $\mathfrak{X}^k(M)$ denote the set of all skew symmetric multi vector fields on M, i.e. the sections of the vector bundle $\Lambda^k TM$. Recall that the Scouten bracket

$$[\cdot,\cdot]:\mathfrak{X}^p(M)\times\mathfrak{X}^q(M)\to\mathfrak{X}^{p+q-1}(M)$$

is the bilinear extension of the Lie derivative which has the following properties

$$\begin{aligned}
[P,Q]&=(-1)^{pq}[Q,P]\\
[P,Q\wedge R]&=[P,Q]\wedge R+(-1)^{pq-q}Q\wedge[P,R]\\
0&=(-1)^{p(r-1)}[P,[Q,R]]+(-1)^{q(p-1)}[Q,[R,P]]+(-1)^{r(p-1)}[R,[P,Q]]
\end{aligned}$$

where $p=\deg P$, $q=\deg Q$ and $r=\deg R$. Moreover one has

$$i_{[P,Q]}\alpha=(-1)^{q(p+1)}i_Pdi_Q\alpha+(-1)^pi_Qdi_P\alpha-i(P\wedge Q)d\alpha \tag{3.1}$$

for $P\in\mathfrak{X}^p(M)$, $Q\in\mathfrak{X}^q(M)$ and $\alpha\in\Omega^{p+q-1}(M)$. See [Vai94] for all this.

In the next lemma we collect a few formulas we will need in the sequel.

3.3.1. Lemma. *Let $\Lambda\in\mathfrak{X}^2(M)$ and $\sigma\in\Omega^1(M)$. Then we have*

$$\tfrac12[\Lambda,\Lambda](\alpha,\beta,\gamma)=\sum_{cycl(\alpha,\beta,\gamma)}\Lambda(d\Lambda(\alpha,\beta),\gamma) \tag{3.2}$$

$$\tfrac12[\Lambda,\Lambda](\alpha,\beta)=\Lambda(d\Lambda(\alpha,\beta))-[\Lambda(\alpha),\Lambda(\beta)] \tag{3.3}$$

$$[\Lambda(\sigma),\Lambda](\alpha,\beta)=(d\sigma)(\Lambda(\alpha),\Lambda(\beta))-\tfrac12[\Lambda,\Lambda](\sigma,\alpha,\beta) \tag{3.4}$$

for closed 1-forms α,β,γ.

Proof. The first equation follows immediately from (3.1). For the second use again (3.1) to compute

$$\begin{aligned}
\gamma\big(\tfrac12[\Lambda,\Lambda](\alpha,\beta)\big)&=i_\Lambda di_\Lambda(\alpha\wedge\beta\wedge\gamma)\\
&=\Lambda(d\Lambda(\alpha,\beta),\gamma)+\Lambda(d\Lambda(\beta,\gamma),\alpha)+\Lambda(d\Lambda(\gamma,\alpha),\beta)\\
&=\gamma\big(\Lambda(d\Lambda(\alpha,\beta))\big)-L_{\Lambda(\alpha)}i_{\Lambda(\beta)}\gamma+i_{\Lambda(\beta)}L_{\Lambda(\alpha)}\gamma\\
&=\gamma\big(\Lambda(d\Lambda(\alpha,\beta))\big)-i_{[\Lambda(\alpha),\Lambda(\beta)]}\gamma\\
&=\gamma\big(\Lambda(d\Lambda(\alpha,\beta))-[\Lambda(\alpha),\Lambda(\beta)]\big)
\end{aligned}$$

Remains to prove the third. On the one hand we have

$$\begin{aligned}(d\sigma)(\Lambda(\alpha),\Lambda(\beta)) &= L_{\Lambda(\alpha)}\sigma(\Lambda(\beta)) - L_{\Lambda(\beta)}\sigma(\Lambda(\alpha)) - \sigma([\Lambda(\alpha),\Lambda(\beta)])\\ &= i_{\Lambda(\alpha)}d\Lambda(\beta,\sigma) - i_{\Lambda(\beta)}d\Lambda(\alpha,\sigma) - \sigma(\Lambda(d\Lambda(\alpha,\beta))) + \sigma(\tfrac{1}{2}[\Lambda,\Lambda](\alpha,\beta))\\ &= \Lambda(\alpha, d\Lambda(\beta,\sigma)) - \Lambda(\beta, d\Lambda(\alpha,\sigma)) - \Lambda(d\Lambda(\alpha,\beta),\sigma) + \tfrac{1}{2}[\Lambda,\Lambda](\alpha,\beta,\sigma)\\ &= -\sum_{\text{cycl}(\alpha,\beta,\sigma)} \Lambda(d\Lambda(\alpha,\beta),\sigma) + \tfrac{1}{2}[\Lambda,\Lambda](\sigma,\alpha,\beta)\end{aligned}$$

and on the other hand

$$\begin{aligned}[\Lambda(\sigma),\Lambda](\alpha,\beta) &= (L_{\Lambda(\sigma)}\Lambda)(\alpha,\beta) = L_{\Lambda(\sigma)}(\Lambda(\alpha,\beta)) - \Lambda(L_{\Lambda(\sigma)}\alpha,\beta) - \Lambda(\alpha, L_{\Lambda(\sigma)}\beta)\\ &= i_{\Lambda(\sigma)}d\Lambda(\alpha,\beta) - \Lambda(di_{\Lambda(\sigma)}\alpha,\beta) - \Lambda(\alpha, di_{\Lambda(\sigma)}\beta)\\ &= \Lambda(\sigma, d\Lambda(\alpha,\beta)) - \Lambda(d\Lambda(\sigma,\alpha),\beta) - \Lambda(\alpha, d\Lambda(\sigma,\beta))\\ &= -\sum_{\text{cycl}(\alpha,\beta,\sigma)} \Lambda(d\Lambda(\alpha,\beta),\sigma)\end{aligned}$$

which proves the third equation. □

3.3.2. Definition. A Jacobi manifold is a manifold together with a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M,\mathbb{R})$ which is local, i.e. $\text{supp}(\{f,g\}) \subseteq \text{supp}(f) \cap \text{supp}(g)$.

One can show (see [GL84] and [DLM91]) that such brackets are in one-to-one correspondence with triples (M,Λ,E), where Λ is a skew symmetric bivector field and E is an ordinary vector field on M which satisfies the following relations

$$[\Lambda,\Lambda] = 2E\wedge\Lambda \quad \text{and} \quad L_E\Lambda = [E,\Lambda] = 0 \tag{3.5}$$

The bracket is then given by:

$$\{f,g\} = \Lambda(df,dg) + f dg(E) - g df(E) \tag{3.6}$$

We only show the following:

3.3.3. Lemma. *Let Λ be a skew symmetric bivector field on M and $E \in \mathfrak{X}(M)$. Then* (3.6) *defines a skew symmetric bilinear bracket on $C^\infty(M,\mathbb{R})$ which is local. It satisfies the Jacobi identity if and only if* (3.5) *are satisfied.*

Proof. Since the Lie derivative commutes with contractions we have

$$(L_E\Lambda)(\alpha,\beta) = L_E(\Lambda(\alpha,\beta)) - \Lambda(L_E\alpha,\beta) - \Lambda(\alpha,L_E\beta) \tag{3.7}$$

for 1-forms α,β. Next we calculate

$$\begin{aligned}\{\{f,g\},h\} &= \Lambda\big(d(\Lambda(df,dg)),dh\big)\\ &\quad - E(df)\Lambda(dg,dh) - E(dg)\Lambda(dh,df) + E(dh)\Lambda(df,dg)\\ &\quad + f\Lambda(L_E dg,dh) + g\Lambda(dh,L_E df) - hL_E(\Lambda(df,dg))\\ &\quad + f dg(E)dh(E) - g dh(E)df(E) + ghL_EL_Ef - hfL_EL_Eg\end{aligned}$$

Taking the sum over all cyclic permutations in (f,g,h) and using (3.2) and (3.7) we obtain

$$\sum_{\text{cycl}(f,g,h)} \{\{f,g\},h\} = \big(\tfrac{1}{2}[\Lambda,\Lambda] - E\wedge\Lambda\big)(df,dg,dh) - \sum_{\text{cycl}(f,g,h)} f(L_E\Lambda)(dg,dh)$$

and hence the Jacobi identity is equivalent to (3.5). □

Suppose $\{\cdot,\cdot\}$ is a bracket as above and a is a nowhere vanishing function on M then one can define a new Jacobi bracket by $\{f,g\}_a := \frac{1}{a}\{af,ag\}$. If $\{f,g\} = \Lambda(df,dg) + fdg(E) - gdf(E)$ then we obtain

$$\begin{aligned}\tfrac{1}{a}\{af,ag\} &= \tfrac{1}{a}\Lambda(fda+adf,gda+adg) + fa(dg)(E) - ga(df)(E)\\ &= (a\Lambda)(df,dg) + f(dg)(aE+\Lambda(da)) - g(df)(aE+\Lambda(da))\end{aligned}$$

and hence the bracket $\{\cdot,\cdot\}_a$ corresponds to (Λ_a, E_a), where $\Lambda_a = a\Lambda$, $E_a = aE + \Lambda(da)$.

A mapping $h \in \mathrm{Diff}(M)$ is called Poisson diffeomorphism iff $\{f,g\}\circ h = \{f\circ h, g\circ h\}_a$ for a nowhere vanishing function a and all $f,g \in C^\infty(M,\mathbb{R})$. Since we have

$$\begin{aligned}\{f,g\}\circ h &= h^*\big(\Lambda(df,dg) + fdg(E) - gdf(E)\big)\\ &= (h^*\Lambda)(dh^*f,dh^*g) + (h^*f)(dh^*g)(h^*E) - (h^*g)(dh^*f)(h^*E)\\ \{f\circ h, g\circ h\}_a &= \Lambda_a(dh^*f,dh^*g) + (h^*f)(dh^*g)(E_a) - (h^*g)(dh^*f)(E_a)\end{aligned}$$

this is the case if and only if $h^*\Lambda = \Lambda_a$ and $h^*E = E_a$.

3.3.4. Definition. If (M,Λ,E) is a Jacobi manifold and $f \in C^\infty(M,\mathbb{R})$ then $X_f := \Lambda(df) + fE$ is called the Hamiltonian vector field to f.

3.3.5. Lemma. *We have* $X_{\{f,g\}} = [X_f, X_g]$.

Proof. Since we have $\frac{1}{2}[\Lambda,\Lambda] = E\wedge\Lambda$ equation (3.3) gives

$$\Lambda(d\Lambda(df,dg)) - [\Lambda(df),\Lambda(dg)] = (L_Ef)\Lambda(dg) - (L_Eg)\Lambda(df) + \Lambda(df,dg)E \tag{3.8}$$

Moreover $L_E\Lambda = 0$ yields $0 = (L_E\Lambda)(df) = L_E(\Lambda(df)) - \Lambda(L_Edf)$ and so

$$[E,\Lambda(df)] = \Lambda(L_Edf) \tag{3.9}$$

Using (3.8) and (3.9) we obtain

$$\begin{aligned}X_{\{f,g\}} &= \Lambda(d\{f,g\}) + \{f,g\}E\\ &= \Lambda(d\Lambda(df,dg)) + \Lambda(d(fL_Eg)) - \Lambda(d(gL_Ef)) + \{f,g\}E\\ &= \Lambda(d\Lambda(df,dg)) + f\Lambda(L_Edg) + (L_Eg)\Lambda(df) - g\Lambda(L_Edf) - (L_Ef)\Lambda(dg) + \{f,g\}E\\ &= [\Lambda(df),\Lambda(dg)] + \Lambda(df,dg)E + f[E,\Lambda(dg)] + g[\Lambda(df),E] + \{f,g\}E\end{aligned}$$

On the other hand we have

$$\begin{aligned}[X_f,X_g] &= [\Lambda(df)+fE, \Lambda(dg)+gE]\\ &= [\Lambda(df),\Lambda(dg)] + g[\Lambda(df),E] + (L_{\Lambda(df)}g)E + f[E,\Lambda(dg)] - (L_{\Lambda(dg)}f)E\\ &\quad + f(L_Eg)E - g(L_Ef)E\\ &= [\Lambda(df),\Lambda(dg)] + g[\Lambda(df),E] + \Lambda(df,dg)E + f[E,\Lambda(dg)] - \Lambda(dg,df)E\\ &\quad + f(L_Eg)E - g(L_Ef)E\\ &= [\Lambda(df),\Lambda(dg)] + g[\Lambda(df),E] + f[E,\Lambda(dg)] + \Lambda(df,dg)E + \{f,g\}E\end{aligned}$$

□

So the Hamiltonian vector fields X_f span a generalized distribution which is involutive. One can show that it is integrable and on every leave there exists a unique induced Jacobi structure. So one is led to the study of so called transitive Jacobi manifolds, that is Jacobi manifolds where this foliation consists only of one leave. A proof of the following proposition (using local coordinates) can be found in [GL84].

3.3.6. Proposition. *There exists a natural one to one correspondence between even dimensional, transitive Jacobi manifolds and locally conformally symplectic manifolds. A diffeomorphism is a Jacobi diffeomorphism iff it preserves the corresponding locally conformally symplectic structure up to conformal change. Moreover if* (M, Ω, ω) *corresponds to* (M, Λ, E) *and a is a nowhere vanishing function on* M *then* (M, Ω_a, ω_a) *corresponds to* (M, Λ_a, E_a)*, where* $\Omega_a = \frac{1}{a}\Omega$*,* $\omega_a = \omega + d\ln|a|$*,* $\Lambda_a = a\Lambda$ *and* $E_a = aE + \Lambda(da)$*.*

Proof. If (Ω, ω) is given then $\sharp^{-1} = \flat : \mathfrak{X}(M) \to \Omega^1(M)$, $\flat X := i_X\Omega$ is a $C^\infty(M, \mathbb{R})$-linear isomorphism. We define $E := \sharp\omega \in \mathfrak{X}(M)$ and $\Lambda \in \mathfrak{X}^2(M)$ by

$$\Lambda(\alpha, \beta) := \Omega(\sharp\beta, \sharp\alpha) = \beta(\sharp\alpha) = -\alpha(\sharp\beta)$$

Notice that $\Lambda(\alpha) = \sharp\alpha$. Suppose conversely (Λ, E) is given. Then $\Lambda(T_x^*M) = T_xM$, for $\Lambda(df) + fE$ span TM, $\Lambda(T_x^*M)$ is even dimensional (Λ is skew symmetric) and M is even dimensional. So $\flat^{-1} = \sharp : \Omega^1(M) \to \mathfrak{X}(M)$, $\sharp\alpha := \Lambda(\alpha)$ is a $C^\infty(M, \mathbb{R})$-linear isomorphism. We define $\omega := \flat(E) \in \Omega^1(M)$ and $\Omega \in \Omega^2(M)$ by $\Omega(X, Y) := \Lambda(\flat Y, \flat X) = Y(\flat X)$. Obviously these constructions are inverse to each other.

Next we show that $d\omega = 0$, $d^\omega\Omega = 0$ are equivalent to $L_E\Lambda = 0$, $[\Lambda, \Lambda] = 2E \wedge \Lambda$. For closed 1-forms α, β, γ we obtain form (3.2) and (3.3)

$$\begin{aligned}(d\Omega)(\sharp\alpha, \sharp\beta, \sharp\gamma) &= L_{\sharp\alpha}\Omega(\sharp\beta, \sharp\gamma) - L_{\sharp\beta}\Omega(\sharp\alpha, \sharp\gamma) + L_{\sharp\gamma}\Omega(\sharp\alpha, \sharp\beta) \\ &\quad - \Omega([\sharp\alpha, \sharp\beta], \sharp\gamma) + \Omega([\sharp\alpha, \sharp\gamma], \sharp\beta) - \Omega([\sharp\beta, \sharp\gamma], \sharp\alpha) \\ &= \sum i_{\sharp\alpha} d\Lambda(\gamma, \beta) + \sum \gamma([\sharp\alpha, \sharp\beta]) \\ &= \sum \Lambda(\alpha, d\Lambda(\gamma, \beta)) + \sum \Lambda(d\Lambda(\alpha, \beta), \gamma) - \sum \tfrac{1}{2}[\Lambda, \Lambda](\alpha, \beta, \gamma) \\ &= [\Lambda, \Lambda](\alpha, \beta, \gamma) - \tfrac{3}{2}[\Lambda, \Lambda](\alpha, \beta, \gamma) = -\tfrac{1}{2}[\Lambda, \Lambda](\alpha, \beta, \gamma)\end{aligned}$$

where the sums are over all cyclic permutations of α, β, γ. Moreover we have

$$(\omega \wedge \Omega)(\sharp\alpha, \sharp\beta, \sharp\gamma) = \sum \omega(\sharp\alpha)\Omega(\sharp\beta, \sharp\gamma) = -\sum \alpha(E)\Lambda(\gamma, \beta) = (E \wedge \Lambda)(\alpha, \beta, \gamma)$$

and so we obtain

$$(d^\omega\Omega)(\sharp\alpha, \sharp\beta, \sharp\gamma) = \big(-\tfrac{1}{2}[\Lambda, \Lambda] + E \wedge \Lambda\big)(\alpha, \beta, \gamma)$$

This shows that $d^\omega\Omega = 0$ is equivalent to $[\Lambda, \Lambda] = 2E \wedge \Lambda$. Finally we have

$$\tfrac{1}{2}[\Lambda, \Lambda](\omega) = (E \wedge \Lambda)(\omega) = E(\omega)\Lambda - E \wedge \Lambda(\omega) = \Omega(E, E)\Lambda - E \wedge E = 0$$

So (3.4) gives

$$[E, \Lambda](\alpha, \beta) = (d\omega)(\sharp\alpha, \sharp\beta)$$

and $L_E\Lambda = 0$ is equivalent to $d\omega = 0$, provided that $[\Lambda, \Lambda] = 2E \wedge \Lambda$.

If a is a nowhere vanishing function then $\flat_a(X) = i_X\Omega_a = \frac{1}{a}i_X\Omega = \frac{1}{a}\flat X$. So $\flat_a = \frac{1}{a}\flat$ and $\sharp_a = a\sharp$. Therefore we have

$$\sharp_a(\omega_a) = a\sharp(\omega + d\ln|a|) = aE + \sharp da = aE + \Lambda(da) = E_a$$

and

$$\Omega_a(\sharp_a\beta, \sharp_a\alpha) = \tfrac{1}{a}\Omega(a\sharp\beta, a\sharp\alpha) = a\Omega(\sharp\beta, \sharp\alpha) = a\Lambda(\alpha, \beta) = \Lambda_a(\alpha, \beta)$$

Let $h \in \mathrm{Diff}^\infty(M)$. From

$$(h^*\Lambda)(h^*\omega) = h^*(\Lambda(\omega)) = h^*E$$
$$(h^*\Lambda)(\alpha, \beta) = h^*\big(\Lambda(h_*\alpha, h_*\beta)\big) = h^*\big(\Omega(\Lambda(h_*\beta), \Lambda(h_*\alpha))\big) = (h^*\Omega)\big((h^*\Lambda)(\beta), (h^*\Lambda)(\alpha)\big)$$

we see that the locally conformally symplectic structure corresponding to $(M, h^*\Lambda, h^*E)$ is $(M, h^*\Omega, h^*\omega)$. Now h is a Jacobi diffeomorphism iff $(M, h^*\Lambda, h^*E) = (M, \Lambda_a, E_a)$ for a nowhere vanishing a and this holds iff $(M, h^*\Omega, h^*\omega) = (M, \Omega_a, \omega_a)$, i.e. h preserves the corresponding locally conformally symplectic structure up to conformal change. □

3.3.7. Remark. On a locally conformally symplectic manifold we have

$$X_f = \Lambda(df) + fE = \sharp(df) + f\sharp\omega = \sharp(df + f\omega) = \sharp(d^\omega f)$$

for the Hamiltonian vector field X_f, and

$$\{f, g\} = \Lambda(df, dg) + fdg(E) - gdf(E) = -\big(\Omega(\sharp df, \sharp dg) + f\omega(\sharp dg) - g\omega(\sharp df))\big)$$

for $f, g \in C^\infty(M, \mathbb{R})$.

Recall that a contact manifold is an odd dimensional manifold together with a 1-form η, such that $\eta(d\eta)^n$ is a volume form. If a is a nowhere vanishing function then $(M, \eta_a := \frac{1}{a}\eta)$ is again a contact manifold. $h \in \mathrm{Diff}^\infty(M)$ is called contact diffeomorphism iff $h^*\eta = \eta_a$ for some a. A proof of the following proposition using local coordinates can be found in [GL84].

3.3.8. Proposition. *There exists a natural one to one correspondence between odd dimensional, transitive Jacobi manifolds and contact manifolds. A diffeomorphism is a Jacobi diffeomorphism iff it is a contact diffeomorphism for the corresponding contact manifold. Moreover if (M, Λ, E) corresponds to (M, η) and a is a nowhere vanishing function then (M, Λ_a, E_a) corresponds to (M, η_a), where $\Lambda_a = a\Lambda$, $E_a = aE + \Lambda(da)$ and $\eta_a = \frac{1}{a}\eta$.*

Proof. Given a contact manifold (M, η) we consider the $C^\infty(M, \mathbb{R})$-linear mapping $\flat : \mathfrak{X}(M) \to \Omega^1(M)$, $\flat X := i_X d\eta + (i_X\eta)\eta$. It is injective since $\flat X = 0$ yields $0 = i_X\flat X = (i_X\eta)^2$, so $i_X\eta = 0$, hence $i_X d\eta = 0$ thus $i_X(\eta(d\eta)^n) = 0$ and finally $X = 0$ since $\eta(d\eta)^n$ is a volume form. Consequently $\sharp^{-1} = \flat : \mathfrak{X}(M) \to \Omega^1(M)$ is an $C^\infty(M, \mathbb{R})$-linear isomorphism and we set $E := \sharp\eta \in \mathfrak{X}(M)$. If one contracts E's defining equation $i_E d\eta + (i_E\eta)\eta = \eta$ with E one obtains $(i_E\eta)^2 = i_E\eta$. So either $i_E\eta = 1$ or $i_E\eta = 0$, but the latter is impossible for then $i_E d\eta = \eta$, hence $i_E(\eta(d\eta)^n) = 0$ and $E = 0$, a contradiction. So E is the unique vector field satisfying $i_E\eta = 1$, $i_E d\eta = 0$ and it is called the Reeb vector field. Moreover we define $\Lambda \in \mathfrak{X}^2(M)$ by

$$\Lambda(\alpha, \beta) := (d\eta)(\sharp\alpha, \sharp\beta) = -i_{\sharp\alpha}\big(\beta - (i_{\sharp\beta}\eta)\eta\big) = \eta(\sharp\beta)\eta(\sharp\alpha) - \beta(\sharp\alpha) = \beta\big(\alpha(E)E - \sharp\alpha\big)$$

where we used $i_E\alpha = i_E(i_{\sharp\alpha}d\eta + (i_{\sharp\alpha}\eta)\eta) = i_{\sharp\alpha}\eta$ for the last equation. So we have $\sharp\alpha = -\Lambda(\alpha) + \alpha(E)E$. From $L_E\eta = di_E\eta + i_Ed\eta = 0$ we obtain

$$\begin{aligned} L_E\alpha &= L_E\big(i_{\sharp\alpha}d\eta + (i_{\sharp\alpha}\eta)\eta\big) \\ &= L_Ei_{\sharp\alpha}d\eta + (L_Ei_{\sharp\alpha}\eta)\eta + (i_{\sharp\alpha}\eta)L_E\eta \\ &= i_{[E,\sharp\alpha]}d\eta + i_{\sharp\alpha}L_Ed\eta + (i_{[E,\sharp\alpha]}\eta)\eta + (i_{\sharp\alpha}L_E\eta)\eta \\ &= i_{[E,\sharp\alpha]}d\eta + (i_{[E,\sharp\alpha]}\eta)\eta \end{aligned}$$

and so $[E, \sharp\alpha] = \sharp L_E\alpha$. From this we get

$$\begin{aligned} 0 &= (dd\eta)(E, \sharp\alpha, \sharp\beta) \\ &= L_E(d\eta(\sharp\alpha, \sharp\beta)) - L_{\sharp\alpha}(d\eta(E, \sharp\beta)) + L_{\sharp\beta}(d\eta(E, \sharp\alpha)) \\ &\quad - d\eta([E, \sharp\alpha], \sharp\beta) + d\eta([E, \sharp\beta], \sharp\alpha) - d\eta([\sharp\alpha, \sharp\beta], E) \\ &= L_E(d\eta(\sharp\alpha, \sharp\beta)) - d\eta(\sharp L_E\alpha, \sharp\beta) + d\eta(\sharp L_E\beta, \sharp\alpha) \\ &= L_E(\Lambda(\alpha, \beta)) - \Lambda(L_E\alpha, \beta) - \Lambda(\alpha, L_E\beta) = (L_E\Lambda)(\alpha, \beta) \end{aligned}$$

and so $L_E\Lambda = 0$. In a similar way one shows $[\Lambda, \Lambda] = 2E \wedge \Lambda$.

Suppose conversely (M, Λ, E) is given. Since it is transitive we have a $C^\infty(M, \mathbb{R})$-linear isomorphism $\flat^{-1} = \sharp : \Omega^1(M) \to \mathfrak{X}(M)$, $\sharp\alpha := -\Lambda(\alpha) + \alpha(E)E$. Moreover we set $\eta := \flat E \in \Omega^1(M)$. Contracting η's defining equation $E = -\Lambda(\eta) + \eta(E)E$ with η yields $\eta(E) = \eta(E)^2$ and hence $\eta(E) = 1$, for $\eta(E) = 0$ would contradict the transversality. So we also get $\Lambda(\eta) = 0$ and

$$\begin{aligned} 0 = L_EE &= L_E\big(- \Lambda(\eta) + \eta(E)E\big) \\ &= -(L_E\Lambda)(\eta) - \Lambda(L_E\eta) + (L_E\eta)(E)E = \sharp L_E\eta = \sharp i_Ed\eta \end{aligned}$$

and therefore $i_Ed\eta = 0$. Together with (3.4) we obtain

$$\begin{aligned} (d\eta)(\sharp\alpha, \sharp\beta) &= d\eta\big(- \Lambda(\alpha) + \alpha(E)E, -\Lambda(\beta) + \beta(E)E\big) \\ &= d\eta(\Lambda(\alpha), \Lambda(\beta)) = [\Lambda(\eta), \Lambda](\alpha, \beta) + \tfrac{1}{2}[\Lambda, \Lambda](\eta, \alpha, \beta) \\ &= (E \wedge \Lambda)(\eta, \alpha, \beta) = \Lambda(\alpha, \beta) \end{aligned}$$

and using $\eta(\sharp\alpha) = \eta(-\Lambda(\alpha) + \alpha(E)E) = \alpha(E)$ we get:

$$i_{\sharp\beta}\big(i_{\sharp\alpha}d\eta + (i_{\sharp\alpha}\eta)\eta\big) = \Lambda(\alpha, \beta) + \alpha(E)\beta(E) = i_{-\Lambda(\beta)}\alpha + i_{\beta(E)E}\alpha = i_{\sharp\beta}\alpha$$

So $i_{\sharp\alpha}d\eta + (i_{\sharp\alpha}\eta)\eta = \alpha$ and the constructions are inverse to each other. For a nowhere vanishing function a, we have

$$\sharp_a(\eta_a) = -a\Lambda(\tfrac{1}{a}\eta) + \tfrac{1}{a}\eta(aE + \Lambda(da))E_a = E_a$$

hence $\eta_a = \flat_aE_a$ and (M, Λ_a, E_a) corresponds to (M, η_a). To show the last assertion let $h \in \mathrm{Diff}(M)$. From

$$-(h^*\Lambda)(h^*\eta) + (h^*\eta)(h^*E)h^*E = h^*\big(- \Lambda(\eta) + \eta(E)E\big) = h^*E$$

one sees that the corresponding contact manifold to $(M, h^*\Lambda, h^*E)$ is $(M, h^*\eta)$. Now h is a Jacobi diffeomorphism iff $(M, h^*\Lambda, h^*E) = (M, \Lambda_a, E_a)$ for some a, and this holds iff $(M, h^*\eta) = (M, \eta_a)$, i.e. h is a contact diffeomorphism. □

3.4 Infinitesimal Invariants

Parts of the following lemma can be found in [GL84].

3.4.1. Lemma. *Let X be a compactly supported vector field on M. Then $X \in \mathfrak{X}_c(M, \Omega, \omega)$ if and only if there exists a locally constant function $c_X \in C^\infty(M, \mathbb{R})$ with $d^\omega(\flat X) = c_X \Omega$. In this case c_X is unique and we have $c_X = i_X\omega - f_X$ where f_X is the function satisfying $L_X\Omega = -f_X\Omega$ and $L_X\omega = df_X$. Moreover $\mathfrak{X}_c(M, \Omega, \omega)$ is a Lie algebra and the mapping*

$$\varphi : \mathfrak{X}_c(M, \Omega, \omega) \to H^0_c(M) \qquad X \mapsto [c_X]$$

is a Lie algebra homomorphism, where $H^0_c(M)$ is considered as abelian Lie algebra. If M is compact it is surjective iff Ω is d^ω-exact.

If $(M, \Omega, \omega) \sim (M, \Omega', \omega')$ then $\mathfrak{X}_c(M, \Omega, \omega) = \mathfrak{X}_c(M, \Omega', \omega')$ and $\varphi = \varphi'$. Let $g \in \mathrm{Diff}^\infty(M)$ and $(M, \Omega'', \omega'') := (M, g^*\Omega, g^*\omega)$. Then $g^* : \mathfrak{X}_c(M, \Omega, \omega) \cong \mathfrak{X}_c(M, \Omega'', \omega'')$ and $\varphi'' \circ g^* = g^* \circ \varphi$.

