Annotated version

JGP - Vol. 2, n. 2, 1985.

A generalization of Hamiltonian mechanics

PETER W. MICHOR

Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Wien, Austria

Abstract. *For a symplectic manifold the Poisson bracket on the space of functions is (uniquely) extended to a graded Lie bracket on the space of differential forms modulo exact forms. A large portion of the Hamiltonian formalism is still working.*

Let (M, ω) be a symplectic manifold. Then there is an exact sequence of Lie algebras and Lie algebra homomorphisms

$$0 \to H^0(M) \to C^\infty(M) \xrightarrow{H} \mathcal{X}_{\theta(.)\omega=0}(M) \xrightarrow{\gamma} H^1(M) \to 0,$$

where $H^0(M)$, $H^1(M)$ are the de Rham cohomology spaces, $\mathcal{X}_{\theta(.)\omega=0}(M)$ is the space of all vector fields X with $\theta(X)\omega = 0$ (Lie derivative), a Lie subalgebra of the space $\mathcal{X}(M)$ of all vector fields. $C^\infty(M)$ is equipped with the Poisson bracket $\{\ ,\ \}$, and $H(f)$ is the Hamiltonian vector field for the generating function f. $\gamma(x)$ is the cohomology class of $i(X)\omega$ (insertion).

We will present the following generalisation: There is an exact sequence

$$0 \to H^*(M) \to \Omega(M)/B(M) \xrightarrow{H} \Omega_{0(.)\omega=0}(M; TM) \xrightarrow{\gamma+P} H^{*+1}(M) \oplus \Gamma(E_\omega) \to 0,$$

Key-Words: Hamiltonian Mechanics, Graded Lie Algebra
1980 Mathematics Subject Classification: *58 A 10, 58 F 05.*

where $\Omega(M)$ is the algebra of all differential forms, $B(M)$ is the subspace of exact forms, $\Omega_{\theta(.)\omega=0}(M; TM)$ is the space of all TM-valued differential forms K with $\theta(K)\omega = 0$ (Lie derivative, see §1), and where E_ω is a sub vector bundle of $\Lambda T^*M \otimes TM$, consisting of all those K with $i(K)\omega = 0$ (insertion, see §1). $\Omega_{\theta(.)\omega=0}(M; TM)$ is a graded Lie algebra with the Frölicher Nijenhuis bracket $[\ ,\ ]$. On $\Omega(M)/B(M)$ there is the generalized Poisson bracket $\{\ ,\ \}$ which makes $\Omega(M)/B(M)$ into a graded Lie algebra. H is the generalized Hamiltonian mapping, which is a homomorphism of graded Lie algebras. On $H^*(M)$ on the left hand side and on $H^{*+1}(M)$ on the right hand side we may put the zero braket, but on $\Gamma(E_\omega)$ there is no compatible graded Lie bracket. For the full statment of the result see theorem 4.6 at the end of the paper.

These results depend heavily on the properties of the Frölicher Nijenhuis bracket. The first section is a short presentation of these, which are due to [1]. There are also some new formulas (1.8, 1.9), which are not essential for the following. The second section presents the usual Hamiltonian formalism with proofs in the form in which it will be generalized later. The reference here is [ ]. Section 3 is devoted to the generalized Hamiltonian mapping. Section 4 contains the generalized Poisson bracket.

I want to thank G. Kainz, J. Kijowski, A. Trautman and H. Urbantke for discussions and hints.

§1. DERIVATIONS ON THE ALGEBRA OF DIFFERENTIAL FORMS

1.1. Let M be a smooth second countable manifold, let $\Omega(M) = \bigoplus_k \Omega^k(M)$ be the graded commutative algebra of differential forms. An R-linear mapping $D : \Omega(M) \to \Omega(M)$ is said to be of degree k if $D(\Omega^h(M)) \subset \Omega^{h+k}(M)$; and D is said to be a (graded) derivation of degree k if furthermore

$$D(\phi \wedge \psi) = D\phi \wedge \psi + (-1)^{hk} \phi \wedge D\psi \quad \text{for } \phi \in \Omega^h(M),\ \psi \in \Omega(M).$$

Let $\mathrm{Der}_k \Omega(M)$ be the linear space of all derivations of degree k and let $\mathrm{Der}\, \Omega(M) = \bigoplus_k \mathrm{Der}_k \Omega(M)$ be the space of all derivations.

PROPOSITION. Der $\Omega(M)$ *becomes a graded Lie algebra with the graded commutator*

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1,\ D_i \in \mathrm{Der}_{k_i} \Omega(M).$$

This means that the bracket is graded anticommutative, $[D_1, D_2] = -(-1)^{k_1 k_2}[D_1, D_2]$ *and satisfies the graded Jacobi identity:* $[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2}[D_2, [D_1, D_3]]$ *(so* ad $(D_1) = [D_1, .\,]$ *is itself a derivation).*

The proof is by computation.

1.2. A derivation $D \in \mathrm{Der}_k \Omega(M)$ is called algebraic, if $D \mid \Omega^0(M) = 0$.

Then $D(f \cdot \phi) = f \cdot D\phi$ for $f \in C^\infty(M)$ and D is tensorial..

Furthermore D is uniquely determined by $D \mid \Omega^1(M) : \Omega^1(M) \to \Omega^{k+1}(M)$, which is induced by a vector bundle mapping $K : T^*M \to \Lambda^{k+1} T^*M$, which we view as an element of $\Omega^{k+1}(M; TM)$, the space of all TM-valued $(k+1)$-froms on M. We write $D = i(K)$ and note the defining equation $D\phi = \phi \circ K$ for $\phi \in \Omega^1(M)$.

PROPOSITION. 1. *For* $\phi \in \Omega^h(M)$ *and* $X_j \in \mathscr{X}(M)$ *(the space of vector fields) we have:*

$$(i(K)\,\phi)\,(X_1, \ldots, X_{k+h}) =$$

$$= \frac{1}{(k+1)!(h-1)!} \sum_{\sigma \in S_{k+h}} \operatorname{sign} \sigma\, \phi(K(X_{\sigma 1}, \ldots, X_{\sigma(k+1)}), X_{\sigma(k+2)}, \ldots, X_{\sigma(k+h)}).$$

Note that this formula makes sense also if $\phi \in \Omega^h(M; E)$ *is a vector bundle valued differential form.*

2. *For* $K_j \in \Omega^{k_j+1}(M; TM)$ *the derivation* $[i(K_1), i(K_2)]$ *is again algebraic, so it is of the form* $i([K_1, K_2]^\wedge)$ *for some unique* $[K_1, K_2]^\wedge \in \Omega^{k_1+k_2+1}(M; TM)$. *With the bracket* $[\ ,\ ]^\wedge$, $\Omega^{*+1}(M; TM)$ *becomes also a graded Lie algebra. We have* $[K_1, K_2]^\wedge = i(K_1)K_2 - (-1)^{k_1 k_2} i(K_2)K_1$ (see 1).

In [1] the expression $i(K)\phi$ is denoted by $\phi \barwedge K$. If $X \in \Omega^0(M; TM)$ is a vector field, then $i(X)$ is the usual insertion operator of degree -1 on $\Omega(M)$.

1.3. The exteriour derivative d is also a derivation of degree 1, which is not algebraic. In view of the well known equation $\theta(X) = i(X)d + di(X)$ ($\theta(X)$ the Lie derivation, X a vector field) we define the derivation $\theta(K) := [i(K), d] \in \mathrm{Der}_k \Omega(M)$ for $K \in \Omega^k(M; TM)$ and call it the Lie derivation along K. Note that $\theta(\mathrm{Id}_{TM}) = d$.

