GAUGE THEORY FOR FIBER BUNDLES

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Introduction

Gauge theory usually investigates the space of principal connections on a principal fiber bundle (P, p, M, G) and its orbit space under the action of the gauge group (called the moduli space), which is the group of all principal bundle automorphisms of P which cover the identity on the base space M. It is the arena for the Yang-Mills-Higgs equations which allows (with structure group $U(1) \times SU(2)$) a satisfactory unified description of electromagnetic and weak interactions, which was developed by Glashow, Salam, and Weinberg. This electro-weak theory predicted the existence of massive vector particles (the intermediate bosons W^+, W^-, and Z), whose experimental verification renewed the interest of physicists in gauge theories.

On the mathematical side the investigation of self dual and anti self $SU(2)$-connections on 4-manifolds and of their image in the orbit space led to stunning topological results in the topology of 4-manifolds by Donaldson. The moduli space of anti self dual connections can be completed and reworked into a 5 dimensional manifold which is a bordism between the base manifold of the bundle and a simple manifold which depends only on the Poincaré duality form in the second dimensional homology space. Combined with results of Freedman this led to the discovery of exotic differential structures on $\mathbb{R}^4$ and on (supposedly all but S^4) compact algebraic surfaces.

In his codification of a principal connection [Ehresmann, 1951] began with a more general notion of connection on a general fiber bundle (E, p, M, S) with standard fiber S. This was called an Ehresmann connection by some authors, we will call it just a connection. It consists of the specification of a complement to the vertical bundle in a differentiable way, called the horizontal distribution.

One can conveniently describe such a connection as a one form on the total space E with values in the vertical bundle, whose kernel is the horizontal distribution. When combined with another venerable notion, the Frölicher-Nijenhuis bracket for vector valued differential forms (see [Frölicher-Nijenhuis, 1956]) one obtains a very convenient way to describe curvature and Bianchi identity for such connections. Parallel transport along curves in the base space is defined only locally, but one may show that each bundle admits complete connections, whose parallel transport is globally defined (theorem 9.10). For such connections one can define holonomy groups and holonomy Lie algebras and one may prove a far-reaching generalization of the Ambrose Singer theorem: A complete connection tells us whether it is induced from a principal connection on a principal fiber bundle (theorem 12.4).

A bundle (E, p, M, S) without structure group can also be viewed as having the whole diffeomorphism group $\operatorname{Diff}(S)$ of the standard fiber S as structure group, as least when S is compact. We define the non linear frame bundle for E which is a principle fiber bundle over M with structure group $\operatorname{Diff}(S)$ and we show that the theory of connections on E corresponds exactly to the theory of principal connections on this non linear frame bundle (see section 13). The gauge group of the non linear frame bundle turns out to be just the group of all fiber respecting diffeomorphisms of E which cover the identity on M, and one may consider the moduli space $\operatorname{Conn}(E)/\operatorname{Gau}(E)$. There is hope that it can be stratified into smooth manifolds corresponding to conjugacy classes of holonomy groups in $\operatorname{Diff}(S)$, but this will be treated in another paper.

The diffeomorphism group $\operatorname{Diff}(S)$ for a compact manifold S admits a smooth classifying space and a classifying connection: the space of all embeddings of S into a Hilbert space ℓ^2, say, is the total space of a principal bundle with structure group $\operatorname{Diff}(S)$, whose base manifold is the non linear Grassmanian of all submanifolds of ℓ^2 of type (diffeomorphic to) S. The action of $\operatorname{Diff}(S)$ on S leads to the (universal or classifying) associated bundle, which admits a classifying connection: Every S-bundle over M can be realized as the pull back of the classifying bundle by a classifying smooth mapping from M into the non linear Grassmanian, which can be arranged in such a way that it pulls back the classifying connection to a given one on E — this is the Narasimhan-Ramadas procedure for $\operatorname{Diff}(S)$.

Characteristic classes owe their existence to invariant polynomials on the Lie algebra of the structure group like characteristic coefficients of matrices. The Lie algebra of $\operatorname{Diff}(S)$ for compact S is the Lie algebra $\mathfrak{X}(S)$ of vector fields on S. Unfortunately it does not admit any invariants. But there are equivariants like $X \mapsto \mathcal{L}_X\omega$ for some closed form ω on S which can be used to give a sort of Chern-Weil construction of characteristic classes in the cohomology of the base M with local coefficients (see sections 16 and 17).

Finally we discuss some self duality and anti self duality conditions which depend on some fiberwise structure on the bundle like a fiberwise symplectic structure.

This booklet starts with a short description of analysis in infinite dimensions along the lines of Frölicher and Kriegl which makes infinite dimensional differential geometry much simpler than it used to be. We treat smoothness, real analyticity and holomorphy. This part is based on [Kriegl-Michor, 1990b] and the forthcoming book [Kriegl-Michor]. Then we give a more detailed exposition of the theory of manifolds of mappings and the diffeomorphism group. This part is adapted from [Michor, 1980] to the use of the Frölicher-Kriegl calculus. Then we present a careful introduction to the Frölicher-Nijenhuis bracket, to fiber bundles and connections, G-structures and principal connection which emphasizes the construction and recognition of induced connections.

The material in the rest of the book, from section 12 onwards, has been published in [Michor, 1988]. The version here is much more detailed and contains more results.

The material in this booklet is a much extended version of a series of lectures held at the Institute of Physics of the University of Napoli, March 28 – April 1, 1988. I want to thank G. Marmo for his hospitality, his interest in this subject, and for the suggestion to publish this booklet.

Wien, August 6, 1990 P. Michor

1. Notations and conventions

1.1. Definition. A *Lie group* G is a smooth manifold and a group such that the multiplication $\mu : G \times G \to G$ is smooth. Then also the inversion $\nu : G \to G$ turns out to be smooth.

We shall use the following notation:
$\mu : G \times G \to G$, multiplication, $\mu(x, y) = x.y$.
$\lambda_a : G \to G$, left translation, $\lambda_a(x) = a.x$.
$\rho_a : G \to G$, right translation, $\rho_a(x) = x.a$.
$\nu : G \to G$, inversion, $\nu(x) = x^{-1}$.
$e \in G$, the unit element.

If $\mathfrak{g} = T_eG$ is the Lie algebra of G, we use the following notation: $\mathrm{Ad}_G : G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ and so on.

1.2. Let $\ell : G \times M \to M$ be a left action, so $\check{\ell} : G \to \mathrm{Diff}(M)$ is a group homomorphism. Then we have partial mappings $\ell_a : M \to M$ and $\ell^x : G \to M$, given by $\ell_a(x) = \ell^x(a) = \ell(a, x) = a.x$.

For any $X \in \mathfrak{g}$ we define the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ by $\zeta_X(x) = T_e(\ell^x).X = T_{(e,x)}\ell.(X, 0_x)$.

Lemma. *In this situation the following assertions hold:*

(1) $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ *is a linear mapping.*
(2) $T_x(\ell_a).\zeta_X(x) = \zeta_{Ad(a)X}(a.x)$.
(3) $R_X \times 0_M \in \mathfrak{X}(G \times M)$ *is ℓ-related to* $\zeta_X \in \mathfrak{X}(M)$.
(4) $[\zeta_X, \zeta_Y] = -\zeta_{[X,Y]}$. $\square$

1.3. Let $r : M \times G \to M$ be a right action, so $\check{r} : G \to Diff(M)$ is a group anti homomorphism. We will use the following notation: $r^a : M \to M$ and $r_x : G \to M$, given by $r_x(a) = r^a(x) = r(x, a) = x.a$.

For any $X \in \mathfrak{g}$ we define the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ by $\zeta_X(x) = T_e(r_x).X = T_{(x,e)}r.(0_x, X)$.

Lemma. *In this situation the following assertions hold:*

(1) $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ *is a linear mapping.*
(2) $T_x(r^a).\zeta_X(x) = \zeta_{Ad(a^{-1})X}(x.a)$.
(3) $0_M \times L_X \in \mathfrak{X}(M \times G)$ *is r-related to* $\zeta_X \in \mathfrak{X}(M)$.
(4) $[\zeta_X, \zeta_Y] = \zeta_{[X,Y]}$. $\square$

2. Calculus of smooth mappings

2.1. The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces a whole flock of different theories were developed, each of them rather complicated and none really convincing. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. This was the original motivation for the development of a whole new field within general topology, convergence spaces.

Then in 1982, Alfred Frölicher and Andreas Kriegl presented independently the solution to the question for the right differential calculus in infinite dimensions. They joined forces in the further development of the theory and the (up to now) final outcome is the book [Frölicher-Kriegl, 1988], which is the general reference for this section. See also the forthcoming book [Kriegl-Michor].

In this section I will sketch the basic definitions and the most important results of the Frölicher-Kriegl calculus.

2.2. The c^∞-topology. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).
(3) $\{E_B \to E : B$ bounded absolutely convex in $E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

2.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent (completeness) conditions is satisfied:

(1) Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges. This is also called c^∞-complete.
(2) If B is bounded closed absolutely convex, then E_B is a Banach space.
(3) Any Lipschitz curve in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1' = c_2$ (existence of antiderivative).

2.4. Lemma. *Let E be a locally convex space. Then the following properties are equivalent:*

(1) *E is c^∞-complete.*
(2) *If $f : \mathbb{R}^k \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.*
(3) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.*
(4) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.*

Here a mapping $f : \mathbb{R}^k \to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}^n$. f scalarwise C^∞ means that $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

This lemma says that a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

2.5. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mapping from E to F.

For E and F finite dimensional this gives the usual notion of smooth mappings: this has been first proved in [Boman, 1967]. Constant mappings are smooth. Multilinear mappings are smooth if and only if they are bounded. Therefore we denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

2.6. Structure on $C^\infty(E, F)$. We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$.

2.7. Lemma. *For locally convex spaces E and F we have:*

(1) *If F is convenient, then also $C^\infty(E, F)$ is convenient, for any*

E. The space $L(E,F)$ is a closed linear subspace of $C^\infty(E,F)$, so it also convenient.

(2) *If E is convenient, then a curve $c:\mathbb{R}\to L(E,F)$ is smooth if and only if $t\mapsto c(t)(x)$ is a smooth curve in F for all $x\in E$.*

2.8. Theorem. Cartesian closedness. *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E\times F,G)\cong C^\infty(E,C^\infty(F,G)),$$

which is even a diffeomorphism.

Of coarse this statement is also true for c^∞-open subsets of convenient vector spaces.