Proof. For any vector field we have $d^\omega(\flat X) = d^\omega i_X\Omega = L_X\Omega + i_X\omega \wedge \Omega - i_X d^\omega\Omega = L_X\Omega + i_X\omega \wedge \Omega$ which yields immediately the first statement. One easily shows

$$\flat[X, Y] = d^\omega(i_X i_Y \Omega) - c_X \flat Y + c_Y \flat X \qquad \forall X, Y \in \mathfrak{X}_c(M, \Omega, \omega) \tag{3.10}$$

hence $d^\omega(\flat[X, Y]) = -c_X c_Y \Omega + c_Y c_X \Omega = 0$ and so $c_{[X,Y]} = 0$. To show equation (3.10) we calculate as follows:

$$\begin{aligned}
i_{[X,Y]}\Omega &= L_X i_Y \Omega - i_Y L_X \Omega = d i_X i_Y \Omega - i_X d i_Y \Omega - i_Y d i_X \Omega - i_Y i_X d\Omega \\
&= d i_X i_Y \Omega - i_X d\flat Y - i_Y d\flat X + i_Y i_X (\omega \wedge \Omega) \\
&= d i_X i_Y \Omega + i_X d\flat Y - i_Y d\flat X + i_X\omega \wedge i_Y\Omega - i_Y\omega \wedge i_X\Omega + \omega \wedge i_Y i_X \Omega \\
&= d i_X i_Y \Omega + i_X(\omega \wedge \flat Y + d\flat Y) - i_Y(d\flat X + \omega \wedge \flat X) + \omega \wedge i_X i_Y \Omega \\
&= d^\omega i_X i_Y \Omega + i_X d^\omega \flat Y - i_Y d^\omega \flat X = d^\omega i_X i_Y \Omega + c_Y \flat X - c_X \flat Y
\end{aligned}$$

If $\Omega' = \frac{1}{a}\Omega$, $\omega' = \omega + \frac{da}{a}$ then $\flat' = \frac{1}{a}\flat$ and $d^{\omega'} \circ \frac{1}{a} = \frac{1}{a} \circ d^\omega$. So the equation $d^\omega(\flat X) = c_X\Omega$ is equivalent to $d^{\omega'}(\flat' X) = c_X \Omega'$. Let $g \in \mathrm{Diff}^\infty(M)$. Then $g^* \circ \flat = \flat'' \circ g^*$ and hence the equation $d^\omega(\flat X) = c_X\Omega$ is equivalent to $d^{\omega''}(\flat''(g^*X)) = (g^*c_X)\Omega''$. But this gives $g^* : \mathfrak{X}_c(M, \Omega, \omega) \cong \mathfrak{X}_c(M, \Omega'', \omega'')$ and $\varphi'' \circ g^* = g^* \circ \varphi$. □

3.4.2. Remark. Notice that the homomorphism φ vanishes if (M, Ω, ω) is conformally equivalent to a symplectic structure, since $H^0_c(M) \neq 0$ only if M has a compact component, but in this case Ω is not d^ω-exact since an exact symplectic structure can only exist on non-compact manifolds. This is because Ω exact implies that the volume form Ω^n is exact too, but this contradicts the fact that $H^{2n}(M) \cong \mathbb{R}$ for connected, compact, orientable, $2n$-dimensional M.

But φ does not vanish in general. For example let $T^4 = S^1 \times S^1 \times S^1 \times S^1$ be the 4-dimensional torus and let dx, dy, dx', dy' denote the generators of $H^1(T^4)$. We take $\omega := dx$, $\alpha := \sin(y)dx' + \cos(y)dy'$ and $\Omega := d^\omega\alpha$. An easy calculation shows $\Omega^2 = 2dx \wedge dy \wedge dx' \wedge dy'$, so (T^4, Ω, ω) is a compact, d^ω-exact locally conformally symplectic manifold, and φ is non-trivial by lemma 3.4.1. Another example of a locally conformally symplectic manifold with non-vanishing φ is the one in example 3.2.5.

3.4.3. Lemma. *We have a surjective Lie algebra homomorphism*

$$\psi : \ker\varphi \to H^1_{d_c^\omega}(M) \qquad X \mapsto [\flat X]$$

where $H^1_{d_c^\omega}(M)$ is considered as abelian Lie algebra. If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then $\ker\varphi = \ker\varphi'$ and $\frac{1}{a}\circ\psi = \psi'$. Let $g \in \mathrm{Diff}^\infty(M)$ and $(M,\Omega'',\omega'') := (M, g^*\Omega, g^*\omega)$. Then $g^* : \ker\varphi \cong \ker\varphi''$ and $\psi''\circ g^* = g^*\circ\psi$.

Proof. ψ is a Lie algebra homomorphism by formula (3.10). If $[\sigma] \in H^1_{d_c^\omega}(M)$ then $\sharp\sigma \in \ker\varphi$ and $\psi(\sharp\sigma) = [\sigma]$, so ψ is onto. If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then $\varphi = \varphi'$ and $\frac{1}{a}\flat = \flat'$. Hence $\ker\varphi = \ker\varphi'$ and $\frac{1}{a}\psi = \psi'$. Let $g \in \mathrm{Diff}^\infty(M)$. From lemma 3.4.1 we get $g^* : \ker\varphi \cong \ker\varphi''$ and since we have $\flat''\circ g^* = g^*\circ\flat$ we also obtain $\psi''\circ g^* = g^*\circ\psi$. □

3.4.4. Lemma. *Let (M,Ω,ω) be a $2n$-dimensional locally conformally symplectic manifold. Then we have another surjective Lie algebra homomorphism*

$$\rho := \ker\psi \to H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big) \qquad X \mapsto [h\Omega^n]$$

where the right hand side is considered as abelian Lie algebra, and h is a compactly supported function on M such that $\flat X = d^\omega h$.

If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then $\ker\psi = \ker\psi'$ and $\frac{1}{a^{n+1}}\circ\rho = \rho'$. Let $g \in \mathrm{Diff}^\infty(M)$ and $(M,\Omega'',\omega'') := (M, g^*\Omega, g^*\omega)$. Then $g^* : \ker\psi \cong \ker\psi''$ and $\rho''\circ g^* = g^*\circ\rho$.

Proof. If h, h' are two functions satisfying $d^\omega h = \flat X = d^\omega h'$ then $d^\omega(h-h') = 0$ and $[(h-h')\Omega^n] \in H^0_{d_c^\omega(M)}(M)\wedge[\Omega^n]$, so

$$[h\Omega^n] = [h'\Omega^n] + [(h-h')\Omega^n] = [h'\Omega^n] \in H^{2n}_{d_c^{(n+1)\omega}}(M)/H^0_{d_c^\omega}(M)\wedge[\Omega^n].$$

This shows that ρ is well defined. Let $\flat X = d^\omega h$ and $\flat Y = d^\omega k$. From formula (3.10) we get $\flat[X,Y] = d^\omega(i_X i_Y\Omega)$ and since

$$(i_X i_Y\Omega)\Omega^n = n i_Y\Omega\wedge i_X\Omega\wedge\Omega^{n-1} = n d^\omega k\wedge d^\omega h\wedge\Omega^{n-1} = n d^{(n+1)\omega}\big(k d^\omega h\wedge\Omega^{n-1}\big)$$

we see that ρ vanishes on brackets. Given any $[\sigma] \in H^{2n}_{d_c^{(n+1)\omega}}(M)$ we may write $\sigma = h\Omega^n$ for some $h \in C^\infty_c(M,\mathbb{R})$, since Ω is non-degenerated. But then $\sharp(d^\omega h) \in \ker\psi$ and $\rho(\sharp(d^\omega h)) = [\sigma]$. So ρ is onto.

If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then lemma 3.4.3 yields $\ker\psi = \ker\psi'$. Moreover $(n+1)\omega' = (n+1)\omega + d(\ln|a^{n+1}|)$ so $\frac{1}{a^{n+1}} : H^{2n}_{d_c^{(n+1)\omega}}(M) \cong H^{2n}_{d_c^{(n+1)\omega'}}(M)$ and $\frac{1}{a^{n+1}} : H^0_{d_c^\omega}(M)\wedge[\Omega^n] \cong H^0_{d_c^{\omega'}}(M)\wedge[\Omega'^n]$. If $\flat X = d^\omega h$ then $\flat' X = \frac{1}{a}\flat X = \frac{1}{a}d^\omega h = d^{\omega'}(\frac{1}{a}h)$ and so $\rho'(X) = [\frac{1}{a}h\Omega'^n] = [\frac{1}{a^{n+1}}h\Omega^n] = \frac{1}{a^{n+1}}\rho(X)$.

Let $g \in \mathrm{Diff}^\infty_c(M)$. From lemma 3.4.3 we get $g^* : \ker\psi \cong \ker\psi''$. If $\flat X = d^\omega h$ then $\flat''(g^*X) = d^{\omega''}(g^*h)$ and hence $\rho''(g^*X) = [g^*h\Omega''^n] = [g^*(h\Omega^n)] = g^*\rho(X)$. □

3.4.5. Remark. Notice that $X \in \ker\rho$ if and only if there exist $h \in \Omega^0_c(M)$ and $\alpha \in \Omega^{2n-1}_c(M)$ such that $\flat X = d^\omega h$ and $h\Omega^n = d^{(n+1)\omega}\alpha$.

3.4.6. Remark. Suppose M is connected. If ω is not exact then $H^{2n}_{d_c^{(n+1)\omega}}(M) = 0$ by example 3.1.7. Moreover, if M is compact then $H^{2n}(M)/(\mathbb{R} \wedge [\Omega^n]) = 0$. For connected M we thus have

$$H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is non-compact and conformally equivalent} \\ & \text{to a symplectic manifold} \\ 0 & \text{otherwise} \end{cases}$$

So for connected M, $\rho \neq 0$ if and only if (M, Ω, ω) is conformally equivalent to a non-compact, symplectic manifold.

The short exact sequence of Lie algebras

$$0 \to \ker \varphi \to \mathfrak{X}_c(M, \Omega, \omega) \xrightarrow{\varphi} \operatorname{Im}(\varphi) \to 0 \tag{3.11}$$

admits a splitting and we obtain a semi direct sum

$$\mathfrak{X}_c(M, \Omega, \omega) \cong \ker \varphi \oplus_\alpha \operatorname{Im}(\varphi)$$

where the action $\alpha : \operatorname{Im}(\varphi) \to \operatorname{Der}(\ker \varphi)$ is given by $\alpha(c)(X) = c[X_0, X]$, where $X_0 \in \mathfrak{X}_c(M, \Omega, \omega)$ is such that $\varphi(X_0) = 1$, i.e. $d^\omega \flat X_0 = \Omega$, on the compact components of M on which Ω is d^ω-exact.

On a locally conformally symplectic manifold (M, Ω, ω) one has the so called symplectic pairing

$$\{\cdot, \cdot\} : H^1_{d_c^\omega}(M) \times H^1_{d_c^\omega}(M) \to H^{2n}_{d_c^{(n+1)\omega}}(M) \qquad \{\alpha, \beta\} := \alpha \wedge \beta \wedge [\Omega^{n-1}].$$

It is non-zero only if M has components which are conformally equivalent to symplectic manifolds (cf. example 3.1.7). If M is a connected symplectic manifold this is the usual symplectic pairing up to Poincaré duality. For $X, Y \in \ker \varphi$ we have $[X, Y] \in \ker \psi$ and $\flat[X, Y] = d^\omega i_X i_Y \Omega$ by equation (3.10). So $\rho([X, Y]) = [(i_X i_Y \Omega)\Omega^n] = n[i_Y \Omega \wedge i_X \Omega \wedge \Omega^{n-1}]$ and

$$\rho([X, Y]) = -n\{\psi(X), \psi(Y)\} \qquad \forall X, Y \in \ker \varphi \tag{3.12}$$

In the symplectic case this is the infinitesimal version of a formula due to G. Rousseau, see [Rou78] and proposition 3.7.20 below. It shows that, in the case where the symplectic pairing $\{\cdot, \cdot\} : H^1_{d_c^\omega}(M) \times H^1_{d_c^\omega}(M) \to H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big)$ is not identically zero, the short exact sequence

$$0 \to \ker \psi \to \ker \varphi \xrightarrow{\psi} H^1_c(M) \to 0$$

does not split, for a section should satisfy $[s(\alpha), s(\beta)] = s([\alpha, \beta]) = s(0) = 0$ and so $0 = \rho([s(\alpha), s(\beta)]) = -n\{\alpha, \beta\}$ by equation (3.12).

The short exact sequence

$$0 \to \ker \rho \to \ker \psi \xrightarrow{\rho} H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big) \to 0 \tag{3.13}$$

splits and we obtain a semidirect sum

$$\ker \psi \cong \ker \rho \oplus_\alpha H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big)$$

where the action $\alpha : H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big) \to \operatorname{Der}(\ker \rho)$ is given by $\alpha(t)(X) = t[X_0, X]$, for a suitable $X_0 \in \ker \psi$.

3.4.7. Lemma. $\ker\psi$ *is an ideal in* $\mathfrak{X}_c(M,\Omega,\omega)$ *and we have a semi direct sum*

$$\mathfrak{X}_c(M,\Omega,\omega)/\ker\psi \cong H^1_{d_c^\omega}(M) \oplus_\alpha \operatorname{Im}(\varphi)$$

where the action $\alpha : \operatorname{Im}(\varphi) \to \operatorname{Der}(H^1_{d_c^\omega}(M))$ *is given by* $\alpha(t)(\beta) = -t \wedge \beta$.

Proof. From (3.10) we obtain immediately

$$\psi([X,Y]) = -\varphi(X) \wedge \psi(Y) \qquad \forall X \in \mathfrak{X}_c(M,\Omega,\omega), Y \in \ker\varphi \tag{3.14}$$

Especially $\ker\psi$ is an ideal in $\mathfrak{X}_c(M,\Omega,\omega)$ and we have commutative diagram

$$\begin{array}{ccccc}
\ker\psi & = & \ker\psi & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\ker\varphi & \hookrightarrow & \mathfrak{X}_c(M,\Omega,\omega) & \xrightarrow{\varphi} & \operatorname{Im}(\varphi) \\
\downarrow{\scriptstyle\psi} & & \downarrow & & \| \\
H^1_{d_c^\omega}(M) & \xrightarrow{\sharp} & \mathfrak{X}_c(M,\Omega,\omega)/\ker\psi & \xrightarrow{\varphi} & \operatorname{Im}(\varphi)
\end{array}$$

with exact rows and columns (for the third row use the nine lemma). It follows from (3.14) that the action is as stated. □

3.4.8. Lemma. $\ker\rho$ *is an ideal in* $\ker\varphi$ *and we have a central extension*

$$\ker\varphi/\ker\rho \cong H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big) \oplus_c H^1_c(M)$$

where the cocycle $c : \bigwedge^2 H^1_c(M) \to H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big)$ *is given by* $c(\alpha,\beta) = -n\{[\alpha],[\beta]\}$.

Proof. Equation (3.12) shows $[\ker\varphi, \ker\psi] \subseteq \ker\rho$. Especially we see that $\ker\rho$ is an ideal in $\ker\varphi$ and we have a commutative diagram

$$\begin{array}{ccccc}
\ker\rho & = & \ker\rho & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\ker\psi & \hookrightarrow & \ker\varphi & \xrightarrow{\psi} & H^1_{d_c^\omega}(M) \\
\downarrow{\scriptstyle\rho} & & \downarrow & & \| \\
H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M) \wedge [\Omega^n]\big) & \xrightarrow{i} & \ker\varphi/\ker\rho & \xrightarrow{\psi} & H^1_{d_c^\omega}(M)
\end{array}$$

where all rows and columns are exact (for the third row use the nine lemma). Since we have $[\ker\varphi, \ker\psi] \subseteq \ker\rho$ the last row is a central extension. The 2-cocycle c corresponding to this extension is $c(\alpha,\beta) = \rho([\sharp\alpha, \sharp\beta] - 0) = -n\{\alpha,\beta\}$ by equation (3.12). □

3.5 Derived Series of $\mathfrak{X}_c(M,\Omega,\omega)$

If (M,Ω,ω) is a locally conformally symplectic manifold and $U \subseteq M$ is an open subset then $(U, \Omega|_U, \omega|_U)$ is a locally conformally symplectic manifold too. In this case we write φ_U,

ψ_U, ρ_U for the invariants of $(U, \Omega|_U, \omega|_U)$. Moreover we identify $\mathfrak{X}_c(U)$ with $\{X \in \mathfrak{X}_c(M) : \operatorname{supp}(X) \subseteq U\}$, via restriction resp. extension by 0. So $\varphi_U(X)$, $\psi_U(X)$, $\rho_U(X)$ do make sense for $X \in \mathfrak{X}_c(M)$ with support in U. Finally let $U \subseteq V \subseteq M$ be two open subsets of M, and let $i : U \to V$ denote the inclusion. Then we have commutative diagrams

$$\begin{array}{ccc} \mathfrak{X}_c(U, \Omega|_U, \omega|_U) & \xrightarrow{\varphi_U} & H^0_c(U) \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ \mathfrak{X}_c(V, \Omega|_V, \omega|_V) & \xrightarrow{\varphi_V} & H^0_c(V) \end{array} \qquad \begin{array}{ccc} \ker \varphi_U & \xrightarrow{\psi_U} & H^1_{d_c^{\omega|_U}}(U) \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ \ker \varphi_V & \xrightarrow{\psi_V} & H^1_{d_c^{\omega|_V}}(V) \end{array}$$

and $\ker \varphi_U \subseteq \ker \varphi_V$, $\ker \psi_U \subseteq \ker \psi_V$, as well as a commutative diagram

$$\begin{array}{ccc} \ker \psi_U & \xrightarrow{\rho_U} & H^{2n}_{d_c^{(n+1)\omega|_U}}(U)/\big(H^0_{d_c^{\omega|_U}}(U) \wedge [\Omega^n|_U]\big) \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ \ker \psi_V & \xrightarrow{\rho_V} & H^{2n}_{d_c^{(n+1)\omega|_V}}(V)/\big(H^0_{d_c^{\omega|_V}}(V) \wedge [\Omega^n|_V]\big) \end{array}$$

and $\ker \rho_U \subseteq \ker \rho_V$. But one should not expect something like $\ker \psi_V \cap \mathfrak{X}_U(M) = \ker \psi_U$.

The following crucial lemma is due to E. Calabi.

3.5.1. Lemma. *Let U, U_1 be open sets in $\mathbb{R}^{2n}$ such that $\bar{U} \subseteq U_1$ and let $\Omega = dx^1 \wedge dx^2 + \cdots$ be the standard symplectic form. Then for all $X \in \ker \rho_U$ there exist $Y_i, Z_i \in \ker \rho_{U_1}$ such that $X = \sum_{i=1}^{2n} [Y_i, Z_i]$. Especially $\ker \rho_U \subseteq [\ker \rho_{U_1}, \ker \rho_{U_1}]$.*

Proof. We follow the proof in [ALDM74]. Since $X \in \ker \rho_U$ we have $i_X \Omega = dh$ and $h\Omega^n = d\alpha$ with $\operatorname{supp} h \subseteq U$ and $\operatorname{supp} \alpha \subseteq U$ (cf. remark 3.4.5). Let A be the vector field defined by $i_A \Omega^n = \alpha$, write $A = \sum_{i=1}^{2n} A^i \frac{\partial}{\partial x^i}$ and define $\tilde{Z}_i := \sharp dA^i$. Obviously we have $\tilde{Z}_i \in \ker \psi_U$. Next we have

$$\begin{aligned} h\Omega^n = d\alpha = L_A \Omega^n &= \textstyle\sum_{i=1}^{2n} L_{A^i \frac{\partial}{\partial x^i}} \Omega^n = \sum_{i=1}^{2n} A^i L_{\frac{\partial}{\partial x^i}} \Omega^n + dA^i \wedge i_{\frac{\partial}{\partial x^i}} \Omega^n \\ &= \textstyle\sum_{i=1}^{2n} \sum_{k=1}^{2n} \frac{\partial A^i}{\partial x^k} dx^k \wedge i_{\frac{\partial}{\partial x^i}} \Omega^n = \sum_{i=1}^{2n} \frac{\partial A^i}{\partial x^i} dx^i \wedge i_{\frac{\partial}{\partial x^i}} \Omega^n = \big(\sum_{i=1}^{2n} \frac{\partial A^i}{\partial x^i}\big) \Omega^n \end{aligned}$$

and hence $h = \sum_{i=1}^{2n} \frac{\partial A^i}{\partial x^i}$. So we obtain

$$\begin{aligned} i_{\sum_{i=1}^{2n} [\frac{\partial}{\partial x^i}, \tilde{Z}_i]} \Omega &= \textstyle\sum_{i=1}^{2n} L_{\frac{\partial}{\partial x^i}} i_{\tilde{Z}_i} \Omega - i_{\tilde{Z}_i} L_{\frac{\partial}{\partial x^i}} \Omega \\ &= \textstyle\sum_{i=1}^{2n} L_{\frac{\partial}{\partial x^i}} dA^i = d \sum_{i=1}^{2n} L_{\frac{\partial}{\partial x^i}} A^i = dh = i_X \Omega \end{aligned}$$

and hence $X = \sum_{i=1}^{2n} [\frac{\partial}{\partial x^i}, \tilde{Z}_i]$. Now choose an open set $U_{\frac{1}{2}}$ such that $\bar{U} \subseteq U_{\frac{1}{2}} \subseteq \bar{U}_{\frac{1}{2}} \subseteq U_1$ and functions y^i supported on $U_{\frac{1}{2}}$ with $\int_{U_1} y^i \Omega^n = 0$ and $\sharp dy^i|_U = \frac{\partial}{\partial x^i}|_U$. Then $Y_i := \sharp dy^i \in \ker \rho_{U_1}$ and $X = \sum_{i=1}^{2n} [Y_i, \tilde{Z}_i]$, since $\operatorname{supp}(\tilde{Z}_i) \subseteq U$. Similarly since $\tilde{Z}_i \in \ker \psi_U \subseteq \ker \psi_{U_1}$ we find functions z^i supported on U_1 with $\int_{U_1} z^i \Omega^n = 0$ and $\sharp dz^i|_{U_{\frac{1}{2}}} = \tilde{Z}_i|_{U_{\frac{1}{2}}}$. Then $Z_i := \sharp dz^i \in \ker \rho_{U_1}$ and $X = \sum_{i=1}^{2n} [Y_i, Z_i]$, since $\operatorname{supp}(Y_i) \subseteq U_{\frac{1}{2}}$. □

3.5.2. Lemma. *Let (M, Ω, ω) be a locally conformally symplectic manifold and let $\mathcal{U}$ be an open covering of M. Then for every $X \in \ker \rho$ there exist $N \in \mathbb{N}$, $U_1, \ldots, U_N \in \mathcal{U}$ and $X_i \in \ker \rho_{U_i}$ such that $X = \sum_{i=1}^{N} X_i$.*

Proof. Since $X \in \ker\rho$ there exist $h \in \Omega^0_c(M)$ and $\alpha \in \Omega^{2n-1}_c(M)$ such that $\flat X = d^\omega h$ and $d^{(n+1)\omega}\alpha = h\Omega^n$, see remark 3.4.5.

Choose $N \in \mathbb{N}$ and $U_1,\dots,U_N \in \mathcal{U}$ which cover $\operatorname{supp}\alpha$ and choose a partition of unity $\{\lambda_0,\dots,\lambda_N\}$ subordinated to $\{M \setminus \operatorname{supp}\alpha, U_1,\dots,U_N\}$. Define $h_i \in C^\infty_c(M,\mathbb{R})$ by $h_i\Omega^n := d^{(n+1)\omega}(\lambda_i\alpha)$ and set $X_i := \sharp d^\omega h_i$. Since we have

$$\textstyle\sum_{i=0}^N h_i\Omega^n = d^{(n+1)\omega}(\sum_{i=0}^N \lambda_i\alpha) = d^{(n+1)\omega}\alpha = h\Omega^n$$

we get $\sum_{i=0}^N h_i = h$ and so $\sum_{i=0}^N X_i = \sharp d^\omega \sum_{i=0}^N h_i = \sharp d^\omega h = X$. Moreover $X_0 = 0$ and $X_i \in \ker\rho_{U_i}$, for $1 \le i \le N$. □

3.5.3. Corollary. *Let (M,Ω,ω) be a locally conformally symplectic manifold. Then $\ker\rho$ is perfect, i.e. $\ker\rho = [\ker\rho,\ker\rho]$.*

Proof. We show $\ker\rho \subseteq [\ker\rho,\ker\rho]$, the other inclusion is trivial. Since we have the fragmentation lemma 3.5.2 it suffices to prove this locally, but the local statement follows from lemma 3.5.1. □

The derived series of a Lie algebra $\mathfrak{g}$ is defined inductively by $D^0\mathfrak{g} := \mathfrak{g}$, $D^1\mathfrak{g} := [\mathfrak{g},\mathfrak{g}]$ and $D^k := [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$, where $[\mathfrak{g},\mathfrak{g}]$ denotes the Lie algebra generated by all commutators.

3.5.4. Corollary. *Let (M,Ω,ω) be a connected locally conformally symplectic manifold and let $\mathfrak{g} := \mathfrak{X}_c(M,\Omega,\omega)$ for the moment. Then we have:*

	$D^0\mathfrak{g} = \mathfrak{g}$	$D^1\mathfrak{g}$	$D^2\mathfrak{g}$	$D^3\mathfrak{g}$
M compact, $[\Omega] = 0 \in H^2_{d^\omega}(M)$	$\mathfrak{g}$	$\ker\varphi$	$\ker\psi = \ker\rho$	$D^2\mathfrak{g}$
M compact, $[\Omega] \neq 0 \in H^2_{d^\omega}(M)$	$\mathfrak{g} = \ker\varphi$	$\ker\psi = \ker\rho$	$D^1\mathfrak{g}$	$D^1\mathfrak{g}$
M not compact, $[\omega] \neq 0 \in H^1(M)$	$\mathfrak{g} = \ker\varphi$	$\ker\psi = \ker\rho$	$D^1\mathfrak{g}$	$D^1\mathfrak{g}$
M not compact, $[\omega] = 0, \{\cdot,\cdot\} = 0$	$\mathfrak{g} = \ker\varphi$	$\ker\rho$	$D^1\mathfrak{g}$	$D^1\mathfrak{g}$
M not compact, $[\omega] = 0, \{\cdot,\cdot\} \neq 0$	$\mathfrak{g} = \ker\varphi$	$\ker\psi$	$\ker\rho$	$D^2\mathfrak{g}$

Proof. Since $\ker\rho$ is perfect (see corollary 3.5.3) we obtain

$$[\ker\psi,\ker\psi] \subseteq \ker\rho = [\ker\rho,\ker\rho] \subseteq [\ker\psi,\ker\psi]$$

and so we always have

$$[\ker\psi,\ker\psi] = \ker\rho. \tag{3.15}$$

Moreover if $\rho = 0$ then $\ker\psi = \ker\rho$ is perfect and we get

$$[\ker\varphi,\ker\varphi] \subseteq \ker\psi = [\ker\psi,\ker\psi] \subseteq [\ker\varphi,\ker\varphi]$$

and we obtain

$$[\ker\varphi,\ker\varphi] = \ker\psi = \ker\rho \qquad \text{if } \rho = 0. \tag{3.16}$$

Now consider the first case, i.e. $\varphi \neq 0$. Remark 3.4.6 gives $\rho = 0$ and so it remains to show $[\mathfrak{g}, \mathfrak{g}] \supseteq \ker \varphi$. To see this choose $X_0 \in \mathfrak{g}$, such that $\varphi(X_0) = 1$. For $X \in \ker \varphi$ we obtain from (3.14) $\psi([X_0, X]) = -\varphi(X_0) \wedge \psi(X) = -\psi(X)$, hence

$$[X_0, X] + X \in \ker \psi = [\ker \psi, \ker \psi] \subseteq [\mathfrak{g}, \mathfrak{g}]$$

and thus $X \in [\mathfrak{g}, \mathfrak{g}]$. In the second and third case we have $\varphi = 0$ and $\rho = 0$ and hence everything follows from equation (3.16). In the forth and fifth case we also have $\varphi = 0$ and

$$\ker \rho = [\ker \psi, \ker \psi] \subseteq [\ker \varphi, \ker \varphi] \subseteq \ker \psi.$$

Since $\ker \rho$ has codimension 1 in $\ker \psi$ we either have $[\ker \varphi, \ker \varphi] = \ker \psi$ or $[\ker \varphi, \ker \varphi] = \ker \rho$. Now use equation (3.12). □

3.5.5. Remark. In the forth case of corollary 3.5.4 we also have $[\ker \psi, \ker \psi] = \ker \rho$, but $\ker \psi \neq \ker \rho$ since $\rho \neq 0$ and $\ker \psi \neq \ker \varphi$ in general. So this is the only case where not all kernels of our invariants do appear in the derived series.

3.6 Pursell-Shanks-Omori like Theorem

A well known theorem of L. E. Pursell and M. E. Shanks see [PS54] states, roughly speaking, that a smooth manifold is completely determined by its Lie algebra of vector fields. More precisely, if there exists an isomorphism of the Lie algebras of vector fields then there exists a unique diffeomorphism between the manifolds, inducing the given Lie algebra isomorphism. Omori proved several generalizations, namely the Lie algebra of vector fields preserving a symplectic form resp. a volume form uniquely determines the manifold together with the symplectic resp. volume structure up to multiplication with a constant, see [Omo74]. We will show an analogous statement for locally conformally symplectic structures, i.e. any of the Lie algebras $\mathfrak{X}_c(M, \Omega, \omega)$, $\ker \varphi$, $\ker \psi$, $\ker \rho$ uniquely determines the locally conformally symplectic manifold (M, Ω, ω) up to conformal equivalence, see Theorem 3.6.8.

3.6.1. Lemma. *Let (M, Ω, ω) be a locally conformally symplectic manifold, $x \in M$ and $X \in \ker \varphi$ with $X(x) \neq 0$. Then there exists a chart (U, u) centered at x such that $X|_U = \frac{\partial}{\partial u^1}$ and $(U, \Omega|_U, \omega|_U) \sim (U, du^1 \wedge du^2 + \cdots + du^{2n-1} \wedge du^{2n}, 0)$.*

Proof. Choose a chart (V, v) centered at x such that $v(V)$ is a ball with center 0 and $X|_V = \frac{\partial}{\partial v^1}$. Since $\omega|_V$ is exact we may assume that $(V, \Omega|_V, \omega|_V)$ is symplectic, that is $\omega|_V = 0$ and $d\Omega|_V = 0$. Since we have $X \in \ker \varphi$ we obtain $L_X \Omega|_V = di_X \Omega|_V = d^\omega \flat X|_V = 0$. Choose $f_{ij} \in C^\infty(V, \mathbb{R})$ with $\Omega|_V = \sum_{i<j} f_{ij} dv^i \wedge dv^j$ and set

$$\sigma_1 := \sum_{j=2}^{2n} f_{1j} dv^j \in \Omega^1(V) \qquad \sigma_2 := \sum_{2 \leq i < j \leq 2n} f_{ij} dv^i \wedge dv^j \in \Omega^2(V)$$

We immediately obtain

$$\Omega|_V = dv^1 \wedge \sigma_1 + \sigma_2, \quad dv^1 \wedge d\sigma_1 = d\sigma_2 \quad \text{and} \quad \sigma_2^{n-1} \neq 0 \text{ on } V.$$

The last statement follows from $0 \neq \Omega^n|_V = (dv^1 \wedge \sigma_1 + \sigma_2)^n = n dv^1 \wedge \sigma_1 \wedge \sigma_2^{n-1}$. Next we have

$$0 = L_X \Omega|_V = L_{\frac{\partial}{\partial v^1}} \sum_{i<j} f_{ij} dv^i \wedge dv^j = \sum_{i<j} (L_{\frac{\partial}{\partial v^1}} f_{ij}) dv^i \wedge dv^j$$

and thus $L_{\frac{\partial}{\partial v^1}} f_{ij} = 0$. So

$$d\sigma_2 = \sum_{k=1}^{2n} \sum_{2 \leq i < j \leq 2n} \tfrac{\partial f_{ij}}{\partial v^k} dv^k \wedge dv^i \wedge dv^j = \sum_{k=2}^{2n} \sum_{2 \leq i < j \leq 2n} \tfrac{\partial f_{ij}}{\partial v^k} dv^k \wedge dv^i \wedge dv^j$$

and

$$dv^1 \wedge d\sigma_1 = \sum_{k=1}^{2n} \sum_{j=2}^{2n} \tfrac{\partial f_{1j}}{\partial v^k} dv^1 \wedge dv^k \wedge dv^j = \sum_{k=2}^{2n} \sum_{j=2}^{2n} \tfrac{\partial f_{1j}}{\partial v^k} dv^1 \wedge dv^k \wedge dv^j$$

and since $d\sigma_2 = dv^1 \wedge d\sigma_1$, both terms have to vanish. That is, $d\sigma_2 = 0$ and $d\sigma_1 = i_{\frac{\partial}{\partial v^1}} (dv^1 \wedge d\sigma_1) = 0$.