PROPOSITION. *Any derivation* $D \in \mathrm{Der}_k \Omega(M)$ *can uniquely be written in the form* $D = \theta(K) + i(L)$ *for* $K \in \Omega^k(M; TM)$ *and* $L \in \Omega^{k+1}(M; TM)$. *D is algebraic if and only if* $K = 0$. $[D, d] = 0$ *if and only if* $L = 0$.

Scetch of proof: Let $X_j \in \mathscr{X}(M)$ be vector fields. Then $f \mapsto (Df)(X_1, \ldots, X_k)$ is a derivation (of degree 0) of $C^\infty(M) = \Omega^0(M)$, so it is given by the action of a vector field $K(X_1, \ldots, X_k)$, which is skew and $C^\infty(M)$-linear in the X_j, so $K \in \Omega^k(M; TM)$. Then $D - \theta(K)$ is algebraic, so equals $i(L)$ for some L.

Note that $[\theta(K), d] = 0$ by the graded Jacobi identity.

1.4. DEFINITION. Let $K_j \in \Omega^{k_j}(M; TM)$. Then clearly $[[\theta(K_1), \theta(K_2)], d] = 0$. So by 1.3 $[\theta(K_1), \theta(K_2)] = \theta([K_1, K_2])$ for some unique $[K_1, K_2] \in \Omega^{k_1 + k_2}(M; TM)$, which is called the Frölicher Nijenhuis bracket of K_1, K_2.

1.5. PROPOSITION. 1. *With the Frölicher Nijenhuis bracket the space $\Omega(M; TM)$ becomes a graded Lie algebra.*

2. *For vector fields X, Y the bracket $[X, Y]$ is the usual Lie bracket of vector fields.*

The proof is clear.

1.6. PROPOSITION. *For $K \in \Omega^k(M; TM)$ and $L \in \Omega^{h+1}(M; TM)$ we have*

$$[\theta(K), i(L)] = i([K, L]) - (-1)^{kh}\theta(i(L)K).$$

Proof. $[\theta(K), i(L)] + (-1)^{hk}\theta(i(L)K)$ vanishes on $\Omega^0(M)$, so is algebraic. By the graded Jacoby identity we get $[[\theta(K), i(L)], d] = [i([K, L]), d]$, and since $[\,.\,, d]$ is injective on algebraic derivations the formula follows. ■

1.7. PROPOSITION. 1. *The space* Der $\Omega(M)$ *is a graded module over the graded commutative algebra $\Omega(M)$ with the action $(\phi \wedge D)\psi = \phi \wedge D\psi$.*

2. *For $D_i \in \mathrm{Der}_{k_i}\Omega(M)$ and $\phi \in \Omega^q(M)$ we have*

$$[\phi \wedge D_1, D_2] = \phi \wedge [D_1, D_2] - (-1)^{(q+k_1)k_2} D_2\phi \wedge D_1.$$

3. *For $L \in \Omega(M; TM)$ we have $i(\phi \wedge L) = \phi \wedge i(L)$.*

4. *For $K \in \Omega^k(M; TM)$ we have $\theta(\phi \wedge K) = \phi \wedge \theta(K) + (-1)^{q+k} d\phi \wedge i(K)$.*

5. *For $L_i \in \Omega^{h_i+1}(M; TM)$ we have*

$$[\phi \wedge L_1, L_2]^\wedge = \phi \wedge [L_1, L_2]^\wedge - (-1)^{(q+h_1)h_2} i(L_2)\,\phi \wedge L_1.$$

6. *For $K_i \in \Omega^{k_i}(M; TM)$ we have*

$$[\phi \wedge K_1, K_2] = \phi \wedge [K_1, K_2] - (-1)^{(q+k_1)k_2}\theta(K_2)\phi \wedge K_1 + \\ + (-1)^{q+k_1} d\phi \wedge i(K_1)\, K_2.$$

7. *For $X, Y \in \mathscr{X}(M)$, $\phi \in \Omega^q(M)$, $\psi \in \Omega(M)$ we have:*

$$[\phi \otimes X, \psi \otimes Y] = \phi \wedge \psi \otimes [X, Y] + \phi \wedge \theta(X)\psi \otimes Y - \theta(Y)\phi \wedge \psi \otimes X \\ + (-1)^q (d\phi \wedge i(X)\psi \otimes Y + i(Y)\phi \wedge d\psi \otimes X).$$

Proof. For 2, 3, 4 just compute. For 5 compute $i([\phi \wedge L_1, L_2]\hat{})$. For 6 compute $\theta([\phi \wedge K_1, K_2])$. For 7 use 6. ∎

Now we include two results which will not be essential for the following, but they probably give some insight into the Frölicher Nijenhuis bracket. We only sketch the proofs.

1.8. PROPOSITION. *For $K \in \Omega^k(M; TM)$, $\phi \in \Omega^h(M)$, $X_i \in \mathscr{X}(M)$ we have*

$$(\theta(K)\phi)(X_1,\dots,X_{k+h}) =$$

$$= \frac{1}{k!h!} \sum_{\sigma \in S_{h+k}} \operatorname{sign}\sigma\, \theta(K(X_{\sigma 1},\dots,X_{\sigma k}))(\phi(X_{\sigma(k+1)},\dots,X_{\sigma(k+h)})) +$$

$$+ \frac{-1}{k!(h-1)!} \sum_{\sigma} \operatorname{sign}\sigma\, \phi([K(X_{\sigma 1},\dots,X_{\sigma k}), X_{\sigma(k+1)}], X_{\sigma(k+2)},\dots,X_{\sigma(k+h)}) +$$

$$+ \frac{(-1)^{k-1}}{(k-1)!(h-1)!2!} \sum_{\sigma} \operatorname{sign}\sigma\, \phi(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3},\dots,X_{\sigma(k+1)}), X_{\sigma(k+2)},\dots,X_{\sigma(k+h)}).$$

This can be proved by combinatorics starting from the formula in 1.2.1 (difficult), or by putting $K = \psi \otimes X$ and using 1.7.4.

1.9. PROPOSITION. *For $K \in \Omega^k(M; TM)$ and $L \in \Omega^h(M; TM)$ we have*

$$[K, L](X_1,\dots,X_{k+h}) =$$

$$= \frac{1}{k!h!} \sum_{\sigma \in S_{k+h}} \operatorname{sign}\sigma\, [K(X_{\sigma 1},\dots,X_{\sigma k}), L(X_{\sigma(k+1)},\dots,X_{\sigma(k+h)})] +$$

$$+ \frac{-1}{k!(h-1)!} \sum_{\sigma} \operatorname{sign}\sigma\, L([K(X_{\sigma 1},\dots,X_{\sigma k}), X_{\sigma(k+1)}], X_{\sigma(k+2)},\dots,X_{\sigma(k+h)}) +$$

$$+ \frac{(-1)^{kh}}{(k-1)!h!} \sum_{\sigma} \operatorname{sign}\sigma\, K([L(X_{\sigma 1},\dots,X_{\sigma h}), X_{\sigma(h+1)}], X_{\sigma(h+2)},\dots,X_{\sigma(h+k)}) +$$

$$+ \frac{(-1)^{k-1}}{(k-1)!(h-1)!2!} \Sigma \operatorname{sign}\sigma\, L(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3},\dots,X_{\sigma(k+1)}), X_{\sigma(k+2)},\dots,X_{\sigma(k+h)}) +$$

$$+ \frac{(-1)^{(k-1)h}}{(k-1)!(h-1)!2!} \Sigma \operatorname{sign}\sigma\, K(L([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3},\dots,X_{\sigma(h+1)}), X_{\sigma(h+2)},\dots,X_{\sigma(h+k)}).$$

For the proof the following concepts are used: let $E \to M$ be a vector bundle. A derivation D of the graded $\Omega(M)$-module $\Omega(M;E)$ is a pair $(D,\tilde{D})$ such that $D(\omega \wedge \Psi) = \tilde{D}\omega \wedge \Psi + (-1)^{kh}\,\omega \wedge D\Psi$ for some $\tilde{D} : \Omega(M) \to \Omega(M)$, $\omega \in \Omega^h(M)$, $\Psi \in \Omega(M;E)$. Then $\tilde{D}$ is uniquely determined and is in $\mathrm{Der}_k\,\Omega(M)$. The graded commutator $[D_1, D_2]$ makes $\mathrm{Der}\,\Omega(M;E)$ into a graded Lie algebra. For $K \in \Omega(M;TM)$ the operator $i(K)$ is in $\mathrm{Der}\,\Omega(M;E)$. Any covariant exteriour derivative ∇ is in $\mathrm{Der}\,\Omega(M;E)$. So is $\Theta_\nabla(K) := [i(K), \nabla]$, the covariant Lie derivative along K.