2.9. Corollary. *Let all spaces be convenient vector spaces. Then the following canonical mappings are smooth.*

$$\begin{aligned}
&\mathrm{ev}: C^\infty(E,F)\times E\to F,\quad \mathrm{ev}(f,x)=f(x)\\
&\mathrm{ins}: E\to C^\infty(F,E\times F),\quad \mathrm{ins}(x)(y)=(x,y)\\
&(\quad)^\wedge: C^\infty(E,C^\infty(F,G))\to C^\infty(E\times F,G)\\
&(\quad)^\vee: C^\infty(E\times F,G)\to C^\infty(E,C^\infty(F,G))\\
&\mathrm{comp}: C^\infty(F,G)\times C^\infty(E,F)\to C^\infty(E,G)\\
&C^\infty(\quad,\quad): C^\infty(F,F')\times C^\infty(E',E)\to C^\infty(C^\infty(E,F),C^\infty(E',F'))\\
&\qquad (f,g)\mapsto (h\mapsto f\circ h\circ g)\\
&\prod: \prod C^\infty(E_i,F_i)\to C^\infty(\prod E_i,\prod F_i)
\end{aligned}$$

2.10. Theorem. *Let E and F be convenient vector spaces. Then the differential operator*

$$\begin{aligned}
d&: C^\infty(E,F)\to C^\infty(E,L(E,F)),\\
df(x)v&:=\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t},
\end{aligned}$$

exists and is linear and bounded (smooth). Also the chain rule holds:

$$d(f\circ g)(x)v=df(g(x))dg(x)v.$$

2.11. Remarks. Note that the conclusion of theorem 2.8 is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more.

If one wants theorem 2.8 to be true and assumes some other obvious properties, then the calculus of smooth functions is already uniquely determined.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

For Fréchet spaces smoothness in the sense described here coincides with the notion C_c^∞ of [Keller, 1974]. This is the differential calculus used by [Michor, 1980], [Milnor, 1984], and [Pressley-Segal, 1986].

A prevalent opinion in contemporary mathematics is, that for infinite dimensional calculus each serious application needs its own foundation. By a serious application one obviously means some application of a hard inverse function theorem. These theorems can be proved, if by assuming enough a priori estimates one creates enough Banach space situation for some modified iteration procedure to converge. Many authors try to build their platonic idea of an a priori estimate into their differential calculus. I think that this makes the calculus inapplicable and hides the origin of the a priori estimates. I believe, that the calculus itself should be as easy to use as possible, and that all further assumptions (which most often come from ellipticity of some nonlinear partial differential equation of geometric origin) should be treated separately, in a setting depending on the specific problem. I am sure that in this sense the Frölicher-Kriegl calculus as presented here and its holomorphic and real analytic offsprings in sections 2 and 3 below are universally usable for most applications.

3. Calculus of holomorphic mappings

3.1. Along the lines of thought of the Frölicher-Kriegl calculus of smooth mappings, in [Kriegl-Nel, 1985] the cartesian closed setting for holomorphic mappings was developed. The right definition of this calculus was already given by [Fantappié, 1930 and 1933]. I will now sketch the basics and the main results. It can be shown that again convenient vector spaces are the right ones to consider. Here we will start with them for the sake of shortness.

3.2. Let E be a complex locally convex vector space whose underlying real space is convenient – this will be called convenient in the sequel. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and let us denote by $C^\omega(\mathbb{D}, E)$ the space of all mappings $c : \mathbb{D} \to E$ such that $\lambda \circ c : \mathbb{D} \to \mathbb{C}$ is holomorphic for each continuous complex-linear functional λ on E. Its elements will be called the *holomorphic curves*.

If E and F are convenient complex vector spaces (or c^∞-open sets therein), a mapping $f : E \to F$ is called *holomorphic* if $f \circ c$ is a holomorphic curve in F for each holomorphic curve c in E. Obviously f is holomorphic if and only if $\lambda \circ f : E \to \mathbb{C}$ is holomorphic for each complex linear continuous functional λ on F. Let $C^\omega(E, F)$ denote the space of all holomorphic mappings from E to F.

3.3. Theorem of Hartogs. *Let E_k for $k = 1, 2$ and F be complex convenient vector spaces and let $U_k \subset E_k$ be c^∞-open. A mapping $f : U_1 \times U_2 \to F$ is holomorphic if and only if it is separably holomorphic (i. e. $f(\quad, y)$ and $f(x, \quad)$ are holomorphic for all $x \in U_1$ and $y \in U_2$).*

This implies also that in finite dimensions we have recovered the usual definition.

3.4 Lemma. *If $f : E \supset U \to F$ is holomorphic then $df : U \times E \to F$ exists, is holomorphic and $\mathbb{C}$-linear in the second variable.*

A multilinear mapping is holomorphic if and only if it is bounded.

3.5 Lemma. *If E and F are Banach spaces and U is open in E, then for a mapping $f : U \to F$ the following conditions are equivalent:*

(1) *f is holomorphic.*
(2) *f is locally a convergent series of homogeneous continuous polynomials.*
(3) *f is $\mathbb{C}$-differentiable in the sense of Fréchet.*

3.6 Lemma. *Let E and F be convenient vector spaces. A mapping $f : E \to F$ is holomorphic if and only if it is smooth and its derivative is everywhere $\mathbb{C}$-linear.*

An immediate consequence of this result is that $C^\omega(E, F)$ is a closed linear subspace of $C^\infty(E_\mathbb{R}, F_\mathbb{R})$ and so it is a convenient vector space if F is one, by 2.7. The chain rule follows from 2.10. The following theorem is an easy consequence of 2.8.

3.7 Theorem. Cartesian closedness. *The category of convenient complex vector spaces and holomorphic mappings between them is cartesian closed, i. e.*

$$C^\omega(E \times F, G) \cong C^\omega(E, C^\omega(F, G)).$$

An immediate consequence of this is again that all canonical structural mappings as in 2.9 are holomorphic.

4. Calculus of real analytic mappings

4.1. In this section I sketch the cartesian closed setting to real analytic mappings in infinite dimension following the lines of the Frölicher-Kriegl calculus, as it is presented in [Kriegl-Michor, 1990a]. Surprisingly enough one has to deviate from the most obvious notion of real analytic curves in order to get a meaningful theory, but again convenient vector spaces turn out to be the right kind of spaces.

4.2. Real analytic curves. Let E be a real convenient vector space with dual E'. A curve $c : \mathbb{R} \to E$ is called *real analytic* if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for each $\lambda \in E'$. It turns out that the set of these curves depends only on the bornology of E.

In contrast a curve is called *topologically real analytic* if it is locally given by power series which converge in the topology of E. They can be extended to germs of holomorphic curves along $\mathbb{R}$ in the complexification $E_{\mathbb{C}}$ of E. If the dual E' of E admits a Baire topology which is compatible with the duality, then each real analytic curve in E is in fact topological real analytic for the bornological topology on E.

4.3. Real analytic mappings. Let E and F be convenient vector spaces. Let U be a c^∞-open set in E. A mapping $f : U \to F$ is called *real analytic* if and only if it is smooth (maps smooth curves to smooth curves) and maps real analytic curves to real analytic curves.

Let $C^\omega(U, F)$ denote the space of all real analytic mappings. We equip the space $C^\omega(U, \mathbb{R})$ of all real analytic functions with the initial topology with respect to the families of mappings

$$C^\omega(U, \mathbb{R}) \xrightarrow{c^*} C^\omega(\mathbb{R}, \mathbb{R}), \text{ for all } c \in C^\omega(\mathbb{R}, U)$$
$$C^\omega(U, \mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R}, \mathbb{R}), \text{ for all } c \in C^\infty(\mathbb{R}, U),$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately as in section 2, and where $C^\omega(\mathbb{R}, \mathbb{R})$ is equipped with the final topology with respect to the embeddings (restriction mappings) of all spaces of holomorphic mappings from a neighborhood V of $\mathbb{R}$ in $\mathbb{C}$ mapping $\mathbb{R}$ to $\mathbb{R}$, and each of these spaces carries the topology of compact convergence.

Furthermore we equip the space $C^\omega(U, F)$ with the initial topology with respect to the family of mappings

$$C^\omega(U, F) \xrightarrow{\lambda_*} C^\omega(U, \mathbb{R}), \text{ for all } \lambda \in F'.$$

It turns out that this is again a convenient space.

4.4. Theorem. *In the setting of 4.3 a mapping $f : U \to F$ is real analytic if and only if it is smooth and is real analytic along each affine line in E.*

4.5. Theorem Cartesian closedness. *The category of convenient spaces and real analytic mappings is cartesian closed. So the equation*

$$C^\omega(U, C^\omega(V), F) \cong C^\omega(U \times V, F)$$

is valid for all c^∞-open sets U in E and V in F, where E, F, and G are convenient vector spaces.

This implies again that all structure mappings as in 2.9 are real analytic. Furthermore the differential operator

$$d : C^\omega(U, F) \to C^\omega(U, L(E, F))$$

exists, is unique and real analytic. Multilinear mappings are real analytic if and only if they are bounded. Powerful real analytic uniform boundedness principles are available.

5. Infinite dimensional manifolds

5.1. In this section I will concentrate on two topics: Smooth partitions of unity, and several kinds of tangent vectors. The material is taken from [Kriegl-Michor, 1990b].

5.2. In the usual way we define manifolds by gluing c^∞-open sets in convenient vector spaces via smooth (holomorphic, real analytic) diffeomorphisms. Then we equip them with the identification topology with respect to the c^∞-topologies on all modeling spaces. We require some properties from this topology, like Hausdorff and regular (which here is not a consequence of Hausdorff).

Mappings between manifolds are smooth (holomorphic, real analytic), if they have this property when composed which any chart mappings.

5.3. Lemma. *A manifold M is metrizable if and only if it is paracompact and modeled on Fréchet spaces.*

5.4. Lemma. *For a convenient vector space E the set $C^\infty(M,E)$ of smooth E-valued functions on a manifold M is again a convenient vector space. Likewise for the real analytic and holomorphic case.*

5.5. Theorem. Smooth partitions of unity. *If M is a smooth manifold modeled on convenient vector spaces admitting smooth bump functions and $\mathcal{U}$ is a locally finite open cover of M, then there exists a smooth partition of unity $\{\varphi_U : U \in \mathcal{U}\}$ with carr/supp$(\varphi_U) \subset U$ for all $U \in \mathcal{U}$.*

If M is in addition paracompact, then this is true for every open cover $\mathcal{U}$ of M.

Convenient vector spaces which are nuclear admit smooth bump functions.

5.6. The tangent spaces of a convenient vector space E. Let $a \in E$. A *kinematic tangent vector* with foot point a is simply a pair (a, X) with $X \in E$. Let $T_aE = E$ be the space of all kinematic tangent vectors with foot point a. It consists of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \to E$ with $c(0) = a$, which explains the choice of the name kinematic.

For each open neighborhood U of a in E (a, X) induces a linear mapping $X_a : C^\infty(U,\mathbb{R}) \to \mathbb{R}$ by $X_a(f) := df(a)(X_a)$, which is continuous for the convenient vector space topology on $C^\infty(U,\mathbb{R})$, and satisfies $X_a(f \cdot g) = X_a(f) \cdot g(a) + f(a) \cdot X_a(g)$, so it is a *continuous derivation over ev_a*. The value $X_a(f)$ depends only on the germ of f at a.

An *operational tangent vector* of E with foot point a is a bounded derivation $\partial : C_a^\infty(E,\mathbb{R}) \to \mathbb{R}$ over ev_a. Let D_aE be the vector space of all these derivations. Any $\partial \in D_aE$ induces a bounded derivation $C^\infty(U,\mathbb{R}) \to \mathbb{R}$ over ev_a for each open neighborhood U of a in E. So the vector space D_aE is a closed linear subspace of the convenient vector space $\prod_U L(C^\infty(U,\mathbb{R}),\mathbb{R})$. We equip D_aE with the induced convenient vector space structure. Note that the spaces D_aE are isomorphic for all $a \in E$.

Example. Let $Y \in E''$ be an element in the bidual of E. Then for each $a \in E$ we have an operational tangent vector $Y_a \in D_aE$, given by $Y_a(f) := Y(df(a))$. So we have a canonical injection $E'' \to D_aE$.