Let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$, $\pi(v^1, \ldots, v^{2n}) = (v^2, \ldots, v^{2n})$ and choose $f \in \Omega^0(\pi(v(V)))$ with $(\pi \circ v)^* df = \sigma_1$ and $\theta \in \Omega^1(\pi(v(V)))$ with $(\pi \circ v)^* d\theta = \sigma_2$ and such that $\theta \wedge d\theta^{n-1} \neq 0$ locally around $0 \in \mathbb{R}^{2n-1}$. This is possible since we have $d\sigma_2^{n-1} \neq 0$, $\frac{\partial f_{ij}}{\partial v^1} = 0$ and $\pi(v(V))$ is a ball, hence contractible.

So θ is a contact form locally around 0 and by Darboux's theorem (see [ABK+92] for example) we find a chart (W, w) centered at $0 \in \mathbb{R}^{2n-1}$ such that $\theta|_W = dw^1 + w^2 dw^3 + \cdots + w^{2n-2} dw^{2n-1}$. We are now able to define the desired chart (U, u) by:

$$(\pi \circ v)^{-1}(W) =: U \to \mathbb{R}^{2n} \qquad u := (v^1, f \circ \pi \circ v, w^2 \circ \pi \circ v, \ldots, w^{2n-1} \circ \pi \circ v)$$

First of all we have $X|_U = \frac{\partial}{\partial v^1}|_U = \frac{\partial}{\partial u^1}$. Moreover

$$\begin{aligned}
\Omega|_U &= (dv^1 \wedge \sigma_1 + \sigma_2)|_U = \big(dv^1 \wedge (\pi \circ v)^* df + (\pi \circ v)^* d\theta\big)|_U \\
&= dv^1 \wedge d(f \circ \pi \circ v)|_U + (\pi \circ v)^* (dw^2 \wedge dw^3 + \cdots + dw^{2n-2} \wedge dw^{2n-1})|_U \\
&= du^1 \wedge du^2 + \cdots + du^{2n-1} \wedge du^{2n}
\end{aligned}$$

But this also shows that $\Omega^n|_U = \frac{1}{n!} du^1 \wedge \cdots \wedge du^{2n}$ and hence u is a local diffeomorphism. □

3.6.2. Lemma. *Let (M, Ω, ω) be a locally conformally symplectic manifold, let $U \subseteq M$ be a small open ball in M and let V be an open subset with $V \subseteq \bar{V} \subseteq U$. Then for every $X \in \ker \varphi$ there exists $Y \in \ker \rho_U \subseteq \ker \rho$ such that $X|_V = Y|_V$. Moreover if $X \in \mathfrak{X}(U)$ with $d^{\omega|_U} \flat X = 0$ then there exists $Y \in \ker \rho_U \subseteq \ker \rho$ such that $Y|_V = X|_V$.*

Proof. Since U is contractible we find in both cases $h \in \Omega^0(U)$ with $d^\omega h = \flat X|_U$. Let λ be a bump function with $\operatorname{supp} \lambda \subseteq U$ and $\lambda|_V = 1$. Then $\lambda h \in \Omega^0_U(M)$, $Y := \flat d^\omega(\lambda h) \in \ker \psi_U$ and $Y|_V = X|_V$. Adding a function with support contained in $U \setminus \bar{V}$ we may assume $[h\Omega^n] = 0 \in H^n_{d_c^{(n+1)\omega|_U}}(U)$, i.e. $Y \in \ker \rho_U$. □

3.6.3. Lemma. *Let (M, Ω, ω) be a locally conformally symplectic manifold, $x \in M$ and let $Z \in \ker \rho$ with $Z(x) \neq 0$. Then there exists a neighborhood V of x such that for every $X \in \ker \rho$ there exists $Y \in \ker \rho$ with $[Z, Y]|_V = X|_V$.*

Proof. By lemma 3.6.1 we find a chart (U,u) centered at x such that $Z|_U = \frac{\partial}{\partial u^1}$ and such that $\Omega|_U = du^1 \wedge du^2 + \cdots + du^{2n-1} \wedge du^{2n}$. For $r > 0$ we let $C_r := \{u \in \mathbb{R}^{2n} : |u^i| < r\}$. Next we may assume that $u(U) = C_{2\varepsilon}$ for some $\varepsilon > 0$, set $V := C_\varepsilon$ and assume that $\operatorname{supp}(X) \subseteq U$ (cf. lemma 3.6.2). We define $\tilde{Y} \in \mathfrak{X}(U)$ by

$$\tilde{Y}(u_1, \ldots, u_{2n}) := \int_{-2\varepsilon}^{u_1} X(t, u_2, \ldots, u_{2n})dt$$

Then we have

$$\begin{aligned} di_{\tilde{Y}}\Omega(u) &= d\textstyle\int_{-2\varepsilon}^{u_1} i_X\Omega(t, u_2, \ldots, u_{2n})dt \\ &= du^1 \wedge i_X\Omega(u) + \textstyle\int_{-2\varepsilon}^{u_1} \sum_{1<i} du^i \wedge \frac{\partial}{\partial u^i}(i_X\Omega)(t, u_2, \ldots, u_{2n})dt \\ &= du^1 \wedge i_X\Omega(u) + \textstyle\int_{-2\varepsilon}^{u_1} \big(di_X\Omega - du^1 \wedge \frac{\partial}{\partial u^1}(i_X\Omega)\big)(t, u_2, \ldots, u_{2n})dt \\ &= du^1 \wedge i_X\Omega(u) - du^1 \wedge \textstyle\int_{-2\varepsilon}^{u_1} \frac{\partial}{\partial u^1}(i_X\Omega)(t, u_2, \ldots, u_{2n})dt = 0 \end{aligned}$$

So we find $Y \in \ker\rho$ with $Y|_V = \tilde{Y}|_V$ and hence we have

$$[Z, Y] = [\tfrac{\partial}{\partial u^1}, \tilde{Y}] = [\tfrac{\partial}{\partial u^1}, \textstyle\sum_{i=1}^{2n} \tilde{Y}^i \frac{\partial}{\partial u^i}] = \sum_{i=1}^{2n} \frac{\partial \tilde{Y}^i}{\partial u^1}\frac{\partial}{\partial u^i} = \sum_{i=1}^{2n} X^i \frac{\partial}{\partial u^i} = X$$

on V. □

3.6.4. Corollary. *Let (M, Ω, ω) be a locally conformally symplectic manifold and let $I \subseteq \ker\rho$ be an ideal such that for every $x \in M$ there exists $Z \in I$ with $Z(x) \neq 0$. Then $I = \ker\rho$.*

Proof. Let $x \in M$, $Z \in I$ with $Z(x) \neq 0$ and let V be the neighborhood of x from lemma 3.6.3. By the fragmentation lemma 3.5.2 it suffices to show $\ker\rho_V \subseteq I$ and since we have $[\ker\rho_V, \ker\rho_V] = \ker\rho_V$ by corollary 3.5.3 it suffices to show $[\ker\rho_V, \ker\rho_V] \subseteq I$. So let $X, Y \in \ker\rho_V$. By lemma 3.6.3 there exists $A \in \ker\rho$ with $[Z, A]|_V = Y|_V$ and

$$[X, Y] = [X, [Z, A]] \in I$$

since $\operatorname{supp} X \subseteq V$. □

3.6.5. Proposition. *Let (M, Ω, ω) be a locally conformally symplectic manifold. For every $x \in M$*

$$I_x := \{X \in \ker\rho : X \textit{ is flat at } x\}$$

is a maximal ideal in $\ker\rho$, and $x \mapsto I_x$ is a bijection between points of M and maximal ideals of $\ker\rho$. Moreover for $X \in \ker\rho$ we have

$$X(x) \neq 0 \quad \Leftrightarrow \quad [X, \ker\rho] + I_x = \ker\rho$$

Proof. Of course I_x are ideals. If $I_x \subseteq I \subseteq \ker\rho$ is an ideal such that there exists $X \in I$ with X not flat at x, then there also exists $Y \in I$ with $Y(x) \neq 0$ and corollary 3.6.4 yields $I = \ker\rho$. So I_x are maximal. Certainly $I_x = I_y$ implies $x = y$ and it remains to check surjectivity. Let I be a maximal ideal in $\ker\rho$. By corollary 3.6.4 there exists a point $x \in M$ such that $X(x) = 0$ for all $X \in I$. Since I is an ideal we also obtain that X is flat at x, that is $I \subseteq I_x$. Since both ideals are maximal we have equality. This shows that $x \mapsto I_x$ is a bijection of M and the maximal ideals in $\ker\rho$.

Suppose $X(x) \neq 0$. We have to show $\ker\rho \subseteq [X, \ker\rho] + I_x$. So let $Y \in \ker\rho$. By lemma 3.6.3 there exists $Z \in \ker\rho$ such that $[X, Z] = Y$ locally around x, so $Y - [X, Z] \in I_x$ and hence $Y = [X, Z] + (Y - [X, Z]) \in [X, \ker\rho] + I_x$.

For the other implication suppose conversely $X(x) = 0$. Choose a chart (U, u) centered at x and define a linear mapping

$$j : \ker\rho \to \mathbb{R}^{2n} \oplus \mathfrak{gl}(2n) \qquad Y = \textstyle\sum_{i=1}^{2n} Y^i \frac{\partial}{\partial u^i} \mapsto \big((Y^i(x))_i, (\frac{\partial Y^i}{\partial u^j}(x))_{ij}\big)$$

Since $X(x) = 0$ we have a commutative diagram

$$\begin{array}{ccc} \ker\rho & \xrightarrow{j} & \mathbb{R}^{2n} \oplus \mathfrak{gl}(2n) \\ \uparrow{\scriptstyle ad_X} & & \uparrow{\scriptstyle a} \\ \ker\rho & \xrightarrow{j} & \mathbb{R}^{2n} \oplus \mathfrak{gl}(2n) \end{array}$$

where $a(b, A) = \big((-\frac{\partial X^i}{\partial u^j}(x))_{ij} b, [A, (\frac{\partial X^i}{\partial u^j}(x))_{ij}] - \sum_{k=1}^{2n} \frac{\partial^2 X^i}{\partial u^k \partial u^j}(x) b^k\big)$. From the commutativity we get $a : \operatorname{Im}(j) \to \operatorname{Im}(j)$, and the assumption $[X, \ker\rho] + I_x = \ker\rho$ shows that this mapping is surjective. So it is an isomorphism, for a is linear. Especially the matrix $(\frac{\partial X^i}{\partial u^j})_{ij} \neq 0$ and consequently $j(X) \neq 0$. But this yields a contradiction since $j(X)$ is in the kernel of a, for $a(j(X)) = j(ad_X(X)) = 0$. □

3.6.6. Theorem. *Let $(M_i, \Omega_i, \omega_i)$, $i = 1, 2$ be two locally conformally symplectic manifolds and let $\kappa : \ker\rho_1 \to \ker\rho_2$ be a Lie algebra isomorphism. Then there exists a unique diffeomorphism $f : M_1 \to M_2$ such that $\kappa = f_*$. Moreover $(M_1, \Omega_1, \omega_1) \sim (M_1, f^*\Omega_2, f^*\omega_2)$.*

Proof. By proposition 3.6.5 we may define a bijection $f : M_1 \to M_2$ by $I_{f(x)} = \kappa(I_x)$. For any $A \subseteq M_i$ we have

$$\bar{A} = \{x \in M_i : \bigcap_{y \in A} I_y \subseteq I_x\}$$

and hence for $A \subseteq M_1$

$$\begin{aligned} f(\bar{A}) &= \{f(x) : \bigcap_{y \in A} I_y \subseteq I_x\} = \{f(x) : \bigcap_{y \in A} \kappa(I_y) \subseteq \kappa(I_x)\} \\ &= \{f(x) : \bigcap_{y \in A} I_{f(y)} \subseteq I_{f(x)}\} = \{p \in M_2 : \bigcap_{q \in f(A)} I_q \subseteq I_p\} = \overline{f(A)} \end{aligned}$$

So f (and similarly f^{-1}) maps closed sets to closed sets. This shows that f is a homeomorphism.

For $X \in \ker\rho_1$ we obtain from the second part of proposition 3.6.5

$$\begin{aligned} X(x) \neq 0 &\Leftrightarrow [X, \ker\rho_1] + I_x = \ker\rho_1 \\ &\Leftrightarrow [\kappa(X), \ker\rho_2] + I_{f(x)} = \kappa\big([X, \ker\rho_1] + I_x\big) = \kappa(\ker\rho_1) = \ker\rho_2 \\ &\Leftrightarrow \kappa(X)(f(x)) \neq 0 \end{aligned}$$

From this we immediately obtain

$$\{X_i\} \text{ linearly independent at } x \Leftrightarrow \{\kappa(X_i)\} \text{ linearly independent at } f(x)$$

for $X_i \in \ker \rho_1$. Moreover if $X, Y_i \in \ker \rho_1$ and h_i are functions on some subset $A \subseteq M_1$ then

$$X|_A = (\textstyle\sum_i h_i Y_i|_A) \quad \Rightarrow \quad \kappa(X)|_{f(A)} = \big(\textstyle\sum_i (h_i \circ f^{-1})\kappa(Y_i)|_{f(A)}\big). \tag{3.17}$$

This can be seen as follows. Let $x \in A$. Then $X - \sum_i h_i(x)Y_i \in \ker \rho_1$ vanishes at x. So $\kappa(X - \sum_i h_i(x)Y_i)$ vanishes at $f(x)$, that is $\kappa(X)(f(x)) - \sum_i (h_i \circ f^{-1})(f(x))\kappa(Y_i)(f(x)) = 0$.

Let $x \in M_1$. Choose a chart (U, u) centered at x and $a_1 \in C^\infty(M_1, \mathbb{R})$ such that $\tilde{\omega}_1 := \omega_1 + d\ln|a_1| = 0$ locally around x and $\tilde{\Omega}_1 := \frac{1}{a_1}\Omega_1 = du^1 \wedge du^2 + \cdots + du^{2n-1} \wedge du^{2n}$ locally around x. Choose $X_i \in \ker \rho_1$ with $X_i = \frac{\partial}{\partial u^i}$ locally around x, cf. lemma 3.6.2. Since X_i is a local frame near x of commuting vector fields, $\kappa(X_i)$ is also a local frame near $f(x)$ of commuting vector fields. So it is possible to choose a chart (V, v) on M_2 centered at $f(x)$ such that $\kappa(X_i) = \frac{\partial}{\partial v^i}$ locally around $f(x)$. Next choose $Y_i \in \ker \rho_1$ with $Y_i = u^{\sigma(i)}\frac{\partial}{\partial u^i}$ locally around x, where

$$\sigma : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\} \qquad \sigma(2i) = 2i - 1, \quad \sigma(2i-1) = 2i.$$

From equation (3.17) we obtain $\kappa(Y_i) = (u^{\sigma(i)} \circ f^{-1})\kappa(X_i)$ locally around $f(x)$ and hence f^{-1} is smooth near $f(x)$, since $\kappa(Y_i)$, $\kappa(X_i)$ are smooth and $\kappa(X_i) \neq 0$ locally around $f(x)$. This shows that f is a diffeomorphism.

Moreover we have $[X_i, Y_j] = \delta_{i\sigma(j)}\frac{\partial}{\partial u^j}$ locally around x and so

$$\delta_{i\sigma(j)}\frac{\partial}{\partial v^j} = \kappa([X_i, Y_j]) = [\kappa(X_i), \kappa(Y_j)] = [\tfrac{\partial}{\partial v^i}, (u^{\sigma(j)} \circ f^{-1})\tfrac{\partial}{\partial v^j}] = \frac{\partial(u^{\sigma(j)} \circ f^{-1})}{\partial v^i}\frac{\partial}{\partial v^j}$$

locally around $f(x)$. So $\frac{\partial(u^j \circ f^{-1})}{\partial v^i} = \delta_{ij}$ which yields $u^j \circ f^{-1} = v^j$ locally around $f(x)$. Especially $f_*\frac{\partial}{\partial u^i} = \frac{\partial}{\partial v^i}$ on some neighborhood W of $f(x)$. If $X \in \ker \rho_1$ with $\operatorname{supp}(X) \subseteq f^{-1}(W)$. Then we have $X = \sum_{i=1}^{2n} \lambda_i X_i$ with $\operatorname{supp} \lambda_i \subseteq f^{-1}(W)$ and thus

$$\begin{aligned} f_*(X) &= f_*(\textstyle\sum_{i=1}^{2n} \lambda_i X_i) = \textstyle\sum_{i=1}^{2n} (\lambda_i \circ f^{-1}) f_*(X_i) \\ &= \textstyle\sum_{i=1}^{2n} (\lambda_i \circ f^{-1})\kappa(X_i) = \kappa(\textstyle\sum_{i=1}^{2n} \lambda_i X_i) = \kappa(X) \end{aligned}$$

Since we have the fragmentation property this shows $f_* = \kappa$. Uniqueness of f_* is obvious.

Choose $a_2 \in C^\infty(M_2, \mathbb{R})$ such that $\tilde{\omega}_2 = \omega_2 + d\ln|a_2| = 0$ near $f(x)$ and $\tilde{\Omega}_2 = \frac{1}{a_2}\Omega_2$ is closed near $f(x)$. Near $f(x)$ we have $\tilde{\Omega}_2 = \sum_{i<j} \lambda_{ij} dv^i \wedge dv^j$. Since we have $\kappa(X_j), \kappa(Y_j) \in \ker \rho_2$ and $\kappa(Y_j) = v^{\sigma(j)}\frac{\partial}{\partial v^j}$ locally around $f(x)$ we obtain

$$\begin{aligned} 0 &= di_{\kappa(Y_j)}\tilde{\Omega}_2 = d(v^{\sigma(j)} i_{\frac{\partial}{\partial v^j}}\tilde{\Omega}_2) \\ &= dv^{\sigma(j)} \wedge i_{\frac{\partial}{\partial v^j}}\tilde{\Omega}_2 + v^{\sigma(j)} di_{\frac{\partial}{\partial v^j}}\tilde{\Omega}_2 = dv^{\sigma(j)} \wedge (\textstyle\sum_{j<i} \lambda_{ji} dv^i - \textstyle\sum_{i<j} \lambda_{ij} dv^i) \end{aligned}$$

thus $\lambda_{ij} = 0$ except $i = \sigma(j)$ or $j = \sigma(i)$. So we get

$$\tilde{\Omega}_2 = \lambda_1 dv^1 \wedge dv^2 + \cdots + \lambda_n dv^{2n-1} \wedge dv^{2n}$$

near $f(x)$. Since $\kappa(X_i) \in \ker \rho_2$ and $\kappa(X_i) = \frac{\partial}{\partial v^i}$ near $f(x)$ we get

$$0 = di_{\frac{\partial}{\partial v^{2i}}}\tilde{\Omega}_2 = -d(\lambda_i dv^{2i-1}) = -d\lambda_i \wedge dv^{2i-1}$$

and similarly $0 = d\lambda_i \wedge dv^{2i}$. These both equations imply that λ_i is constant near $f(x)$. Finally choose $Z_{ij} \in \ker \rho_1$ such that $Z_{ij} = u^{2i}\frac{\partial}{\partial u^{2j-1}} - u^{2j}\frac{\partial}{\partial u^{2i-1}}$ near x. Then $\kappa(Z_{ij}) \in \ker \rho_2$ and $\kappa(Z_{ij}) = v^{2i}\frac{\partial}{\partial v^{2j-1}} - v^{2j}\frac{\partial}{\partial v^{2i-1}}$ near $f(x)$. This gives

$$0 = di_{\kappa(Z_{ij})}\tilde{\Omega}_2 = d\big(v^{2i}\lambda_j dv^{2j} + v^{2j}\lambda_i dv^{2i}\big) = (\lambda_j - \lambda_i) dv^{2i} \wedge dv^{2j}$$

and so $\lambda := \lambda_1 = \cdots = \lambda_n$ locally around $f(x)$. So there exists a locally around x defined function $a = a_1 \frac{1}{\lambda}\frac{1}{a_2 \circ f}$ with

$$f^*\Omega_2 = f^*(a_2\tilde{\Omega}_2) = (a_2 \circ f) f^*\tilde{\Omega}_2 = (a_2 \circ f)\lambda\tilde{\Omega}_1 = (a_2 \circ f)\lambda\frac{1}{a_1}\Omega_1 = \big(a_1 \tfrac{1}{\lambda}\tfrac{1}{a_2\circ f}\big)^{-1}\Omega_1 = \tfrac{1}{a}\Omega_1$$

and

$$f^*\omega_2 = f^*(-d\ln|a_2|) = -d\ln|a_2 \circ f| = \omega_1 + d\ln|a_1| - d\ln|a_2 \circ f| = \omega_1 + d\ln|a|$$

locally around x. Since the function a is unique it is globally defined and it is smooth, for the defining equation $f^*\Omega_2 = \frac{1}{a}\Omega_1$ is smooth and $f^*\Omega_2 \neq 0$ and $\Omega_1 \neq 0$. □

3.6.7. Lemma. *Let $\mathfrak{g}$ be a Lie algebra such that $ad : \mathfrak{g} \to \mathfrak{gl}([\mathfrak{g},\mathfrak{g}])$ is injective and let $\lambda : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra homomorphism such that $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = \mathrm{id}$. Then $\lambda = \mathrm{id}$.*

Proof. For $X \in \mathfrak{g}$ we have

$$[X,Y] = \lambda([X,Y]) = [\lambda(X),\lambda(Y)] = [\lambda(X),Y] \qquad \forall Y \in [\mathfrak{g},\mathfrak{g}]$$

hence $ad(X - \lambda(X)) = 0 \in \mathfrak{gl}([\mathfrak{g},\mathfrak{g}])$ and hence $\lambda(X) = X$. □

3.6.8. Corollary. *Let (M_i,Ω_i,ω_i), $i = 1,2$ be two locally conformally symplectic manifolds and assume that κ is a Lie algebra isomorphism from one of the Lie algebras $\mathfrak{X}_c(M_1,\Omega_1,\omega_1)$, $\ker\varphi_1$, $\ker\psi_1$, $\ker\rho_1$ onto one of the Lie algebras $\mathfrak{X}_c(M_2,\Omega_2,\omega_2)$, $\ker\varphi_2$, $\ker\psi_2$, $\ker\rho_2$. Then there exists a unique diffeomorphism $f : M_1 \to M_2$ such that $\kappa = f_*$. Moreover we have $(M_1,\Omega_1,\omega_1) \sim (M_1, f^*\Omega_2, f^*\omega_2)$.*

Proof. We have a Lie algebra isomorphism $\kappa : \mathfrak{g}_1 \to \mathfrak{g}_2$. Hence the restriction of κ is an isomorphism $\kappa|_{D^2\mathfrak{g}_1} : D^2\mathfrak{g}_1 \to D^2\mathfrak{g}_2$. But in any case $D^2\mathfrak{g}_i = \ker\rho_i$ for $i = 1,2$ by corollary 3.5.4. So we can apply theorem 3.6.6 and obtain a unique diffeomorphism $f : M_1 \to M_2$ such that $\kappa|_{D^2\mathfrak{g}_1} = f_*|_{D^2\mathfrak{g}_1}$. Moreover $(M_1,\Omega_1,\omega_1) \sim (M_1, f^*\Omega_2, f^*\omega_2)$. So $f^*\mathfrak{g}_2$ is one of the Lie algebras $\mathfrak{X}_c(M,\Omega_1,\omega_1)$, $\ker\varphi_1$, $\ker\psi_1$, $\ker\rho_1$ and we either have $f^*\mathfrak{g}_2 \subseteq \mathfrak{g}_1$ or $f^*\mathfrak{g}_2 \supseteq \mathfrak{g}_1$. Assume we are in the first case (for the second consider f^{-1}). Then $\lambda := f_*^{-1} \circ \kappa = f^* \circ \kappa : \mathfrak{g}_1 \to \mathfrak{g}_1$ is a Lie algebra homomorphism and we know that $\lambda|_{D^2\mathfrak{g}_1} = \mathrm{id}$. Moreover we obviously have for every vector field $Z \in \mathfrak{X}(M_1)$ the following property:

$$[Z,X] = 0 \quad \forall X \in \ker\rho_1 \quad \Rightarrow \quad Z = 0$$

Using $\ker\rho_1 \subseteq D^{i+1}\mathfrak{g}_1$ we obtain $ad : D^i\mathfrak{g}_1 \to \mathfrak{gl}([D^i\mathfrak{g}_1, D^i\mathfrak{g}_1]) = \mathfrak{gl}(D^{i+1}\mathfrak{g}_1)$ is injective for all i. So we can apply lemma 3.6.7 inductively and obtain successively $\lambda|_{D^2\mathfrak{g}_1} = \mathrm{id}$, $\lambda|_{D^1\mathfrak{g}_1} = \mathrm{id}$ and finally $\lambda = \lambda|_{D^0\mathfrak{g}_1} = \mathrm{id}$, that is $f_* = \kappa$. □

3.7 Integrating the Invariants

The aim of this section is to integrate the infinitesimal invariants from section 3.4. For a brief summary see section 3.8 below.

3.7.1. Lemma. *φ is $Ad(\mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ)$ invariant, i.e. for all $X \in \mathfrak{X}_c(M,\Omega,\omega)$ and $g \in \mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ$ we have $\varphi(Ad(g)\cdot X) = \varphi((g^{-1})^*X) = \varphi(X)$.*

Proof. From lemma 3.4.1 we obtain a commutative diagram:

$$\begin{array}{ccc} \mathfrak{X}_c(M,\Omega,\omega) & \xrightarrow{g^*} & \mathfrak{X}_c(M,\Omega,\omega) = \mathfrak{X}_c(M,g^*\Omega,g^*\omega) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ H_c^0(M) & \xrightarrow{g^*} & H_c^0(M) \end{array}$$

Since g is isotopic to id we have $\mathrm{id} = g^* : H_c^0(M) \to H_c^0(M)$. ☐

3.7.2. Proposition. *The Lie algebra homomorphism φ integrates to a group homomorphism $\tilde\Phi : \widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ \to H_c^0(M)$, i.e.*

$$\begin{array}{ccc} \mathfrak{X}_c(M,\Omega,\omega) & \xrightarrow{\varphi} & H_c^0(M) \\ \downarrow{\scriptstyle\exp=\mathrm{Fl}} & & \downarrow{\scriptstyle\exp=\mathrm{id}} \\ \widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\tilde\Phi} & H_c^0(M) \end{array}$$

commutes. We have the following formulas:

$$\tilde\Phi(g) = \textstyle\int_I \varphi_*(\delta^r g) = \int_0^1 \varphi(\dot g_t)dt = \big[\int_0^1 c_{\dot g_t}dt\big] = \big[\int_0^1 g_t^* c_{\dot g_t}dt\big]$$

If $(M,\Omega,\omega) \sim (M,\Omega',\omega')$ then $\widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ = \widetilde{\mathrm{Diff}}_c^\infty(M,\Omega',\omega')_\circ$ and $\tilde\Phi = \tilde\Phi'$.

Proof. Notice that $\varphi_*(\delta^r g) \in \Omega^1(I; H_c^0(M))$ where $H_c^0(M)$ is a separated, complete locally convex vector space (cf. theorem 3.1.11) and hence integration is well defined. Obviously the various formulas for $\tilde\Phi$ are equal. We have to check that they do only depend on the homotopy type relative endpoints of g. So let $G : D^2 \to \mathrm{Diff}_c^\infty(M,\Omega,\omega)$ and denote by $i : S^1 \hookrightarrow D^2$ the inclusion of the unit circle into the unit disk. Using Stokes and the Maurer Cartan equation (1.2) for $\delta^r G$ we obtain

$$\textstyle\int_{S^1} \varphi_*(\delta^r(i^*G)) = \int_{S^1} i^*\varphi_*(\delta^r G) = \int_{D^2} d\varphi_*(\delta^r G) = \int_{D^2} \varphi_*(\frac{1}{2}[\delta^r G, \delta^r G])$$

but the right hand side is zero since φ vanishes on brackets.

Let $f, g : (I,0) \to (\mathrm{Diff}_c^\infty(M,\Omega,\omega), \mathrm{id})$. Using the Leibniz rule (1.5), the fact that $f(t) \in \mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ$ for every $t \in I$ and lemma 3.7.1 we obtain

$$\varphi_*(\delta^r(fg))(t) = \varphi(\dot f_t) + \varphi((f(t)^{-1})^*\dot g_t) = \varphi(\dot f_t) + \varphi(\dot g_t) = (\varphi_*(\delta^r f) + \varphi_*(\delta^r g))(t)$$

So $\varphi_*(\delta^r(fg)) = \varphi_*(\delta^r f) + \varphi_*(\delta^r g)$ and hence $\tilde\Phi(fg) = \tilde\Phi(f) + \tilde\Phi(g)$. The rest follows from $\delta^r(\mathrm{Fl}^X) = X dt$. ☐

The homomorphism $\tilde\Phi$ has the following geometrical interpretation:

3.7.3. Proposition. *Let* $g : (I,0) \to \big(\mathrm{Diff}_c^\infty(M,\Omega,\omega),\mathrm{id}\big)$ *and denote by* a_t *the functions satisfying* $g_t^*\Omega = \frac{1}{a_t}\Omega$, $g_t^*\omega = \omega + d(\ln|a_t|)$. *Then for* $x \in M$ *we have*

$$\tilde\Phi(g)(x) = \textstyle\int_I (g^x)^*\omega - \ln|a_1(x)|$$

where $g^x : I \to M$ *is the path* $t \mapsto g_t(x)$.