If ∇ is a torsionfree covariant exteriour derivative on the tangent bundle TM, then $[K,L] = \Theta_\nabla(K)L - (-1)^{kh}\,\Theta_\nabla(L)K$. Via this formula, the analogue of formula 1.8 for $\Theta_\nabla(K)L$ implies 1.9.

§2. HAMILTONIAN MECHANICS

2.1. Let (M, ω) be a symplectic manifold, so $\omega \in \Omega(M)$ is a 2-form, $d\omega = 0$, and $\check{\omega} : TM \to T^*M$ is a fibrewise linear isomorphism, where $\langle \check{\omega}(X), Y \rangle = \omega(X, Y)$. We have $(\check{\omega})^* = -\check{\omega}$, so $\dim M = 2n$ and $\omega \wedge \ldots \wedge \omega$ (n times) is a volume form.

2.2. Put $(\check{\omega})^{-1} =: \rho : T^*M \to TM$, a fibrewise linear isomorphism.

For a function $f \in C^\infty(M)$ define $H_f := \rho\, df$, the Hamiltonian vector field with generating function f. H_f is uniquely determined by the equation $\omega(H_f, X) = df(X) = Xf$ for any vector field X, or $i(H_f)\omega = df$.

We also have $H(fg) = H_f \cdot g + f \cdot H_g$.

2.3. Define the Poisson bracket $\{f, g\}$ of two functions by

$$\{f,g\} := \omega(H_g, H_f) = i(H_f)i(H_g)\omega = i(H_f)dg = \Theta(H_f)g.$$

Then $(C^\infty(M), \{\,,\})$ is a Lie algebra (see 2.4) and $\{f, gh\} = \{f,g\} \cdot h + g \cdot \{f, h\}$, so $\mathrm{ad}: (C^\infty(M), \{\,,\}) \to \mathrm{Der}\,(C^\infty(M), .\,)$ is a Lie algebra homomorphism.

2.4. PROPOSITION. *Let* $\mathscr{X}_{\theta(.)\omega=0}(M) := \{X \in \mathscr{X}(M) : \Theta(X)\omega = 0\}$. *Then this is a Lie subalgebra of* $\mathscr{X}(M)$ *and we have the following exact sequence of Lie algebras and Lie algebra homomorphisms with the indicated Lie brackets:*

$$\begin{array}{ccccccccc} 0 \to & H^0(M) & \xrightarrow{i} & C^\infty(M) & \xrightarrow{H} & \mathscr{X}_{\theta(.)\omega=0}(M) & \xrightarrow{\gamma} & H^1(M) & \to 0 \\ & 0 & & \{\,,\} & & [\,,] & & 0. & \end{array}$$

Proof. i is the embedding of the locally constant functions. If $H_f = \rho\, df = 0$, then $df = 0$, so f is locally constant, $f \in H^0(M)$.

$\Theta(H_f)\omega = i(H_f)\omega + di(H_f)\omega = 0 + ddf = 0$, so H takes values in $\mathscr{X}_{\theta(.)\omega=0}(M)$. Now $H(\{f,g\}) = [H_f, H_g]$ follows from

$$i(H\{f,g\})\omega = d\{f,g\} = d\Theta(H_f)g = \Theta(H_f)\,dg =$$

$$= \Theta(H_f)i(H_g)\omega - i(H_g)\Theta(H_f)\omega, \text{ since } \Theta(H_f)\omega = 0,$$

$$= [\Theta(H_f), i(H_g)]\omega = i([H_f, H_g])\omega + 0 \text{ by 1.6.}$$

For $X \in \mathscr{X}(M)$ we have $\Theta(X)\omega = 0 + di(X)\omega$, so $i(X)\omega$ is closed if and and only if $\Theta(X)\omega = 0$. Thus $\gamma : \mathscr{X}_{\theta(.)\omega=0}(M) \to H^1(M)$ is well defined, where $\gamma(X)$ is the (de Rham) cohomology class of $i(X)\omega$. Furthermore $\gamma(X) = 0$ if $i(X)\omega$ is exact, that is $i(X)\omega = df$ for some f, but then $X = H_f$. So $\ker\gamma =$ $= \operatorname{im} H$.

For $X, Y \in \mathscr{X}_{\theta(.)\omega=0}(M)$ we have by 1.6:

$$i([X,Y])\omega = [\Theta(X), i(Y)]\omega = \Theta(X)i(Y)\omega - 0 =$$

$$= i(X)di(Y)\omega + di(X)i(Y)\omega = 0 + di(X)i(Y)\omega$$

so $\gamma([X,Y]) = 0$.

$H^0(M)$ is in the center of $(C^\infty(M), \{\ ,\ \})$, for $df = 0$ implies $\{f,g\} = \Theta(H_f)g =$ $= \Theta(0)g = 0$.

Finally we show that $\{\ ,\ \}$ is a Lie bracket:

$$\{f,g\} = \omega(H_g, H_f) = -\omega(H_f, H_g) = -\{g,f\}.$$

$$\{\{f,g\},h\} = \Theta(H\{f,g\})h = \Theta([H_f,H_g])h = [\Theta(H_f),\Theta(H_g)]h =$$

$$= \Theta(H_f)\Theta(H_g)h - \Theta(H_g)\Theta(H_f)h = \{f,\{g,h\}\} - \{g,\{f,h\}\}. \quad \blacksquare$$

§3. THE GENERALIZED HAMILTONIAN MAPPING

3.1. Let (M,ω) be a symplectic manifold. Recall the fibrewise isomorphism $\rho = \check{\omega}^{-1} : T^*M \to TM$ from 2.2. We extend ρ to a module valued derivation of degree -1 (see 3.2.1 below) $\rho : \Omega(M) \to \Omega(M; TM)$ by putting:

$$\rho(\psi_1\wedge\ldots\wedge\psi_k) = \sum_{i=1}^{k} (-1)^{i-1}\psi_1\wedge\ldots\wedge\psi_{i-1}\wedge\rho(\psi_i)\wedge\psi_{i+1}\wedge\ldots\wedge\psi_k =$$

$$= \Sigma(-1)^{i-1}\psi_1\wedge\ldots\hat{\psi}_i\ldots\wedge\psi_k\otimes\rho(\psi_i), \qquad \psi_i \in \Omega^1(M);$$

$$\rho \,|\, \Omega^0(M) = 0.$$

3.2. $\Omega(M;TM)$ is a graded left $\Omega(M)$-module and also a graded right $\Omega(M)$-module by $\phi\wedge K = (-1)^{qk} K\wedge\phi$ for $\phi \in \Omega^q(M)$ and $K \in \Omega^k(M;TM)$.