Example. Let $\ell : L^2(E;\mathbb{R}) \to \mathbb{R}$ be a bounded linear functional which vanishes on the subset $E' \otimes E'$. Then for each $a \in E$ we have an operational tangent vector $\partial_\ell^2|_a \in D_aE$ given by $\partial_\ell^2|_a(f) := \ell(d^2f(a))$, since

$$\begin{aligned}
&\ell(d^2(fg)(a)) = \\
&\qquad = \ell(d^2f(a)g(a) + df(a)\otimes dg(a) + dg(a)\otimes df(a) + f(a)d^2g(a)) \\
&\qquad = \ell(d^2f(a))g(a) + 0 + f(a)\ell(d^2g(a)).
\end{aligned}$$

5.7. Lemma. *Let $\ell \in L^k_{sym}(E;\mathbb{R})'$ be a bounded linear functional which vanishes on the subspace*

$$\sum_{i=1}^{k-1} L^i_{sym}(E;\mathbb{R}) \vee L^{k-i}_{sym}(E;\mathbb{R})$$

of decomposable elements of $L^k_{sym}(E;\mathbb{R})$. Then ℓ defines an operational tangent vector $\partial_\ell^k|_a \in D_aE$ for each $a \in E$ by

$$\partial_\ell^k|_a(f) := \ell(d^kf(a)).$$

The linear mapping $\ell \mapsto \partial_\ell^k|_a$ is an embedding onto a topological direct summand $D_a^{(k)}E$ of D_aE. Its left inverse is given by $\partial \mapsto (\Phi \mapsto \partial((\Phi \circ diag)(a+\quad)))$. The sum $\sum_{k>0} D_a^{(k)}E$ in D_aE is a direct one.

5.8. Lemma. *If E is an infinite dimensional Hilbert space, all operational tangent space summands $D_0^{(k)}E$ are not zero.*

5.9. Definition. A convenient vector space is said to have the *(bornological) approximation property* if $E' \otimes E$ is dense in $L(E,E)$ in the bornological locally convex topology.

The following spaces have the bornological approximation property: $\mathbb{R}^{(\mathbb{N})}$, nuclear Fréchet spaces, nuclear (LF)spaces.

5.10 Theorem. Tangent vectors as derivations. *Let E be a convenient vector space which has the approximation property. Then we have $D_aE = D_a^{(1)}E \cong E''$. So if E is in addition reflexive, each operational tangent vector is a kinematic one.*

5.11. The kinematic tangent bundle TM of a manifold M is constructed by gluing all the kinematic tangent bundles of charts with the help of the kinematic tangent mappings (derivatives) of the chart changes. $TM \to M$ is a vector bundle and $T : C^\infty(M,N) \to C^\infty(TM,TN)$ is well defined and has the usual properties.

5.12. The operational tangent bundle DM of a manifold M is constructed by gluing all operational tangent spaces of charts. Then $\pi_M : DM \to M$ is again a vector bundle which contains the kinematic tangent bundle TM embeds as a splitting subbundle. Also for each $k \in \mathbb{N}$ the same gluing construction as above gives us tangent bundles $D^{(k)}M$ which are splitting subbundles of DM. The mappings $D^{(k)} : C^\infty(M,N) \to C^\infty(D^{(k)}M, D^{(k)}N)$ are well defined for all k including ∞ and have the usual properties.

Note that for manifolds modeled on reflexive spaces having the bornological approximation property the operational and the kinematic tangent bundles coincide.

6. Manifolds of mappings

6.1. Theorem. Manifold structure of $C^\infty(M,N)$. *Let M and N be smooth finite dimensional manifolds. Then the space $C^\infty(M,N)$ of all smooth mappings from M to N is a smooth manifold, modeled on spaces $C^\infty(f^*TN)$ of smooth sections of pullback bundles along $f : M \to N$ over M.*

A careful description of this theorem (but without the help of the Frölicher-Kriegl calculus) can be found in [Michor, 1980]. I include a proof of this result here because the result is important and the proof is much simpler now with the help of the Frölicher Kriegl-calculus.

Proof. Choose a smooth Riemannian metric on N. Let $\exp : TN \supseteq U \to N$ be the smooth exponential mapping of this Riemannian metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_N, \exp) : U \to N \times N$ is a smooth diffeomorphism onto an open neighborhood V of the diagonal.

For $f \in C^\infty(M,N)$ we consider the pullback vector bundle

$$\begin{tikzcd} M \times_N TN \arrow[r, equal] & f^*TN \arrow[r, "\pi_N^* f"] \arrow[d, "f^*\pi_N"'] & TN \arrow[d, "\pi_N"] \\ & M \arrow[r, "f"] & N. \end{tikzcd}$$

Then $C^\infty(f^*TN)$ is canonically isomorphic to the space $C^\infty_f(M,TN) := \{h \in C^\infty(M,TN) : \pi_N \circ h = f\}$ via $s \mapsto (\pi_N^* f) \circ s$ and $(Id_M, h) \leftarrow h$. We consider the space $C^\infty_c(f^*TN)$ of all smooth sections with compact support and equip it with the inductive limit topology

$$C^\infty_c(f^*TN) = \operatorname{inj}\lim_K C^\infty_K(f^*TN),$$

where K runs through all compact sets in M and each of the spaces $C^\infty_K(f^*TN)$ is equipped with the topology of uniform convergence (on K) in all derivatives separately. For $f,\ g \in C^\infty(M,N)$ we write $f \sim g$ if f and g agree off some compact subset in M. Now let

$$\begin{gathered} U_f := \{g \in C^\infty(M,N) : (f(x), g(x)) \in V \text{ for all } x \in M, g \sim f\}, \\ u_f : U_f \to C^\infty_c(f^*TN), \\ u_f(g)(x) = (x, \exp^{-1}_{f(x)}(g(x))) = (x, ((\pi_N, \exp)^{-1} \circ (f,g))(x)). \end{gathered}$$

Then u_f is a bijective mapping from U_f onto the set $\{s \in C_c^\infty(f^*TN) : s(M) \subseteq f^*U\}$, whose inverse is given by $u_f^{-1}(s) = \exp \circ (\pi_N^* f) \circ s$, where we view $U \to N$ as fiber bundle. The set $u_f(U_f)$ is open in $C_c^\infty(f^*TN)$ for the topology described above.

Now we consider the atlas $(U_f, u_f)_{f \in C^\infty(M,N)}$ for $C^\infty(M,N)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C_c^\infty(g^*TN)$ by

$$\begin{aligned}(u_f \circ u_g^{-1})(s) &= (Id_M, (\pi_N, \exp)^{-1} \circ (f, \exp \circ (\pi_N^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s),\end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)})))$ is a smooth diffeomorphism $\tau_g : g^*TN \supseteq g^*U \to (g \times Id_N)^{-1}(V) \subseteq M \times N$ which is fiber respecting over M.

Smooth curves in $C_c^\infty(f^*TN)$ are just smooth sections of the bundle $pr_2^* f^*TN \to \mathbb{R} \times M$, which have compact support in M locally in $\mathbb{R}$. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and obviously maps smooth curves to smooth curves, therefore it is also smooth.

Finally we put the identification topology from this atlas on the space $C^\infty(M,N)$, which is obviously finer than the compact open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the smooth structure does not depend on the choice of the smooth Riemannian metric on N. □

6.2. Lemma. Smooth curves in $C^\infty(M,N)$. *Let M and N be smooth finite dimensional manifolds.*

Then the smooth curves c in $C^\infty(M,N)$ correspond exactly to the smooth mappings $\hat{c} \in C^\infty(\mathbb{R} \times M, N)$ which satisfy the following property:

(1) *For each compact interval $[a,b] \subset \mathbb{R}$ there is a compact subset $K \subset M$ such that the restriction $\hat{c}|([a,b] \times (M \setminus K))$ is constant.*

Proof. Since $\mathbb{R}$ is locally compact property (1) is equivalent to

(2) For each $t \in \mathbb{R}$ there is an open neighborhood U of t in $\mathbb{R}$ and a compact $K \subset M$ such that the restriction $\hat{c}|(U \times (M \setminus K))$ is constant.

Since this is a local condition on $\mathbb{R}$ and since smooth curves in $C^\infty(M,N)$ locally take values in charts as in the proof of theorem 6.1 it suffices to describe the smooth curves in the space $C_c^\infty(E)$ of sections with compact support of a vector bundle (E, p, M, V). It is equipped with the inductive limit topology

$$C_c^\infty(E) = \operatorname{inj\,lim}_K C_K^\infty(E),$$

where K runs through all compact subsets of M, and where $C^\infty_K(E)$ denotes the space of sections of E with support in K, equipped with the compact C^∞-topology (the topology of uniform convergence on compact subsets, in all derivatives separately). Since this injective limit is strict (each bounded subset is contained and bounded in some step) smooth curves in $C^\infty_c(E)$ locally factor to smooth maps in $C^\infty_K(E)$ for some K.

Let $(U_\alpha, \psi_\alpha : E \to U_\alpha \times V)$ be a vector bundle atlas for E. Then

$$C^\infty_K(E) \ni s \mapsto (pr_2 \circ \psi_\alpha \circ s|U_\alpha)_\alpha \in \prod_\alpha C^\infty(U_\alpha, V)$$

is a linear embedding onto a closed linear subspace of the product, where the factor spaces on the right hand side carry the topology used in 2.6. So a curve in $C^\infty_K(E)$ is smooth if and only if each its "coordinates" in $C^\infty(U_\alpha, V)$ is smooth. By cartesian closedness 2.8 this is the case if and only if their canonically associates are smooth $\mathbb{R} \times U_\alpha \to V$. these fit together to a smooth section of the pullback bundle $pr_2^* E \to \mathbb{R} \times M$ with support in $\mathbb{R} \times K$. □

6.3. Theorem. C^ω-manifold structure of $C^\omega(M,N)$. *Let M and N be real analytic manifolds, let M be compact. Then the space $C^\omega(M,N)$ of all real analytic mappings from M to N is a real analytic manifold, modeled on spaces $C^\omega(f^*TN)$ of real analytic sections of pullback bundles along $f : M \to N$ over M.*

The proof is a variant of the proof of 6.1, using a real analytic Riemannian metric. It can also be found in [Kriegl-Michor, 1990a].

6.4. Theorem. C^ω-manifold structure on $C^\infty(M,N)$. *Let M and N be real analytic manifolds, with M compact. Then the smooth manifold $C^\infty(M,N)$ with the structure from 6.1 is even a real analytic manifold.*

Proof. For a fixed real analytic exponential mapping on N the charts (U_f, u_f) from 6.1 for $f \in C^\omega(M,N)$ form a smooth atlas for $C^\infty(M,N)$, since $C^\omega(M,N)$ is dense in $C^\infty(M,N)$ by [Grauert, 1958, Proposition 8]. The chart changings $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ are real analytic: this follows from a careful description of the set of real analytic curves into $C^\infty(f^*TN)$. See again [Kriegl-Michor, 1990a] for more details. □

6.5 Remark. If M is not compact, $C^\omega(M,N)$ is dense in $C^\infty(M,N)$ for the Whitney-C^∞-topology by [Grauert, 1958, Prop. 8]. This is not the case for the $\mathcal{D}$-topology from [Michor, 1980], in which $C^\infty(M,N)$ is a smooth manifold. The charts U_f for $f \in C^\infty(M,N)$ do not cover $C^\infty(M,N)$.