Proof. Differentiating the equation $g_t^*\Omega = \frac{1}{a_t}\Omega$ with respect to t we get $\frac{\partial}{\partial t}(\ln|a_t|) = g_t^* f_{\dot g_t}$, where $f_{\dot g_t}$ are the functions satisfying $L_{\dot g_t}\Omega = -f_{\dot g_t}\Omega$ and $L_{\dot g_t}\omega = df_{\dot g_t}$, and therefore

$$\ln|a_1| = \ln|a_1| - \ln|a_0| = \textstyle\int_0^1 \frac{\partial}{\partial t}(\ln|a_t|)dt = \int_0^1 g_t^* f_{\dot g_t} dt$$

Next we have

$$\textstyle\int_I (g^x)^*\omega = \int_0^1 \omega(\frac{\partial}{\partial s}|_t g_s(x))dt = \int_0^1 \omega(\dot g_t(g_t(x))dt = \int_0^1 (g_t^* i_{\dot g_t}\omega)dt(x)$$

Putting these two equations together we obtain

$$\textstyle\int_I (g^x)^*\omega - \ln|a_1(x)| = \int_0^1 g_t^*(i_{\dot g_t}\omega - f_{\dot g_t})dt(x) = \int_0^1 g_t^* c_{\dot g_t} dt(x) = \tilde\Phi(g)(x)$$

and we are done. □

We define $\Delta := \tilde\Phi\big(\boldsymbol{\pi}_1(\mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ)\big)$. Then $\tilde\Phi$ descends to a homomorphism Φ

$$\begin{array}{ccc} \widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\tilde\Phi} & H_c^0(M) \\ \downarrow & & \downarrow \pi \\ \mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\Phi} & H_c^0(M)/\Delta \end{array}$$

If M is compact then Φ is surjective iff Ω is d^ω-exact. If $(M,\Omega,\omega) \sim (M,\Omega',\omega')$ then $\mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ = \mathrm{Diff}_c^\infty(M,\Omega',\omega')_\circ$, $\Delta = \Delta'$ and $\Phi = \Phi'$.

3.7.4. Corollary. *If* M *is connected and compact then* $H_c^0(M) \cong \mathbb{R}$ *and*

$$\Delta \subseteq \mathrm{Per}(\omega) := \{\langle \omega, c\rangle : c \in H_1(M;\mathbb{Z})\} \subseteq \mathbb{R}.$$

Especially $\Delta \subseteq H_c^0(M)$ *is always countable.*

3.7.5. Example. Recall the locally conformally symplectic manifold $\big(S^1 \times S^3, \Omega, \omega\big)$ from example 3.2.5. In this example one has $\sharp\alpha = \partial_t$ and $g \in \boldsymbol{\pi}_1\big(\mathrm{Diff}_c^\infty(M,\Omega,\omega)\big)$, where $g_t := \mathrm{Fl}_t^{\sharp\alpha}$. Moreover $\tilde\Phi(g) = \varphi(\sharp\alpha) = 1$, hence $\mathbb{Z} \subseteq \Delta$. Since $\mathrm{Per}(\omega) = \mathbb{Z}$, corollary 3.7.4 yields $\Delta = \mathbb{Z} \subseteq \mathbb{R}$.

3.7.6. Example. Recall the example $\big(S^1 \times S^1 \times S^1 \times S^1, \Omega, \omega\big)$ in remark 3.4.2. We have $\sharp\alpha = \partial_x$ and $g \in \boldsymbol{\pi}_1\big(\mathrm{Diff}_c^\infty(M,\Omega,\omega)\big)$, where $g_t := \mathrm{Fl}_t^{\sharp\alpha}$. Moreover $\tilde\Phi(g) = \varphi(\sharp\alpha) = 1$ and hence $\mathbb{Z} \subseteq \Delta$. Since $\mathrm{Per}(\omega) = \mathbb{Z}$, corollary 3.7.4 yields $\Delta = \mathbb{Z} \subseteq \mathbb{R}$.

3.7.7. Corollary. *Let* $g \in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}_c^\infty(M,\Omega,\omega),\mathrm{id})\big)$. *Then*

$$g \in C^\infty(\mathbb{R},\ker\Phi) \quad\Leftrightarrow\quad \delta^r g \in \Omega^1(\mathbb{R};\ker\varphi) \quad\Leftrightarrow\quad \dot g_t \in \ker\varphi$$

Especially $\mathrm{Fl}^X \in C^\infty(\mathbb{R},\ker\Phi)$ *iff* $X \in \ker\varphi$.

Proof. By lemma 3.2.6 we may assume that g has values in $\operatorname{Diff}_c^\infty(M,\Omega,\omega)$ and $\delta^r g \in \Omega^1\big(\mathbb{R};\mathfrak{X}_c(M,\Omega,\omega)\big)$. For $s\in\mathbb{R}$ let $\mu_s : I \to \mathbb{R}$, $\mu_s(t) := ts$. We then have

$$\Phi(g_s) = \pi(\tilde\Phi(\mu_s^* g)) = \pi\big(\textstyle\int_I \mu_s^*\varphi_*(\delta^r g)\big) = \pi\big(\int_{\mu_s(I)} \varphi_*(\delta^r g)\big) = \pi\big(\int_0^s \varphi(\dot g_t)dt\big)$$

So the implication $\Leftarrow$ follows immediately. So let us assume that g has values in $\ker\Phi$. Then $\int_0^s \varphi(\dot g_t)dt \in \Delta$ for all $s \in I$. Since this depends continuously on s and has values in a countable subset of a separated topological vector space it has to be constant, i.e. $\int_0^s \varphi(\dot g_t)dt = 0$ for all $s\in I$. Differentiating with respect to s we obtain $\dot g_s \in \ker\varphi$ for all $s \in \mathbb{R}$. □

3.7.8. Lemma. $\ker\Phi$ *is connected by smooth arcs, and the natural inclusion induces an isomorphism of groups* $i : \widetilde{\ker\Phi} \cong \ker\tilde\Phi$, *such that* $\operatorname{ev}_1\circ i = \operatorname{ev}_1$.

Proof. Consider $i : \widetilde{\ker\Phi} \to \ker\tilde\Phi$, $i(g) = g$. Notice that $i(g)\in\ker\tilde\Phi$, since $\dot g_t \in \ker\varphi$ by lemma 3.7.7, and i is well defined, for two curves which are homotopic relative endpoints in $\ker\Phi$ are obviously homotopic relative endpoints in $\operatorname{Diff}_c^\infty(M,\Omega,\omega)$. Next we show that i is onto. Let $g\in\ker\tilde\Phi$. For any $s\in I$ we define $h_s \in C^\infty\big((I,0),(\operatorname{Diff}_c^\infty(M,\Omega,\omega),\mathrm{id})\big)$ by $\delta^r(h_s) = s\delta^r g$ (cf. lemma 3.2.6). Then $h_0(t)=\mathrm{id}$ and $h_1(t) = g(t)$. Moreover

$$\tilde\Phi(h_s) = \textstyle\int_I \varphi_*(\delta^r(h_s)) = s\int_I \varphi_*(\delta^r g) = s\tilde\Phi(g) = 0,$$

so $\Phi(h_s(1)) = 0$ for all $s\in I$, and g is homotopic relative endpoints to $s\mapsto h_s(1)$, which is a curve in $\ker\Phi$ (cf. figure 3.1). In order two show injectivity of i let $g\in\ker i$, i.e. there

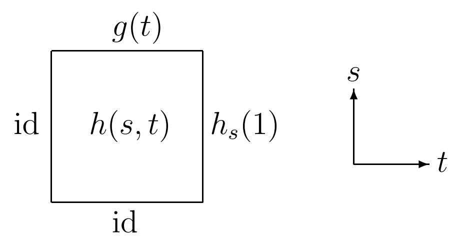


Figure 3.1: A homotopy relative endpoints from $t\mapsto g(t)$ to $t\mapsto h_t(1)$

exists $G\in C^\infty\big(I\times I, \operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ\big)$ with $G(0,t)=\mathrm{id}$, $G(1,t) = g(t)$ and $G(s,0) = G(s,1) = \mathrm{id}$. For $(s,u)\in I\times I$ we define $H(s,\cdot,u) \in C^\infty\big((I,0),(\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ,\mathrm{id})\big)$ by $\delta^r H(s,\cdot,u) = u\delta^r G(s,\cdot)$. We have $G(1,t) = g(t)\in\ker\Phi$, so $\delta^r G(1,\cdot)\in\Omega^1(I;\ker\varphi)$, hence $\delta^r H(1,\cdot,u)\in\Omega^1(I;\ker\varphi)$ and thus $H(1,t,u)\in\ker\Phi$ for all $t,u\in I$. So g is homotopic relative endpoints in $\ker\Phi$ to $u\mapsto H(1,1,u)$, for we have $H(s,t,0)=\mathrm{id}$, $H(s,0,u)=\mathrm{id}$ and $H(s,t,1) = G(s,t)$. Moreover $(s,u)\mapsto H(s,1,u)$ is a smooth homotopy relative endpoints from id to $H(1,1,\cdot)$. We claim that it has values in $\ker\Phi$. Indeed, since $\tilde\Phi\big(G(s,\cdot)\big) = 0$ we have

$$\tilde\Phi\big(H(s,\cdot,u)\big) = \textstyle\int_I \varphi_*\delta^r H(s,\cdot,u) = u\int_I \varphi_*\delta^r G(s,\cdot) = u\tilde\Phi\big(G(s,\cdot)\big) = 0$$

and hence $\Phi(H(s,1,u)) = 0$. So g is homotopic relative endpoints in $\ker\Phi$ to id, i.e. i is one-to-one.

If $f\in\ker\Phi$ then there exists $g\in\widetilde{\operatorname{Diff}}_c^\infty(M,\Omega,\omega)_\circ$ with $\pi(g) = g(1) = f$ and $\tilde\Phi(g)\in\Delta$. By multiplying g with something in $\boldsymbol{\pi}_1(\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ)$ we may assume that $\tilde\Phi(g) = 0$. So $\operatorname{ev}_1 : \ker\tilde\Phi \to \ker\Phi$ is onto, and since $\widetilde{\ker\Phi}\cong\ker\tilde\Phi$ we get $\operatorname{ev}_1 : \widetilde{\ker\Phi}\to\ker\Phi$ is onto, too. So $\ker\Phi$ is connected by smooth arcs. □

From lemma 3.7.8 we obtain a commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{\pi}_1(\ker\Phi) & \hookrightarrow & \boldsymbol{\pi}_1\big(\widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ\big) & \xrightarrow{\tilde\Phi} & \Delta \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\ker}\,\Phi \cong \ker\tilde\Phi & \hookrightarrow & \widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\tilde\Phi} & \mathrm{Im}(\varphi) = \mathrm{Im}(\tilde\Phi) \\
\downarrow{\scriptstyle \pi=\mathrm{ev}_1} & & \downarrow{\scriptstyle \pi=\mathrm{ev}_1} & & \downarrow{\scriptstyle \pi} \\
\ker\Phi & \hookrightarrow & \mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\Phi} & \mathrm{Im}(\varphi)/\Delta
\end{array}$$

where all rows and columns are exact. Moreover the middle row splits and we have a semi direct product

$$\widetilde{\mathrm{Diff}}_c^\infty(M,\Omega,\omega)_\circ \cong \ker\tilde\Phi \times_\alpha \mathrm{Im}(\varphi)$$

cf. the extension (3.11) on page 58.

3.7.9. Lemma. *ψ is $Ad(\ker\Phi)$ invariant, i.e. for all $X \in \ker\varphi$ and $g \in \ker\Phi$ we have $\psi(Ad(g)\cdot X) = \psi((g^{-1})^* X) = \psi(X)$.*

Proof. From lemma 3.4.3 we obtain a commutative diagram:

$$\begin{array}{ccccc}
\ker\varphi & \xrightarrow{g^*} & \ker\varphi = \ker\varphi' & & \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi'} & \searrow^{\psi} & \\
H^1_{d_c^\omega}(M) & \xrightarrow{g^*} & H^1_{d_c^{g^*\omega}}(M) & \xrightarrow{a} & H^1_{d_c^\omega}(M)
\end{array}$$

where a is such that $(M,\Omega,\omega) \overset{a}{\sim} (M, g^*\Omega, g^*\omega)$ and φ', ψ' correspond to $(M, g^*\Omega, g^*\omega)$. So it remains to show that $ag^* : H^1_{d_c^\omega}(M) \to H^1_{d_c^\omega}(M)$ is the identity. Since $\ker\Phi$ is connected by smooth arcs (lemma 3.7.8) there exists a curve $g_t \in \ker\Phi$ with $g_0 = \mathrm{id}$ and $g_1 = g$. We define a_t by $(M,\Omega,\omega) \overset{a_t}{\sim} (M, g_t^*\Omega, g_t^*\omega)$, so $a_0 = 1$ and $a_1 = a$. Since $g_t \in \ker\Phi$ we have $\dot g_t \in \ker\varphi$ by corollary 3.7.7 and hence $f_{\dot g_t} = i_{\dot g_t}\omega$ by lemma 3.4.1. Differentiating $g_t^*\Omega = \frac{1}{a_t}\Omega$ with respect to t and using $L_{\dot g_t}\Omega = -f_{\dot g_t}\Omega$ we obtain

$$\tfrac{\dot a_t}{a_t} = g_t^* f_{\dot g_t} = g_t^* i_{\dot g_t}\omega = \mathrm{inc}_t^*\, i_{\partial_t} g^*\omega$$

and so a_t satisfies the same differential equation as a_t in lemma 3.1.1 and are thus equal. But then lemma 3.1.1 yields $ag^* = a_1 g_1^* = a_0 g_0^* = \mathrm{id}$. □

3.7.10. Proposition. *The Lie algebra homomorphism ψ integrates to a surjective group homomorphism $\tilde\Psi : \widetilde{\ker}\,\Phi \to H^1_{d_c^\omega}(M)$, i.e.*

$$\begin{array}{ccc}
\ker\varphi & \xrightarrow{\psi} & H^1_{d_c^\omega}(M) \\
{\scriptstyle \exp=\mathrm{Fl}}\downarrow & & \downarrow{\scriptstyle \exp=\mathrm{id}} \\
\widetilde{\ker}\,\Phi & \xrightarrow{\tilde\Psi} & H^1_{d_c^\omega}(M)
\end{array}$$

commutes. We have the following formulas:

$$\tilde\Psi(g) = \textstyle\int_I \psi_*(\delta^r g) = \int_0^1 \psi(\dot g_t)dt = \big[\int_0^1 i_{\dot g_t}\Omega dt\big] = \big[\int_0^1 a_t g_t^* i_{\dot g_t}\Omega dt\big]$$

where $g_t^*\Omega = \frac{1}{a_t}\Omega$. If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then $\widetilde{\ker}\,\Phi' = \widetilde{\ker}\,\Phi$ and $\frac{1}{a}\circ\tilde\Psi' = \tilde\Psi$.

Proof. The proof is exactly the same as the proof of proposition 3.7.2. Integration is well defined since $\delta^r g \in \Omega^1(I; \ker\varphi)$ by corollary 3.7.7 and since $H^1_{d^\omega_c}(M)$ is a separated, complete locally convex vector space by theorem 3.1.11. To see that the formulas do only depend on the homotopy type relative endpoints in $\ker\Phi$ of the curve g one does the same argument, but now one has to use that ψ vanishes on brackets, cf. lemma 3.4.3. Also the proof that $\tilde\Psi$ is a homomorphism is similar, but one has to use that ψ is $Ad(\ker\Phi)$-invariant, cf. lemma 3.7.9. ∎

We let $\Gamma := \tilde\Psi(\boldsymbol{\pi}_1(\ker\Phi))$. Then $\tilde\Psi$ factors to a surjective homomorphism Ψ

$$\begin{array}{ccc} \widetilde{\ker\Phi} & \xrightarrow{\tilde\Psi} & H^1_{d^\omega_c}(M) \\ \big\downarrow & & \big\downarrow \pi \\ \ker\Phi & \xrightarrow{\Psi} & H^1_{d^\omega_c}(M)/\Gamma \end{array}$$

If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then $\ker\Phi' = \ker\Phi$, $\frac{1}{a}\Gamma = \Gamma'$ and $\frac{1}{a}\circ\Psi' = \Psi$.

Let $H^{\text{loc. fin.}}_*(M, B_\omega)$ denote the homology based on locally finite chains with values in the bundle of coefficients B_ω, cf. the discussion above corollary 3.1.10. The k-chains of this homology theory are formal infinite linear combinations $\sum_i \lambda_i(\sigma_i, f_i)$, where $\lambda_i \in \mathbb{R}$, $\sigma_i : \Delta^k \to M$, $f_i : \Delta^k \to \mathbb{R}$ satisfying $f_i\sigma_i^*\omega + df_i = 0$ and $\{\sigma_i\}$ is locally finite. Using proposition 3.1.6, theorem 3.1.9 and Poincaré duality for sheaf (co)homology (see [Bre67]) we obtain

$$(H^k_{d^\omega_c}(M))^* \cong H^{n-k}_{d^{-\omega}}(M) \cong H^{n-k}(M; \mathcal{F}_{-\omega}) \cong H_k(M, \mathcal{F}_{-\omega}) \cong H^{\text{loc. fin.}}_k(M, B_{-\omega})$$

and the isomorphism $P_\omega : H^{\text{loc. fin.}}_k(M, B_{-\omega}) \to (H^k_{d^\omega_c}(M))^*$ comes from the pairing $\langle\cdot,\cdot\rangle_\omega : C^{\text{loc. fin}}_k(M, B_{-\omega}) \times \Omega^k_c(M) \to \mathbb{R}$ given by the formula:

$$\langle \textstyle\sum_i \lambda_i(\sigma_i, f_i), \alpha\rangle_\omega = \sum_i \lambda_i \int_{\Delta^k} f_i\sigma_i^*\alpha$$

Notice that only finitely many summands are non-zero since $\{\sigma_i\}$ is locally finite and α has compact support. We have: $\langle\partial c, \alpha\rangle_\omega = \langle c, d^\omega\alpha\rangle_\omega$. Indeed it suffices to check this for a k-simplex $c = (\sigma, f)$. Using Stokes and $-f\sigma^*\omega + df = 0$ we get

$$\begin{aligned} \langle\partial(\sigma,f),\alpha\rangle_\omega &= \big\langle \textstyle\sum_{i=0}^k (-1)^i(\sigma\circ\delta_i, f\circ\delta_i), \alpha\big\rangle_\omega \\ &= \textstyle\sum_{i=0}^k (-1)^i \int_{\Delta^{k-1}} (f\circ\delta_i)(\sigma\circ\delta_i)^*\alpha \\ &= \textstyle\sum_{i=0}^k (-1)^i \int_{\Delta^{k-1}} \delta_i^*(f\sigma^*\alpha) \\ &= \textstyle\int_{\Delta^k} d(f\sigma^*\alpha) = \int_{\Delta^k} df\wedge\sigma^*\alpha + f\sigma^* d\alpha \\ &= \textstyle\int_{\Delta^k} f\sigma^*(\omega\wedge\alpha + d\alpha) = \langle(\sigma,f), d^\omega\alpha\rangle_\omega \end{aligned}$$

Moreover if $g : M_1 \to M_2$ is proper and satisfies $g^*\omega_2 = \omega_1$ for closed 1-forms ω_i then

$$g_* : C^{\text{loc. fin.}}_*(M, B_{\omega_1}) \to C^{\text{loc. fin.}}_*(M, B_{\omega_2}) \qquad g_*\big(\textstyle\sum_i \lambda_i(\sigma_i, f_i)\big) := \sum_i \lambda_i(g\circ\sigma_i, f_i)$$

is well defined. Indeed, the image is locally finite since g is proper, and $f(g\circ\sigma)^*\omega_2 = f\sigma^*g^*\omega_2 = f\sigma^*\omega_1 = df$. Moreover g_* is a chain map and we have an induced mapping in homology:

$$g_* : H^{\text{loc. fin.}}_*(M_1, B_{\omega_1}) \to H^{\text{loc. fin.}}_*(M_2, B_{\omega_2})$$

The pairing is natural, i.e. we have:

$$\langle g_*(\sigma,f),\alpha\rangle_{\omega_2}=\int_{\Delta^k}f(g\circ\sigma)^*\alpha=\int_{\Delta^k}f\sigma^*g^*\alpha=\langle(\sigma,f),g^*\alpha\rangle_{\omega_1}$$

Moreover it behaves very nice under conformal change. Recall that if $\omega'=\omega+d\ln|a|$ then $\frac{1}{a}:H^*_{d_c^\omega}(M)\to H^*_{d_c^{\omega'}}(M)$ is an isomorphism. Moreover

$$B_a:B_{\omega'}=M\times\mathbb{R}\to M\times\mathbb{R}=B_\omega\qquad(x,t)\mapsto(x,at)$$

is an isomorphism of bundles of coefficients, for we have:

$$B_a^*(t\omega+dt)=at\omega+d(at)=a(t\omega+dt+\tfrac{1}{a}tda)=a(t\omega'+dt)$$

We have an induced mapping

$$(B_a)_*:H_*^{\text{loc. fin.}}(M,B_{\omega'})\ \to H_*^{\text{loc. fin.}}(M,B_\omega)\qquad(\sigma,f)\mapsto(\sigma,(\sigma^*a)f)$$

Since $-\omega'=-\omega+d\ln|\frac{1}{a}|$ we get $(B_{\frac{1}{a}})_*:H_*^{\text{loc. fin.}}(M,B_{-\omega'})\to H_*^{\text{loc. fin.}}(M,B_{-\omega})$ and

$$\langle(B_{\frac{1}{a}})_*c,\alpha\rangle_\omega=\langle c,\tfrac{1}{a}\alpha\rangle_{\omega'}$$

as one easily sees from a short calculation.

We have the following geometric interpretation of $\tilde\Psi$:

3.7.11. Lemma. *Let $g:(I,0)\to(\ker\Phi,\mathrm{id})$ and $g_t^*\Omega=\frac{1}{a_t}\Omega$. For $c=\sum_i\lambda_i(\sigma_i,f_i)\in C_1^{\text{loc. fin.}}(M,B_{-\omega})$ we have*

$$\langle c,\tilde\Psi(g)\rangle_\omega=\langle g*c,\Omega\rangle_\omega$$

where $g*c=\sum_i\lambda_i(g*\sigma_i,g*f_i)\in C_2^{\text{loc. fin.}}(M,B_{-\omega})$ and $g*\sigma_i:I\times I\to M$, $(g*\sigma_i)(s,t)=g_t(\sigma_i(s))$, $g*f_i:I\times I\to\mathbb{R}$, $(g*f_i)(s,t)=f_i(s)a_t(\sigma_i(s))$.

Proof. It suffices to check everything for a simplex $c=(\sigma,f)\in C_1^{\text{loc. fin.}}(M,B_{-\omega})$, that is $-f\sigma^*\omega+df=0$. Now $(g*f,g*\sigma)$ defines a 2-chain in $C_2^{\text{loc. fin.}}(M,B_{-\omega})$ iff $\beta:=-(g*f)(g*\sigma)^*\omega+d(g*f)=0$. Since $g_t\in\ker\Phi$ we have $a_tg_t^*i_{\dot g_t}\omega=\dot a_t$ and therefore

$$\begin{aligned}(i_{\partial_t}\beta)(s,t)&=-f(s)a_t(\sigma(s))\omega\big(\tfrac{\partial}{\partial t}g_t(\sigma(s))\big)+\tfrac{\partial}{\partial t}\big(f(s)a_t(\sigma(s))\big)\\&=-f(s)a_t(\sigma(s))(i_{\dot g_t}\omega)\big(g_t(\sigma(s))\big)+f(s)\dot a_t(\sigma(s))\\&=-f(s)(a_tg_t^*i_{\dot g_t}\omega)(\sigma(s))+f(s)\dot a_t(\sigma(s))=0\end{aligned}$$

Using $g_t^*\omega=\omega+d\ln|a_t|$ and $-f\sigma^*\omega+df=0$ we obtain

$$\begin{aligned}\mathrm{inc}_t^*\beta&=-f(\sigma^*a_t)\sigma^*g_t^*\omega+d(f\sigma^*a_t)\\&=-f(\sigma^*a_t)\sigma^*\omega-f(\sigma^*a_t)\sigma^*d\ln|a_t|+f\sigma^*da_t+(df)\sigma^*a_t\\&=-f(\sigma^*a_t)\tfrac{1}{\sigma^*a_t}\sigma^*da_t+f\sigma^*da_t=0\end{aligned}$$

So $\beta=0$, i.e. $g*c$ defines a 2-chain in $C_2^{\text{loc. fin.}}(M,B_{-\omega})$. Next we have

$$\begin{aligned}\int_If\sigma^*\tilde\Psi(g)&=\int_0^1f(s)\big(\int_0^1a_tg_t^*i_{\dot g_t}\Omega dt\big)(\tfrac{\partial}{\partial s}\sigma(s))ds\\&=\int_0^1\int_0^1f(s)a_t(\sigma(s))\Omega\big(\dot g_t(g_t(\sigma(s))),Tg_t\cdot\tfrac{\partial}{\partial s}\sigma(s)\big)dsdt\\&=\int_0^1\int_0^1(g*f)(s,t)\Omega\big(\tfrac{\partial}{\partial t}g_t(\sigma(s)),\tfrac{\partial}{\partial s}g_t(\sigma(s))\big)dsdt\\&=\int_0^1\int_0^1(g*f)(s,t)((g*\sigma)^*\Omega)(\partial_t,\partial_s)dsdt=\int_{I\times I}(g*f)(g*\sigma)^*\Omega\end{aligned}$$

but this is $\langle(\sigma,f),\tilde\Psi(g)\rangle_\omega=\langle(g*\sigma,g*f),\Omega\rangle_\omega$. ∎

3.7.12. Corollary. *Every continuous curve in Γ is constant.*

Proof. Let $c = \sum_i \lambda_i(\sigma_i, f_i) \in C_1^{\text{loc. fin.}}(M, B_{-\omega})$ and let $S \subseteq B_{-\omega}$ be the smallest subbundle of coefficients such that $(\sigma_i, f_i) \in C_1^{\text{loc. fin.}}(M, S)$. The fibers of S are countable and hence $H_k(M, S)$ is countable. This can be shown using a countable good covering of M, Mayer Vietoris sequence and the fact that $H_*(U, S|_U)$ is countable for a contractible $U \subseteq M$. Suppose $g \in \pi_1(\ker\Phi)$ and $\partial c = 0$. Then $g * c \in C_2^{\text{loc. fin.}}(M, S)$ and $\partial(g * c) = 0$. Moreover $g * c - \operatorname{id} * c$ is a finite chain and thus $[g * c] \in [\operatorname{id} * c] + j(H_2(M, S))$, where $j : H_k(M, S) \to H_k^{\text{loc. fin.}}(M, S)$ is the map induced from the inclusion of finite chains into locally finite chains. From lemma 3.7.11 we obtain

$$\langle c, \tilde{\Psi}(g)\rangle_\omega = \langle g * c, \Omega\rangle_\omega = P_\omega([g * c])([\Omega]) \subseteq P_\omega\big([\operatorname{id} * c] + j(H_2(M, S))\big)([\Omega]) \subseteq \mathbb{R}$$

Since the latter is countable we see that $P_\omega([c])(\Gamma) \subseteq \mathbb{R}$ is countable for every $[c]$. So, if h is a continuous curve in Γ, then $P_\omega([c]) \circ h$ is constant for every $[c]$. But $\{P_\omega([c]) : [c] \in H_1^{\text{loc. fin.}}(M, B_{-\omega})\}$ is point separating since P_ω is onto $(H^1_{d_c^\omega}(M))^*$, thus h has to be constant. □

3.7.13. Lemma. *Let $g \in C^\infty\big((\mathbb{R}, 0), (\operatorname{Diff}_c^\infty(M, \Omega, \omega), \operatorname{id})\big)$. Then*

$$g \in C^\infty(\mathbb{R}, \ker\Psi) \quad\Leftrightarrow\quad \delta^r g \in \Omega^1(\mathbb{R}; \ker\psi) \quad\Leftrightarrow\quad \dot{g}_t \in \ker\psi$$

Especially $\mathrm{Fl}^X \in C^\infty(\mathbb{R}, \ker\Psi)$ *iff* $X \in \ker\psi$.

Proof. By corollary 3.7.7 we may assume $g \in C^\infty(\mathbb{R}, \ker\Phi)$ and $\delta^r \in \Omega^1(\mathbb{R}, \ker\varphi)$. As in the proof of 3.7.7 one shows $\Psi(g_s) = \pi\big(\int_0^s \psi(\dot{g}_t)dt\big)$. Again the implication $\Leftarrow$ is now obvious. Moreover if g has values in $\ker\Psi$ this equation shows $\int_0^s \psi(\dot{g}_t)dt \in \Gamma$. By corollary 3.7.12 it has to be constant $= 0$. Differentiating with respect to s yields $\dot{g}_s \in \ker\psi$ for all $s \in \mathbb{R}$. □

3.7.14. Lemma. $\ker\Psi$ *is connected by smooth arcs, and the natural inclusion induces an isomorphism of groups* $i : \widetilde{\ker\Psi} \cong \ker\tilde{\Psi}$, *such that* $\operatorname{ev}_1 \circ i = \operatorname{ev}_1$.

Proof. The proof is similar to the proof of lemma 3.7.8. □

So we have again a commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{\pi}_1(\ker\Psi) & \hookrightarrow & \boldsymbol{\pi}_1(\ker\Phi) & \xrightarrow{\tilde{\Psi}} & \Gamma \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\ker\Psi} \cong \ker\tilde{\Psi} & \hookrightarrow & \widetilde{\ker\Phi} & \xrightarrow{\tilde{\Psi}} & H^1_{d_c^\omega}(M) \\
\downarrow{\scriptstyle \pi=\operatorname{ev}_1} & & \downarrow{\scriptstyle \pi=\operatorname{ev}_1} & & \downarrow{\scriptstyle \pi} \\
\ker\Psi & \hookrightarrow & \ker\Phi & \xrightarrow{\Psi} & H^1_{d_c^\omega}(M)/\Gamma
\end{array}$$

with exact rows and columns. The middle row does not split in general, cf. (3.12) on page 58 and remark 3.7.21.

3.7.15. Proposition. *Suppose (M, Ω, ω) is an exact locally conformally symplectic manifold, i.e. $\Omega = d^\omega\alpha$. Then for $g \in \widetilde{\ker\Phi}$ we have*

$$\tilde{\Psi}(g) = [a_1 g_1^*\alpha - \alpha] \in H^1_{d_c^\omega}(M)$$

where $g_t^*\Omega = \frac{1}{a_t}\Omega$ and $g_t^*\omega = \omega + d(\ln|a_t|)$. Especially $\Gamma = 0$ in this situation.