LEMMA. 1. $\rho(\phi \wedge \psi) = \rho(\phi) \wedge \psi + (-1)^q \phi \wedge \rho(\psi)$ *for* $\phi, \psi \in \Omega(M)$, ϕ *of degree* q.

2. $i(\rho\phi)\omega = (-1)^{q-1} \cdot q \cdot \phi$ *for* $\phi \in \Omega^q(M)$.

3. $[\rho, i(\phi \otimes X)] = \rho i(\phi \otimes X) - (-1)^{q-1} i(\phi \otimes X)\rho = \rho\phi \wedge i(X)$, $\phi \in \Omega^q(M)$, $X \in \mathscr{X}(M)$.

Proof. 1. is clear. 2. take $\phi = \psi_1 \wedge \ldots \wedge \psi_q$ for $\psi_i \in \Omega^1(M)$ and compute. 3. ρ is a derivation $\Omega(M) \to \Omega(M; TM)$ of degree -1 by 1, $i(\phi \otimes X): \Omega(M) \to \Omega(M)$ is a derivation of degree $q-1$, and $i(\phi \otimes X): \Omega(M; TM) \to \Omega(M; TM)$ is also a derivation of degree $q-1$ of the $\Omega(M)$-module in the sense indicated in 1.9. So the graded commutator in 3 makes sense and is a derivation $\Omega(M) \to \Omega(M; TM)$ of degree $q-2$ which vanishes on $\Omega^0(M)$ and it is therefore uniquely determined by its action on $\Omega^1(M)$. The same is true for $\rho\phi \wedge i(X): \Omega(M) \to \Omega(M; TM)$. So it suffices to check the equation on 1-forms: $\psi \in \Omega^1(M)$.

$\rho i(\phi \otimes X)\psi - (-1)^{q-1} i(\phi \otimes X)\rho\psi = \rho(\phi \wedge i(X)\psi) + 0$

since $\rho\psi$ is a vector field and $i(X)\psi \in \Omega^0(M)$ since $= \rho\phi \wedge i(X)\psi + 0$. ■

3.3. Define $\bar{\rho}: \Omega^k(M) \to \Omega^{k-1}(M; TM)$ by $\bar{\rho}(\phi) = (-1)^{k-1} \frac{1}{k} \rho\phi$ for $k > 0$ and $\bar{\rho} \mid \Omega^0(M) = 0$. Then $i(\bar{\rho}\phi)\omega = \phi$ for degree $\phi > 0$, and $\bar{\rho}: \Omega^+(M) = \bigoplus_{k>0} \Omega^k(M) \to \Omega(M; TM)$ is a right inverse to $i(\,.\,)\omega: \Omega(M; TM) \to \Omega^+(M)$.

Since both operators are algebraic and hence of constant rank we may consider the sub vector bundle $E^k_\omega := \ker i(\,.\,)\omega \hookrightarrow \Lambda^k T^*M \otimes TM$ over M and we put $E_\omega = \bigoplus_{k>0} E^k_\omega$. Note that $E^0_\omega = 0$ and $E^{\dim M}_\omega = \Lambda^{\dim M} T^*M \otimes TM$.

Furthermore we have $\Lambda^k T^*M \otimes TM = E^k_\omega \oplus \operatorname{im} \bar{\rho} \mid \Lambda^{k+1} T^*M$, and in turn

$$\Omega^k(M; TM) = \Gamma(E^k_\omega) \oplus \operatorname{im} \bar{\rho} \mid \Omega^{k+1}(M) = \Gamma(E^k_\omega) \oplus \Omega^{k+1}(M),$$

$$\Omega(M; TM) = \Gamma(E_\omega) \oplus \operatorname{im} \bar{\rho} = \Gamma(E_\omega) \oplus \Omega^+(M).$$

The projection onto $\operatorname{im} \bar{\rho}$ is $\bar{\rho} \circ i(\,.\,)\omega$, that onto $\Omega^+(M)$ is $i(\,.\,)\omega$, than onto $\Gamma(E_\omega)$ is $\mathrm{Id} - \bar{\rho} \circ i(\,.\,)\omega =: P$. $\Gamma(E_\omega)$ is a graded $\Omega(M)$-module, since $i(K)\omega = 0$ implies $i(\phi \wedge K)\omega = \phi \wedge i(K)\omega = 0$. Note that $\bar{\rho}$ is no longer a derivation.

3.4. Now we define the *generalized Hamiltonian mapping* $H: \Omega(M) \to \Omega(M; TM)$ by $H_\phi = H(\phi) := \rho d\phi \in \Omega^k(M; TM)$ for $\phi \in \Omega^k(M)$. For $f \in C^\infty(M)$ we get the usual Hamiltonian vector field H_f discussed in §2.

LEMMA. 1. $i(H_\phi)\omega = (-1)^k \cdot (k+1) \cdot d\phi$ *for* $\phi \in \Omega^k(M)$.

2. $\Theta(H_\phi)\omega = 0$

3. $H(\phi \wedge \psi) = H_\phi \wedge \psi + \phi \wedge H_\psi + (-1)^k(\rho\phi \wedge d\psi - d\phi \wedge \rho\psi)$ *for* ϕ *of degree* k.
4. $H(d\phi \wedge \psi) = d\phi \wedge H_\psi - (-1)^{kq} d\psi \wedge H_\phi$ *for* $\deg(\phi) = k$ *and* $\deg(\psi) = q$.
5. $H(f_0 df_1 \wedge \ldots \wedge df_k) = \sum_{j=0}^{k} (-1)^j df_0 \wedge \ldots \widehat{df_j} \ldots \wedge df_k \otimes H_{f_j}$.

Proof. 1. $i(H_\phi)\omega = i(\rho d\phi)\omega = (-1)^k \cdot (k+1) \cdot d\phi$ by 3.2.2.

2. $\Theta(H_\phi)\omega = [i(H_\phi), d]\omega = i(H_\phi)d\omega - (-1)^{k-1} di(H_\phi)\omega = 0 + (k+1)dd\phi = 0$ by 1.

3.
$$\begin{aligned} H(\phi \wedge \psi) &= \rho d(\phi \wedge \psi) = \rho(d\phi \wedge \psi + (-1)^k \phi \wedge d\psi) \\ &= \rho d\phi \wedge \psi + (-1)^{k+1} d\phi \wedge \rho\psi + (-1)^k \rho\phi \wedge d\psi + \phi \wedge \rho d\psi \\ &= H_\phi \wedge \psi + \phi \wedge H_\psi + (-1)^k(\rho\phi \wedge d\psi - d\phi \wedge \rho\psi). \end{aligned}$$

4.
$$\begin{aligned} H(d\phi \wedge \psi) &= 0 + d\phi \wedge H_\psi + (-1)^{k+1}(\rho d\phi \wedge d\psi - 0) \text{ by 3} \\ &= d\phi \wedge H_\psi - (-1)^k H_\phi \wedge d\psi = d\phi \wedge H_\psi - (-1)^{kq} d\psi \wedge H_\phi. \end{aligned}$$

5. $H(f_0 df_1 \wedge \ldots \wedge df_k) = H_{f_0} \wedge df_1 \wedge \ldots \wedge df_k + 0 + (-1)^0(0 - df_0 \wedge \rho(df_1 \ldots df_k))$ by 3

$$= df_1 \wedge \ldots \wedge df_k \otimes H_{f_0} - df_0 \wedge \sum_{j=1}^{k} (-1)^{j-1} df_1 \wedge \ldots \widehat{df_j} \ldots \wedge df_k \otimes df_j$$

by 3.1

$$= \sum_{j=1}^{k} (-1)^j df_0 \wedge \ldots \widehat{df_j} \ldots \wedge df_k \otimes H_{f_j}.$$ ∎

3.5. We consider the following sequnece:

$$1. \quad 0 \to Z(M) \xrightarrow{i} \Omega(M) \xrightarrow{H} \Omega_{\theta(.)\omega = 0}(M; TM) \xrightarrow{\gamma + P} H^{*+1}(M) \oplus \Gamma(E_\omega) \to 0.$$

Here $Z(M)$ is the space of closed forms, i is the embedding, $\Omega_{\theta(.)\omega=0}(M; TM) := \{K \in \Omega(M; TM) : \Theta(K)\omega = 0\}$, $\gamma(K)$ is the cohomology class of $i(K)\omega$ (see 3.6 below), γ has degree $+1$ as indicated in the sequence, and P is the fibre projection $\mathrm{Id} - \bar{\rho} \circ i(.)\omega$ of 3.3.