6.6. Theorem. Smoothness of composition. *If M, N are finite dimensional smooth manifolds, then the evaluation mapping* $\mathrm{ev} : C^\infty(M,N) \times M \to N$ *is smooth.*

If P is another smooth manifold, then the composition mapping

$$\mathrm{comp} : C^\infty(M,N) \times C^\infty_{\mathrm{prop}}(P,M) \to C^\infty(P,N)$$

is smooth, where $C^\infty_{\mathrm{prop}}(P,M)$ denote the space of all proper smooth mappings $P \to M$ (i.e. compact sets have compact inverse images). This is open in $C^\infty(P,M)$.

In particular $f_* : C^\infty(M,N) \to C^\infty(M,N')$ and $g^* : C^\infty(M,N) \to C^\infty(P,N)$ are smooth for $f \in C^\infty(N,N')$ and $g \in C^\infty_{\mathrm{prop}}(P,M)$.

The corresponding statement for real analytic mappings is also true, but P and M have to be compact.

Proof. Using the description of smooth curves in $C^\infty(M,N)$ given in 6.2 we immediately see that $(\mathrm{ev} \circ (c_1,c_2)) = \hat{c}_1(t, c_1(t))$ is smooth for each smooth $(c_1,c_2) : \mathbb{R} \to C^\infty(M,N) \times M$, so ev is smooth as claimed.

The space of proper mappings $C^\infty_{\mathrm{prop}}(P,M)$ is open in the manifold $C^\infty(P,M)$ since changing a mapping only on a compact set does not change its property of being proper. Let $(c_1,c_2) : \mathbb{R} \to C^\infty(M,N) \times C^\infty_{\mathrm{prop}}(P,M)$ be a smooth curve. Then we have $(\mathrm{comp} \circ (c_1,c_2))(t)(p) = \hat{c}_1(t, \hat{c}_2(t,p))$ and one may check that this has again property (1), so it is a smooth curve in $C^\infty(P,N)$. Thus comp is smooth.

The proof for real analytic manifolds is similar. $\square$

6.7. Theorem. *Let M and N be smooth manifolds. Then the infinite dimensional smooth vector bundles $TC^\infty(M,N)$ and $C^\infty_c(M,TN) \subset C^\infty(M,TN)$ over $C^\infty(M,N)$ are canonically isomorphic. The same assertion is true for $C^\omega(M,N)$, if M is compact.*

Proof. Here by $C^\infty_c(M,TN)$ we denote the space of all smooth mappings $f : M \to TN$ such that $f(x) = 0_{\pi_M f(x)}$ for $x \notin K_f$, a suitable compact subset of M (equivalently, such that the associated section of the pull back bundle $(\pi_M \circ f)^* TN \to M$ has compact support).

One can check directly that the atlas from 6.1 for $C^\infty(M,N)$ induces an atlas for $TC^\infty(M,N)$ which is equivalent to that for $C^\infty(M,TN)$ via some natural identifications in TTN. This is carried out in great detail in [Michor, 1980, 10.19]. $\square$

6.8. Theorem. Exponential law. *Let $\mathcal{M}$ be a (possibly infinite dimensional) smooth manifold, and let M and N be finite dimensional smooth manifolds.*

Then we have a canonical embedding

$$C^\infty(\mathcal{M}, C^\infty(M,N)) \subseteq C^\infty(\mathcal{M}\times M \to N),$$

where the image in the right hand side consists of all smooth mappings $f : \mathcal{M}\times M \to N$ *which satisfy the following property*

(1) *For each point in* $\mathcal{M}$ *there is an open neighborhood* $\mathcal{U}$ *and a compact set* $K \subset M$ *such that the restriction* $f|(\mathcal{U}\times(M\setminus K))$ *is constant.*

We have equality if and only if M *is compact.*

If M *and* N *are real analytic manifolds with* M *compact we have*

$$C^\omega(\mathcal{M}, C^\omega(M,N)) = C^\omega(\mathcal{M}\times M, N)$$

for each real analytic (possibly infinite dimensional) manifold.

Proof. This follows directly from the description of all smooth curves in $C^\infty(M,N)$ given in the proof of 6.6. $\square$

7. Diffeomorphism groups

7.1. Theorem. Diffeomorphism group. *For a smooth manifold M the group* $\operatorname{Diff}(M)$ *of all smooth diffeomorphisms of M is an open submanifold of $C^\infty(M,M)$, composition and inversion are smooth.*

The Lie algebra of the smooth infinite dimensional Lie group $\operatorname{Diff}(M)$ *is the convenient vector space $C^\infty_c(TM)$ of all smooth vector fields on M with compact support, equipped with the negative of the usual Lie bracket. The exponential mapping* $\operatorname{Exp} : C^\infty_c(TM) \to \operatorname{Diff}^\infty(M)$ *is the flow mapping to time 1, and it is smooth.*

For a compact real analytic manifold M the group $\operatorname{Diff}^\omega(M)$ *of all real analytic diffeomorphisms is a real analytic Lie group with Lie algebra $C^\omega(TM)$ and with real analytic exponential mapping.*

Proof. It is well known that the space $\operatorname{Diff}(M)$ of all diffeomorphisms of M is open in $C^\infty(M,M)$ for the Whitney C^∞-topology. Since the topology we use on $C^\infty(M,M)$ is finer, $\operatorname{Diff}(M)$ is an open submanifold of $C^\infty_{\text{prop}}(M,M)$. The composition is smooth by theorem 6.6. To show that the inversion inv is smooth, we consider a smooth curve $c : \mathbb{R} \to \operatorname{Diff}(M) \subset C^\infty(M,M)$. Then the mapping $\hat{c} : \mathbb{R} \times M \to M$ satisfies 6.6.(1) and $(\operatorname{inv} \circ c)\hat{}$ fulfills the finite dimensional implicit equation $\hat{c}(t, (\operatorname{inv} \circ c)\hat{}(t,m)) = m$ for all $t \in \mathbb{R}$ and $m \in M$. By the finite dimensional implicit function theorem $(\operatorname{inv} \circ c)\hat{}$ is smooth in (t,m). Property 6.6.(1) is obvious. So inv maps smooth curves to smooth curves and is thus smooth.

This proof is by far simpler than the original one, see [Michor, 1980], and shows the power of the Frölicher-Kriegl calculus.

By the chart structure from 6.1, or directly from theorem 6.7we see that the tangent space $T_e \operatorname{Diff}(M)$ equals the space $C^\infty_c(TM)$ of all vector fields with compact support. Likewise $T_f \operatorname{Diff}(M) = C^\infty_c(f^*TM)$ which we identify with the space of all vector fields with compact support along the diffeomorphism f. Right translation ρ_f is given by $\rho_f(g) = f^*(g) = g \circ f$, thus $T(\rho_f).X = X \circ f$ and for the flow Fl^X_t of the vector field with compact support X we have $\frac{d}{dt} Fl^X_t = X \circ Fl^X_t = T(\rho_{\operatorname{Fl}^X_t}).X$. So the one parameter group $t \mapsto \operatorname{Fl}^X_t \in \operatorname{Diff}(M)$ is the integral curve of the right invariant vector field $R_X : f \mapsto T(\rho_f).X = X \circ f$ on $\operatorname{Diff}(M)$. Thus the exponential mapping of the diffeomorphism group is given by $\operatorname{Exp} = \operatorname{Fl}_1 : \mathfrak{X}_c(M) \to \operatorname{Diff}(M)$. To show that is smooth we consider a smooth curve in $\mathfrak{X}_c(M)$, i. e. a time dependent vector field with compact support X_t. We may view it as a complete vector field $(0_t, X_t)$ on $\mathbb{R} \times M$ whose smooth flow respects the level surfaces $\{t\} \times M$ and is smooth. Thus Exp maps smooth curves to smooth curves and is smooth itself. Again one may compare this simple proof with the original one in [Michor, 1983, section 4].

Let us finally compute the Lie bracket on $\mathfrak{X}_c(M)$ viewed as the Lie algebra of $\operatorname{Diff}(M)$.

$$\begin{aligned}
\operatorname{Ad}(\operatorname{Exp}(sX))Y &= \tfrac{\partial}{\partial t}\big|_0 \operatorname{Exp}(sX)\circ\operatorname{Exp}(tY)\circ\operatorname{Exp}(-sX)\\
&= T(\operatorname{Fl}^X_t)\circ Y\circ \operatorname{Fl}^X_{-t}\\
&= (\operatorname{Fl}^X_{-t})^*Y, \qquad \text{thus}\\
\tfrac{\partial}{\partial t}\big|_0 \operatorname{Ad}(\operatorname{Exp}(tX))Y &= \tfrac{\partial}{\partial t}\big|_0 (\operatorname{Fl}^X_{-t})^*Y\\
&= -[X,Y],
\end{aligned}$$

the negative of the usual Lie bracket on $\mathfrak{X}_c(M)$. To understand this contemplate 1.2.(4). $\square$

7.2. Remarks. The group $\operatorname{Diff}(M)$ of smooth diffeomorphisms does not carry any real analytic Lie group structure by [Milnor, 1984, 9.2], and it has no complexification in general, see [Pressley-Segal, 1986, 3.3]. The mapping

$$Ad\circ\operatorname{Exp}: C^\infty_c(TM)\to\operatorname{Diff}(M)\to L(C^\infty(TM),C^\infty(TM))$$

is not real analytic, see [Michor, 1983, 4.11].

For $x\in M$ the mapping $\operatorname{ev}_x\circ\operatorname{Exp}: C^\infty_c(TM)\to\operatorname{Diff}(M)\to M$ is not real analytic since $(ev_x\circ\operatorname{Exp})(tX)=Fl^X_t(x)$, which is not real analytic in t for general smooth X.

The exponential mapping $\operatorname{Exp}: C^\infty_c(TM)\to\operatorname{Diff}(M)$ is in a very strong sense not surjective: In [Grabowski, 1988] it is shown, that $\operatorname{Diff}(M)$ contains an arcwise connected free subgroup on $2^{\aleph_0}$ generators which meets the image of Exp only at the identity.

The real analytic Lie group $\operatorname{Diff}^\omega(M)$ is *regular* in the sense of [Milnor, 1984. 7.6], who weakened the original concept of [Omori, 1982]. This condition means that the mapping associating the evolution operator to each time dependent vector field on M is smooth. It is even real analytic, compare the proof of theorem 7.2.

By theorem 6.6 the left action $\operatorname{ev}:\operatorname{Diff}(M)\times M\to M$ is smooth. If $\dim M\ge 2$ this action is *n-transitive* for each $n\in\mathbb{N}$, i. e. for two ordered finite n-tupels $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ of pair wise different elements of M there is a diffeomorphism $f\in\operatorname{Diff}(M)$ with $f(x_i)=y_i$ for each i. To see this connect x_i with y_i by a simple smooth curve $c_i:[0,1]\to M$ which is an embedding, such that the images are pair wise disjoint. Using tubular neighborhoods or submanifold charts one can show that then there is a vector field with compact support $X \in \mathfrak{X}_c(M)$ which restricts to $c_i'(t)$ along $c_i(t)$. The diffeomorphism $\operatorname{Exp}(X) = \operatorname{Fl}_1^X$ then maps x_i to y_i.