Proof. First of all we have

$$i_{\dot g_t}\Omega = i_{\dot g_t} d^\omega \alpha = L_{\dot g_t}\alpha + i_{\dot g_t}\omega\wedge\alpha - d^\omega(i_{\dot g_t}\alpha)$$

Since $g_t \in \ker\Phi$ the a_t are the same as the a_t of lemma 3.1.1 (cf. proof of lemma 3.7.9). So we get

$$[a_t g_t^* i_{\dot g_t}\Omega] = [a_t g_t^*(L_{\dot g_t}\alpha + i_{\dot g_t}\omega\wedge\alpha + d^\omega(i_{\dot g_t}\alpha))] = [a_t g_t^*(L_{\dot g_t}\alpha + i_{\dot g_t}\omega\wedge\alpha)]$$

Since $\dot g_t \in \ker\varphi$ we have $\frac{\partial}{\partial t}a_t = a_t g_t^* f_{\dot g_t} = a_t g_t^* i_{\dot g_t}\omega$ and hence

$$a_t g_t^*(L_{\dot g_t}\alpha + i_{\dot g_t}\omega\wedge\alpha) = a_t\tfrac{\partial}{\partial t}(g_t^*\alpha) + (\tfrac{\partial}{\partial t}a_t)g_t^*\alpha = \tfrac{\partial}{\partial t}(a_t g_t^*\alpha)$$

Putting all together we obtain

$$\tilde\Psi(g) = \int_0^1 [a_t g_t^* i_{\dot g_t}\Omega]dt = \Big[\int_0^1 \tfrac{\partial}{\partial t}(a_t g_t^*\alpha)dt\Big] = [a_1 g_1^*\alpha - a_0 g_0^*\alpha] = [a_1 g^*\alpha - \alpha]$$

□

3.7.16. Lemma. *ρ is $Ad(\ker\Psi)$ invariant, i.e. for all $X \in \ker\psi$ and $g \in \ker\Psi$ we have $\rho(Ad(g)\cdot X) = \rho((g^{-1})^*X) = \rho(X)$.*

Proof. From lemma 3.4.4 we obtain a commutative diagram:

$$\begin{array}{ccc}
\ker\psi & \xrightarrow{\rho} & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big) \\
\downarrow g^* & & \downarrow g^* \\
\ker\psi = \ker\psi' & \xrightarrow{\rho'} & H^{2n}_{d_c^{(n+1)g^*\omega}}(M)/\big(H^0_{d_c^{g^*\omega}}(M)\wedge[g^*\Omega^n]\big) \\
& \searrow^{\rho} & \downarrow a^{n+1} \\
& & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)
\end{array}$$

where a is such that $(M,\Omega,\omega) \overset{a}{\sim} (M, g^*\Omega, g^*\omega)$ and ψ', ρ' correspond to $(M, g^*\Omega, g^*\omega)$. So it remains to show that $a^{n+1}g^* : H^{2n}_{d_c^{(n+1)\omega}}(M) \to H^{2n}_{d_c^{(n+1)\omega}}(M)$ is the identity. Since $\ker\Psi$ is connected by smooth arcs (lemma 3.7.14) there exists a curve $g_t \in \ker\Psi$ with $g_0 = \mathrm{id}$ and $g_1 = g$. We define a_t by $(M,\Omega,\omega) \overset{a_t}{\sim} (M, g_t^*\Omega, g_t^*\omega)$. As in the proof of lemma 3.7.9 one sees that a_t is a_t from lemma 3.1.1 and hence a_t^{n+1} is a_t from lemma 3.1.1 with $(n+1)\omega$. But then lemma 3.1.1 yields $a^{n+1}g^* = a_1^{n+1}g_1^* = a_0^{n+1}g_0^* = \mathrm{id} : H^{2n}_{d_c^{(n+1)\omega}}(M) \to H^{2n}_{d_c^{(n+1)\omega}}(M)$. □

3.7.17. Proposition. *Let (M,Ω,ω) be a $2n$-dimensional locally conformally symplectic manifold. Then the Lie algebra homomorphism ρ integrates to a surjective group homomorphism $\tilde R : \widetilde{\ker\Psi} \to H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)$, i.e.*

$$\begin{array}{ccc}
\ker\psi & \xrightarrow{\rho} & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big) \\
\downarrow \exp=\mathrm{Fl} & & \downarrow \exp=\mathrm{id} \\
\widetilde{\ker\Psi} & \xrightarrow{\tilde R} & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)
\end{array}$$

commutes. We have the following formulas:

$$\tilde{R}(g) = \int_I \rho_*(\delta^r g) = \int_0^1 \rho(\dot{g}_t)dt = \left[\int_0^1 h_t \Omega^n dt\right] = \left[\int_0^1 a_t (g_t^* h_t)\Omega^n dt\right]$$

where $g_t^*\Omega = \frac{1}{a_t}\Omega$ *and* $d^\omega h_t = \flat \dot{g}_t$. *If* $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ *then* $\widetilde{\ker}\,\Psi = \widetilde{\ker}\,\Psi'$ *and* $\frac{1}{a^{n+1}} \circ \tilde{R} = \tilde{R}'$.

Proof. The proof is exactly the same as the proof of proposition 3.7.2. ☐

We let $\Lambda := \tilde{R}(\boldsymbol{\pi}_1(\ker\Psi))$. Then $\tilde{R}$ descends to a surjective homomorphism R

$$\begin{array}{ccc}
\widetilde{\ker}\,\Psi & \xrightarrow{\tilde{R}} & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big) \\
\downarrow & & \downarrow \\
\ker\Psi & \xrightarrow{R} & \Big(H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)\Big)/\Lambda
\end{array}$$

If $(M,\Omega,\omega) \overset{a}{\sim} (M,\Omega',\omega')$ then $\ker\Psi = \ker\Psi'$, $\frac{1}{a^{n+1}}\Lambda = \Lambda'$ and $\frac{1}{a^{n+1}} \circ R = R'$. The homomorphisms $\tilde{R}$ and R are due to G. Rousseau, see [Rou78], where it is also shown that Λ is countable, especially every continuous curve in Λ is constant.

3.7.18. Lemma. *Let* (M,Ω,ω) *be a* $2n$*-dimensional locally conformally symplectic manifold. Then for every* $g \in C^\infty\big((\mathbb{R},0),(\mathrm{Diff}^\infty_c(M,\Omega,\omega),\mathrm{id})\big)$ *we have*

$$g \in C^\infty(\mathbb{R},\ker R) \quad\Leftrightarrow\quad \delta^r g \in \Omega^1(\mathbb{R};\ker\rho) \quad\Leftrightarrow\quad \dot{g}_t \in \ker\rho$$

Especially $\mathrm{Fl}^X \in C^\infty(\mathbb{R},\ker R)$ *iff* $X \in \ker\rho$. *For the implication* $\Leftarrow$ *the assumption on* Λ *is superfluous.*

Proof. The proof is similar to the proof of corollary 3.7.7. ☐

3.7.19. Lemma. $\ker R$ *is connected by smooth arcs, and the natural inclusion induces an isomorphism of groups* $i : \widetilde{\ker}\,R \cong \ker\tilde{R}$, *such that* $\mathrm{ev}_1 \circ i = \mathrm{ev}_1$.

Proof. The proof is similar to the proof of lemma 3.7.8. ☐

In the situation of lemma 3.7.19 we have a commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{\pi}_1(\ker R) & \hookrightarrow & \boldsymbol{\pi}_1(\ker\Psi) & \xrightarrow{\tilde{R}} & \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\ker}\,R \cong \ker\tilde{R} & \hookrightarrow & \widetilde{\ker}\,\Psi & \xrightarrow{\tilde{R}} & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big) \\
\downarrow{\scriptstyle \pi=\mathrm{ev}_1} & & \downarrow{\scriptstyle \pi=\mathrm{ev}_1} & & \downarrow{\scriptstyle \pi} \\
\ker R & \hookrightarrow & \ker\Psi & \xrightarrow{R} & \Big(H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)\Big)/\Lambda
\end{array}$$

with exact rows and columns. The middle row splits and gives rise to a semi direct product

$$\widetilde{\ker}\,\Psi \cong \widetilde{\ker}\,R \times_\alpha H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)$$

cf. the extension (3.13) on page 58.

The following formula is due to Rousseau, see [Rou78].

3.7.20. Proposition. *For $g, h \in \widetilde{\ker \Phi}$ we have $[g,h] \in \widetilde{\ker \Psi}$ and*

$$\tilde{R}([g,h]) = n\{\tilde{\Psi}(g), \tilde{\Psi}(h)\}$$

where $\{\cdot,\cdot\}$ is the symplectic pairing, cf. formula (3.12) on page 58. Moreover the symplectic pairing descends to

$$\{\cdot,\cdot\} : \big(H^1_{d^\omega_c}(M)/\Gamma\big) \times \big(H^1_{d^\omega_c}(M)/\Gamma\big) \to \big(H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d^\omega_c}(M) \wedge [\Omega^n]\big)\big)/\Lambda \tag{3.18}$$

and for $g, h \in \ker \Phi$ we have $R([g,h]) = n\{\Psi(g), \Psi(h)\}$.

Proof. Notice first that $(s,t) \mapsto [g_{s+(1-s)t}, h_t]$ is a homotopy relative endpoints in $\ker \Psi$ from $[g_t, h_t]$ to $[g_1, h_t]$, so $\tilde{R}([g,h]) = \tilde{R}([g_1,h])$. Since both sides of the equation in question transform in the same way under conformal change, and since both sides vanish if (M,Ω,ω) is not conformally equivalent to a symplectic manifold, we may assume that M is symplectic. Using $\delta^r(gh)(\partial_t)(t) = \dot{g}_t + (g_t^{-1})^*\dot{h}_t$ and $\delta^r(g^{-1})(\partial_t)(t) = -(g_t)^*\dot{g}_t$ (which is an immediate consequence of the first equation) one obtains

$$\delta^r([g_1,h])(\partial_t)(t) = (g_1^{-1})^*\big(\dot{h}_t - (h_t g_1 h_t^{-1})^*\dot{h}_t\big)$$

and from lemma 3.1.1 for $\omega = 0$

$$\begin{aligned} i_{\delta^r([g_1,h])(\partial_t)(t)}\Omega &= -(g_1^{-1})^*\big((h_t g_1 h_t^{-1})^* i_{\dot{h}_t}\Omega - i_{\dot{h}_t}\Omega\big) \\ &= -(g_1^{-1})^* d\big(\textstyle\int_0^1 (h_t g_s h_t^{-1})^* i_{(h_t^{-1})^*\dot{g}_s} i_{\dot{h}_t}\Omega ds\big) \end{aligned}$$

Using $(i_X i_Y \Omega)\Omega^n = -n i_X\Omega \wedge i_Y\Omega \wedge \Omega^{n-1}$ we get

$$\begin{aligned} \rho\big(\delta^r([g_1,h])(\partial_t)(t)\big) &= -\textstyle\int_0^1 \big[(g_1^{-1})^*(h_t g_s h_t^{-1})^*(i_{(h_t^{-1})^*\dot{g}_s} i_{\dot{h}_t}\Omega)\Omega^n\big] ds \\ &= n \textstyle\int_0^1 [i_{\dot{g}_s}\Omega] \wedge [i_{\dot{h}_t}\Omega] \wedge [\Omega^{n-1}] ds = n\tilde{\Psi}(g) \wedge \psi(\dot{h}_t) \wedge [\Omega^{n-1}] \end{aligned}$$

So

$$\tilde{R}([g,h]) = \tilde{R}([g_1,h]) = n\tilde{\Psi}(g) \wedge \tilde{\Psi}(h) \wedge [\Omega^{n-1}] = n\{\tilde{\Psi}(g), \tilde{\Psi}(h)\}$$

Remains to check (3.18). We will show a little more, namely the symplectic pairing induces a mapping:

$$\{\cdot,\cdot\} : \big(H^1_{d^\omega_c}(M)/\Gamma\big) \times \big(H^1_{d^\omega_c}(M)/\Gamma\big) \to H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d^\omega_c}(M) \wedge [\Omega^n]\big)$$

Indeed, if $\alpha \in \Gamma$ and $\beta \in H^1_{d^\omega_c}(M)$ there exist $g \in \boldsymbol{\pi}_1(\ker \Phi)$ and $h \in \widetilde{\ker \Phi}$ with $\tilde{\Psi}(g) = \alpha$ and $\tilde{\Psi}(h) = \beta$. Hence

$$\{\alpha,\beta\} = \{\tilde{\Psi}(g), \tilde{\Psi}(h)\} = \tfrac{1}{n}\tilde{R}([g,h]) = \tfrac{1}{n}\tilde{R}(\mathrm{id}) = 0$$

since $[g,h]$ is homotopic relative endpoints in $\ker \Psi$ to $[g_1,h] = [\mathrm{id},h] = \mathrm{id}$. □

3.7.21. Remark. Proposition 3.7.20 shows that the short exact sequence

$$0 \to \ker \tilde{\Psi} \to \ker \tilde{\Phi} \xrightarrow{\tilde{\Psi}} H^1_{d^\omega_c}(M) \to 0$$

does not split in general, since a section s should satisfy $[s(\alpha), s(\beta)] = \mathrm{id}$ and hence $0 = \tilde{R}([s(\alpha), s(\beta)]) = n\{\alpha,\beta\}$.

3.7.22. Corollary. *Let (M,Ω,ω) be a $2n$-dimensional locally conformally symplectic manifold. Then $\ker R$ is an ideal in $\ker\Phi$ and $\ker\Phi/\ker R$ is a central extension of $H^1_c(M)/\Gamma$ by $\big(H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)\big)/\Lambda$.*

Proof. From proposition 3.7.20 we get $[\ker\Phi,\ker\Psi]\subseteq\ker R$. Especially $\ker R$ is an ideal in $\ker\Phi$. We have the following commutative diagram

$$\begin{array}{ccccc}
\ker R & = & \ker R & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\ker\Psi & \hookrightarrow & \ker\Phi & \overset{\Psi}{\twoheadrightarrow} & H^1_{d_c^\omega}(M)/\Gamma \\
\downarrow{\scriptstyle R} & & \downarrow & & \| \\
\big(H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d_c^\omega}(M)\wedge[\Omega^n]\big)\big)/\Lambda & \overset{i}{\hookrightarrow} & \ker\Phi/\ker R & \overset{\Psi}{\twoheadrightarrow} & H^1_{d_c^\omega}(M)/\Gamma
\end{array}$$

with exact rows and columns. Since $[\ker\Phi,\ker\Psi]\subseteq\ker R$ the last row is a central extension. □

3.7.23. Proposition. *Let (M,Ω,ω) be an exact, $2n$-dimensional locally conformally symplectic manifold, $\Omega=d^\omega\alpha$. For $g\in\widetilde{\ker\Psi}$ we have*

$$\tilde R(g)=\tfrac{1}{n+1}[k\Omega^n]=\tfrac{1}{n+1}[(a_1g_1^*\alpha)\wedge\alpha\wedge\Omega^{n-1}]\in H^{2n}_{d_c^{(n+1)\omega}}(M)$$

where $k\in C^\infty_c(M,\mathbb{R})$ is the unique function satisfying $d^\omega k=a_1g_1^*\alpha-\alpha$ (cf. proposition 3.7.15). Especially we have $\Lambda=0$ in this situation.

Proof. If ω is not exact then by example 1.6 $H^0_{d_c^\omega}(M)=0$, and if ω is exact then (M,Ω,ω) is conformally equivalent to a symplectic structure and it is well known that this can only happen if M is not compact, i.e. $0=H^0_c(M)\cong H^0_{d_c^\omega}(M)$. So we always have $H^0_{d_c^\omega}(M)=0$ and so k is unique.

Let h_t be the functions satisfying $\flat\dot g_t=d^\omega h_t$ and recall the homotopy operator from lemma 3.1.1. Then we have

$$a_1g_1^*\alpha-\alpha=H(d^\omega\alpha)+d^\omega H(\alpha)=H(\Omega)+d^\omega H(\alpha)$$

and

$$H(\Omega)=\int_0^1 a_tg_t^*i_{\dot g_t}\Omega dt=\int_0^1 a_tg_t^*(d^\omega h_t)dt=\int_0^1 d^\omega(a_tg_t^*h_t)dt$$

Together this yields

$$a_1g_1^*\alpha-\alpha=d^\omega\big(\int_0^1 a_tg_t^*h_tdt\big)+d^\omega\big(\int_0^1 a_tg_t^*i_{\dot g_t}\alpha dt\big)$$

and so

$$k=\int_0^1 a_tg_t^*h_tdt+\int_0^1 a_tg_t^*i_{\dot g_t}\alpha dt=:k_1+k_2$$

Next we have

$$\begin{aligned}
(a_tg_t^*i_{\dot g_t}\alpha)\wedge\Omega^n &= a_t^{n+1}g_t^*(i_{\dot g_t}\alpha\wedge\Omega^n)=na_t^{n+1}g_t^*(\alpha\wedge i_{\dot g_t}\Omega\wedge\Omega^{n-1})\\
&= na_t^{n+1}g_t^*(\alpha\wedge d^\omega h_t\wedge\Omega^{n-1})\\
&= na_t^{n+1}g_t^*(h_t\Omega^n)-na_t^{n+1}g_t^*d^{(n+1)\omega}(\alpha h_t\Omega^{n-1})\\
&= na_t^{n+1}g_t^*(h_t\Omega^n)-d^{(n+1)\omega}(na_t^{n+1}g_t^*(\alpha h_t\Omega^{n-1}))
\end{aligned}$$

and therefore $[k_2\Omega^n] = n[k_1\Omega^n] \in H^{2n}_{d_c^{(n+1)\omega}}(M)$. So

$$\tfrac{1}{n+1}[k\Omega^n] = \int_0^1 [a_t^{n+1} g_t^*(h_t\Omega^n)]dt = \int_0^1 [h_t\Omega^n]dt = \int_0^1 \rho(\dot g_t)dt = \tilde R(g)$$

The second expression follows now easily:

$$\begin{aligned}[][k\Omega^n] &= [kd^\omega\alpha\wedge\Omega^{n-1}] = [d^{(n+1)\omega}(k\wedge\alpha\wedge\Omega^{n-1}) - (d^\omega k)\wedge\alpha\wedge\Omega^{n-1}]\\ &= [(a_1g_1^*\alpha - \alpha)\wedge\alpha\wedge\Omega^{n-1}] = [(a_1g_1^*\alpha)\wedge\alpha\wedge\Omega^{n-1}]\end{aligned}$$

□

3.7.24. Lemma. *For $g \in \widetilde{\operatorname{Diff}}_c^\infty(M,\Omega,\omega)_\circ$ and $X \in \ker\varphi$ we have $g_1^*X \in \ker\varphi$ and*

$$\psi(g_1^*X) = e^{-\tilde\Phi(g)}\psi(X).$$

Proof. Recall $a_t = \exp\big(\int_0^t \operatorname{inc}_s^* i_{\partial_t} g^*\omega ds\big) = \exp\big(\int_0^t g_s^* i_{\dot g_s}\omega ds\big)$ from lemma 3.1.1. Moreover let b_t denote the functions satisfying $g_t^*\Omega = \frac{1}{b_t}\Omega$ and $g_t^*\omega = \omega + d\ln b_t$. Differentiating the first we obtain $g_t^* f_{\dot g_t} = \frac{\partial}{\partial t}\ln b_t$, where $f_{\dot g_t}$ is the function satisfying $L_{\dot g_t}\Omega = -f_{\dot g_t}\Omega$. From lemma 3.4.1 we obtain

$$\begin{aligned}a_t &= \exp\big(\textstyle\int_0^t g_s^* i_{\dot g_s}\omega ds\big) = \exp\big(\int_0^t g_s^* f_{\dot g_s} ds + \int_0^t g_s^* c_{\dot g_s} ds\big)\\ &= \exp\big(\textstyle\int_0^t \frac{\partial}{\partial s}\ln b_s ds\big)\cdot\exp\big(\int_0^t \varphi(\dot g_s)ds\big) = b_t\exp\big(\int_0^t\varphi(\dot g_s)ds\big)\end{aligned}$$

and hence $a_1 = b_1 e^{\tilde\Phi(g)}$. So we get

$$\psi(g_1^*X) = b_1 g_1^*\psi(X) = a_1 e^{-\tilde\Phi(g)} g_1^*\psi(X) = e^{-\tilde\Phi(g)} a_0 g_0^*\psi(X) = e^{-\tilde\Phi(g)}\psi(X)$$

where we used lemma 3.4.3 for the first equality and lemma 3.1.1 for the third one. □

3.7.25. Corollary. *If (M,Ω,ω) is a connected locally conformally symplectic manifold and $H^1_{d_c^\omega}(M) \neq 0$, then $\Delta = 0$.*

Proof. Suppose conversely $\Delta \neq 0$ and choose $g \in \boldsymbol{\pi}_1\big(\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ\big)$ with $\tilde\Phi(g) \neq 0$ and $X \in \ker\varphi$ with $\psi(X) \neq 0$. Then lemma 3.7.24 yields

$$\psi(X) = \psi(\mathrm{id}^*X) = \psi(g_1^*X) = e^{-\tilde\Phi(g)}\psi(X)$$

a contradiction since $\psi(X) \neq 0$ and $e^{-\tilde\Phi(g)} \neq 1$. □

3.7.26. Corollary. $\ker\Psi$ *is an ideal in* $\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ$ *and we have a semi direct product*

$$\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ/\ker\Psi \cong (H^1_{d_c^\omega}(M)/\Gamma) \times_\alpha (\operatorname{Im}(\varphi)/\Delta)$$

where the action $\alpha : \operatorname{Im}(\varphi)/\Delta \to \operatorname{Aut}\big(H^1_{d_c^\omega}(M)/\Gamma\big)$ is given by $\alpha(c)(\beta) = e^c\beta$.

Proof. We first show:

$$\tilde\Psi(ghg^{-1}) = e^{\tilde\Phi(g)}\tilde\Psi(h) \qquad \forall g \in \widetilde{\operatorname{Diff}}_c^\infty(M,\Omega,\omega)_\circ, h \in \widetilde{\ker}\,\Phi \tag{3.19}$$

Notice that $(s,t) \mapsto g_{s+(1-s)t} h_t g_{s+(1-s)t}^{-1}$ is a homotopy relative endpoints in $\ker \Phi$ from $t \mapsto g_t h_t g_t^{-1}$ to $t \mapsto g_1 h_t g_1^{-1}$ and so we obtain from lemma 3.7.24

$$\begin{aligned}\tilde{\Psi}(ghg^{-1}) = \tilde{\Psi}(g_1 h g_1^{-1}) &= \textstyle\int_0^1 \psi\big(\delta^r(g_1 h g_1^{-1})(\partial_t)(t)\big)dt \\ &= \textstyle\int_0^1 \psi\big((g_1^{-1})^* \dot{h}_t\big)dt = \int_0^1 e^{\tilde{\Phi}(g)}\psi(\dot{h}_t)dt = e^{\tilde{\Phi}(g)}\tilde{\Psi}(h)\end{aligned}$$

which is precisely (3.19). From (3.19) we immediately obtain:

$$\Psi(ghg^{-1}) = e^{\Phi(g)}\Psi(h) \qquad \forall g \in \operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ, h \in \ker \Phi \tag{3.20}$$

The latter equation makes sense, because on the components of M on which $\operatorname{Im}(\varphi) \neq 0$ and $H^1_{d_c^\omega}(M) \neq 0$ we have $\Gamma = 0$ by proposition 3.7.15 and $\Delta = 0$ by corollary 3.7.25. Especially $\ker \Psi$ is an ideal in $\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ$ and we have a commutative diagram

$$\begin{array}{ccccc} \ker \Psi & = & \ker \Psi & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \ker \Phi & \hookrightarrow & \operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\Phi} & \operatorname{Im}(\varphi)/\Delta \\ \downarrow{\scriptstyle\Psi} & & \downarrow & & \| \\ H^1_{d_c^\omega}(M)/\Gamma & \hookrightarrow & \operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ/\ker\Psi & \xrightarrow{\Phi} & \operatorname{Im}(\varphi)/\Delta \end{array}$$

with exact rows and columns. If $H^1_{d_c^\omega}(M) = 0$ we are done. So assume $H^1_{d_c^\omega}(M) \neq 0$. Then $\Delta = 0$ by corollary 3.7.25 and the middle row splits since the middle row (and thus the bottom row) of the big diagram on page 71 splits. So the bottom row of the diagram above splits too, and $\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ/\ker\Psi$ is a semi direct product of $H^1_{d_c^\omega}(M)/\Gamma$ and $\operatorname{Im}(\varphi)/\Delta$. The corresponding action is as stated, for we have (3.20). □

3.8 Summary of the Various Invariants

In this section we give a brief summary of the invariants we have considered up to now. We have seen that a vector field $X \in \mathfrak{X}_c(M)$ is an infinitesimal automorphism of the locally conformally symplectic manifold (M,Ω,ω) iff there exists a locally constant function $c_X \in C_c^\infty(M,\mathbb{R})$ such that $d^\omega(\flat X) = c_X\Omega$. If $c_X = 0$ then $\flat X$ defines a cohomology class in $H^1_{d_c^\omega}(M)$, and if in addition this cohomology class vanishes, then there exists a function $h_X \in C_c^\infty(M,\mathbb{R})$ with $\flat X = d^\omega h_X$. We have shown in section 3.4 that the following are well defined homomorphisms of Lie algebras:

$$\begin{aligned} \varphi &: \mathfrak{X}_c(M,\Omega,\omega) \to H^0_c(M) & \varphi(X) &= [c_X] \\ \psi &: \ker\varphi \to H^1_{d_c^\omega}(M) & \psi(X) &= [\flat X] \\ \rho &: \ker\psi \to H^{2n}_{d_c^{(n+1)\omega}}(M)/(H^0_{d_c^\omega}(M)\wedge[\Omega^n]) & \rho(X) &= [h_X\Omega^n] \end{aligned}$$

If $g : I \to \operatorname{Diff}_c^\infty(M,\Omega,\omega)$ is a smooth curve then $\delta^r g \in \Omega^1\big(I;\mathfrak{X}_c(M,\Omega,\omega)\big)$ and the following are well defined homomorphisms of groups, integrating φ:

$$\begin{array}{ccc} \widetilde{\operatorname{Diff}}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\tilde{\Phi}} & H^0_c(M) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ & \xrightarrow{\Phi} & H^0_c(M)/\Delta \end{array} \qquad \begin{aligned} \tilde{\Phi}(g) &= \textstyle\int_I \varphi_*(\delta^r g) \\ \Delta &= \tilde{\Phi}\big(\boldsymbol{\pi}_1(\operatorname{Diff}_c^\infty(M,\Omega,\omega)_\circ)\big) \end{aligned}$$

If g has values in $\ker\Phi$ then $\delta^r g \in \Omega^1(I;\ker\varphi)$ and the following are well defined homomorphisms of groups, integrating ψ:

$$\begin{array}{ccc} \widetilde{\ker\Phi} & \xrightarrow{\tilde\Psi} & H^1_{d^\omega_c}(M) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \ker\Phi & \xrightarrow{\Psi} & H^1_{d^\omega_c}(M)/\Gamma \end{array} \qquad \begin{aligned} \tilde\Psi(g) &= \textstyle\int_I \psi_*(\delta^r g) \\ \Gamma &= \tilde\Psi(\boldsymbol{\pi}_1(\ker\Phi)) \end{aligned}$$

If g has values in $\ker\Psi$ then $\delta^r g \in \Omega^1(I;\ker\psi)$ and the following are well defined homomorphisms of groups, integrating ρ:

$$\begin{array}{ccc} \widetilde{\ker\Psi} & \xrightarrow{\tilde R} & H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d^\omega_c}(M)\wedge[\Omega^n]\big) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \ker\Psi & \xrightarrow{R} & \big(H^{2n}_{d_c^{(n+1)\omega}}(M)/\big(H^0_{d^\omega_c}(M)\wedge[\Omega^n]\big)\big)/\Lambda \end{array} \qquad \begin{aligned} \tilde R(g) &= \textstyle\int_I \rho_*(\delta^r g) \\ \Lambda &= \tilde R(\boldsymbol{\pi}_1(\ker\Psi)) \end{aligned}$$

All this can be found in section 3.7.

3.9 A Chart for $\mathrm{Diff}_c^\infty(M,\Omega,\omega)$

3.9.1. Theorem. *Let (M,Ω,ω) be a connected locally conformally symplectic manifold and assume that Ω is not d^ω_c-exact. Then $\mathrm{Diff}_c^\infty(M,\Omega,\omega)=\ker\Phi$ is a Lie group in the sense of [KM97] modeled on the convenient vector space $\mathfrak{X}_c(M,\Omega,\omega)=\ker\varphi$.*

Proof. Notice first that the assumption "Ω is not d^ω_c-exact" is satisfied iff $\varphi=0$ (see lemma 3.4.1). We consider the locally conformally symplectic manifold (T^*M,Ω',ω'), where $\omega'=\pi^*\omega$, $\Omega'=d^{\omega'}\Theta$ and $\pi:T^*M\to M$ is the projection (see example 3.2.4). For $\alpha\in\Omega^1_c(M)$ we have $\alpha^*\Omega'=d^\omega\alpha$ and hence

$$\alpha^*\Omega'=0 \quad\Leftrightarrow\quad d^\omega\alpha=0. \tag{3.21}$$

Let $p_1,p_2:M\times M\to M$ denote the projections on the first and second factor, and let $\Delta\subseteq M\times M$ be the diagonal. Since $p_2^*\omega-p_1^*\omega$ is closed and vanishes when pulled back to Δ, there exists a function λ, defined locally around Δ, such that

$$p_2^*\omega-p_1^*\omega=d\ln\lambda \quad\text{and}\quad \lambda|_\Delta=1.$$

On a neighborhood of Δ we consider the locally conformally symplectic structure $(\tilde\Omega,\tilde\omega)$, where $\tilde\omega:=p_1^*\omega$ and $\tilde\Omega:=p_1^*\Omega-\lambda p_2^*\Omega$. Indeed we have

$$d^{\tilde\omega}\tilde\Omega=d^{p_1^*\omega}\big(p_1^*\Omega-\lambda p_2^*\Omega\big)=0-d^{p_1^*\omega}(\lambda p_2^*\Omega)=-\lambda d^{p_2^*\omega}p_2^*\Omega=0$$

and $\tilde\Omega$ is of course non-degenerated. We claim that for $g\in\mathrm{Diff}_c^\infty(M)$ near the identity we have

$$g\in\mathrm{Diff}_c^\infty(M,\Omega,\omega) \quad\Leftrightarrow\quad (\mathrm{id},g)^*\tilde\Omega=0 \tag{3.22}$$

where $(\mathrm{id},g):M\to M\times M$. Indeed from $(\mathrm{id},g)^*\tilde\Omega=0$ we get

$$0=(\mathrm{id},g)^*\tilde\Omega=(\mathrm{id},g)^*\big(p_1^*\Omega-\lambda p_2^*\Omega\big)=\Omega-((\mathrm{id},g)^*\lambda)g^*\Omega$$

i.e. $g^*\Omega = \frac{1}{(\mathrm{id},g)^*\lambda}\Omega$. Moreover

$$g^*\omega = (\mathrm{id},g)^* p_2^*\omega = (\mathrm{id},g)^*(p_1^*\omega + d\ln\lambda) = \omega + d\ln((\mathrm{id},g)^*\lambda)$$

and hence $g \in \mathrm{Diff}_c^\infty(M,\Omega,\omega)$. Suppose conversely $g \in \mathrm{Diff}_c^\infty(M,\Omega,\omega)$ with $g^*\Omega = \frac{1}{a}\Omega$ and $g^*\omega = \omega + d\ln a$. From the last equation and $g^*\omega = \omega + d\ln((\mathrm{id},g)^*\lambda)$ we obtain $\frac{a}{(\mathrm{id},g)^*\lambda} = c$, where c is a constant. It remains to show $c = 1$, for then

$$(\mathrm{id},g)^*\tilde{\Omega} = \Omega - ((\mathrm{id},g)^*\lambda)g^*\Omega = \Omega - ag^*\Omega = 0.$$

Notice that outside a compact set $c = 1$ and therefore we are done if M is non-compact. So assume M compact and $c \neq 1$. Since $\tilde{\Omega}$ is $d^{\tilde{\omega}}$-closed and vanishes when pulled back to Δ we obtain from corollary 3.1.2 a 1-form β, locally defined around Δ, such that $d^{\tilde{\omega}}\beta = \tilde{\Omega}$. Then we have

$$\Omega = ag^*\Omega = c((\mathrm{id},g)^*\lambda)g^*\Omega = -c(\mathrm{id},g)^*\tilde{\Omega} + c\Omega$$

and hence

$$\Omega = \tfrac{c}{c-1}(\mathrm{id},g)^*\tilde{\Omega} = \tfrac{c}{c-1}(\mathrm{id},g)^*d^{\tilde{\omega}}\beta = \tfrac{c}{c-1}d^{\omega}\big((\mathrm{id},g)^*\beta\big) = d^{\omega}\big(\tfrac{c}{c-1}(\mathrm{id},g)^*\beta\big)$$

a contradiction to the assumption that Ω is not d_c^{ω}-exact.