Now let $B(M)$ be the space of exact forms, so $Z(M)/B(M) = H(M)$. We consider also the following sequence:

$$2. \quad 0 \to H(M) \xrightarrow{i} \Omega(M)/B(M) \xrightarrow{H} \Omega_{\theta(.)\omega = 0}(M; TM) \xrightarrow{\gamma + P} H^{*+1}(M) \oplus \Gamma(E_\omega) \to 0.$$

Note that the «degree 0 part» of both sequences is the «usual Hamiltonian sequence» of 2.4.

3.6. LEMMA. *The sequences 3.5.1 and 3.5.2 are both exact.*

Proof. We only have to show that the first sequence is exact.

Exactness at $Z(M)$ is clear.

Exactness at $\Omega(M) : H_\phi = \rho d\phi = 0$ if and only if $d\phi = 0$ since $\rho \mid \Omega^+(M)$ is injective.

Exactness at $\Omega_{\theta(.)\omega=0}(M;TM)$:

$$PH(\phi) = (\mathrm{Id} - \bar{p}\circ i(.)\omega)p\,d\phi = p\,d\phi - \bar{p}i(p\,d\phi)\omega =$$

$$= p\,d\phi - p\,i(\bar{p}d\phi)\omega = p\,d\phi - p\,d\phi = 0 \quad \text{by 3.3.}$$

Let $K \in \Omega^k_{\theta(.)\omega=0}(M;TM)$. Then $0 = \Theta(K)\omega = 0 - (-1)^{k-1}di(K)\omega$, so $i(K)\omega$ is closed and $\gamma(K) = [i(K)\omega] \in H^{k+1}(M)$ is well defined. Since $i(H_\phi)\omega = (-1)^k \cdot (k+1)\cdot d\phi$ is exact we have $\gamma(H_\phi) = 0$. Now suppose that $\gamma(K) = 0$ and $P(K) = 0$. Then $0 = P(K) = K - \bar{p}i(K)\omega$, so $K = \bar{p}i(K)\omega$; furthermore $\gamma(K) = 0$ means $i(K)\omega = d\psi$ for some $\psi \in \Omega^k(M)$, so $K = \bar{p}i(K)\omega = \bar{p}d\psi = (-1)^k \dfrac{1}{k+1} H_\psi$.

Exactness at $H^{*+1}(M) \oplus \Gamma(E_\omega)$: Let $[\phi] \in H^{k+1}(M)$, so $\phi \in \Omega^{k+1}(M)$, $d\phi = 0$. $\Theta(\bar{p}\phi)\omega = 0 - (-1)^{k-1}di(\bar{p}\phi)\omega = (-1)^k d\phi = 0$, so $\bar{p}\phi \in \Omega^k_{\theta(.)\omega=0}(M;TM)$, and $i(\bar{p}\phi)\omega = \phi$, so $\gamma(\bar{p}\phi) = [\phi]$ and γ is surjective.

Let $K \in \Omega(M;TM)$. Then $K \in \Gamma(E_\omega)$ if and only if $i(K)\omega = 0$. But then clearly $\Theta(K)\omega = 0 - (-1)^{k-1}di(K)\omega = 0$, so $K \in \Omega_{\theta(.)\omega=0}(M;TM)$ and $P(K) = K$, $\gamma(K) = 0$. So finally $P(\bar{p}\phi + K) = K$ by 3.3 and $\gamma(\bar{p}\phi + K) = [\phi]$, thus $\gamma + P$ is surjective. ■

3.7. LEMMA. *$\Omega_{\theta(.)\omega=0}(M;TM)$ is a graded Lie subalgebra of $\Omega(M;TM)$. For K, L in this space we have $i([K,L])\omega = dA(K,L)\omega$, where*

$$A(K,L) = (-1)^k i(K)i(L) - (-1)^{(k-1)l}\, i(i(L)K) \in \mathrm{End}_{k+l-2}\Omega(M)$$

$$= -(-1)^{(k-1)l}\, i(L)i(K) + (-1)^k\, i(i(K)L) =$$

$$= \frac{(-1)^k}{2}\,(i(K)i(L) + (-1)^{k-1)(l-1)}i(L)i(K) +$$

$$+ i(i(K)L + (-1)^{(k-1)(l-1)}i(L)K)).$$

Proof. $\Theta(K)\omega = 0$, $\Theta(L)\omega = 0$ implies $\Theta([K,L]) = [\Theta(K),\Theta(L)]\omega = 0$.
$i([K,L])\omega = [\Theta(K), i(L)]\omega + (-1)^{k(l-1)}\Theta(i(L)K)\omega$ by 1.6
$= \Theta(K)i(L)\omega - (-1)^{k(l-1)}i(L)\Theta(K)\omega + (-1)^{k(l-1)}(0 - (-1)^{k+l-2}di(i(L)K)\omega)$
$= i(K)di(L)\omega - (-1)^{k-1}di(K)i(L)\omega - (-1)^{(k-1)l}di(i(L)K)\omega$
$= d((-1)^k i(K)i(L) - (-1)^{(k-1)l}i(i(L)K))\omega$
$= dA(K,L)\omega$.

Now recall that $[i(K), i(L)] = i([K,L]\hat{\ })$ and $[K,L]\hat{\ } = i(K)L - (-1)^{(k-1)(l-1)}i(L)K$ from 1.2.2 and use this to transform the first expression for $A(K,L)$ into the second one. The third expression is then the arithmetic mean of the first and the second. ■

Remark. 1. It follows, that the mapping γ in the sequences of 3.5 induces the bracket 0 on $H^{*+1}(M)$.

2. Note the graded anticommutator $i(K)i(L) + (-1)^{(k-1)(l-1)} i(L)i(K)$ and similar things in the expression of $A(K, L)$. Here graded Jordan algebras make their appearence.

3.8. *Remark.* $\Gamma(E_\omega) \subseteq \Omega_{\theta(.)\omega=0}(M; TM)$ is not a graded Lie subalgebra for the Frölicher Nijenhuis bracket, if $\dim M > 2$.

Proof. $K = i(H_f)\omega \otimes H_f = df \otimes H_f$ and $L = dg \otimes H_g$, then $K, L \in \Gamma(E^1_\omega)$, but $i([K, L])\omega = dA(K, L)\omega = d\{f, g\} \wedge df \wedge dg$ which is not 0 in general if $\dim M > 2$. Choose $f = \frac{1}{2} p_1^2 + p_2$, $g = \frac{1}{2} q_1^2$ ■

3.9. *Remark.* The kernel of $P : \Omega_{\theta(.)\omega=0}(M; TM) \to \Gamma(E_\omega)$ is not an ideal for the Frölicher Nijenhuis bracket.