In contrast to this one knows from [Omori, 1978] that a Banach Lie group acting effectively on a finite dimensional manifold is necessarily finite dimensional. So there is no way to model the diffeomorphism group on Banach spaces as a manifold. There is, however, the possibility to view $\operatorname{Diff}(M)$ as an ILH-group (i. e. inverse limit of Hilbert manifolds) which sometimes permits to use an implicit function theorem. See [Omori, 1974] for this.

7.3. Theorem. Principal bundle of embeddings. *Let M and N be smooth manifolds.*

Then the set $\operatorname{Emb}(M,N)$ of all smooth embeddings $M \to N$ is an open submanifold of $C^\infty(M,N)$. It is the total space of a smooth principal fiber bundle with structure group $\operatorname{Diff}(M)$, whose smooth base manifold is the space $B(M,N)$ of all submanifolds of N of type M.

The open subset $\operatorname{Emb}_{\text{prop}}(M,N)$ of proper (equivalently closed) embeddings is saturated under the $\operatorname{Diff}(M)$-action and is thus the total space of the restriction of the principal bundle to the open submanifold $B_{\text{closed}}(M,N)$ of $B(M,N)$ consisting of all closed submanifolds of N of type M.

In the real analytic case this theorem remains true provided that M is supposed to be compact, see [Kriegl-Michor, 1990a], section 6.

Proof. Let us fix an embedding $i \in \operatorname{Emb}(M,N)$. Let g be a fixed Riemannian metric on N and let $\exp^N$ be its exponential mapping. Then let $p : \mathcal{N}(i) \to M$ be the *normal bundle* of i, defined in the following way: For $x \in M$ let $\mathcal{N}(i)_x := (T_x i(T_x M))^\perp \subset T_{i(x)}N$ be the g-orthogonal complement in $T_{i(x)}N$. Then

$$\begin{array}{ccc} \mathcal{N}(i) & \xrightarrow{\bar{i}} & TN \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[i]{} & N \end{array}$$

is an injective vector bundle homomorphism over i.

Now let $U^i = U$ be an open neighborhood of the zero section of $\mathcal{N}(i)$ which is so small that $(\exp^N \circ \bar{i})|U : U \to N$ is a diffeomorphism onto its image which describes a tubular neighborhood of the submanifold $i(M)$. Let us consider the mapping

$$\tau = \tau^i := (\exp^N \circ \bar{i})|U : \mathcal{N}(i) \supset U \to N,$$

a diffeomorphism onto its image, and the open set in $\operatorname{Emb}(M,N)$ which will serve us as a saturated chart,

$$\mathcal{U}(i) := \{j \in \operatorname{Emb}(M,N) : j(M) \subseteq \tau^i(U^i), j \sim i\},$$

where $j \sim i$ means that $j = i$ off some compact set in M. Then by [Michor, 1980, section 4] the set $\mathcal{U}(i)$ is an open neighborhood of i in $\operatorname{Emb}(M,N)$. For each $j \in \mathcal{U}(i)$ we define

$$\begin{aligned}
&\varphi_i(j) : M \to U^i \subseteq \mathcal{N}(i),\\
&\varphi_i(j)(x) := (\tau^i)^{-1}(j(x))
\end{aligned}$$

Then $\varphi_i = ((\tau^i)^{-1})_* : \mathcal{U}(i) \to C^\infty(M,\mathcal{N}(i))$ is a smooth mapping which is bijective onto the open set

$$\mathcal{V}(i) := \{h \in C^\infty(M,\mathcal{N}(i)) : h(M) \subseteq U^i, h \sim 0\}$$

in $C^\infty(M,\mathcal{N}(i))$. Its inverse is given by the smooth mapping $\tau^i_* : h \mapsto \tau^i \circ h$.

We have $\tau^i_*(h \circ f) = \tau^i_*(h) \circ f$ for those $f \in \operatorname{Diff}(M)$ which are near enough to the identity so that $h \circ f \in \mathcal{V}(i)$. We consider now the open set

$$\{h \circ f : h \in \mathcal{V}(i), f \in \operatorname{Diff}(M)\} \subseteq C^\infty((M,U^i)).$$

Obviously we have a smooth mapping from it into $C^\infty_c(U^i) \times \operatorname{Diff}(M)$ given by $h \mapsto (h \circ (p \circ h)^{-1}, p \circ h)$, where $C^\infty_c(U^i)$ is the space of sections with compact support of $U^i \to M$. So if we let $\mathcal{Q}(i) := \tau^i_*(C^\infty_c(U^i) \cap \mathcal{V}(i)) \subset \operatorname{Emb}(M,N)$ we have

$$\mathcal{W}(i) := \mathcal{U}(i) \circ \operatorname{Diff}(M) \cong \mathcal{Q}(i) \times \operatorname{Diff}(M) \cong (C^\infty_c(U^i) \cap \mathcal{V}(i)) \times \operatorname{Diff}(M),$$

since the action of $\operatorname{Diff}(M)$ on i is free. Furthermore $\pi|\mathcal{Q}(i) : \mathcal{Q}(i) \to \operatorname{Emb}(M,N)/\operatorname{Diff}(M)$ is bijective onto an open set in the quotient.

We now consider $\varphi_i \circ (\pi|\mathcal{Q}(i))^{-1} : \pi(\mathcal{Q}(i)) \to C^\infty(U^i)$ as a chart for the quotient space. In order to investigate the chart change let $j \in \operatorname{Emb}(M,N)$ be such that $\pi(\mathcal{Q}(i)) \cap \pi(\mathcal{Q}(j)) \neq \emptyset$. Then there is an immersion $h \in \mathcal{W}(i) \cap \mathcal{Q}(j)$, so there exists a unique $f_0 \in \operatorname{Diff}(M)$ (given by $f_0 = p \circ \varphi_i(h)$) such that $h \circ f_0^{-1} \in \mathcal{Q}(i)$. If we consider $j \circ f_0^{-1}$ instead of j and call it again j, we have $\mathcal{Q}(i) \cap \mathcal{Q}(j) \neq \emptyset$ and consequently $\mathcal{U}(i) \cap \mathcal{U}(j) \neq \emptyset$. Then the chart change is given as follows:

$$\begin{gathered}
\varphi_i \circ (\pi|\mathcal{Q}(i))^{-1} \circ \pi \circ (\tau^j)_* : C^\infty_c(U^j) \to C^\infty_c(U^i)\\
s \mapsto \tau^j \circ s \mapsto \varphi_i(\tau^j \circ s) \circ (p^i \circ \varphi_i(\tau^j \circ s))^{-1}.
\end{gathered}$$

This is of the form $s \mapsto \beta \circ s$ for a locally defined diffeomorphism $\beta : \mathcal{N}(j) \to \mathcal{N}(i)$ which is not fiber respecting, followed by $h \mapsto h \circ (p^i \circ h)^{-1}$. Both composants are smooth by the general properties of manifolds of mappings. So the chart change is smooth.

We show that the quotient space $B(M,N) = \operatorname{Emb}(M,N)/\operatorname{Diff}(M)$ is Hausdorff. Let i, $j \in \operatorname{Emb}(M,N)$ with $\pi(i) \neq \pi(j)$. Then $i(M) \neq j(M)$ in N for otherwise put $i(M) = j(M) =: L$, a submanifold of N; the mapping $i^{-1} \circ j : M \to L \to M$ is then a diffeomorphism of M and $j = i \circ (i^{-1} \circ j) \in i \circ \operatorname{Diff}(M)$, so $\pi(i) = \pi(j)$, contrary to the assumption.

Now we distinguish two cases.

Case 1. We may find a point $y_0 \in i(M) \setminus j(M)$, say, which is not a cluster point of $j(M)$. We choose an open neighborhood V of y_0 in N and an open neighborhood W of $j(M)$ in N such that $V \cap W = \emptyset$. Let $\mathcal{V} := \{k \in \operatorname{Emb}(M,N) : k(M) \cap V\}$ $\mathcal{W} := \{k \in \operatorname{Emb}(M,N) : k(M) \subset W\}$. Then $\mathcal{V}$ is obviously open in $\operatorname{Emb}(M,N)$ and $\mathcal{V}$ is even open in the coarser compact open topology. Both $\mathcal{V}$ and $\mathcal{W}$ are $\operatorname{Diff}(M)$ saturated, $i \in \mathcal{W}$, $j \in \mathcal{V}$, and $\mathcal{V} \cap \mathcal{W} = \emptyset$. So $\pi(\mathcal{V})$ and $\pi(\mathcal{W})$ separate $\pi(i)$ and $\pi(j)$ in $B(M,N)$.

Case 2 Let $i(M) \subset \overline{j(M)}$ and $j(M) \subset \overline{i(M)}$. Let $y \in i(X)$, say. Let (V,v) be a chart of N centered at y which maps $i(M) \cap V$ into a linear subspace, $v(i(M) \cap V) \subseteq \mathbb{R}^m \cap v(V) \subset \mathbb{R}^n$, where $n = \dim M$, $n = \dim N$. Since $j(M) \subseteq \overline{i(M)}$ we conclude that we have also $v((i(M) \cup j(M)) \cap V) \subseteq \mathbb{R}^m \cap v(V)$. So we see that $L := i(M) \cup j(M)$ is a submanifold of N of the same dimension as N. Let (W_L, p_L, L) be a tubular neighborhood of L. Then $W_L|i(M)$ is a tubular neighborhood of $i(M)$ and $W_L|j(M)$ is one of $j(M)$. $\square$

7.4. Theorem. *Let M and N be smooth manifolds. Then the diffeomorphism group $\operatorname{Diff}(M)$ acts smoothly from the right on the smooth manifold $\operatorname{Imm}_{\text{prop}}(M,N)$ of all smooth proper immersions $M \to N$, which is an open subset of $C^\infty(M,N)$.*

Then the space of orbits $\operatorname{Imm}_{prop}(M,N)/\operatorname{Diff}(M)$ is Hausdorff in the quotient topology.

Let $\operatorname{Imm}_{\text{free, prop}}(M,N)$ be set of all proper immersions, on which $\operatorname{Diff}(M)$ acts freely. Then this is open in $C^\infty(M,N)$ and is the total space of a smooth principal fiber bundle

$$\operatorname{Imm}_{\text{free,prop}}(M,N) \to \operatorname{Imm}_{\text{free,prop}}(M,N)/\operatorname{Diff}(M).$$

This is proved in [Cervera-Mascaro-Michor, 1989], where also the existence of smooth transversals to each orbit is shown and the stratification of the orbit space into smooth manifolds is given.

8. The Frölicher-Nijenhuis Bracket

8.1. In this section let M be a smooth manifold. We consider the graded commutative algebra $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$ of differential forms on M. The space $\operatorname{Der}_k \Omega(M)$ consists of all *(graded) derivations* of degree k, i.e. all linear mappings $D : \Omega(M) \to \Omega(M)$ with $D(\Omega^\ell(M)) \subset \Omega^{k+\ell}(M)$ and $D(\varphi \wedge \psi) = D(\varphi) \wedge \psi + (-1)^{k\ell}\varphi \wedge D(\psi)$ for $\varphi \in \Omega^\ell(M)$.