If $\exp : TM \to M \times M$ denotes the exponential mapping of a Riemannian metric on M we obtain a diffeomorphism

$$\exp\circ\sharp : T^*M \supseteq V \to W \subseteq M \times M$$

where V is an open neighborhood of the zero-section and W is an open neighborhood of Δ which maps the zero-section identically (in the natural way) onto Δ. Now (V,Ω',ω') and $\big(V,(\exp\circ\sharp)^*\tilde{\Omega},(\exp\circ\sharp)^*\tilde{\omega}\big)$ are two locally conformally symplectic structures, the zero-section is a common Lagrangian submanifold, and the 1-forms ω', $(\exp\circ\sharp)^*\tilde{\omega}$ equal when pulled back to the zero-section. So we may apply lemma 3.2.10 to obtain a diffeomorphism, mapping the first structure to the second up to conformal change. Summing up we obtain possibly smaller neighborhoods V, W of the zero-section resp. Δ and a diffeomorphism

$$\gamma : T^*M \supseteq V \to W \subseteq M \times M$$

which maps the zero section identically onto Δ, and such that $(V,\gamma^*\tilde{\Omega},\gamma^*\tilde{\omega})$ is conformally equivalent to (V,Ω',ω'). It is well known (see [KM97]) that there exists an open neighborhood U of the $\mathrm{id} \in \mathrm{Diff}_c^\infty(M)$ such that

$$u : \mathrm{Diff}_c^\infty(M) \supseteq U \to u(U) \subseteq \Omega_c^1(M) \qquad u(g) := \gamma^{-1}\circ(\mathrm{id},g)\circ\big(\pi\circ\gamma^{-1}\circ(\mathrm{id},g)\big)^{-1}$$

is a chart for $\mathrm{Diff}_c^\infty(M)$, centered at id. Its inverse is:

$$u^{-1} : \Omega_c^1(M) \supseteq u(U) \to U \subseteq \mathrm{Diff}_c^\infty(M) \qquad u^{-1}(\alpha) = p_2\circ\gamma\circ\alpha\circ\big(p_1\circ\gamma\circ\alpha\big)^{-1}$$

For $g \in U$ we obtain from the equations (3.21) and (3.22)

$$\begin{aligned} g \in \mathrm{Diff}_c^\infty(M,\Omega,\omega) &\Leftrightarrow (\mathrm{id},g)^*\tilde{\Omega} = 0 \Leftrightarrow (\gamma^{-1}\circ(\mathrm{id},g))^*\Omega' = 0 \\ &\Leftrightarrow (u(g))^*\Omega' = 0 \Leftrightarrow d^{\omega}(u(g)) = 0. \end{aligned}$$

Therefore
$$u\big(U\cap \mathrm{Diff}_c^\infty(M,\Omega,\omega)\big)=u(U)\cap\{\alpha\in\Omega_c^1(M):d^\omega\alpha=0\}$$
and so u is a submanifold chart for $\mathrm{Diff}_c^\infty(M,\Omega,\omega)\subseteq\mathrm{Diff}_c^\infty(M)$. Especially $\mathrm{Diff}_c^\infty(M,\Omega,\omega)$ is a Lie group modeled on the convenient vector space of d^ω-closed 1-forms, but via $\sharp$ this is isomorphic to $\ker\varphi=\mathfrak{X}_c(M,\Omega,\omega)$. □

3.9.2. Remark. Notice that the assumption in theorem 3.9.1 is satisfied if and only if $\varphi=0$. So, if (M,Ω,ω) is conformally equivalent to a symplectic manifold the assumption is always satisfied, see remark 3.4.2. In this case the chart constructed in the proof of theorem 3.9.1 is precisely the Weinstein chart, see [Wei71], [Wei77] or [KM97]. Moreover if M is not compact then the assumption of theorem 3.9.1 is always satisfied too.

3.9.3. Theorem. *Let (M,Ω,ω) be a connected, locally conformally symplectic manifold such that Ω is d_c^ω-exact. Then M is compact and $\mathrm{Diff}^\infty(M,\Omega,\omega)$ is a Lie group in the sense of [KM97] modeled on the convenient vector space $\mathfrak{X}(M,\Omega,\omega)$.*

Proof. By assumption there exists $\beta\in\Omega_c^1(M)$ with $d^\omega\beta=\Omega$, especially M is compact. Recall from the proof of theorem 3.9.1 that there exist open neighborhoods V, W of the zero-section resp. the diagonal Δ and a diffeomorphism
$$\gamma:T^*M\supseteq V\to W\subseteq M\times M$$
such that $\gamma^*(p_1^*\Omega-\lambda p_2^*\Omega)$ equals $d^{\pi^*\omega}\Theta$ up to multiplication with a nowhere vanishing function. Let $\Omega_2:=d^{\pi^*\omega}\Theta\in\Omega^2(V\times\mathbb{R})$ and $\kappa_2:=dt\in\Omega^1(V\times\mathbb{R})$. Here, and from now on, π denotes the projection $T^*M\times\mathbb{R}\to M$ and Θ is the pull back of the canonical 1-form on T^*M to $T^*M\times\mathbb{R}$. Next we define $\Omega_3:=p_1^*\Omega-\lambda p_2^*\Omega\in\Omega^2(W\times\mathbb{R})$, $\kappa_3:=dt\in\Omega^1(W\times\mathbb{R})$, where $p_1,p_2:M\times M\times\mathbb{R}\to M$ denote the two projections onto M. The diffeomorphism
$$\rho_2:=\gamma\times\mathrm{id}_\mathbb{R}:V_2:=V\times\mathbb{R}\to W\times\mathbb{R}=:V_3$$
has the property that $\rho_2^*\kappa_3=\kappa_2$ and $\rho_2^*\Omega_3$ equals Ω_2 up to multiplication with a nowhere vanishing function. Moreover for $\alpha\in\Omega^1(M)$ consider the diffeomorphism $\tau_\alpha:T^*M\to T^*M$, $\tau_\alpha(e)=e+\alpha(\pi(e))$, let
$$\rho_1:=(\tau_{-t\beta},t):T^*M\times\mathbb{R}\to T^*M\times\mathbb{R}$$
and set $V_1:=\rho_1^{-1}(V_2)$. We have $\rho_1^*\Omega_2=\Omega_1:=d^{\pi^*\omega}\Theta-t\pi^*\Omega\in\Omega^2(V_1)$ and $\rho_1^*\kappa_2=\kappa_1:=dt\in\Omega^1(V_1)$. Indeed $\rho_1^*\Omega_2=(\tau_{-t\beta},t)^*d^{\pi^*\omega}\Theta=d^{\pi^*\omega}(\tau_{-t\beta}^*\Theta)=d^{\pi^*\omega}(\Theta+\pi^*(-t\beta))=d^{\pi^*\omega}\Theta-t\pi^*d^\omega\beta=\Omega_1$, cf. lemma 4.3.2. Next we consider the diffeomorphism
$$\rho_3:M\times M\times\mathbb{R}\to M\times M\times\mathbb{R}\qquad\rho_3(x,y,t):=\big(x,\mathrm{Fl}_t^{\sharp\beta}(y),t\big)$$
and set $V_4:=\rho_3^{-1}(V_3)$. We have $\rho_3^*\Omega_3=\Omega_4:=p_1^*\Omega-e^t\lambda p_2^*\Omega\in\Omega^2(V_4)$ and $\rho_3^*\kappa_3=\kappa_4:=dt\in\Omega^1(V_4)$. Indeed $\rho_3^*\Omega_3=\rho_3^*(p_1^*\Omega-\lambda p_2^*\Omega)=p_1^*\Omega-(\rho_3^*\lambda)p_2^*(\mathrm{Fl}_t^{\sharp\beta})^*\Omega$ and thus it suffices to show $e^t\lambda p_2^*\Omega=(\rho_3^*\lambda)p_2^*(\mathrm{Fl}_t^{\sharp\beta})^*\Omega$. For $t=0$ this is obviously true and one easily shows that both sides satisfy the same differential equation with respect to t. Finally let
$$\rho_4:M\times M\times\mathbb{R}\to M\times M\times(0,\infty)\qquad\rho_4(x,y,t):=\big(x,y,e^t\lambda(x,y)\big)$$
and $V_5:=\rho_4(V_4)$. A simple calculation shows $\rho_4^*\Omega_5=\Omega_4$ and $\rho_4^*\kappa_5=\kappa_4$, where $\Omega_5:=p_1^*\Omega-tp_2^*\Omega\in\Omega^2\big(M\times M\times(0,\infty)\big)$ and $\kappa_5:=p_1^*\omega-p_2^*\omega+d\ln t\in\Omega^1\big(M\times M\times(0,\infty)\big)$.

Summing up we have open neighborhoods V_1, V_5 of the zero-section in $T^*M \times \mathbb{R}$ resp. $\Delta \times \{1\}$ and a diffeomorphism

$$\rho := \rho_4 \circ \rho_3^{-1} \circ \rho_2 \circ \rho_1 : T^*M \times \mathbb{R} \supseteq V_1 \to V_5 \subseteq M \times M \times (0, \infty)$$

which maps the zero section identically onto the diagonal, i.e. $\rho(0_x, 0) = (x, x, 1)$. Moreover $\rho^*\kappa_5 = \kappa_1$ and $\rho^*\Omega_5$ equals Ω_1 up to multiplication with a nowhere vanishing function. Now consider the semi direct product

$$\mathrm{Diff}^\infty(M) \times C^\infty(M, \mathbb{R}^*) \qquad (g, a) \cdot (h, b) := \big(g \circ h, (h^*a)b\big)$$

where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. With the help of ρ we obtain a chart for this group

$$u : \mathrm{Diff}^\infty(M) \times C^\infty(M, \mathbb{R}^*) \supseteq U \to u(U) \subseteq \Gamma(T^*M \times \mathbb{R} \to M)$$
$$u(g, a) := \rho^{-1} \circ (\mathrm{id}, g, a) \circ \big(\pi \circ \rho^{-1} \circ (\mathrm{id}, g, a)\big)^{-1}$$

where U is an open, sufficiently small neighborhood of $(\mathrm{id}, 1)$. Its inverse is:

$$u^{-1}(s) = \big(p_2 \circ \rho \circ s \circ (p_1 \circ \rho \circ s)^{-1}, p_3 \circ \rho \circ s \circ (p_1 \circ \rho \circ s)^{-1}\big)$$

We have a homomorphism of groups

$$i : \mathrm{Diff}^\infty(M, \Omega, \omega) \to \mathrm{Diff}^\infty(M) \times C^\infty(M, \mathbb{R}^*) \qquad i(g) := (g, a)$$

where $g^*\Omega = \frac{1}{a}\Omega$, which is a homeomorphism onto its image. Moreover

$$j : \mathfrak{X}(M, \Omega, \omega) \to \Gamma(T^*M \times \mathbb{R} \to M) \qquad j(X) := (\flat X, c)$$

where the constant c is defined by $d^\omega(\flat X) = c\Omega$ (cf. lemma 3.4.1), is a linear homeomorphism onto its image. For $(g, a) \in U$ we have

$$\begin{aligned}
(g, a) \in \mathrm{Im}(i) &\Leftrightarrow g^*\Omega = \tfrac{1}{a}\Omega \text{ and } g^*\omega = \omega + d \ln a \\
&\Leftrightarrow (\mathrm{id}, g, a)^*\Omega_5 = 0 \text{ and } (\mathrm{id}, g, a)^*\kappa_5 = 0 \\
&\Leftrightarrow (\rho^{-1} \circ (\mathrm{id}, g, a))^*\Omega_1 = 0 \text{ and } (\rho^{-1} \circ (\mathrm{id}, g, a))^*\kappa_1 = 0 \\
&\Leftrightarrow (u(g, a))^*\Omega_1 = 0 \text{ and } (u(g, a))^*\kappa_1 = 0 \\
&\Leftrightarrow d^\omega(u_1(g, a)) = u_2(g, a)\Omega \text{ and } d(u_2(g, a)) = 0 \\
&\Leftrightarrow u(g, a) \in \mathrm{Im}(j)
\end{aligned}$$

where $u_1(g, a) \in \Omega^1(M)$, $u_2(g, a) \in C^\infty(M, \mathbb{R})$ denote the two components of $u(g, a) \in \Gamma(T^*M \times \mathbb{R}) \cong \Omega^1(M) \times C^\infty(M, \mathbb{R})$. So $u(U \cap \mathrm{Im}(i)) = u(U) \cap \mathrm{Im}(j)$ and u is a submanifold chart for $\mathrm{Diff}^\infty(M, \Omega, \omega) \subseteq \mathrm{Diff}^\infty(M) \times C^\infty(M, \mathbb{R}^*)$. Especially $\mathrm{Diff}^\infty(M, \Omega, \omega)$ is a Lie group modeled on the convenient vector space $\mathrm{Im}(j) \cong \mathfrak{X}(M, \Omega, \omega)$. □

3.10 Fragmentation Lemmas

Let $i : U \to V$ denote the inclusion of two open subsets in M. Similar to the discussion at the beginning of section 3.5 we have commutative diagrams

$$\begin{array}{ccc}
\widetilde{\mathrm{Diff}}_c^\infty(U, \Omega|_U, \omega|_U)_\circ & \xrightarrow{\tilde\Phi_U} & H_c^0(U) \\
\downarrow & & \downarrow{\scriptstyle i_*} \\
\widetilde{\mathrm{Diff}}_c^\infty(V, \Omega, \omega)_\circ & \xrightarrow{\tilde\Phi_V} & H_c^0(V)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Diff}_c^\infty(U, \Omega|_U, \omega|_U)_\circ & \xrightarrow{\Phi_U} & H_c^0(U)/\Delta_U \\
\downarrow & & \downarrow{\scriptstyle i_*} \\
\mathrm{Diff}_c^\infty(V, \Omega, \omega)_\circ & \xrightarrow{\tilde\Phi_V} & H_c^0(V)/\Delta_V
\end{array}$$

and hence $\ker \Phi_U \subseteq \ker \Phi_V$. Moreover we have commutative diagrams

$$\begin{array}{ccc} \widetilde{\ker}\,\Phi_U & \xrightarrow{\tilde\Psi_U} & H^1_{d^\omega_c}(U) \\ \downarrow & & \downarrow i_* \\ \widetilde{\ker}\,\Phi_V & \xrightarrow{\tilde\Psi_V} & H^1_{d^\omega_c}(V) \end{array} \qquad \begin{array}{ccc} \ker \Phi_U & \xrightarrow{\Psi_U} & H^1_{d^\omega_c}(U)/\Gamma_U \\ \downarrow & & \downarrow i_* \\ \ker \Phi_V & \xrightarrow{\Psi_V} & H^1_{d^\omega_c}(V)/\Gamma_V \end{array}$$

and hence $\ker \Psi_U \subseteq \ker \Psi_V$. Finally the diagrams

$$\begin{array}{ccc} \widetilde{\ker}\,\Psi_U & \xrightarrow{\tilde R_U} & H^{2n}_{d_c^{(n+1)\omega}}(U)/\big(H^0_{d^\omega_c}(U)\wedge[\Omega^n|_U]\big) \\ \downarrow & & \downarrow i_* \\ \widetilde{\ker}\,\Psi_V & \xrightarrow{\tilde R_V} & H^{2n}_{d_c^{(n+1)\omega}}(V)/\big(H^0_{d^\omega_c}(V)\wedge[\Omega^n|_V]\big) \end{array}$$

and

$$\begin{array}{ccc} \ker \Psi_U & \xrightarrow{R_U} & \big(H^{2n}_{d_c^{(n+1)\omega}}(U)/\big(H^0_{d^\omega_c}(U)\wedge[\Omega^n|_U]\big)\big)/\Lambda_U \\ \downarrow & & \downarrow i_* \\ \ker \Psi_V & \xrightarrow{R_V} & \big(H^{2n}_{d_c^{(n+1)\omega}}(V)/\big(H^0_{d^\omega_c}(V)\wedge[\Omega^n|_V]\big)\big)/\Lambda_V \end{array}$$

commute and so $\ker R_U \subseteq \ker R_V$.

3.10.1. Lemma. *Let (M,Ω,ω) be a locally conformally symplectic manifold and let $\mathcal{U}$ be an open covering of M. Then any $g \in C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ has a decomposition $g = g_1\cdots g_n$, where each g_i is supported in some $U_i \in \mathcal{U}$ and $g_i \in C^\infty\big((I,0),(\ker\Psi_{U_i},\mathrm{id})\big)$*

Proof. Fix a compact set $K \subseteq M$ and define

$$H_K : C^\infty\big(I,\Omega^0_K(M)\big) \to C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big) \qquad \alpha \mapsto \mathrm{Evol}\,\big((\sharp\circ d^\omega)_*\alpha\big)$$

that is the defining equation for $g = H_K(\alpha)$ is $\flat\dot g_t = d^\omega\alpha_t$ with initial condition $g_0 = \mathrm{id}$, cf. lemma 3.7.13. We define the structure of a topological group on the left hand side space such that H_K becomes a continuous homomorphism. Namely we set

$$(\alpha\beta)(t) := \alpha(t) + (H_K(\alpha)(t)^{-1})^*(\tfrac{1}{a_t}\beta(t)) \tag{3.23}$$

where $H_K(\alpha)(t)^*\Omega = \frac{1}{a_t}\Omega$. If $\alpha,\beta \in C^\infty\big(I,\Omega^0_K(M)\big)$ and $g = H_K(\alpha)$, $h = H_K(\beta)$ we have

$$\begin{aligned} d^\omega\big(\alpha_t + (g_t^{-1})^*(\tfrac{1}{a_t}\beta_t)\big) &= \flat\dot g_t + (g_t^{-1})^*(\tfrac{1}{a_t}d^\omega\beta_t) = \flat\dot g_t + (g_t^{-1})^*(\tfrac{1}{a_t}\flat\dot h_t) \\ &= \flat\dot g_t + \flat\big((g_t^{-1})^*\dot h_t\big) = \flat\big(\delta^r(gh)(\partial_t)(t)\big) \end{aligned}$$

so H_K is a homomorphism, provided (3.23) defines a group structure on $C^\infty\big(I,\Omega^0_K(M)\big)$. To see this notice first that $0 \in C^\infty\big(I,\Omega^0_K(M)\big)$ is the neutral element and $(\alpha^{-1})(t) := -a_t(H_K(\alpha)(t))^*(\alpha(t))$ is the inverse of α. So the only non-trivial thing to check is associativity. So let $\alpha,\beta,\gamma \in C^\infty\big(I,\Omega^0_K(M)\big)$, $g = H_K(\alpha)$, $h = H_K(\beta)$, $k = H_K(\gamma)$ and let $g_t^*\Omega = \frac{1}{a_t}\Omega$, $h_t^*\Omega = \frac{1}{b_t}\Omega$. Since we already know that $H_K(\alpha\beta)(t) = g_th_t$ we get

$$((\alpha\beta)\gamma)(t) = \alpha_t + (g_t^{-1})^*(\tfrac{1}{a_t}\beta_t) + ((g_th_t)^{-1})^*(\tfrac{1}{(h_t^*a_t)b_t}\gamma_t)$$

which is equal to:

$$(\alpha(\beta\gamma))(t) = \alpha_t + (g_t^{-1})^*\big(\tfrac{1}{a_t}\big(\beta_t + (h_t^{-1})^*(\tfrac{1}{b_t}\gamma_t)\big)\big)$$

From lemma 3.7.13 and corollary 3.1.12 we get $\bigcup_K \operatorname{Im} H_K = C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ and so we only have to show that every $g \in \operatorname{Im}(H_K)$ has the desired decomposition.

Now choose $U_1,\dots,U_n \in \mathcal{U}$ covering K, open sets V_i, W_i with $\bar W_i \subseteq V_i \subseteq \bar V_i \subseteq U_i$ such that W_i still cover K and a partition of unity $\{\lambda_0,\dots,\lambda_n\}$ subordinated to $\{M\setminus K, W_1,\dots,W_n\}$. Consider the open neighborhoods $\mathcal{W}_i$ of the identity

$$\mathcal{W}_i := \big\{g \in C^\infty\big((I,0),(\mathrm{Diff}_c^\infty(M,\Omega,\omega),\mathrm{id})\big) : g_t(M\setminus \bar V_i) \subseteq M \setminus \bar W_i \quad \forall t \in I\big\}$$

and define an open neighborhood of $0 \in C^\infty\big(I,\Omega^0_K(M)\big)$ by

$$\mathcal{W}_K := \big\{\alpha \in C^\infty\big(I,\Omega^0_K(M)\big) : H_K(\textstyle\sum_{j=0}^{i-1}\lambda_j\alpha) \in \mathcal{W}_i \quad \forall 1 \le i \le n\big\}$$

Since $\mathcal{W}_K$ is open it generates $C^\infty\big(I,\Omega^0_K(M)\big)$ as group and so $H_K(\mathcal{W}_K)$ generates $\operatorname{Im}(H_K)$. Consequently it suffices to show that every $g \in H_K(\mathcal{W}_K)$ has the desired decomposition.

For $\alpha \in \mathcal{W}_K$ we set $f_i := H_K(\sum_{j=0}^{i}\lambda_j\alpha)$, $i = 0,\dots,n$. Then we have $f_0 = \mathrm{id}$, $f_n = H_K(\alpha)$, and if we let $g_i := f_{i-1}^{-1}f_i$, $i = 1,\dots,n$, we obtain $H_K(\alpha) = g_1\cdots g_n$. It remains to show that $g_i \in C^\infty\big((I,0),(\ker\Psi_{U_i},\mathrm{id})\big)$, but this follows from

$$\begin{aligned} g_i &= f_{i-1}^{-1}f_i \\ &= H_K\big(t \mapsto -a_{i-1}(t)f_{i-1}(t)^*(\textstyle\sum_{j=0}^{i-1}\lambda_j\alpha_t) + f_{i-1}(t)^*\big(f_{i-1}^{-1}(t)^*(a_{i-1}(t))\sum_{j=0}^{i}\lambda_j\alpha_t\big)\big) \\ &= H_K\big(t \mapsto a_{i-1}(t)f_{i-1}(t)^*(\lambda_i\alpha_t)\big) \end{aligned}$$

where, $f_i^*\Omega = \frac{1}{a_i}\Omega$, for we have $\operatorname{supp}\big(t \mapsto a_{i-1}(t)f_{i-1}(t)^*(\lambda_i\alpha_t)\big) \subseteq \bar V_i \subseteq U_i$. □

3.10.2. Corollary. *Let (M,Ω,ω) be a locally conformally symplectic manifold and let $\mathcal{U}$ be an open covering of M. Then every $g \in \ker\Psi$ has a decomposition $g = g_1\cdots g_n$, where every g_i is supported in some $U_i \in \mathcal{U}$ and $g_i \in \ker\Psi_{U_i}$.*

Proof. This is an immediate consequence of lemma 3.10.1 and the fact that $\ker\Psi$ is connected by smooth arcs, see lemma 3.7.14. □

3.10.3. Lemma. *Let (M,Ω,ω) be a locally conformally symplectic manifold and let $\mathcal{U}$ be an open covering of M. Then every $g \in C^\infty\big((I,0),(\ker R,\mathrm{id})\big)$ has a decomposition $g = g_1\cdots g_n$, where every g_i is supported in some $U_i \in \mathcal{U}$ and $g_i \in C^\infty\big((I,0),(\ker R_{U_i},\mathrm{id})\big)$.*

Proof. Fix a compact set $K \subseteq M$ and define

$$H_K : C^\infty\big(I,\Omega_K^{2n-1}(M)\big) \to C^\infty\big((I,0),(\ker R,\mathrm{id})\big) \qquad \alpha \mapsto \mathrm{Evol}\,\big((\sharp\circ d^\omega)_* u\big)$$

where $u \in C^\infty\big(I,\Omega^0_K(M)\big)$ is the unique function satisfying $d^{(n+1)\omega}\alpha_t = u_t\Omega^n$. So the defining equation for $g = H_K(\alpha)$ is $\flat\dot g_t = d^\omega u_t$ with initial condition $g_0 = \mathrm{id}$, cf. lemma 3.7.18 and remark 3.4.5. We define the structure of a topological group on the left hand side space such that H_K becomes a continuous homomorphism. Namely we set

$$(\alpha\beta)(t) := \alpha_t + (H_K(\alpha)(t)^{-1})^*\big(\tfrac{1}{a_t^{n+1}}\beta_t\big) \tag{3.24}$$

where $H_K(\alpha)(t)^*\Omega = \frac{1}{a_t}\Omega$. Let $\alpha, \beta \in C^\infty\big(I, \Omega_K^{2n-1}(M)\big)$, $u, v \in C^\infty\big(I, \Omega_K^0(M)\big)$ such that $d^{(n+1)\omega}\alpha_t = u_t\Omega^n$, $d^{(n+1)\omega}\beta_t = v_t\Omega^n$ and $g := H_K(\alpha)$, $h := H_K(\beta)$. Then we have

$$\begin{aligned} d^{(n+1)\omega}\big((\alpha\beta)(t)\big) &= u_t\Omega^n + (g_t^{-1})^*\big(\tfrac{1}{a_t^{n+1}}d^{(n+1)\omega}\beta_t\big) \\ &= u_t\Omega^n + (g_t^{-1})^*\big(\tfrac{1}{a_t}v_t g_t^*\Omega^n\big) \\ &= \big(u_t + (g_t^{-1})^*(\tfrac{1}{a_t}v_t)\big)\Omega^n \end{aligned}$$

and since we have $\flat\big(\delta^r(gh)(\partial_t)(t)\big) = d^\omega\big(u_t + (g_t^{-1})^*(\frac{1}{a_t}v_t)\big)$ from the proof of lemma 3.10.1 we see that H_K is a homomorphism, provided (3.24) defines a group structure, but this follows as in the proof of lemma 3.10.1.

From lemma 3.7.18, remark 3.4.5 and corollary 3.1.12 we immediately get $\bigcup_K \operatorname{Im} H_K = C^\infty\big((I,0), (\ker R, \mathrm{id})\big)$ and so we only have to show that every $g \in \operatorname{Im}(H_K)$ has the desired decomposition. From now on the proof is similar to the proof of lemma 3.10.1. □

3.10.4. Corollary. *Let (M, Ω, ω) be a locally conformally symplectic manifold and let $\mathcal{U}$ be an open covering of M. Then every $g \in \ker R$ has a decomposition $g = g_1 \cdots g_n$, where every g_i is supported in some $U_i \in \mathcal{U}$ and $g_i \in \ker R_{U_i}$.*

Proof. This is an immediate consequence of lemma 3.10.3 and the fact that $\ker R$ is connected by smooth arcs, see lemma 3.7.19. □

3.10.5. Remark. There is no fragmentation lemma for $\ker \Phi$. Indeed let $g \in \ker \Phi \setminus \ker \Psi$. If there would be a fragmentation lemma we would find contractible U_i and $g_i \in \ker \Phi_{U_i}$ with $g = g_1 \cdots g_n$. Since U_i is contractible we have $H^1_{d_c^\omega}(U_i) = 0$ and thus $\Psi_{U_i}(g_i) = 0$. So $g_i \in \ker \Psi_{U_i} \subseteq \ker \Psi$ and thus $g \in \ker \Psi$, a contradiction. So $\Psi(g)$ is the obstruction to fragmentation in $\ker \Phi$. A similar argument shows that $g \in \mathrm{Diff}_c^\infty(M, \Omega, \omega)_\circ$ can be fragmented, with respect to arbitrary (contractible) coverings, iff $\Phi(g) = 0$ and $\Psi(g) = 0$.

3.10.6. Lemma. *Let (M, Ω, ω) be a locally conformally symplectic manifold and $U \subseteq V$ open subsets such that V is contractible. If $g \in C^\infty\big((I,0), (\ker \Phi, \mathrm{id})\big)$ with $\overline{\bigcup_{t\in I} g_t(U)} \subseteq V$ then there exists $h \in C^\infty\big((I,0), (\ker R_V, \mathrm{id})\big)$ satisfying $g_t|_U = h_t|_U$ for all $t \in I$.*

Proof. Since g is a curve in $\ker \Phi$ we get $d^\omega \flat \dot{g}_t = 0$. Since V is contractible we find $u_t \in C^\infty(V, \mathbb{R})$ with $\flat \dot{g}_t|_V = d^\omega u_t$. Now choose a bump function λ with $\operatorname{supp} \lambda \subseteq V$, $\lambda = 1$ on $\overline{\bigcup_{t\in I} g_t(U)}$ and define h such that $\dot{h}_t = d^\omega(\lambda u_t) \in \ker \psi_V$. Then $h \in C^\infty\big((I,0), (\ker \Psi_V, \mathrm{id})\big)$ and $g_t = h_t$ on U. To see that h can be chosen to have values in $\ker R_V$ one simply multiplies h with a curve f supported in $V \setminus \overline{\bigcup_{t\in I} g_t(U)}$ which satisfies $R_V(h_t) = -R_V(f_t)$. □

3.11 The Symplectic Torus

Consider the torus T^{2n} with the symplectic structure:

$$\Omega = dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n}$$

We have $\partial_i := \frac{\partial}{\partial x_i} \in \mathfrak{X}(T^{2n}, \Omega)$ and $\psi(\partial_i) = [(-1)^{i-1} dx^{\sigma(i)}] \in H^1(T^n) \cong \mathbb{R}^{2n}$, where

$$\sigma : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\} \qquad \sigma(2i) = 2i - 1, \quad \sigma(2i-1) = 2i.$$

Since $Fl_1^{\partial_i} = \mathrm{id}$ we have $\mathrm{Fl}_t^{\partial_i} \in \boldsymbol{\pi}_1\big(\operatorname{Diff}^\infty(T^{2n},\Omega)_\circ\big)$ and so

$$[(-1)^{i-1}dx^{\sigma(i)}] = \psi(\partial_i) = \tilde\Psi(\mathrm{Fl}_t^{\partial_i}) \in \Gamma \subseteq H^1(T^{2n}),$$

i.e. $\mathbb{Z}^{2n} \subseteq \Gamma \subseteq H^1(T^{2n}) \cong \mathbb{R}^{2n}$. On the other hand for $g \in \boldsymbol{\pi}_1\big(\operatorname{Diff}^\infty(T^{2n},\Omega)_\circ\big)$ lemma 3.7.11 yields $\langle c, \tilde\Psi(g)\rangle = \langle g * c, \Omega\rangle \in \operatorname{Per}(\Omega) = \mathbb{Z}$, for all $c \in H_1(T^{2n};\mathbb{Z})$, where $\operatorname{Per}(\Omega)$ denotes the periods of Ω. Consequently $\Gamma \subseteq \mathbb{Z}^{2n} \cong H^1(T^{2n};\mathbb{Z}) \subseteq H^1(T^{2n})$. So we have shown:

$$\mathbb{Z}^{2n} = \Gamma \subseteq H^1(T^{2n}) \cong \mathbb{R}^{2n}$$

If we consider T^{2n} as subgroup of $\operatorname{Diff}^\infty(T^{2n},\Omega)$, via $\alpha \mapsto R_\alpha$, where R_α denotes rotation by α, the preceding also shows that $\Psi|_{T^{2n}} : T^{2n} \to H^1(T^{2n})/\Gamma \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong T^{2n}$ is given by:

$$\Psi|_{T^{2n}}(x_1,\dots,x_n) = (x_2,-x_1,\dots,x_{2n},x_{2n-1})$$

Notice that $\Phi = 0$ since (T^{2n},Ω) is a symplectic manifold and $R = 0$ since (T^{2n},Ω) is compact, cf. remark 3.4.6.