Proof. Let $K = df \otimes H_f \in \Gamma(E^1_\omega)$. Then $i([K, H_g])\omega = dA(K, H_g)\omega = ~~df \otimes H\{f, g\}~~ + ~~d\{f, g\} \otimes H_f~~$ which is not 0 in general, if $\dim M > 0$. ■

$= ?\ df \wedge d\{f, g\}$ Prove, that not an ideal

$H_g \in \ker P$; $i([K, H_g])\omega = df \otimes H\{f, g\} - d\{f, g\} \otimes H_f + df \otimes H\{g, f\} = df \otimes [H_g, H_f] = [K, H_g]$

3.10. LEMMA. 1. $i(\phi \otimes X)H_\psi = -\phi \wedge \rho i(X) d\psi$ *for* $\phi, \psi \in \Omega(M)$, $X \in \mathfrak{X}(M)$.

2. $i(i(K)H_\psi) = (-1)^k \cdot k \cdot i(K) d\psi$ *for* $\psi \in \Omega^k(M)$, $K \in \Omega(M; TM)$.

Proof. 1. $i(\phi \otimes X)H_\psi = i(\phi \otimes X)\rho d\psi = [i(\phi \otimes X), \rho] d\psi + (-1)^{k-1} \rho\, i(\phi \otimes X) d\psi$

$= (-1)^k \rho \phi \wedge i(X) d\psi - (-1)^k \rho(\phi \wedge i(X) d\psi)$ by 3.2.3

$= (-1)^k \rho\phi \wedge i(X) d\psi - (-1)^k \rho\phi \wedge i(X) d\psi - \phi \wedge \rho i(X) d\psi$

by 3.2.1

$= -\phi \wedge \rho i(X) d\psi.$

2. $i(i(\phi \otimes X)H_\psi)\omega = i(-\phi \wedge \rho i(X) d\psi)\omega$ by 1

$= -\phi \wedge i(\rho i(X) d\psi)\omega = -\phi \wedge (-1)^{k-1} \cdot k \cdot i(X) d\psi$

by 3.2.2

$= (-1)^k \cdot k \cdot i(\phi \otimes X) d\psi.$ ■

3.11. LEMMA. $[X, H_\phi] = H(\Theta(X)\phi)$, *for* $\phi \in \Omega(M)$, $X \in \Omega^0_{\theta(.)\omega=0}(M; TM) = \mathfrak{X}_{\theta(.)\omega=0}(M)$. *If the degree of X is not 0, then this is wrong.*

Proof. Let us first take $\phi = f \in C^\infty(M)$. Then by using 3.7 we get $i([H_f, X])\omega = dA(H_f, X)\omega = di(H_f)i(X)\omega - 0 = -di(X)i(H_f)\omega = -di(X)df = -d(Xf)$. So by 2.2 we see that $[H_f, X] = -H(Xf)$.

Now in general we may take $\phi = f_0 df_1 \wedge \ldots \wedge df_k$. Then we use 3.4.5 and 1.7.6 to get:

$[H_\phi, X] = [\Sigma(-1)^j\, df_0\wedge\ldots\hat{df_j}\ldots\wedge df_k \otimes H_{f_j}, X] =$
$= \Sigma\,(-1)^j (df_0\wedge\ldots\hat{df_j}\ldots\wedge df_k \otimes [H_{f_j}, X] - \Theta(X)(df_0\wedge\ldots\hat{df_j}\ldots\wedge df_k)\otimes H_{f_j} + 0).$

This turns out to be $-H(\Theta(X)\phi)$ by using the derivation property of $\Theta(X)$ and the equation above. ∎

§4. THE GENERALIZED POISSON BRACKET

4.1. Let (M, ω) be again a symplectic manifold. Consider the generalized Hamiltonian mapping $H:\Omega(M)\to\Omega(M; TM)$. We want to find a graded Lie bracket $\{\ ,\ \}$ on $\Omega(M)$, generalizing the Poisson bracket of §2 ~~on~~ $\Omega^0(M)$ such that H becomes a homomorphism. It will turn out that this is not possible, but that there is a generalized Poisson bracket on $\Omega(M)/B(M)$.

Suppose that there is a generalized Poisson bracket $\{\ ,\ \}$ on $\Omega(M)$. Then we have $i(H\{\phi,\psi\})\omega = (-1)^{k+l}\cdot(k+l+1)d\{\phi,\psi\}$ by 3.4.1 for $\deg(\phi)=k$, $\deg(\psi)=l$. On the other hand $i(H\{\phi,\psi\})\omega = i([H_\phi, H_\psi])\omega = dA(H_\phi, H_\psi)\omega$ by 3.7.

So we get $\{\phi,\psi\} = \dfrac{(-1)^{k+l}}{k+l+1} A(H_\phi, H_\psi)\omega$ + something closed.

One can show that this equals $i(H_\phi)d\psi$ + something closed, so we start our investigation with $i(H_\phi)d\psi$.

4.2. Define $\{\phi,\psi\}^1 := i(H_\phi)d\psi$ for $\phi,\psi\in\Omega(M)$. For $f, g\in C^\infty(M)$ the function $\{f,g\}^1$ coincides with the usual Poisson bracket of §2.

LEMMA. 1. $\{d\phi\wedge\psi,\tau\}^1 = d\phi\wedge\{\psi,\tau\}^1 - (-1)^{kl}\, d\psi\wedge\{\phi,\tau\}^1$ *for* $\deg(\phi)=k$, $\deg(\psi)=l$.

2. $\{\phi,\psi\}^1 = -(-1)^{kl}\{\psi,\phi\}^1$
3. $H(\{\phi,\psi\}^1) = [H_\phi, H_\psi]$.
4. $\{f_0\, df_1\wedge\ldots\wedge df_k, g_0\, dg_1\wedge\ldots\wedge dg_l\}^1 =$
$= \sum_{i,j}(-1)^{i+j}\{f_i, f_j\} df_0\wedge\ldots\hat{df_i}\ldots\wedge df_k\wedge dg_0\wedge\ldots\hat{dg_j}\ldots\wedge dg_l.$
5. *The bracket* $\{\ ,\ \}^1$ *does not satisfy the graded Jacobi identity. See 4.4 below.*

Proof. 1. $\{d\phi\wedge\psi,\tau\}^1 = i(H(d\phi\wedge\psi))d\tau = i(d\phi\wedge H_\psi - (-1)^{kl}\, d\psi\wedge H_\phi)d\tau$ by 3.4.4
$= d\phi\wedge i(H_\psi)d\tau - (-1)^{kl}\, d\psi\wedge i(H_\phi)d\tau$ by 1.7.3
$= d\phi\wedge\{\psi,\tau\}^1 - (-1)^{kl}\, d\psi\wedge\{\phi,\tau\}^1.$

2. We use induction on $k+l$: $\{f,g\} = -\{g,f\}$ holds by §2.

For the general induction step it suffices to consider $\{fd\phi,\psi\}^1$, since the bracket is local.

$$\{fd\phi, \psi\}^1 = \{d\phi\wedge f, \psi\}^1 = d\phi\wedge\{f, \psi\}^1 - df\wedge\{\phi, \psi\}^1 \qquad \text{by 1,}$$
$$= -d\phi\wedge\{\psi, f\}^1 + (-1)^{kl}\, df\wedge\{\psi, \phi\}^1 \qquad \text{by induction.}$$
$$\{\psi, fd\phi\}^1 = i(H_\psi)d(fd\phi) = i(H_\psi)(df\wedge d\phi) = i(H_\psi)df\wedge d\phi + (-1)^{l-1} df\wedge i(H_\psi)d\phi$$
$$= \{\psi, f\}^1\wedge d\phi - (-1)^l df\wedge\{\psi, \phi\}^1 = (-1)^{(k+1)l} d\phi\wedge\{\psi, f\}^1 - (-1)^l df\wedge\{\psi, \phi\}^1$$
$$= -(-1)^{(k+1)l}\{fd\phi, \psi\}^1 \qquad \text{by the expression above.}$$