Lemma. *Then the space* $\operatorname{Der}\Omega(M) = \bigoplus_k \operatorname{Der}_k \Omega(M)$ *is a graded Lie algebra with the graded commutator* $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ *as bracket. This means that the bracket is graded anticommutative,* $[D_1, D_2] = -(-1)^{k_1 k_2}[D_2, D_1]$*, and satisfies the graded Jacobi identity* $[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^[D_2, [D_1, D_3]]$ *(so that* $ad(D_1) = [D_1, \quad]$ *is itself a derivation).*

Proof. Plug in the definition of the graded commutator and compute. □

From the basics of differential geometry one is already familiar with some graded derivations: for a vector field X on M the derivation i_X is of degree -1, $\mathcal{L}_X$ is of degree 0, and d is of degree 1. Note also that the important formula $\mathcal{L}_X = d\, i_X + i_X\, d$ translates to $\mathcal{L}_X = [i_X, d]$.

8.2. A derivation $D \in \operatorname{Der}_k \Omega(M)$ is called *algebraic* if $D \mid \Omega^0(M) = 0$. Then $D(f.\omega) = f.D(\omega)$ for $f \in C^\infty(M, \mathbf{R})$, so D is of tensorial character and induces a derivation $D_x \in \operatorname{Der}_k \Lambda T_x^* M$ for each $x \in M$. It is uniquely determined by its restriction to 1-forms $D_x | T_x^* M : T_x^* M \to \Lambda^{k+1} T^* M$ which we may view as an element $K_x \in \Lambda^{k+1} T_x^* M \otimes T_x M$ depending smoothly on $x \in M$. To express this dependence we write $D = i_K = i(K)$, where $K \in C^\infty(\Lambda^{k+1} T^* M \otimes TM) =: \Omega^{k+1}(M; TM)$. We call $\Omega(M, TM) = \bigoplus_{k=0}^{\dim M} \Omega^k(M, TM)$ the space of all *vector valued differential forms.*

Theorem. (1) *For* $K \in \Omega^{k+1}(M, TM)$ *the formula*

$$(i_K \omega)(X_1, \dots, X_{k+\ell}) =$$
$$= \tfrac{1}{(k+1)!\,(\ell-1)!} \sum_{\sigma \in \S_{k+\ell}} \operatorname{sign}\sigma\, \omega(K(X_{\sigma 1}, \dots, X_{\sigma(k+1)}), X_{\sigma(k+2)}, \dots)$$

for $\omega \in \Omega^\ell(M)$, $X_i \in \mathfrak{X}(M)$ *(or* $T_x M$*) defines an algebraic graded derivation* $i_K \in \operatorname{Der}_k \Omega(M)$ *and any algebraic derivation is of this form.*

(2) *By* $i([K, L]\widehat{\ }) := [i_K, i_L]$ *we get a bracket* $[\quad,\quad]\widehat{\ }$ *on the space* $\Omega^{*-1}(M, TM)$ *which defines a graded Lie algebra structure with the grading as indicated, and for* $K \in \Omega^{k+1}(M, TM)$, $L \in \Omega^{\ell+1}(M, TM)$ *we have*

$$[K, L]\widehat{\ } = i_K L - (-1)^{k\ell} i_L K.$$

$[\ ,\ ]^\wedge$ is called the *algebraic bracket* or also the *Nijenhuis-Richardson bracket*, see [Nijenhuis-Richardson, 1967].

Proof. Since $\Lambda^* T_x^* M$ is the free graded commutative algebra generated by the vector space $T_x^* M$ any $K \in \Omega^{k+1}(M, TM)$ extends to a graded derivation. By applying it to an exterior product of 1-forms one can derive the formula in (1). The graded commutator of two algebraic derivations is again algebraic, so the injection $i : \Omega^{*+1}(M, TM) \to \mathrm{Der}_k(\Omega(M))$ induces a graded Lie bracket on $\Omega^{*+1}(M, TM)$ whose form can be seen by applying it to a 1-form. $\square$

8.3. The exterior derivative d is an element of $\mathrm{Der}_1 \Omega(M)$. In view of the formula $\mathcal{L}_X = [i_X, d] = i_X\, d + d\, i_X$ for vector fields X, we define for $K \in \Omega^k(M; TM)$ the *Lie derivation* $\mathcal{L}_K = \mathcal{L}(K) \in \mathrm{Der}_k \Omega(M)$ by $L_K := [i_k, d]$.

Then the mapping $\mathcal{L} : \Omega(M, TM) \to \mathrm{Der}\, \Omega(M)$ is injective, since $\mathcal{L}_K f = i_K df = df \circ K$ for $f \in \mathcal{C}^\infty(M, \mathbf{R})$.

Theorem. *For any graded derivation $D \in \mathrm{Der}_k \Omega(M)$ there are unique $K \in \Omega^k(M; TM)$ and $L \in \Omega^{k+1}(M; TM)$ such that*

$$D = \mathcal{L}_K + i_L.$$

We have $L = 0$ if and only if $[D, d] = 0$. D is algebraic if and only if $K = 0$.

Proof. Let $X_i \in \mathfrak{X}(M)$ be vector fields. Then $f \mapsto (Df)(X_1, \dots, X_k)$ is a derivation $C^\infty(M, \mathbf{R}) \to C^\infty(M, \mathbf{R})$, so there is a unique vector field $K(X_1, \dots, X_k) \in \mathfrak{X}(M)$ such that

$$(Df)(X_1, \dots, X_k) = K(X_1, \dots, X_k) f = df(K(X_1, \dots, X_k)).$$

Clearly $K(X_1, \dots, X_k)$ is $C^\infty(M, \mathbf{R})$-linear in each X_i and alternating, so K is tensorial, $K \in \Omega^k(M; TM)$.

The defining equation for K is $Df = df \circ K = i_K df = \mathcal{L}_K f$ for $f \in C^\infty(M, \mathbf{R})$. Thus $D - \mathcal{L}_K$ is an algebraic derivation, so $D - \mathcal{L}_K = i_L$ by 8.2 for unique $L \in \Omega^{k+1}(M; TM)$.

Since we have $[d, d] = 2d^2 = 0$, by the graded Jacobi identity we obtain $0 = [i_K, [d, d]] = [[i_K, d], d] + (-1)^{k-1}[d, [i_K, d]] = 2[\mathcal{L}_K, d]$. The mapping $L \mapsto [i_L, d] = \mathcal{L}_K$ is injective, so the last assertions follow. $\square$

8.4. Applying $i(Id_{TM})$ on a k-fold exterior product of 1-forms we see that $i(Id_{TM})\omega = k\omega$ for $\omega \in \Omega^k(M)$. Thus we have $\mathcal{L}(Id_{TM}) = \omega = i(Id_{TM})d\omega - d\, i(Id_{TM})\omega = (k+1)d\omega - kd\omega = d\omega$. Thus $\mathcal{L}(Id_{TM}) = d$.

8.5. Let $K \in \Omega^k(M;TM)$ and $L \in \Omega^\ell(M;TM)$. Then obviously $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K, L])$$

for a uniquely defined $[K, L] \in \Omega^{k+\ell}(M;TM)$. This vector valued form $[K, L]$ is called the *Frölicher-Nijenhuis bracket* of K and L.

Theorem. *The space $\Omega(M;TM) = \bigoplus_{k=0}^{\dim M} \Omega^k(M;TM)$ with its usual grading is a graded Lie algebra. $Id_{TM} \in \Omega^1(M;TM)$ is in the center, i.e. $[K, Id_{TM}] = 0$ for all K.*

$\mathcal{L} : (\Omega(M;TM), [\quad,\quad]) \to \operatorname{Der}\Omega(M)$ *is an injective homomorphism of graded Lie algebras. For vector fields the Frölicher-Nijenhuis bracket coincides with the Lie bracket.*

Proof. $df \circ [X, Y] = \mathcal{L}([X, Y])f = [\mathcal{L}_X, \mathcal{L}_Y]f$. The rest is clear. $\square$

8.6. Lemma. *For $K \in \Omega^k(M;TM)$ and $L \in \Omega^{\ell+1}(M;TM)$ we have*

$$[\mathcal{L}_K, i_L] = i([K, L]) - (-1)^{k\ell}\mathcal{L}(i_L K), \text{ or}$$
$$[i_L, \mathcal{L}_K] = \mathcal{L}(i_L K) - (-1)^k\, i([L, K]).$$

This generalizes the usual formula for vector fields.

Proof. For $f \in C^\infty(M, \mathbf{R})$ we have $[i_L, \mathcal{L}_K]f = i_L\, i_K\, df - 0 = i_L(df \circ K) = df \circ (i_L K) = \mathcal{L}(i_L K)f$. So $[i_L, \mathcal{L}_K] - \mathcal{L}(i_L K)$ is an algebraic derivation.

$$\begin{aligned}[][[i_L, \mathcal{L}_K], d] &= [i_L, [\mathcal{L}_K, d]] - (-1)^{k\ell}[\mathcal{L}_K, [i_L, d]] = \\ &= 0 - (-1)^{k\ell}\mathcal{L}([K, L]) = (-1)^k[i([L, K]), d]).\end{aligned}$$

Since $[\quad, d]$ kills the "$\mathcal{L}$'s" and is injective on the "i's", the algebraic part of $[i_L, \mathcal{L}_K]$ is $(-1)^k\, i([L, K])$. $\square$

8.7. The space $\operatorname{Der}\Omega(M)$ is a graded module over the graded algebra $\Omega(M)$ with the action $(\omega \wedge D)\varphi = \omega \wedge D(\varphi)$.

Theorem. *Let the degree of ω be q, of φ be k, and of ψ be ℓ. Let the other degrees be as indicated. Then we have:*

(1) $\quad [\omega \wedge D_1, D_2] = \omega \wedge [D_1, D_2] - (-1)^{(q+k_1)k_2} d_2(\omega) \wedge D_1.$

(2) $\quad i(\omega \wedge L) = \omega \wedge i(L)$

(3) $\quad \omega \wedge \mathcal{L}_K = \mathcal{L}(\omega \wedge K) + (-1)^{q+k-1} i(d\omega \wedge K).$

(4) $\quad [\omega \wedge L_1, L_2]\widehat{\ } = \omega \wedge [L_1, L_2]\widehat{\ } -$

$$- (-1)^{(q+\ell_1-1)(\ell_2-1)} i(L_2)\omega \wedge L_1.$$

(5) $$[\omega \wedge K_1, K_2] = \omega \wedge [K_1, K_2] - (-1)^{(q+k_1)k_2}\mathcal{L}(K_2)\omega \wedge K_1 + (-1)^{q+k_1} d\omega \wedge i(K_1)K_2.$$

(6) $$\begin{aligned}[\varphi \otimes X, \psi \otimes Y] &= \varphi \wedge \psi \otimes [X,Y] \\ &- \big(i_Y d\varphi \wedge \psi \otimes X - (-1)^{k\ell} i_X d\psi \wedge \varphi \otimes Y\big) \\ &- \big(d(i_Y\varphi \wedge \psi) \otimes X - (-1)^{k\ell} d(i_X\psi \wedge \varphi) \otimes Y\big) \\ &= \varphi \wedge \psi \otimes [X,Y] + \varphi \wedge \mathcal{L}_X\psi \otimes Y - \mathcal{L}_Y\varphi \wedge \psi \otimes X \\ &+ (-1)^k \left(d\varphi \wedge i_X\psi \otimes Y + i_Y\varphi \wedge d\psi \otimes X\right).\end{aligned}$$

Proof. For (1) , (2) , (3) write out the definitions. For (4) compute $i([\omega \wedge L_1, L_2]\widehat{\ })$. For (5) compute $\mathcal{L}([\omega \wedge K_1, K_2])$. For (6) use (5) . □