3.11.1. Theorem. *Consider the symplectic torus (T^{2n},Ω). Then $\widetilde{\ker}\,\Psi = \widetilde{\ker}\,R$ is perfect.*

Proof. We have to show $\widetilde{\ker}\,\Psi \subseteq [\widetilde{\ker}\,\Psi, \widetilde{\ker}\,\Psi]$. Choose $\gamma \in T^{2n}$ satisfying a diophantic equation. If $g \in C^\infty\big((I,0),(\widetilde{\ker}\,\Psi,\mathrm{id})\big)$ is sufficiently close to id theorem 1.5.3 yields $\lambda \in C^\infty\big((I,0),(T^{2n},0)\big)$ and $f \in C^\infty\big((I,0),(\operatorname{Diff}^\infty(T^{2n})_\circ,\mathrm{id})\big)$ with

$$R_\gamma g = R_\lambda f^{-1} R_\gamma f \quad \text{i.e.} \quad g = R_\lambda[R_\gamma^{-1}, f^{-1}]$$

From $g_t \in \ker\Psi$ we obtain

$$\Omega = g_t^*\Omega = f_t^* R_\gamma^* (f_t^{-1})^* (R_\gamma^{-1})^* (R_{\lambda_t})^*\Omega = f_t^* R_\gamma^* (f_t^{-1})^*\Omega$$

and so $(f_t^{-1})^*\Omega$ is R_γ-invariant. If $(f_t^{-1})^*\Omega = \sum_{i<j} a_{ij} dx^i \wedge dx^j$ we thus obtain $a_{ij} \circ R_\gamma = a_{ij}$ and since R_γ generates a dense subgroup of T^{2n} the a_{ij} are constant. Moreover since f_t^{-1} is homotopic to id we must have $[(f_t^{-1})^*\Omega] = [\Omega] \in H^2(T^{2n})$ and so $(f_t^{-1})^*\Omega = \Omega$. Hence $f \in C^\infty\big((I,0),(\operatorname{Diff}^\infty(T^{2n},\Omega)_\circ,\mathrm{id})\big)$ and we get:

$$0 = \Psi(g_t) = \Psi(R_{\lambda_t}) - \Psi(R_\gamma) - \Psi(f_t) + \Psi(R_\gamma) + \Psi(f_t) = \Psi(R_{\lambda_t})$$

Since $\Psi|_{T^{2n}} : T^{2n} \to H^1(T^{2n})/\Gamma$ is one-to-one this yields $\lambda_t = 0 \in T^{2n}$ and we have $g = [R_\gamma^{-1}, f^{-1}]$. Now choose a path α in T^{2n} from 0 to γ. Then $(s,t) \mapsto [R^{-1}_{\alpha(s+(1-s)t)}, f_t^{-1}]$ is a homotopy relative endpoints in $\ker\Psi$ from $t \mapsto [R_{\alpha_t}^{-1}, f_t^{-1}]$ to g and so:

$$g = [R_\gamma^{-1}, f^{-1}] = [R_\alpha^{-1}, f^{-1}] \in \widetilde{\ker}\,\Psi$$

Up to now we have shown: $\widetilde{\ker}\,\Psi \subseteq [\widetilde{\operatorname{Diff}}^\infty(T^{2n},\Omega)_\circ, \widetilde{\operatorname{Diff}}^\infty(T^{2n},\Omega)_\circ]$. Now choose a path $\beta \in C^\infty\big((I,0),(T^{2n},0)\big)$ with $\Psi(R_{\beta_t}) = \Psi(f_t)$. Then $h := f^{-1}R_\beta \in C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ and

$$g = [R_\alpha^{-1}, f^{-1}] = [R_\alpha^{-1}, f^{-1}R_\beta] = [R_\alpha^{-1}, h] \in \widetilde{\ker}\,\Psi$$

since rotations commute. Next choose open balls $U_i, V_i \subseteq T^{2n}$ such that $\overline{\bigcup_{t\in I} R_{\alpha_t}^{-1}(U_i)} \subseteq V_i$ and such that U_i cover T^{2n}. This is possible since α can be chosen close to the constant path 0 if γ was close to $0 \in T^{2n}$. From the cut-off lemma 3.10.6 we obtain

$k_i \in C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ with $k_i(t)|_{U_i} = R_{\alpha t}^{-1}|_{U_i}$ for all $t \in I$. Moreover the fragmentation lemma 3.10.1 yields $h_j \in C^\infty\big((I,0),(\ker\Psi,\mathrm{id})\big)$ with $\operatorname{supp} h_j \subseteq U_{i(j)}$ and $h = h_1 \cdots h_l$. From $k_i(t)|_{U_i} = R_{\alpha t}^{-1}|_{U_i}$ we obtain $k_i^{-1}(t) = R_{\alpha_t}$ on $R_{\alpha_t}^{-1}(U_i)$ and therefore $R_{\alpha_t}^{-1} h_j(t) R_{\alpha_t} = k_{i(j)}(t) h_j(t) k_{i(j)}^{-1}(t)$ on T^{2n}. So

$$\begin{aligned} g = [R_\alpha^{-1}, h] &= (R_\alpha^{-1} h_1 R_\alpha) \cdots (R_\alpha^{-1} h_l R_\alpha) h_l^{-1} \cdots h_1^{-1} \\ &= (k_{i(1)} h_1 k_{i(1)}^{-1}) \cdots (k_{i(l)} h_l k_{i(l)}^{-1}) h_l^{-1} \cdots h_1^{-1} \\ &= 0 \in \widetilde{\ker}\,\Psi / [\widetilde{\ker}\,\Psi, \widetilde{\ker}\,\Psi] \end{aligned}$$

since every factor is in $\widetilde{\ker}\,\Psi$ and up to the ordering the product is the identity. So we have shown $g \in [\widetilde{\ker}\,\Psi, \widetilde{\ker}\,\Psi]$. □

3.11.2. Corollary. *For the symplectic torus (T^{2n}, Ω) we have*

$$H_1\big(B\overline{\ker\Psi}; \mathbb{Z}\big) = H_1\big(B\overline{\ker R}; \mathbb{Z}\big) = 0$$

and $\ker\Psi = \ker R$ *is perfect too.*

Proof. This is an immediate consequence of theorem 3.11.1 and proposition 1.4.5. □

3.12 Derived Series of $\mathrm{Diff}^\infty_c(M,\Omega,\omega)_\circ$

A well known theorem of W. P. Thurston states that $\mathrm{Diff}^\infty_c(M)_\circ$ is a simple group, cf. [Thu74]. His proof used a theorem of Epstein (see theorem 1.3.2 or [Eps70]) and a theorem due to Herman (see corollary 1.5.5 or [Her73]). Mather proved that $\mathrm{Diff}^r_c(M)_\circ$ is simple for $\infty > r \neq \dim(M)+1$, see [Mat74] and [Mat75]. As far as I know it is still unsolved if this holds for $r = \dim(M)+1$ too. Mather's proof is 'elementary' but very tricky. Epstein managed to generalize Mather's construction and reproved the simplicity of $\mathrm{Diff}^\infty_c(M)_\circ$, see [Eps84]. The group of volume preserving diffeomorphisms is not simple in general, but there exists a homomorphism and its kernel is simple. This was shown by Thurston, see [Ban97] for a proof. Banyaga showed an analogous statement in the symplectic case.

The difficult part of such theorems is the perfectness, simplicity then follows either from Epstein's theorem or proposition 1.3.1, roughly speaking. There doesn't seem to exist a way to obtain perfectness of the group from perfectness of the corresponding Lie algebra, which is much more easier to show.

In the sequel we will show a simplicity theorem for locally conformally symplectic manifolds, see theorem 3.12.3, and compute the derived series of $\mathrm{Diff}^\infty_c(M,\Omega,\omega)_\circ$, see corollary 3.12.4.

3.12.1. Lemma. *Let (M,Ω,ω) be a connected locally conformally symplectic manifold. Then* $\ker R$ *acts k-transitive for all $k \in \mathbb{N}$.*

Proof. From lemma 3.6.2 we obtain immediately that $\ker\rho$ acts infinitesimal k-transitive for all $k \in \mathbb{N}$, cf. definition 1.2.5. Since we have lemma 3.7.18 the statement follows from proposition 1.2.6. □

Using a Weinstein chart one can identify simplices of $S_p(B\overline{\ker}R)$ with 1-forms on $\Delta^p \times M$. To these one can apply the fragmentation mapping from section 2.2, and so the next proposition follows from corollary 3.11.2, see [Ban97].

3.12.2. Proposition. *Consider the symplectic manifold $(U, \Omega, 0)$, where $U \subseteq \mathbb{R}^{2n}$ is an open ball equipped with the standard symplectic form $\Omega = dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n}$. Then $\ker R$ is perfect, i.e. $\ker R = [\ker R, \ker R]$.*

The following theorem is due to A. Banyaga in the symplectic case, see [Ban78] or [Ban97].

3.12.3. Theorem. *Let (M, Ω, ω) be a connected locally conformally symplectic manifold. Then $\ker R$ is simple, i.e. $\ker R$ has no non-trivial normal subgroups. Especially there do not exist non-trivial homomorphisms defined on $\ker R$.*

Proof. We want to apply proposition 1.3.1 for $G = \ker R$. Let $\mathcal{U}$ be the set of all symplectic balls in M, i.e. open sets U in M such that there exists a diffeomorphism onto an open ball in $\mathbb{R}^{2n}$ mapping the locally conformally symplectic structure of M to the standard symplectic structure on $\mathbb{R}^{2n}$, up to conformal change. A locally conformally symplectic structure is locally conformally equivalent to a symplectic structure. So $\mathcal{U}$ is a basis of the topology, for we have Darboux's theorem for symplectic manifolds. For $U \in \mathcal{U}$ we let $G_U := \ker R_U \subseteq \ker R$. Then G_U is perfect by proposition 3.12.2. Remains to check the three assumptions in proposition 1.3.1. The first is a special case of lemma 3.12.1. The second is precisely corollary 3.10.4. The third assumption is obvious, but see the discussion at the beginning of section 3.10 and recall that $\ker R$ remains the same if one changes the locally conformally symplectic structure conformally. □

The derived series D^iG of a group G is defined inductively, $D^0G := G$, $D^1G = [G, G]$, $D^iG := [D^{i-1}G, D^{i-1}G]$, where $[G, G]$ denotes the subgroup generated by all commutators of G.

3.12.4. Corollary. *Let (M, Ω, ω) be a connected locally conformally symplectic manifold and let $G := \mathrm{Diff}_c^\infty(M, \Omega, \omega)_\circ$ for the moment. Then we have:*

	$D^0G = G$	D^1G	D^2G	D^3G
M compact, $[\Omega] = 0 \in H^2_{d^\omega}(M)$	G	$\ker \Phi$	$\ker \Psi = \ker R$	D^2G
M compact, $[\Omega] \neq 0 \in H^2_{d^\omega}(M)$	$G = \ker \Phi$	$\ker \Psi = \ker R$	D^1G	D^1G
M not compact, $[\omega] \neq 0 \in H^1(M)$	$G = \ker \Phi$	$\ker \Psi = \ker R$	D^1G	D^1G
M not compact, $[\omega] = 0, \{\cdot,\cdot\} = 0$	$G = \ker \Phi$	$\ker R$	D^1G	D^1G
M not compact, $[\omega] = 0, \{\cdot,\cdot\} \neq 0$	$G = \ker \Phi$	$\ker \Psi$	$\ker R$	D^2G

Proof. Since $\ker R$ is simple (theorem 3.12.3) it is perfect too and we get:

$$[\ker \Psi, \ker \Psi] \subseteq \ker R = [\ker R, \ker R] \subseteq [\ker \Psi, \ker \Psi]$$

So we always have:

$$[\ker \Psi, \ker \Psi] = \ker R \tag{3.25}$$

Moreover, if $\rho = 0$ then $R = 0$, hence $\ker \Psi = \ker R$ is perfect and we get

$$[\ker \Phi, \ker \Phi] \subseteq \ker \Psi = [\ker \Psi, \ker \Psi] \subseteq [\ker \Phi, \ker \Phi]$$

and hence

$$[\ker \Phi, \ker \Phi] = \ker \Psi = \ker R \qquad \text{if } \rho = 0. \tag{3.26}$$

In the first case we have $\varphi \neq 0$ and $\rho = 0$ by remark 3.4.6. So in this case it remains to show $[G, G] \supseteq \ker \Phi$. To see this choose $g \in G$, such that $\Phi(g) = \ln 2$. From (3.20) on page 80 we obtain $\Psi(ghg^{-1}) = 2\Psi(h)$ for all $h \in \ker \Phi$, and hence $\Psi([g, h]h^{-1}) = \Psi(ghg^{-1}) - 2\Psi(h) = 0$. Using (3.26) this gives

$$[g, h]h^{-1} \in \ker \Psi = [\ker \Phi, \ker \Phi] \subseteq [G, G] \qquad \forall h \in \ker \Phi$$

and thus $h \in [G, G]$. In the second and the third case we have $\rho = 0$, $\varphi = 0$, thus $R = 0$, $\Phi = 0$ and everything follows from (3.26). In the forth and fifth case we also have $\Phi = 0$ and

$$\ker R = [\ker \Psi, \ker \Psi] \subseteq [\ker \Phi, \ker \Phi] \subseteq \ker \Psi. \tag{3.27}$$

The fourth case now follows immediately from proposition 3.7.20. In the fifth case it remains to check $[\ker \Phi, \ker \Phi] \supseteq \ker \Psi$. So see this let $g \in \ker \Psi$. Since the symplectic pairing is non-zero it is surjective and so there exist $h, k \in \ker \Phi$ with $R([h, k]) = R(g)$, by proposition 3.7.20 and the fact that Ψ is onto. Using (3.27) we obtain $[h, k]g^{-1} \in \ker R \subseteq [\ker \Phi, \ker \Phi]$ and thus $g \in [\ker \Phi, \ker \Phi]$. □

3.12.5. Remark. Notice that corollary 3.12.4 is precisely the integral counterpart of corollary 3.5.4

3.12.6. Remark. In the fourth case of corollary 3.12.4 we also have $[\ker \Psi, \ker \Psi] = \ker R$, but $\ker \Psi \neq \ker R$ since $R \neq 0$ and in general $\ker \Psi \neq \ker \Phi$. So this is the only case where not all kernels of the various invariants do appear in the derived series.

3.13 Filipkiewicz type Theorem

Filipkiewicz showed that a smooth manifold is uniquely determined by its group of diffeomorphisms. That is, if two manifold have isomorphic diffeomorphism groups then the underlying manifolds are diffeomorphic, see [Fil82]. He used techniques developed in [Whi63] and [Tak79] who proved an analogous statement in the topological setting. There were many generalizations to other geometric structures, see [Ban86], [Ban88], [BM95] and [Ryb95b] for some non-transitive geometric structures. In the sequel we will show the analogous statement for locally conformally symplectic manifolds, see theorem 3.13.1 and corollary 3.13.3.

3.13.1. Theorem. *Let $(M_i, \Omega_i, \omega_i)$, $i = 1, 2$ be two locally conformally symplectic manifolds and suppose $\kappa : \ker R_1 \to \ker R_2$ is an isomorphism of groups. Then there exists a unique homeomorphism $f : M_1 \to M_2$ such that $\kappa(g) = f \circ g \circ f^{-1}$ for all $g \in \ker R_1$. Moreover f is a diffeomorphism and $(M_1, \Omega_1, \omega_1) \sim (M_1, f^*\Omega_2, f^*\omega_2)$.*

Proof. This is an application of a theorem due to T. Rybicki, see [Ryb95b]. We have to verify that $\ker R$ satisfies the four axioms of [Ryb95b]. The first is a fragmentation property, see corollary 3.10.4. The second axiom states that for every sufficiently small open ball U in M and $x \in U$ there exists $g \in \ker R$ such that $\mathrm{Fix}(g) = (M \setminus U) \cup \{x\}$. Such a g is easily constructed using a Darboux chart. The third axiom states that $\ker R$ acts 3-transitive on M, we have shown this in lemma 3.12.1. The fourth axiom requires the existence of a Pursell-Shanks-Omori like theorem, see theorem 3.6.6. □

3.13.2. Lemma. *Let G be a group such that $\mathrm{conj} : G \to \mathrm{Aut}([G,G])$ is injective and let $\lambda : G \to G$ be a homomorphism of groups, such that $\lambda|_{[G,G]} = \mathrm{id}$. Then $\lambda = \mathrm{id}$. (cf. lemma 3.6.7).*

Proof. For $g \in G$ we have

$$[g,h] = \lambda([g,h]) = [\lambda(g),\lambda(h)] = [\lambda(g),h] \qquad \forall h \in [G,G]$$

hence $\mathrm{conj}_{g^{-1}\lambda(g)} = \mathrm{id} \in \mathrm{Aut}([G,G])$, and by injectivity $\lambda(g) = g$. □

3.13.3. Corollary. *Consider two locally conformally symplectic manifolds $(M_i, \Omega_i, \omega_i)$. Let G_1 be one of the groups $\mathrm{Diff}_c^\infty(M_1,\Omega_1,\omega_1)_\circ$, $\ker \Phi_1$, $\ker \Psi_1$, $\ker R_1$ and G_2 be one of the groups $\mathrm{Diff}_c^\infty(M_2,\Omega_2,\omega_2)_\circ$, $\ker \Phi_2$, $\ker \Psi_2$, $\ker R_2$, and assume that $\kappa : G_1 \to G_2$ is an isomorphism of groups. Then there exists a unique homeomorphism $f : M_1 \to M_2$ such that $\kappa(g) = f \circ g \circ f^{-1}$ for all $g \in G_1$. Moreover f is a diffeomorphism and $(M_1,\Omega_1,\omega_1) \sim (M_1, f^*\Omega_2, f^*\omega_2)$.*

Proof. The restriction of κ is an isomorphism $\kappa|_{D^2G_1} : D^2G_1 \to D^2G_2$. In any case $D^2G_i = \ker R_i$ for $i = 1,2$, by corollary 3.12.4. So we may apply theorem 3.13.1 and obtain a unique homeomorphism $f : M_1 \to M_2$ such that $\kappa(g) = fgf^{-1}$ for all $g \in \ker R_1 = D^2G_1$. Moreover f is a diffeomorphism and $(M_1,\Omega_1,\omega_2) \sim (M_1, f^*\Omega_2, f^*\omega_2)$. So it remains to show that $\kappa(g) = \mathrm{conj}_f(g) := fgf^{-1}$ for all $g \in G_1$. From $(M_1,\Omega_1,\omega_2) \sim (M_1, f^*\Omega_2, f^*\omega_2)$ we see that $\mathrm{conj}_{f^{-1}}(G_2) \subseteq G_1$ or $\mathrm{conj}_{f^{-1}}(G_2) \supseteq G_1$. Assume we are in the first case (for the second consider f^{-1}). Then $\lambda := \mathrm{conj}_{f^{-1}} \circ \kappa : G_1 \to G_1$ is a homomorphism and $\lambda|_{D^2G_1} = \mathrm{id}$. Moreover, for $g \in \mathrm{Diff}^\infty(M_1)$ we have:

$$[g,h] = \mathrm{id} \quad \forall h \in \ker R_1 \quad \Rightarrow \quad g = \mathrm{id}$$

since $\ker R_1$ acts 2-transitive on M_1. Using $\ker R_1 \subseteq D^{i+1}G_1$ we obtain $\mathrm{conj} : D^iG_1 \to \mathrm{Aut}([D^iG_1, D^iG_1]) = \mathrm{Aut}(D^{i+1}G_1)$ is injective for all i. So we may apply lemma 3.13.2 inductively and obtain successively $\lambda|_{D^2G_1} = \mathrm{id}$, $\lambda|_{D^1G_1} = \mathrm{id}$ and finally $\lambda = \lambda_{D^0G_1} = \mathrm{id}$, i.e. $\kappa = (\mathrm{conj}_f)|_G 1$. □

3.13.4. Remark. Since $\ker \Psi$ also satisfies all four axioms in [Ryb95b], we could derive corollary 3.13.3 for $\ker \Psi$ from Rybickis theorem, too. But $\ker \Phi$ and $\mathrm{Diff}_c^\infty(M,\Omega,\omega)_\circ$ do not have the fragmentation property (see remark 3.10.5) and are therefore not covered by [Ryb95b].

4. Extension and Transgression of the Flux

4.1 Cohomology of Groups

Let G be a group and let M be a G-module. We recall briefly the definition of cohomology groups $H^*(G;M)$. Choose a projective resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z}$$

of the trivial G-module $\mathbb{Z}$, consider the induced cochain complex

$$0 \to \operatorname{Hom}_G(F_0, M) \to \operatorname{Hom}_G(F_1, M) \to \operatorname{Hom}_G(F_2, M) \to \cdots$$

and define the cohomology groups $H^*(M;G)$ to be the cohomology groups of the complex above. It is well known that this does not depend on the choice of the resolution. Since the Hom_G functor is left-exact we immediately obtain $H^0(G;M) = \operatorname{Hom}_G(\mathbb{Z}, M) = M^G := \{m \in M : gm = m \quad \forall g \in G\}$.

Let $C_p(G)$ denote the free G-module with generators $[g_1|\cdots|g_p]$ and define a G-module homomorphism $\partial : C_p(G) \to C_{p-1}(G)$ by:

$$\partial([g_1|\cdots|g_p]) := g_1[g_2|\cdots|g_p] - \textstyle\sum_{i=1}^{p-1}[g_1|\cdots|g_ig_{i+1}|\cdots|g_p] + (-1)^p[g_1|\cdots|g_{p-1}]$$

Moreover let $\varepsilon : C_0(G) \to \mathbb{Z}$ be the G-module homomorphism, defined by $\varepsilon([]) := 1$, the usual augmentation mapping. For this notice that $C_0(G)$ is generated by $[]$. It is well known that

$$\cdots \xrightarrow{\partial} C_2(G) \xrightarrow{\partial} C_1(G) \xrightarrow{\partial} C_0(G) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a free (hence projective) resolution of $\mathbb{Z}$. It is called the bar resolution. Consequently, if we set $C^p(G;M) := \operatorname{Map}(G^p, M) \cong \operatorname{Hom}_G(C_p(G), M)$, then the complex

$$0 \to C^0(G;M) \xrightarrow{\delta} C^1(G;M) \xrightarrow{\delta} C^2(G;M) \xrightarrow{\delta} \cdots$$

computes the cohomology groups $H^*(G;M)$. Here $\delta : C^{p-1}(G;M) \to C^p(G;M)$ is induced from ∂, i.e.

$$\delta(c)(g_1,\dots,g_p) = g_1c(g_2,\dots,g_p) + \textstyle\sum_{i=1}^{p-1} c(g_1,\dots,g_ig_{i+1},\dots,g_p) + (-1)^p c(g_1,\dots,g_{p-1})$$

for $c \in C^{p-1}(G;M)$.

4.1.1. Example. Let $c \in C^1(G;M)$. Then we have $(\delta c)(g_1,g_2) = g_1c(g_2) - c(g_1g_2) + c(g_1)$ and so $\delta c = 0$ iff $c \in \operatorname{Der}(G,M) := \{d \in \operatorname{Map}(G,M) : d(gh) = gd(h) + d(g) \quad \forall g,h \in G\}$. Moreover $c = \delta u$ if and only if $c(g) = gu - u$ for some $u \in M$. These c are called inner derivations ($\operatorname{Inn}(G,M)$). Summing up we have seen $H^1(G;M) = \operatorname{Der}(G,M)/\operatorname{Inn}(G,M)$.

4.1.2. Example. Let $0 \to A \xrightarrow{i} G \xrightarrow{p} H \to 0$ be an extension of H by A and assume that A is abelian. Since $i(A) = \ker p$ is a normal subgroup, G acts by conjugation on A. Moreover since A is abelian this action descends to an H-action on A. We call two such extensions $0 \to A \to G_1 \to H \to 0$ and $0 \to A \to G_2 \to H \to 0$ equivalent if there exists a homomorphism $\varphi : G_1 \to G_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & G_2 & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

commutes. Then φ turns out to be an isomorphism by the five-lemma. We want to describe the set of equivalence classes of extensions of H by A which give rise to this H-action on A. Choose a set theoretic section of p, such that $s(e) = e$ and define $c \in C^2(H; A)$ by $s(h)s(k) = i(c(h,k))s(hk)$. Moreover define $\varphi : A \times H \to G$, $\varphi(a,h) := i(a)s(h)$. Then φ is bijective and the group multiplication of G on $A \times H$ is the following

$$(a,h)(b,k) = (a + hb + c(h,k), hk) \tag{4.1}$$

for we have $i(a)s(h)i(b)s(k) = i(a)i(hb)s(h)s(k) = i(a + hb + c(h,k))s(hk)$. An easy calculation shows that (4.1) defines a group multiplication on $A \times H$ iff $\delta c = 0$ and $c(e,e) = 0$. So, for every such c we have an extension we denote by $A \times_c H$. Moreover the cohomology class $[c] \in H^2(H; M)$ does not depend on the choice of s, for if s_1, s_2 are two sections there exists $u \in \mathrm{Map}(H, A)$ with $s_2(h) = i(u(h))s_1(h)$, therefore

$$\begin{aligned} i\big(c_2(h,k) - c_1(h,k)\big) &= s_2(h)s_2(k)s_2(hk)^{-1}s_1(hk)s_1(k)^{-1}s_1(h)^{-1} \\ &= i(u(h))s_1(h)i(u(k))s_1(k)\big(i(u(hk))s_1(hk)\big)^{-1}s_1(hk)s_1(k)^{-1}s_1(h)^{-1} \\ &= i\big(u(h) + h\big(u(k) - k(hk)^{-1}u(hk)\big)\big) \\ &= i\big(u(h) + hu(k) - u(hk)\big) = i\big((\delta u)(h,k)\big) \end{aligned}$$

and so $c_2 = c_1 + \delta u$. Next one shows that equivalent extensions give rise to the same cohomology class. Indeed let $\varphi : G_1 \to G_2$ be an isomorphism of extensions. If s_1 is a section of p_1 then $s_2 := \varphi \circ s_1$ is a section of p_2. Since φ is the identity on A we obtain

$$\begin{aligned} i(c_2(h,k)) &= s_2(h)s_2(k)s_2(hk)^{-1} \\ &= \varphi(s_1(h)s_1(k)s_1(hk)^{-1}) = \varphi\big(i(c_1(h,k))\big) = i(c_1(h,k)) \end{aligned}$$

and thus $[c_1] = [c_2] \in H^2(H; A)$. So we can associate a cohomology class $[c] \in H^2(M; A)$ to every equivalence class of extensions, and this mapping is onto since every class in $H^2(M; A)$ has a representative satisfying $c(e,e) = 0$. Finally we show that this mapping is one-to-one. So suppose G_1 and G_2 are two extensions which give rise to the same cohomology class, i.e. there exists $u \in C^1(H; A)$ such that $c_1 - c_2 = \delta u$. It suffices to show that the extensions $A \times_{c_1} H$ and $A \times_{c_2} H$ are equivalent. An equivalence is given by:

$$\varphi : A \times_{c_1} H \to A \times_{c_2} H \qquad \varphi((a,h)) = (a + u(h), h)$$

Notice that $u(e) = 0$ since $c_i(e,e) = 0$ and we have $c_1 - c_2 = \delta u$.

Summing up we have a natural one-to-one correspondence of $H^2(H; A)$ and the set of equivalence classes of extensions of H by A which give rise to the fixed H-action on A. Obviously the semi direct product of H and A defined by this action of H on A corresponds to $0 \in H^2(G; A)$, since it possesses a section, which is a homomorphism.

We will make use of the following so called five-term exact sequence, due to Hochschild and Serre.

4.1.3. Theorem. *Let $0 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 0$ be an arbitrary extension and let M be a Q-module. Then the following sequence is exact and natural:*

$$0 \to H^1(Q;M) \xrightarrow{p^*} H^1(G;M) \xrightarrow{i^*} \mathrm{Hom}_Q(H_{ab},M) \xrightarrow{t} H^2(Q;M) \xrightarrow{p^*} H^2(G;M)$$

Here M is considered as G-module via p and as trivial H-module. Moreover $H_{ab} := H/[H,H]$ denotes the abelianization of H, considered as Q-module via the conjugate action. Finally the mapping t is given by $t(\varphi) = \varphi_*([c])$, where $[c] \in H^2(Q;H_{ab})$ is the cohomology class corresponding to the extension $0 \to H_{ab} \to G/i([H,H]) \to Q \to 0$, cf. example 4.1.2.

A proof can be found for example in [Bro82]. In fact one shows that there exists a spectral sequence converging to $H^*(G;M)$ with E^2-term $E_2^{pq} = H^p(Q;H^q(H;M))$. The five-term exact sequence is an immediate consequence of this and the fact that $H^1(H;M) \cong \mathrm{Hom}(H_{ab},M)$.

4.2 The Flux on Loops

The following is well known and can be found in [Ban97] for example.

4.2.1. Proposition. *Let $\theta \in \Omega^p(M)$ be closed. Then $S_\theta(g) := \int_0^1 g_t^* i_{\dot g_t}\theta dt$ defines a homomorphism*

$$S_\theta : \boldsymbol{\pi}_1\big(\mathrm{Diff}_c^\infty(M)_\circ\big) \to H_c^{p-1}(M).$$

Moreover S_θ only depends on the cohomology class $[\theta] \in H^p(M)$, in particular $S_\theta = 0$ if θ is exact.