3. We use again induction on $k + l$: For $k + l = 0$ this is true by §2. Since both sides are local, it suffices again to consider $H(\{fd\phi, \psi\}^1)$.

$$H(\{fd\phi, \psi\}^1) = H(d\phi\wedge\{f, \psi\}^1 - df\wedge\{\phi, \psi\}^1) \qquad \text{by 1}$$
$$(3.4.4) \qquad = d\phi\wedge H(\{f, \psi\}^1) - (-1)^{kl} d\{f, \psi\}^1\wedge H_\phi - df\wedge H(\{\phi, \psi\}^1) + d\{\phi, \psi\}^1\wedge H_f$$
$$= d\phi\wedge[H_f, H_\psi] - (-1)^{kl} d\{f, \psi\}^1\wedge H_\phi - df\wedge[H_\phi, H_\psi] + d\{\phi, \psi\}^1\wedge H_f,$$
by induction.

$$[H(fd\phi), H_\psi] = [d\phi\wedge H_f - df\wedge H_\phi, H_\psi] \qquad \text{by 3.4.4}$$
$$= d\phi\wedge[H_f, H_\psi] - (-1)^{(k+1)l}\,\Theta(H_\psi)d\phi\wedge H_f + 0 \qquad \text{by 1.7.6}$$
$$- df\wedge[H_\phi, H_\psi] + (-1)^{(1+k)l}\,\Theta(H_\psi)df\wedge H_\phi + 0$$
$$= d\phi\wedge[H_f, H_\psi] - (-1)^{kl}\, di(H_\psi)d\phi\wedge H_f$$
$$- df\wedge[H_\phi, H_\psi] + (-1)^{kl}\, di(H_\psi)df\wedge H_\phi$$
$$= d\phi\wedge[H_f, H_\psi] - (-1)^{kl}\, d\{\psi, \phi\}^1\wedge H_f$$
$$- df\wedge[H_\phi, H_\psi] + (-1)^{kl}\, d\{\psi, f\}^1\wedge H_\phi$$
$$= d\phi\wedge[H_f, H_\psi] + d\{\phi, \psi\}^1\wedge H_f - df\wedge[H_\phi, H_\psi] - (-1)^{kl} d\{f, \psi\}^1\wedge H_\phi \qquad \text{by 2}$$
$$= H(\{fd\phi, \psi\}^1).$$

4. $\{f_0 df_1\wedge\ldots\wedge df_k, g_0 dg_1\wedge\ldots\wedge dg_l\}^1 = i(H(f_0 df_1\wedge\ldots\wedge df_k))d(g_0 dg_1\wedge\ldots\wedge dg_l)$

$$= i(\sum_i (-1)^i df_0\wedge\ldots\hat{df_i}\ldots\wedge df_k\otimes H_{f_i})(dg_0\wedge\ldots\wedge dg_l) \qquad \text{by 3.4.5}$$
$$= \sum_i (-1)^i df_0\wedge\ldots\hat{df_i}\ldots\wedge df_k\wedge i(H_{f_i})(dg_0\wedge\ldots\wedge dg_l) \qquad \text{by 1.7.3}$$
$$= \sum_i (-1)^i df_0\wedge\ldots\hat{df_i}\ldots\wedge df_k\wedge\sum_j (-1)^j dg_0\wedge\ldots\wedge i(H_{f_i})dg_j\wedge\ldots\wedge dg_l$$
$$= \sum_{i,j} (-1)^{i+j}\{f_i, g_j\}\, df_0\wedge\ldots\hat{df_i}\ldots\wedge df_k\wedge dg_0\wedge\ldots\hat{dg_j}\ldots\wedge dg_l. \qquad \blacksquare$$

4.3. Define $\{\phi, \psi\}^2 := \Theta(H_\phi)\psi = i(H_\phi)d\psi + (-1)^k di(H_\phi)\psi = \{\phi, \psi\}^1 + (-1)^k di(H_\phi)\psi$ for $\phi\in\Omega^k(M)$, $\psi\in\Omega(M)$. For $f, g\in C^\infty(M)$ the bracket $\{f, g\}^2$ is again the usual Poisson bracket of §2.

LEMMA. 1. $\{\phi, \psi\wedge\tau\}^2 = \{\phi, \psi\}^2\wedge\tau + (-1)^{kl}\psi\wedge\{\phi, \tau\}^2$ *for* $\deg(\phi) = k$ *and* $\deg(\psi) = l$.

2. $H(\{\phi, \psi\}^2) = [H_\phi, H_\psi]$.

3. $\{\ ,\ \}^2$ *satisfies the graded Jacobi identity in the form*

$$\{\phi, \{\psi, \tau\}^2\}^2 = \{\{\phi, \psi\}^2, \tau\}^2 + (-1)^{kl}\{\psi, \{\phi, \tau\}^2\}^2.$$

4. $\{f_0 df_1\wedge\ldots\wedge df_k, g_0 dg_1\wedge\ldots\wedge dg_l\}^2 =$

$$= \sum_i (-1)^i \{f_i, g_0\} df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge dg_1 \wedge \dots \wedge dg_l$$
$$-(-1)^k \sum_{i=0}^{k} \sum_{j=1}^{l} (-1)^{i+j} d\{f_i, g_j\} df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge g_0 \wedge dg_1 \wedge \dots dg_j \dots \wedge \widehat{dg_l}$$

5. $\{\ ,\ \}^2$ *is not graded anticommutative, we have*

$$\{\phi, \psi\}^2 + (-1)^{kl} \{\psi, \phi\}^2 = (-1)^k d(i(H_\phi)\psi - (-1)^{(k-1)(l-1)} i(H_\psi)\phi).$$

Proof. 1. $\{\phi, \psi \wedge \tau\}^2 = \Theta(H_\phi)(\psi \wedge \tau) = \Theta(H_\phi)\psi \wedge \tau + (-1)^{kl} \psi \wedge \Theta(H_\phi)\tau$
$= \{\phi, \psi\}^2 \wedge \tau + (-1)^{kl} \psi \wedge \{\phi, \tau\}^2.$

2. $H(\{\phi, \psi\}^2) = H(\{\phi, \psi\}^1 + (-1)^k di(H_\phi)\psi) = H(\{\phi, \psi\}^1) + 0 = [H_\phi, H_\psi]$ by 4.2.3.