8.8. Theorem. *For $K \in \Omega^k(M;TM)$ and $\omega \in \Omega^\ell(M)$ the Lie derivative of ω along K is given by the following formula, where the X_i are vector fields on M.*

$$\begin{aligned}&(\mathcal{L}_K\omega)(X_1, \dots, X_{k+\ell}) = \\ &= \tfrac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\ \mathcal{L}(K(X_{\sigma 1}, \dots, X_{\sigma k}))(\omega(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})) \\ &+ \tfrac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\ \omega([K(X_{\sigma 1}, \dots, X_{\sigma k}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \dots) \\ &+ \tfrac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\ \omega(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots), X_{\sigma(k+2)}, \dots).\end{aligned}$$

Proof. It suffices to consider $K = \varphi \otimes X$. Then by 8.7.3 we have $\mathcal{L}(\varphi \otimes X) = \varphi \wedge \mathcal{L}_X - (-1)^{k-1} d\varphi \wedge i_X$. Now use the usual global formulas to expand this. □

8.9. Theorem. Global formula for the Frölicher-Nijenhuis bracket. *For $K \in \Omega^k(M;TM)$ and $L \in \Omega^\ell(M;TM)$ we have for the Frölicher-Nijenhuis bracket $[K,L]$ the following formula, where the X_i are vector fields on M.*

$$\begin{aligned}&[K,L](X_1, \dots, X_{k+\ell}) = \\ &= \tfrac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\ [K(X_{\sigma 1}, \dots, X_{\sigma k}), L(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})] \\ &+ \tfrac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\ L([K(X_{\sigma 1}, \dots, X_{\sigma k}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \dots)\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{(-1)^{k\ell}}{(k-1)!\,\ell!} \sum_\sigma \operatorname{sign}\sigma\, K([L(X_{\sigma 1},\dots,X_{\sigma\ell}),X_{\sigma(\ell+1)}],X_{\sigma(\ell+2)},\dots)\\
&+ \tfrac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!} \sum_\sigma \operatorname{sign}\sigma\, L(K([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(k+2)},\dots)\\
&+ \tfrac{(-1)^{(k-1)\ell}}{(k-1)!\,(\ell-1)!\,2!} \sum_\sigma \operatorname{sign}\sigma\, K(L([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots),X_{\sigma(\ell+2)},\dots).
\end{aligned}
$$

Proof. It suffices to consider $K = \varphi\otimes X$ and $L=\psi\otimes Y$, then for $[\varphi\otimes X,\psi\otimes Y]$ we may use 8.7.6 and evaluate that at $(X_1,\dots,X_{k+\ell})$. After some combinatorial computation we get the right hand side of the above formula for $K=\varphi\otimes X$ and $L=\psi\otimes Y$. $\square$

There are more illuminating ways to prove this formula, see [Michor, 1987].

8.10. Local formulas. In a local chart (U,u) on the manifold M we put $K\mid U=\sum K^i_\alpha d^\alpha\otimes\partial_i$, $L\mid U=\sum L^j_\beta d^\beta\otimes\partial_j$, and $\omega\mid U=\sum\omega_\gamma d^\gamma$, where $\alpha=(1\le\alpha_1<\alpha_2<\dots<\alpha_k\le\dim M)$ is a form index, $d^\alpha=du^{\alpha_1}\wedge\ldots\wedge du^{\alpha_k}$, $\partial_i=\frac{\partial}{\partial u^i}$ and so on.

Plugging $X_j=\partial_{i_j}$ into the global formulas 8.2, 8.8, and 8.9, we get the following local formulas:

$$
\begin{aligned}
i_K\omega\mid U &= \sum K^i_{\alpha_1\dots\alpha_k}\omega_{i\alpha_{k+1}\dots\alpha_{k+\ell-1}}\,d^\alpha\\
[K,L]\widehat{}\mid U &= \sum\Big(K^i_{\alpha_1\dots\alpha_k}\,L^j_{i\alpha_{k+1}\dots\alpha_{k+\ell}}\\
&\qquad -(-1)^{(k-1)(\ell-1)}L^i_{\alpha_1\dots\alpha_\ell}\,K^j_{i\alpha_{\ell+1}\dots\alpha_{k+\ell}}\Big)\,d^\alpha\otimes\partial_j\\
\mathcal{L}_K\omega\mid U &= \sum\Big(K^i_{\alpha_1\dots\alpha_k}\,\partial_i\omega_{\alpha_{k+1}\dots\alpha_{k+\ell}}\\
&\qquad +(-1)^k(\partial_{\alpha_1}K^i_{\alpha_1\dots\alpha_k})\,\omega_{i\alpha_{k+2}\dots\alpha_{k+\ell}}\Big)d^\alpha\\
[K,L]\mid U &= \sum\Big(K^i_{\alpha_1\dots\alpha_k}\,\partial_iL^j_{\alpha_{k+1}\dots\alpha_{k+\ell}}\\
&\qquad -(-1)^{k\ell}L^i_{\alpha_1\dots\alpha_\ell}\,\partial_iK^j_{\alpha_{\ell+1}\dots\alpha_{k+\ell}}\\
&\qquad -kK^j_{\alpha_1\dots\alpha_{k-1}i}\,\partial_{\alpha_k}L^i_{\alpha_{k+1}\dots\alpha_{k+\ell}}\\
&\qquad +(-1)^{k\ell}\ell L^j_{\alpha_1\dots\alpha_{\ell-1}i}\,\partial_{\alpha_\ell}K^i_{\alpha_{\ell+1}\dots\alpha_{k+\ell}}\Big)\,d^\alpha\otimes\partial_j
\end{aligned}
$$

8.11. Theorem. *For $K_i\in\Omega^{k_i}(M;TM)$ and $L_i\in\Omega^{k_i+1}(M;TM)$ we have*

(1)
$$
\begin{aligned}
&[\mathcal{L}_{K_1}+i_{L_1},\mathcal{L}_{K_2}+i_{L_2}]=\\
&\quad=\mathcal{L}\left([K_1,K_2]+i_{L_1}K_2-(-1)^{k_1k_2}i_{L_2}K_1\right)\\
&\qquad+i\left([L_1,L_2]\widehat{}+[K_1,L_2]-(-1)^{k_1k_2}[K_2,L_1]\right).
\end{aligned}
$$

Each summand of this formula looks like a semidirect product of graded Lie algebras, but the mappings

$$\begin{aligned} i &: \Omega(M;TM) \to \operatorname{End}(\Omega(M;TM), [\quad,\quad]) \\ ad &: \Omega(M;TM) \to \operatorname{End}(\Omega(M;TM), [\quad,\quad]\hat{}\,) \end{aligned}$$

do not take values in the subspaces of graded derivations. We have instead for $K \in \Omega^k(M;TM)$ and $L \in \Omega^{\ell+1}(M;TM)$ the following relations:

$$\begin{aligned} (2)\qquad & i_L[K_1,K_2] = [i_L K_1, K_2] + (-1)^{k_1\ell}[K_1, i_L K_2] \\ & \qquad - \Big((-1)^{k_1\ell} i([K_1,L])K_2 - (-1)^{(k_1+\ell)k_2} i([K_2,L])K_1\Big) \end{aligned}$$

$$\begin{aligned} (3)\qquad & [K,[L_1,L_2]\hat{}\,] = [[K,L_1],L_2]\hat{}\, + (-1)^{kk_1}[L_1,[K,L_2]]\hat{}\, - \\ & \qquad - \Big((-1)^{kk_1}[i(L_1)K, L_2] - (-1)^{(k+k_1)k_2}[i(L_2)K, L_1]\Big) \end{aligned}$$

The algebraic meaning of the relations of this theorem and its consequences in group theory have been investigated in [Michor, 1989b]. The corresponding product of groups is well known to algebraists under the name "Zappa-Szep"-product.

Proof. Equation (1) is an immediate consequence of 8.6. Equations (2) and (3) follow from (1) by writing out the graded Jacobi identity, or as follows: Consider $\mathcal{L}(i_L[K_1,K_2])$ and use 8.6 repeatedly to obtain $\mathcal{L}$ of the right hand side of (2). Then consider $i([K,[L_1,L_2]\hat{}\,])$ and use again 8.6 several times to obtain i of the right hand side of (3). $\square$

8.12. Corollary (of 8.9). *For K, $L \in \Omega^1(M;TM)$ we have*

$$\begin{aligned} [K,L](X,Y) &= [KX,LY] - [KY,LX] \\ &\quad - L([KX,Y] - [KY,X]) \\ &\quad - K([LX,Y] - [LY,X]) \\ &\quad + (LK + KL)[X,Y]. \end{aligned}$$

8.13. Curvature. Let $P \in \Omega^1(M;TM)$ be a fiber projection, i.e. $P \circ P = P$. This is the most general case of a (first order) *connection*. We may call $\ker P$ the *horizontal space* and $\operatorname{im} P$ the *vertical space* of the connection. If P is of constant rank, then both are sub vector bundles of TM. If $\operatorname{im} P$ is some primarily fixed sub vector bundle or (tangent bundle of) a foliation, P can be called a connection for it. Special cases of this will be treated extensively later on. For the general theory see [Michor, 1989a] The following result is immediate from 8.12.

Lemma. *We have*

$$[P, P] = 2R + 2\bar{R},$$

where R, $\bar{R} \in \Omega^2(M; TM)$ are given by $R(X, Y) = P[(Id - P)X, (Id - P)Y]$ and $\bar{R}(X, Y) = (Id - P)[PX, PY]$.

If P has constant rank, then R is the obstruction against integrability of the horizontal bundle $\ker P$, and $\bar{R}$ is the obstruction against integrability of the vertical bundle $\operatorname{im} P$. Thus we call R the *curvature* and $\bar{R}$ the *cocurvature* of the connection P. We will see later, that for a principal fiber bundle R is just the negative of the usual curvature.

8.14. Lemma. General Bianchi identity. *If $P \in \Omega^1(M; TM)$ is a connection (fiber projection) with curvature R and cocurvature $\bar{R}$, then we have*

$$[P, R + \bar{R}] = 0$$
$$[R, P] = i_R \bar{R} + i_{\bar{R}} R.$$

Proof. We have $[P, P] = 2R + 2\bar{R}$ by 8.13 and $[P, [P, P]] = 0$ by the graded Jacobi identity. So the first formula follows. We have $2R = P \circ [P, P] = i_{[P,P]} P$. By 8.11.2 we get $i_{[P,P]}[P, P] = 2[i_{[P,P]} P, P] - 0 = 4[R, P]$. Therefore $[R, P] = \frac{1}{4} i_{[P,P]}[P, P] = i(R+\bar{R})(R+\bar{R}) = i_R \bar{R} + i_{\bar{R}} R$ since R has vertical values and kills vertical vectors, so $i_R R = 0$; likewise for $\bar{R}$. $\square$

8.15. Naturality of the Frölicher-Nijenhuis bracket. Let $f : M \to N$ be a smooth mapping between manifolds. Two vector valued forms $K \in \Omega^k(M; TM)$ and $K' \in \Omega^k(N; TN)$ are called *f-related* or *f-dependent*, if for all $X_i \in T_x M$ we have

(1) $$K'_{f(x)}(T_x f \cdot X_1, \dots, T_x f \cdot X_k) = T_x f \cdot K_x(X_1, \dots, X_k).$$

Theorem.