Proof. First of all the formula defines a cohomology class since we have:

$$d\big(\int_0^1 g_t^* i_{\dot g_t}\theta dt\big) = \int_0^1 g_t^* d i_{\dot g_t}\theta dt = \int_0^1 g_t^* L_{\dot g_t}\theta dt = \int_0^1 \tfrac{\partial}{\partial t} g_t^*\theta dt = g_1^*\theta - g_0^*\theta = 0$$

Next we have to show that $S_\theta(g)$ does only depend on the homotopy type relative endpoints of g. So let $G : I \times I \to \mathrm{Diff}_c^\infty(M)_\circ$ be such a homotopy, i.e. $G(s,0) = G(s,1) = \mathrm{id}$. We have to show:

$$\int_0^1 G_{1,t}^* i_{\delta^r G(\partial_t)}\theta dt = \int_0^1 G_{0,t}^* i_{\delta^r G(\partial_t)}\theta dt$$

Using equation 1.4 and lemma 1.2.3 we get

$$\begin{aligned}
\tfrac{\partial}{\partial s} G_{s,t}^* i_{\delta^r G(\partial_t)}\theta &= G_{s,t}^* L_{\delta^r G(\partial_s)} i_{\delta^r G(\partial t)}\theta + G_{s,t}^* \tfrac{\partial}{\partial s}\big(i_{\delta^r G(\partial_t)}\theta\big)\\
&= G_{s,t}^* i_{[\delta^r G(\partial_s),\delta^r G(\partial_t)]}\theta + G_{s,t}^* i_{\delta^r G(\partial_t)} L_{\delta^r G(\partial_s)}\theta + G_{s,t}^* i_{\frac{\partial}{\partial_s}\delta^r G(\partial_t)}\theta\\
&= G_{s,t}^* i_{\frac{\partial}{\partial t}\delta^r G(\partial_s)}\theta + G_{s,t}^* i_{\delta^r G(\partial_t)} d i_{\delta^r G(\partial_s)}\theta\\
&= G_{s,t}^* \tfrac{\partial}{\partial t}\big(i_{\delta^r G(\partial_s)}\theta\big) + G_{s,t}^* L_{\delta^r G(\partial_t)} i_{\delta^r G(\partial_s)}\theta - G_{s,t}^* d i_{\delta^r G(\partial_t)} i_{\delta^r G(\partial_s)}\theta\\
&= \tfrac{\partial}{\partial t}\big(G_{s,t}^* i_{\delta^r G(\partial_s)}\theta\big) - d\big(G_{s,t}^* i_{\delta^r G(\partial_t)} i_{\delta^r G(\partial_s)}\theta\big)
\end{aligned}$$

and so

$$\begin{aligned}
\int_0^1 G_{1,t}^* i_{\delta^r G(\partial_t)}\theta dt - \int_0^1 G_{0,t}^* i_{\delta^r G(\partial_t)}\theta dt &= \int_0^1\int_0^1 \tfrac{\partial}{\partial s} G_{s,t}^* i_{\delta^r G(\partial_t)}\theta dt ds\\
&= \int_0^1\int_0^1 \tfrac{\partial}{\partial t} G_{s,t}^* i_{\delta^r G(\partial_s)}\theta dt ds - d\big(\int_0^1\int_0^1 G_{s,t}^* i_{\delta^r G(\partial_t)} i_{\delta^r G(\partial_s)}\theta ds dt\big)\\
&= \int_0^1 G_{s,1}^* i_{\delta^r G(\partial_s)}\theta ds - \int_0^1 G_{s,0}^* i_{\delta^r G(\partial_s)}\theta ds - d(\cdots) = -d(\cdots)
\end{aligned}$$

since the endpoints are fixed and thus $\delta^r G(\partial_s)(s,t) = 0$ for $t = 0$ and $t = 1$. Recall that the product of $g, h \in \boldsymbol{\pi}_1\big(\operatorname{Diff}^\infty_c(M)_\circ\big)$ is represented by the loop

$$k(t) = \begin{cases} g(2t) & 0 \le t \le \frac{1}{2} \\ h(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

since this is homotopic relative endpoints to $t \mapsto g_t h_t$. To avoid smoothness difficulties at $t = \frac{1}{2}$ one can reparametrize g, h such that $g_t = h_t = \mathrm{id}$ for $t \le \frac{1}{3}$ and $t \ge \frac{2}{3}$. Then we have

$$\dot{k}_t = \begin{cases} 2\dot{g}_{2t} & 0 \le t \le \frac{1}{2} \\ 2\dot{h}_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$$

and thus

$$\begin{aligned} S_\theta(gh) = S_\theta(k) &= \textstyle\int_0^{\frac{1}{2}} g_{2t}^* i_{2\dot{g}_{2t}} \theta dt + \int_{\frac{1}{2}}^1 h_{2t-1}^* i_{2\dot{h}_{2t-1}} \theta dt \\ &= \textstyle\int_0^1 g_t^* i_{\dot{g}_t} \theta dt + \int_0^1 h_t^* \dot{h}_t \theta dt = S_\theta(g) + S_\theta(h) \end{aligned}$$

So S_θ is a homomorphism. Finally notice that S_θ depends linearly on θ, and for $\theta = d\alpha$ we have

$$\begin{aligned} S_\theta(g) &= \textstyle\int_0^1 g_t^* i_{\dot{g}_t} d\alpha dt = \int_0^1 g_t^* L_{\dot{g}_t} \alpha dt - \int_0^1 g_t^* d i_{\dot{g}_t} \alpha dt \\ &= \textstyle\int_0^1 \frac{\partial}{\partial t} g_t^* \alpha dt - d\big(\int_0^1 g_t^* i_{\dot{g}_t} \alpha dt\big) = g_1^* \alpha - g_0^* \alpha = 0 \in H_c^{p-1}(M) \end{aligned}$$

and so S_θ only depends on the cohomology class of θ. □

4.2.2. Proposition. *Let (M, Ω, ω) be a locally conformally symplectic manifold and let $i : \operatorname{Diff}^\infty_c(M, \Omega, \omega)_\circ \to \operatorname{Diff}^\infty_c(M)_\circ$ denote the inclusion. Then the diagram*

$$\begin{array}{ccc} \boldsymbol{\pi}_1\big(\operatorname{Diff}^\infty_c(M,\Omega,\omega)_\circ\big) & \xrightarrow{\tilde{\Phi}} & H_c^0(M) \\ \downarrow{\scriptstyle i_\sharp} & \nearrow{\scriptstyle S_\omega} & \\ \boldsymbol{\pi}_1\big(\operatorname{Diff}^\infty_c(M)_\circ\big) & & \end{array}$$

commutes.

Proof. Let g be a closed loop in $\operatorname{Diff}^\infty_c(M, \Omega, \omega)_\circ$ starting at id. Let a_t be the functions satisfying $g_t^*\Omega = \frac{1}{a_t}\Omega$ and $f_{\dot{g}_t}$ the functions satisfying $L_{\dot{g}_t} = -f_{\dot{g}_t}\Omega$. Recall that we have $\frac{\partial}{\partial t} \ln|a_t| = g_t^* f_{\dot{g}_t}$ and therefore

$$\begin{aligned} \tilde{\Phi}(g) &= \textstyle\int_0^1 g_t^* c_{\dot{g}_t} dt = \int_0^1 g_t^*(i_{\dot{g}_t}\omega - f_{\dot{g}_t}) dt = \int_0^1 g_t^* i_{\dot{g}_t} \omega dt - \int_0^1 g_t^* f_{\dot{g}_t} dt \\ &= \textstyle S_\omega(i_\sharp(g)) - \int_0^1 \frac{\partial}{\partial t} \ln|a_t| dt = S_\omega(i_\sharp(g)) - \ln|a_1| + \ln|a_0| = S_\omega(i_\sharp(g)) \end{aligned}$$

□

4.2.3. Proposition. *Let ω be a closed 1-form and $\theta \in \Omega^p(M)$ be d^ω-closed. Then $S_\theta(g) := \int_0^1 a_t g_t^* i_{\dot{g}_t} \theta dt$ defines a homomorphism*

$$S_\theta : \boldsymbol{\pi}_1\big(\operatorname{Diff}^\infty_c(M)_\circ\big) \supseteq \ker S_\omega \to H^{p-1}_{d^\omega_c}(M)$$

where $a_t := \exp\big(\int_0^t \operatorname{inc}_s^* i_{\partial_s} g^*\omega ds\big) = \exp\big(\int_0^t g_s^* i_{\dot{g}_s}\omega ds\big)$. Moreover S_θ only depends on the d^ω-cohomology class $[\theta] \in H^p_{d^\omega}(M)$, especially $S_\theta = 0$ if θ is d^ω-exact.

Proof. Again we first check that the formula defines a d^ω-cohomology class. Using $d^\omega i_X\alpha + i_X d^\omega\alpha = L_X\alpha + i_X\omega\wedge\alpha$ and $\frac{\partial}{\partial t}a_t = a_t g_t^* i_{\dot g_t}\omega$ we obtain:

$$\begin{aligned} d^\omega\Big(\int_0^1 a_t g_t^* i_{\dot g_t}\theta dt\Big) &= \int_0^1 a_t g_t^* d^\omega i_{\dot g_t}\theta dt = \int_0^1 a_t g_t^* L_{\dot g_t}\theta dt + \int_0^1 a_t g_t^*(i_{\dot g_t}\omega\wedge\theta)dt \\ &= \int_0^1 a_t \tfrac{\partial}{\partial t}(g_t^*\theta)dt + \int_0^1 (\tfrac{\partial}{\partial t}a_t)\wedge g_t^*\theta dt \\ &= \int_0^1 \tfrac{\partial}{\partial t}(a_t g_t^*\theta)dt = a_1 g_1^*\theta - a_0 g_0^*\theta = \big(e^{S_\omega(g)}-1\big)\theta = 0 \end{aligned}$$

Let $G : I\times I \to \mathrm{Diff}_c^\infty(M)_\circ$ be a homotopy relative endpoints, such that $g_s := G(s,\cdot)\in \ker S_\omega$. In fact it suffices that $g_0\in\ker S_\omega$, for then $g_s\in\ker S_\omega$ by proposition 4.2.1. Set $a(s,t) := \exp\big(\int_0^t G_{s,u}^* i_{\delta^r G(\partial t)}\omega du\big)$. A calculation very similar to the corresponding one in the proof of proposition 4.2.1 yields:

$$\tfrac{\partial}{\partial s}\big(a_{s,t}G_{s,t}^* i_{\delta^r G(\partial t)}\theta\big) = \tfrac{\partial}{\partial t}\big(a_{s,t}G_{s,t}^* i_{\delta^r G(\partial s)}\theta\big) - d^\omega(\cdots)$$

Consequently

$$\begin{aligned} \int_0^1 a_{1,t}G_{1,t}^* i_{\delta^r G(\partial_t)}\theta dt - \int_0^1 a_{0,t}G_{0,t}^* i_{\delta^r G(\partial_t)}\theta dt &= \int_0^1\int_0^1 \tfrac{\partial}{\partial s} a_{s,t}G_{s,t}^* i_{\delta^r G(\partial_t)}\theta dt ds \\ &= \int_0^1\int_0^1 \tfrac{\partial}{\partial t} a_{s,t}G_{s,t}^* i_{\delta^r G(\partial_s)}\theta dt ds - d^\omega(\cdots) \\ &= \int_0^1 a_{s,1}G_{s,1}^* i_{\delta^r G(\partial_s)}\theta ds - \int_0^1 a_{s,0}G_{s,0}^* i_{\delta^r G(\partial_s)}\theta ds - d^\omega(\cdots) = d^\omega(\cdots) \end{aligned}$$

and so $S_\theta(g)$ does only depend on the homotopy type relative endpoints of g. Next we show that S_θ is a homomorphism. Let g,h be closed curves and k their product, as in the proof of proposition 4.2.1. Moreover let a_t, b_t, c_t correspond to g,h,k respectively. Then one easily shows

$$c_t = \begin{cases} a_{2t} & 0\le t\le \frac12 \\ e^{S_\omega(g)}b_{2t-1} = b_{2t-1} & \frac12 \le t\le 1 \end{cases}$$

and we obtain:

$$S_\theta(gh) = S_\theta(k) = \int_0^{\frac12} a_{2s}g_{2s}^* i_{2\dot g_{2s}}\theta ds + \int_{\frac12}^1 b_{2s-1}h_{2s-1}^* i_{2\dot h_{2s-1}}\theta ds = S_\theta(g) + S_\theta(h)$$

Finally for $\theta = d^\omega\alpha$ we have

$$\begin{aligned} S_\theta(g) &= \int_0^1 a_t g_t^* i_{\dot g_t} d^\omega\alpha dt = \int_0^1 a_t g_t^*\big(L_{\dot g_t}\alpha + i_{\dot g_t}\omega\wedge\alpha - d^\omega i_{\dot g_t}\alpha\big)dt \\ &= \int_0^1 \big(a_t \tfrac{\partial}{\partial t} g_t^*\alpha + a_t g_t^* i_{\dot g_t}\omega\wedge g_t^*\alpha\big)dt - d^\omega(\cdots) \\ &= \int_0^1 \tfrac{\partial}{\partial t}\big(a_t g_t^*\alpha\big)dt = a_1 g_1^*\alpha - a_0 g_0^*\alpha = 0 \in H^{p-1}_{d_c^\omega}(M) \end{aligned}$$

and so S_θ only depends on the d^ω-cohomology class of θ. □

4.2.4. Proposition. *Let (M,Ω,ω) be a locally conformally symplectic manifold. Then the diagram*

$$\begin{array}{ccc} \boldsymbol{\pi}_1(\ker\Phi) & \xrightarrow{\tilde\Psi} & H^1_{d_c^\omega}(M) \\ {\scriptstyle i_\sharp}\downarrow & \nearrow_{S_\omega} & \\ \ker S_\omega & & \end{array}$$

commutes, where $i : \ker\Phi \to \mathrm{Diff}_c^\infty(M)_\circ$ denotes the inclusion.

Proof. The inclusion i induces a mapping $i_\sharp : \boldsymbol{\pi}_1(\ker\Phi) \to \ker S_\omega \subseteq \boldsymbol{\pi}_1\big(\mathrm{Diff}_c^\infty(M)_\circ\big)$, for we have proposition 4.2.2 and since a closed curve with values in $\ker\Phi$ is contained in $\ker\tilde\Phi$. We have $\tilde\Psi(g) = \int_0^1 a_t g_t^* i_{\dot g_t}\Omega dt$, where $g_t^*\Omega = \frac{1}{a_t}\Omega$, but since $g_t\in\ker\Phi$ we also have $a_t = \exp\big(\int_0^t g_s^* i_{\dot g_s}\omega ds\big)$, as mentioned several times. □

4.3 Extensions of Diffeomorphism Groups

Recall that there exists a canonical p-form $\Theta_p \in \Omega^p\big(\bigwedge^p T^*M\big)$, generalizing the canonical 1-form on T^*M. It is given by

$$\Theta_p(e)(X_1,\dots,X_p) := e(T_e\pi\cdot X_1,\dots,T_e\pi\cdot X_p)$$

where $\pi : \bigwedge^p T^*M \to M$ denotes the projection. Let $\mathrm{Aff}_c(\bigwedge^p T^*M)$ denote the fiber wise affine diffeomorphisms g of $\pi : \bigwedge^p T^*M \to M$ such that $\pi(\mathrm{supp}(g))$ is compact.

4.3.1. Lemma. *For $\alpha \in \Omega^p(M)$, considered as mapping $\alpha : M \to \bigwedge^p T^*M$, one has $\alpha^*\Theta_p = \alpha$. Moreover for $f \in \mathrm{Diff}^\infty_c(M)$ we have $\tilde f := \bigwedge^p(Tf^{-1})^* \in \mathrm{Aff}_c(\bigwedge^p T^*M)$ and $\tilde f^*\Theta_p = \Theta_p$. Finally for $\alpha \in \Omega^p(M)$ and $f \in \mathrm{Diff}^\infty(M)$ we have $f^*\alpha = \tilde f^{-1}\circ\alpha\circ f$.*

Proof. To show the first assertion we calculate as follows:

$$\begin{aligned}(\alpha^*\Theta_p)(x)(X_1,\dots,X_p) &= \Theta_p(\alpha(x))\big(T_x\alpha\cdot X_1,\dots,T_x\alpha\cdot X_p\big)\\ &= \alpha(x)\big(T_{\alpha(x)}\pi T_x\alpha\cdot X_1,\dots,T_{\alpha(x)}\pi T_x\alpha\cdot X_p\big) = \alpha(x)(X_1,\dots,X_p)\end{aligned}$$

In order to see the second assertion we have

$$\begin{aligned}(\tilde f^*\Theta_p)(e)(X_1,\dots,X_p) &= \Theta_p(\tilde f(e))\big(T_e\tilde f\cdot X_1,\dots,T_e\tilde f\cdot X_p\big)\\ &= \tilde f(e)\big(T_{\tilde f(e)}\pi T_e\tilde f\cdot X_1,\dots,T_{\tilde f(e)}\pi T_e\tilde f\cdot X_p\big)\\ &= \big(\textstyle\bigwedge^p(Tf^{-1})^*\big)(e)\big(T_{\pi(e)}fT_e\pi\cdot X_1,\cdots,T_{\pi(e)}fT_e\pi\cdot X_p\big)\\ &= e\big(T_{f(\pi(e))}f^{-1}T_{\pi(e)}fT_e\pi\cdot X_1,\cdots,T_{f(\pi(e))}f^{-1}T_{\pi(e)}fT_e\pi\cdot X_p\big)\\ &= e\big(T_e\pi\cdot X_1,\dots,T_e\pi\cdot X_p\big) = \Theta_p(e)(X_1,\dots,X_p)\end{aligned}$$

where we used $\pi\circ\tilde f = f\circ\pi$. The third assertion now follows easily:

$$\tilde f^{-1}\circ\alpha\circ f = (\tilde f^{-1}\circ\alpha\circ f)^*\Theta_p = f^*\alpha^*(\tilde f^{-1})^*\Theta_p = f^*\alpha^*\Theta_p = f^*\alpha$$

□

4.3.2. Lemma. *Consider the mapping $\tau : \Omega^p_c(M) \to \mathrm{Aff}_c(\bigwedge^p T^*M)$ given by $\tau_\alpha(e) := \tau(\alpha)(e) = e + \alpha(\pi(e))$. Then one has $\tau_\alpha^*\Theta_p = \Theta_p + \pi^*\alpha$ for all $\alpha \in \Omega^p_c(M)$.*

Proof. Indeed we have

$$\begin{aligned}(\tau_\alpha^*\Theta_p)(e)(X_1,\dots,X_p) &= \Theta_p(\tau_\alpha(e))\big(T_e\tau_\alpha\cdot X_1,\dots,T_e\tau_\alpha\cdot X_p\big)\\ &= (\tau_\alpha(e))\big(T_{\tau_\alpha(e)}\pi T_e\tau_\alpha\cdot X_1,\dots,T_{\tau_\alpha(e)}\pi T_e\tau_\alpha\cdot X_p\big)\\ &= \big(e+\alpha(\pi(e))\big)\big(T_e\pi\cdot X_1,\dots,T_e\pi\cdot X_p\big)\\ &= (\Theta_p + \pi^*\alpha)(X_1,\dots,X_p)\end{aligned}$$

since $\pi\circ\tau_\alpha = \pi$. □

Since every element of $\mathrm{Aff}_c(\bigwedge^p T^*M)$ preserves the fibers we obtain a homomorphism $q : \mathrm{Aff}_c(\bigwedge^p T^*M) \to \mathrm{Diff}^\infty_c(M)$.

4.3.3. Proposition. *Let $\vartheta \in \Omega^p(M)$ be a closed form. Then the sequence*

$$0 \to Z_c^{p-1}(M) \xrightarrow{\tau} \mathrm{Aff}_c\left(\textstyle\bigwedge^{p-1} T^*M, d\Theta_{p-1} + \pi^*\vartheta\right) \xrightarrow{q} \mathrm{Diff}_c^\infty(M, [\vartheta]) \to 0$$

is exact, where $\mathrm{Aff}_c(\bigwedge^{p-1} T^*M, d\Theta_{p-1} + \pi^*\vartheta)$ consists of those $g \in \mathrm{Aff}_c(\bigwedge^{p-1} T^*M)$ which in addition preserve the closed p-form $d\Theta_{p-1} + \pi^*\vartheta$ and $\mathrm{Diff}_c^\infty(M, [\vartheta])$ denotes the group of all $g \in \mathrm{Diff}_c^\infty(M)$ such that $g^*\vartheta - \vartheta = d\alpha$ for some $\alpha \in \Omega_c^{p-1}(M)$. Moreover the action of $g \in \mathrm{Diff}_c^\infty(M, [\vartheta])$ on $Z_c^{p-1}(M)$ defined by this extension is simply pullback by g^{-1}.

Proof. For $\alpha \in \Omega_c^{p-1}(M)$ we obtain from lemma 4.3.2

$$\tau_\alpha^*(d\Theta_{p-1} + \pi^*\vartheta) = d\tau_\alpha^*\Theta_{p-1} + (\pi \circ \tau_\alpha)^*\vartheta = d(\Theta_{p-1} + \pi^*\alpha) + \pi^*\vartheta$$

and so τ_α preserves $d\Theta_{p-1} + \pi^*\vartheta$ if and only if $\alpha \in Z_c^{p-1}(M)$, for π^* is injective. Moreover τ is obviously one-to-one. Let $g \in \mathrm{Aff}_c\left(\bigwedge^{p-1} T^*M, d\Theta_{p-1} + \pi^*\vartheta\right)$. We have to show that $q(g)$ preserves the cohomology class $[\vartheta] \in H^p(M)$. We have $g^*(d\Theta_{p-1} + \pi^*\vartheta) = d\Theta_{p-1} + \pi^*\vartheta$ and therefore

$$\pi^*\big(q(g)^*\vartheta - \vartheta\big) = g^*\pi^*\vartheta - \pi^*\vartheta = d\Theta_{p-1} - g^*d\Theta_{p-1} = d\big(\Theta_{p-1} - g^*\Theta_{p-1}\big).$$

So $\pi^*\big(q(g)^*\vartheta - \vartheta\big)$ is exact and since $\pi^* : H^*(M) \to H^*(\bigwedge^{p-1} T^*M)$ is an isomorphism $q(g)^*\vartheta - \vartheta$ is exact too. Moreover it is clear that $q \circ \tau = \mathrm{id}$. Next we check $\ker q \subseteq \mathrm{Im}\,\tau$. Let $g \in \ker q$ and let X be a vertical vector field on $\bigwedge^{p-1} T^*M$. Then we have $i_X(g^*\Theta_{p-1} - \Theta_{p-1}) = 0$, for g_*X is vertical as well. Moreover we have

$$d(g^*\Theta_{p-1} - \Theta_{p-1}) = g^*d\Theta_{p-1} - d\Theta_{p-1} = \pi^*\vartheta - g^*\pi^*\vartheta = \pi^*\vartheta - \pi^*\vartheta = 0$$

and thus $L_X(g^*\Theta_{p-1} - \Theta_{p-1}) = 0$. Hence there exists $\alpha \in \Omega_c^{p-1}(M)$ such that $g^*\Theta_{p-1} - \Theta_{p-1} = \pi^*\alpha$. If s denotes any section of $\bigwedge^{p-1} T^*M$, i.e. $s \in \Omega^{p-1}(M)$ we get

$$g \circ s - s = (g \circ s)^*\Theta_{p-1} - s^*\Theta_{p-1} = s^*\pi^*\alpha = \alpha$$

and consequently $g = \tau_\alpha \in \mathrm{Im}\,\tau$. Next we will show that q is onto. Let $f \in \mathrm{Diff}_c^\infty(M, [\vartheta])$ and choose $\alpha \in \Omega_c^{p-1}(M)$ such that $f^*\vartheta - \vartheta = d\alpha$. From lemma 4.3.1 and lemma 4.3.2 we obtain

$$\begin{aligned}(\tilde f \circ \tau_{-\alpha})^*(d\Theta_{p-1} + \pi^*\vartheta) &= d\tau_{-\alpha}^*\tilde f^*\Theta_{p-1} + \tau_{-\alpha}^*\pi^* f^*\vartheta = d\tau_{-\alpha}^*\Theta_{p-1} + \pi^* f^*\vartheta \\ &= d(\Theta_{p-1} - \pi^*\alpha) + \pi^*(\vartheta + d\alpha) = d\Theta_{p-1} + \pi^*\vartheta\end{aligned}$$

so $\tilde f \circ \tau_{-\alpha} \in \mathrm{Aff}_c\left(\bigwedge^{p-1} T^*M, d\Theta_{p-1} + \pi^*\vartheta\right)$ and obviously $q(\tilde f \circ \tau_{-\alpha}) = f$. At last we want to show that the action of $\mathrm{Diff}_c^\infty(M, [\vartheta])$ on $Z_c^{p-1}(M)$ induced from this extension is simply $g \cdot \beta = (g^{-1})^*\beta$. For this we have to show

$$\tilde f \circ \tau_{-\alpha} \circ \tau_\beta \circ \tau_{-\alpha}^{-1} \circ \tilde f^{-1} = \tilde f \circ \tau_\beta \circ \tilde f^{-1} = \tau_{(f^{-1})^*\beta}$$

for all $\beta \in Z_c^{p-1}(M)$, where α is as above. Indeed we have

$$\begin{aligned}(\tilde f \circ \tau_\beta \circ \tilde f^{-1})(e) &= \tilde f\big(\tilde f^{-1}(e) + \beta(\pi(\tilde f^{-1}(e)))\big) = e + (\tilde f \circ \beta \circ \pi \circ \tilde f^{-1})(e) \\ &= e + (\tilde f \circ \beta \circ f^{-1} \circ \pi)(e) = e + ((f^{-1})^*\beta)(\pi(e)) = \tau_{(f^{-1})^*\beta}(e)\end{aligned}$$

where we used again lemma 4.3.1. □

4.4 Transgression of the Flux

The following theorem can be found in [Ban97].

4.4.1. Theorem. *Let ϑ be a closed p-form. Then we have*

$$-t(S_\vartheta) = p_* j^* c \in H^2\big(\operatorname{Diff}_c^\infty(M)_\circ; H_c^{p-1}(M)\big)$$

where

$$t : \operatorname{Hom}\big(\boldsymbol{\pi}_1\big(\operatorname{Diff}_c^\infty(M)_\circ\big); H_c^{p-1}(M)\big) \to H^2\big(\operatorname{Diff}_c^\infty(M)_\circ; H_c^{p-1}(M)\big)$$

is the transgression homomorphism associated to the central extension

$$0 \to \boldsymbol{\pi}_1\big(\operatorname{Diff}_c^\infty(M)_\circ\big) \to \widetilde{\operatorname{Diff}}_c^\infty(M)_\circ \xrightarrow{\pi} \operatorname{Diff}_c^\infty(M)_\circ \to 0 \tag{4.2}$$

and the trivial $\operatorname{Diff}_c^\infty(M)_\circ$-module $H_c^{p-1}(M)$, defined in theorem 4.1.3, c is the cohomology class corresponding to the extension constructed in proposition 4.3.3, $j : \operatorname{Diff}_c^\infty(M)_\circ \to \operatorname{Diff}_c^\infty(M, [\vartheta])$ denotes the inclusion and $p : Z_c^{p-1}(M) \to H_c^{p-1}(M)$ the usual projection.

Proof. Choose a set theoretic section s of (4.2) such that $s(\mathrm{id}) = \mathrm{id}$. By theorem 4.1.3 $t(S_\vartheta)$ is represented by the 2-cocycle:

$$t(S_\vartheta)(g,h) = S_\vartheta\big(s(g)s(h)s(gh)^{-1}\big)$$

Since $t \mapsto \big(s(g)s(h)s(gh)^{-1}\big)(t)$ is homotopic relative endpoints to

$$t \mapsto \begin{cases} s(h)(3t) & 0 \le t \le \frac{1}{3} \\ s(g)(3t-1)h & \frac{1}{3} \le t \le \frac{2}{3} \\ s(gh)(3-3t) & \frac{2}{3} \le t \le 1 \end{cases}$$

we obtain

$$\begin{aligned} t(S_\vartheta)(g,h) &= S_\vartheta\big(s(g)s(h)s(gh)^{-1}\big) \\ &= \Big[\int_0^{\frac{1}{3}} s(h)_{3t}^* i_{3\delta^r(s(h))(\partial_t)(3t)} \vartheta dt + \int_{\frac{1}{3}}^{\frac{2}{3}} h^* s(g)_{3t-1}^* i_{3\delta^r(s(g))(\partial_t)(3t-1)} \vartheta dt \\ &\quad + \int_{\frac{2}{3}}^1 s(gh)_{3-3t}^* i_{-3\delta^r(s(gh))(\partial_t)(3-3t)} \vartheta dt \Big] \\ &= \Big[\int_0^1 s(h)_t^* i_{\delta^r(s(h))\partial_t} \vartheta dt + h^*\big(\int_0^1 s(g)_t^* i_{\delta^r(s(g))\partial_t} \vartheta dt \big) - \int_0^1 s(gh)_t^* i_{\delta^r(s(gh))\partial_t} \vartheta dt \Big] \\ &= [\alpha(h) + h^*\alpha(g) - \alpha(gh)] \end{aligned}$$

with $\alpha(g) := \int_0^1 s(g)_t^* i_{\delta^r(s(g))\partial_t} \vartheta dt$. An easy calculation shows $d(\alpha(g)) = g^*\vartheta - \vartheta$ and so

$$\begin{aligned} \sigma : \operatorname{Diff}_c^\infty(M)_\circ &\to \operatorname{Aff}_c\big(\textstyle\bigwedge^{p-1} T^*M, d\Theta_{p-1} + \pi^*\vartheta\big) \\ \sigma(g) &:= \tilde{g} \circ \tau_{-\alpha(g)} \end{aligned}$$

is a set theoretic section of the extension from proposition 4.3.3 restricted to $\operatorname{Diff}_c^\infty(M)_\circ$. Moreover we have

$$\begin{aligned} \tilde{g}\tau_{-\alpha(g)} \tilde{h} \tau_{-\alpha(h)} \big(\widetilde{gh}\tau_{-\alpha(gh)}\big)^{-1} &= \tilde{g}\tau_{-\alpha(g)} \tilde{h} \tau_{\alpha(gh)-\alpha(h)} \tilde{h}^{-1} \tilde{g}^{-1} \\ &= \tilde{g} \tau_{(h^{-1})^*(\alpha(gh)-\alpha(h))-\alpha(g)} \tilde{g}^{-1} \\ &= \tau\big(((gh)^{-1})^*(\alpha(gh) - \alpha(h) - h^*\alpha(g))\big) \end{aligned}$$

and so p_*j^*c is represented by

$$\begin{aligned}(g,h) \mapsto & p\big(\tau^{-1}\big(\sigma(g)\sigma(h)\sigma(gh)^{-1}\big)\big)\\ &= [((gh)^{-1})^*(\alpha(gh) - \alpha(h) - h^*\alpha(g))]\\ &= [\alpha(gh) - \alpha(h) - h^*\alpha(g)] = -t(S_\vartheta)(g,h)\end{aligned}$$

□

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