3. $\{\phi, \{\psi, \tau\}^2\}^2 = \Theta(H_\phi)\Theta(H_\psi)\tau = [\Theta(H_\phi), \Theta(H_\psi)]\tau + (-1)^{kl}\Theta(H_\psi)\Theta(H_\phi)\tau$
$= \Theta([H_\phi, H_\psi]) + (-1)^{kl}\Theta(H_\psi)\Theta(H_\phi)\tau$ by 1.4
$= \Theta(H(\{\phi, \psi\}^2))\tau + (-1)^{kl}\Theta(H_\psi)\Theta(H_\phi)\tau$ by 2
$= \{\{\phi, \psi\}^2, \tau\}^2 + (-1)^{kl}\{\psi, \{\phi, \tau\}^2\}^2.$

4. $\{f_0 df_1 \wedge \dots \wedge df_k, g_0 dg_1 \wedge \dots \wedge dg_l\}^2 =$
$= \{f_0 df_1 \wedge \dots \wedge df_k, g_0 dg_1 \wedge \dots \wedge dg_l\}^1 + (-1)^k di(H(f_0 df_1 \wedge \dots \wedge df_k))(g_0 dg_1 \wedge \dots \wedge dg_l)$
$= \sum_{i,j} (-1)^{i+j} \{f_i, f_j\} df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge dg_0 \wedge \dots \widehat{dg_j} \dots \wedge dg_l$ by 4.2.4
$+ (-1)^k d(\sum_i (-1)^i df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge i(H_{f_i})(g_0 dg_1 \wedge \dots \wedge dg_l))$ by 3.4.5

The second expression equals in turn

$$(-1)^k d\left(\sum_i (-1)^i df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge g_0 \wedge \sum_{j=1} (-1)^{j-1} dg_1 \wedge \dots \wedge i(H_{f_i}) dg_j \wedge \dots \wedge dg_l\right)$$

$$= (-1)^{k-1} \sum_{i=0}^{k} \sum_{j=1}^{l} (-1)^{i+j} d(g_0\{f_i, g_j\}) df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge dg_1 \wedge \dots \widehat{dg_j} \dots \wedge dg_l$$

$$= -\sum_{i=0}^{k} \sum_{j=1}^{l} (-1)^{i+j} \{f_i, g_j\} df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge dg_0 \wedge \dots \widehat{dg_j} \dots \wedge dg_l$$

$$-(-1)^k \sum_{i=0}^{k} \sum_{j=1}^{l} (-1)^{i+j} d\{f_i, g_j\} df_0 \wedge \dots \widehat{df_i} \dots \wedge df_k \wedge g_0 \wedge dg_1 \wedge \dots \widehat{dg_j} \dots \wedge dg_l.$$

Putting this back we get the formula in the lemma.

5. $\{\phi, \psi\}^2 + (-1)^{kl}\{\psi, \phi\}^2 = \{\phi, \psi\}^1 + (-1)^k di(H_\phi)\psi + (-1)^{kl}(\{\psi, \phi\}^1 + (-1)^l di(H_\psi)\phi)$
$= 0 + d((-1)^k i(H_\phi)\psi + (-1)^{(k-1)l} i(H_\psi)\phi)$ by 4.2.2
$= (-1)^k d(i(H_\phi)\psi - (-1)^{(k-1)(l-1)} i(H_\psi)\phi).$ ∎

4.4. COROLLARY. $\{\phi, \{\psi, \tau\}^1\}^1 - \{\{\phi, \psi\}^1, \tau\}^1 - (-1)^{kl}\{\psi, \{\phi, \tau\}^1\}^1 = -dA(H_\phi, H_\psi)d\tau$, *where* $A(K, L)$ *is from 3.7, and this is not 0 in general if* $\dim M > 2$.

Proof. $\{\phi,\{\psi,\tau\}^2\}^2=\{\phi,\{\psi,\tau\}^2\}^1+(-1)^k di(H_\phi)(\{\psi,\tau\}^2)$
$=\{\phi,\{\psi,\tau\}^1\}^1+\{\phi,(-1)^l di(H_\psi)\tau\}^1+(-1)^k di(H_\phi)\Theta(H_\psi)\tau$
$=\{\phi,\{\psi,\tau\}^1\}^1+(-1)^k di(H_\phi)\Theta(H_\psi)\tau.$
Plug this and similar expressions into formula 4.3.3 and use 1.6, 4.3.2 and 3.7 to derive the result.

To see that this does not vanish in general we use canonical (Darboux-) coordinates (q^i,p_i) on M and choose $\phi=\sum_i p_i\,dq^i$, $\psi=f$, $\tau=g$. Then $H(\phi)=Id_{TU}\in\Omega^1(U,TU)$, where U is the coordinate domain, and $-A(H_\phi,H_f)dg=d\{f,g\}$, which is not 0 in general. ■

4.5. COROLLARY. *If we try to make* $\{\ ,\ \}^2$ *graded anticommutative by force, i.e. if we put* $\{\phi,\psi\}^3:=\{\phi,\psi\}^2-(-1)^{kl}\{\psi,\phi\}^2=\Theta(H_\phi)\psi-(-1)^{kl}\Theta(H_\psi)\phi$, *then* $\{\phi,\{\psi,\tau\}^3\}^3-\{\{\phi,\psi\}^3,\tau\}^3-(-1)^{kl}\{\psi,\{\phi,\tau\}^3\}^3=-(-1)^{kl+km}\{\{\psi,\tau\}^2,\phi\}^2+$ $+(-1)^{kl}\{\{\psi,\phi\}^2,\tau\}^2-(-1)^{km+lm}\{\{\tau,\phi\}^2,\psi\}^2.$

The second expression is a form of the graded Jacobi identity which would be equivalent to 4.3.3 if $\{\ ,\}^2$ *were graded anticommutative; however it is not 0 in general.*

Proof. Write out the definition of $\{\ ,\}^3$ and use three times the graded Jacobi identity 4.3.3 to get the formula. For the last assertion choose again canonical coordinates (q^i,p_i) and $\phi=p_1$, $\psi=\sum_i p_i\,dq^i$, $\tau=(q^1)^2$. Then the first two terms in the second expression vanish and the third one gives $2dq^1$ up to a sign. ■

4.6. THEOREM. *Let* (M,ω) *be a symplectic manifold.*

1. *The following sequence is exact and consists of graded Lie algebras:*

$$\begin{array}{ccccccc} 0\to & H^*(M) & \to & \Omega(M)/B(M) & \xrightarrow{H} & \Omega_{\theta(.)\omega=0}(M;TM), \\ & 0 & & \{\ ,\} & & [\ ,] \end{array}$$

where the brackets are written below the spaces. The graded Poisson bracket $\{\ ,\ \}$ *is induced from* $\{\ ,\ \}^1$ *or* $\{\ ,\}^2$ *on* $\Omega(M)$. *All mappings are homomorphisms of graded Lie algebras.*

2. $$\Omega(M)/B(M)\xrightarrow{H}\Omega_{\theta(.)\omega=0}(M;TM)\xrightarrow{\gamma+P}H^{*+1}(M)\oplus\Gamma(E_\omega)\to 0$$

is also exact, but there is no compatible structure of a graded Lie algebra on $\Gamma(E_\omega)$.

3. *Any* $K\in\Omega^k(M;TM)$ *induces an operator* $\Theta(K):\Omega(M)/B(M)\to\Omega(M)/B(M)$ *of degree k. For* $\bar\phi,\bar\psi\in\Omega(M)/B(M)$ *we have* $\Theta(H_{\bar\phi})\bar\psi=\{\bar\phi,\bar\psi\}$, *so* $\Theta(H_{\bar\phi})\bar\phi=\{\bar\phi,\bar\phi\}$ *(= 0 for even degree of* $\bar\phi$*; «conservation of energy» in this case).*

4. *$\Omega(M)/B(M)$ is no longer an algebra, not even a $\Omega(M)$-module. But it is a graded $Z(M)$-module.*

Proof. 1. See 3.6 and §4.

2. See 3.6, 4.4 and 4.5.

3. Since $[\Theta(K), d] = 0$, $\Theta(K)$ factors to $\Omega(M)/B(M)$.

4. $B(M)$ is a graded ideal in $Z(M)$, but not in $\Omega(M)$. ∎

REFERENCES

[1] A. FROLICHER, A. NIJENHUIS, *Theory of vector valued differential forms*, part I, Indagationes Math. 18 (1956), 338 - 359.

[2] A. WEINSTEIN, *Lectures on symplectic manifolds*, Regional conference series in Math. 29. Amer. Math. Soc. 1977.

Manuscript received: June 10, 1985