(2) *If K and K' as above are f-related then $i_K \circ f^* = f^* \circ i_{K'} : \Omega(N) \to \Omega(M)$.*

(3) *If $i_K \circ f^* \mid B^1(N) = f^* \circ i_{K'} \mid B^1(N)$, then K and K' are f-related, where B^1 denotes the space of exact 1-forms.*

(4) *If K_j and K'_j are f-related for $j = 1, 2$, then $i_{K_1} K_2$ and $i_{K'_1} K'_2$ are f-related, and also $[K_1, K_2]\widehat{\ }$ and $[K'_1, K'_2]\widehat{\ }$ are f-related.*

(5) *If K and K' are f-related then $\mathcal{L}_K \circ f^* = f^* \circ \mathcal{L}_{K'} : \Omega(N) \to \Omega(M)$.*

(6) *If* $\mathcal{L}_K \circ f^* \mid \Omega^0(N) = f^* \circ \mathcal{L}_{K'} \mid \Omega^0(N)$, *then* K *and* K' *are* f*-related.*
(7) *If* K_j *and* K'_j *are* f*-related for* $j = 1, 2$, *then their Frölicher-Nijenhuis brackets* $[K_1, K_2]$ *and* $[K'_1, K'_2]$ *are also* f*-related.*

Proof. (2) By 8.2 we have for $\omega \in \Omega^q(N)$ and $X_i \in T_xM$:

$$\begin{aligned}
&(i_K f^*\omega)_x(X_1, \dots, X_{q+k-1}) = \\
&\quad = \tfrac{1}{k!\,(q-1)!} \sum_\sigma \operatorname{sign}\sigma\, (f^*\omega)_x(K_x(X_{\sigma 1}, \dots, X_{\sigma k}), X_{\sigma(k+1)}, \dots) \\
&\quad = \tfrac{1}{k!\,(q-1)!} \sum_\sigma \operatorname{sign}\sigma\, \omega_{f(x)}(T_xf \cdot K_x(X_{\sigma 1}, \dots), T_xf \cdot X_{\sigma(k+1)}, \dots) \\
&\quad = \tfrac{1}{k!\,(q-1)!} \sum_\sigma \operatorname{sign}\sigma\, \omega_{f(x)}(K'_{f(x)}(T_xf \cdot X_{\sigma 1}, \dots), T_xf \cdot X_{\sigma(k+1)}, \dots) \\
&\quad = (f^* i_{K'}\omega)_x(X_1, \dots, X_{q+k-1})
\end{aligned}$$

(3) follows from this computation, since the df, $f \in C^\infty(M, \mathbf{R})$ separate points. (4) follows from the same computation for K_2 instead of ω, the result for the bracket then follows 8.2.2.

(5) The algebra homomorphism f^* intertwines the operators i_K and $i_{K'}$ by (2), and f^* commutes with the exterior derivative d. Thus f^* intertwines the commutators $[i_K, d] = \mathcal{L}_K$ and $[i_{K'}, d] = \mathcal{L}_{K'}$.

(6) For $g \in \Omega^0(N)$ we have $\mathcal{L}_K\, f^*\, g = i_K\, d\, f^*\, g = i_K\, f^*\, dg$ and $f^*\, \mathcal{L}_{K'}\, g = f^*\, i_{K'}\, dg$. By (3) the result follows.

(6) The algebra homomorphism f^* intertwines $\mathcal{L}_{K_j}$ and $\mathcal{L}_{K'_j}$, so also their graded commutators which equal $\mathcal{L}([K_1, K_2])$ and $\mathcal{L}([K'_1, K'_2])$, respectively. Now use (6) . $\square$

8.16. Let $f : M \to N$ be a local diffeomorphism. Then we can consider the *pullback operator* $f^* : \Omega(M; TM) \to \Omega(M; TM)$, given by

$$(1) \quad (F^*K)_x(X_1, \dots, X_k) = (T_xf)^{-1} K_{f(x)}(T_xf \cdot X_1, \dots, T_xf \cdot X_k).$$

Clearly K and f^*K are then f-related.

Theorem. *In this situation we have:*
(2) $f^*\,[K, L] = [f^*K, f^*L]$.
(3) $f^*\, i_K L = i_{f^*K} f^*L$.
(4) $f^*\,[K, L]\hat{\ } = [f^*K, f^*L]\hat{\ }$.
(5) *For a vector field* $X \in \mathfrak{X}(M)$ *and* $K \in \Omega(M; TM)$ *the Lie derivative* $\mathcal{L}_X K = \frac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^X_t)^* K$ *satisfies* $\mathcal{L}_X K = [X, K]$, *the Frölicher-Nijenhuis-bracket.*

We may say that the Frölicher-Nijenhuis bracket, $[\quad,\quad]\hat{}$, etc. are *natural bilinear concomitants*.

Proof. (2) – (4) are obvious from 8.16. (5) Obviously $\mathcal{L}_X$ is $\mathbf{R}$-linear, so it suffices to check this formula for $K = \psi \otimes Y$, $\psi \in \Omega(M)$ and $Y \in \mathfrak{X}(M)$. But then

$$\begin{aligned} \mathcal{L}_X(\psi \otimes Y) &= \mathcal{L}_X\psi \otimes Y + \psi \otimes \mathcal{L}_X Y \\ &= \mathcal{L}_X\psi \otimes Y + \psi \otimes [X,Y] \\ &= [X, \psi \otimes Y] \quad \text{by 8.7.6.} \quad \square \end{aligned}$$

8.17. Remark. At last we mention the best known application of the Frölicher-Nijenhuis bracket, which also led to its discovery. A vector valued 1-form $J \in \Omega^1(M;TM)$ with $J \circ J = -Id$ is called a *nearly complex structure*; if it exists, $\dim M$ is even and J can be viewed as a fiber multiplication with $\sqrt{-1}$ on TM. By 8.12 we have

$$[J,J](X,Y) = 2([JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]).$$

The vector valued form $\frac{1}{2}[J,J]$ is also called the *Nijenhuis tensor* of J. For it the following result is true:

A manifold M with a nearly complex structure J is a complex manifold (i.e., there exists an atlas for M with holomorphic chart-change mappings) if and only if $[J,J] = 0$. See [Newlander-Nirenberg, 1957].

9. Fiber Bundles and Connections

9.1. Definition. A *(fiber) bundle* (E, p, M, S) consists of manifolds E, M, S, and a smooth mapping $p : E \to M$; furthermore each $x \in M$ has an open neighborhood U such that $E \mid U := p^{-1}(U)$ is diffeomorphic to $U \times S$ via a fiber respecting diffeomorphism:

$$\begin{array}{ccc} E \mid U & \xrightarrow{\psi} & U \times S \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ U & = & U \end{array}$$

E is called the *total space*, M is called the *base space* or *basis*, p is a surjective submersion, called the *projection*, and S is called *standard fiber*. (U, ψ) as above is called a *fiber chart*.

A collection of fiber charts (U_α, ψ_α), such that (U_α) is an open cover of M, is called a *(fiber) bundle atlas*. If we fix such an atlas, then $\psi_\alpha \circ {\psi_\beta}^{-1}(x, s) = (x, \psi_{\alpha\beta}(x, s))$, where $\psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times S \to S$ is smooth and $\psi_{\alpha\beta}(x, \ )$ is a diffeomorphism of S for each $x \in U_{\alpha\beta} := U_\alpha \cap U_\beta$. We may thus consider the mappings $\psi_{\alpha\beta} : U_{\alpha\beta} \to \operatorname{Diff}(S)$ with values in the group $\operatorname{Diff}(S)$ of all diffeomorphisms of S; their differentiability is a subtle question, which will be discussed in 13.1 below. In either form these mappings $\psi_{\alpha\beta}$ are called the *transition functions* of the bundle. They satisfy the *cocycle condition*: $\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x)$ for $x \in U_{\alpha\beta\gamma}$ and $\psi_{\alpha\alpha}(x) = Id_S$ for $x \in U_\alpha$. Therefore the collection $(\psi_{\alpha\beta})$ is called a *cocycle of transition functions*.

Given an open cover (U_α) of a manifold M and a cocycle of transition functions $(\psi_{\alpha\beta})$ we may construct a fiber bundle (E, p, M, S) similarly as in the following way: On the disjoint union $\bigsqcup_\alpha \{\alpha\} \times U_\alpha \times S$ we consider the equivalence relation:

$$(\alpha, x, s) \sim (\beta, y, t) \text{ if and only if } x = y \text{ and } \psi_{\alpha\beta}(y, t) = s.$$

The quotient space then turns out to be a fiber bundle, and we may use the mappings $\psi_\alpha([(\alpha, x, s)]) := (x, s)$ as a bundle atlas whose cocycle of transition functions is exactly the one we started from.

9.2. Lemma. *Let $p : N \to M$ be a surjective submersion (a fibred manifold) such that $p^{-1}(x)$ is compact for each $x \in M$ and let M be connected. Then (N, p, M) is a fiber bundle.*

Proof. We have to produce a fiber chart at each $x_0 \in M$. So let (U, u) be a chart centered at x_0 on M such that $u(U) \cong \mathbf{R}^m$. For each $x \in U$ let $\xi_x(y) := (T_y u)^{-1}.u(x)$, then $\xi_x \in \mathfrak{X}(U)$, depending smoothly on $x \in U$, such that $u(\mathrm{Fl}^{\xi_x}_t u^{-1}(z)) = z + t.u(x)$, so each ξ_x is a complete vector field on U. Since p is a submersion, with the help of a partition of unity on $p^{-1}(U)$ we may construct vector fields $\eta_x \in \mathfrak{X}(p^{-1}(U))$ which depend smoothly on $x \in U$ and are p-related to ξ_x: $Tp.\eta_x = \xi_x \circ p$. Thus $p \circ \mathrm{Fl}^{\eta_x}_t = \mathrm{Fl}^{\xi_x}_t \circ p$ and $\mathrm{Fl}^{\eta_x}_t$ is fiber respecting, and since each fiber is compact and ξ_x is complete, η_x has a global flow too. Denote $p^{-1}(x_0)$ by S. Then $\varphi : U \times S \to p^{-1}(U)$, defined by $\varphi(x, y) = \mathrm{Fl}^{\eta_x}_1(y)$, is a diffeomorphism and is fiber respecting, so (U, φ^{-1}) is a fiber chart. Since M is connected, the fibers $p^{-1}(x)$ are all diffeomorphic.

9.3. Let (E, p, M, S) be a fiber bundle; we consider the fiber linear mapping $Tp : TE \to TM$ and its kernel $\ker Tp =: VE$ which is called the *vertical bundle* of E. The following is special case of 8.13.

Definition A *connection* on the fiber bundle (E, p, M, S) is a vector valued 1-form $\Phi \in \Omega^1(E; VE)$ with values in the vertical bundle VE such that $\Phi \circ \Phi = \Phi$ and $\mathrm{Im}\Phi = VE$; so Φ is just a projection $TE \to VE$.

Then $\ker \Phi$ is of constant rank, so $\ker \Phi$ is a sub vector bundle of TE, it is called the space of *horizontal vectors* or the *horizontal bundle* and it is denoted by HE Clearly $TE = HE \oplus VE$ and $T_uE = H_uE \oplus V_uE$ for $u \in E$.

Now we consider the mapping $(Tp, \pi_E) : TE \to TM \times_M E$. Then $(Tp, \pi_E)^{-1}(0_{p(u)}, u) = V_uE$ by definition, so $(Tp, \pi_E) \mid HE : HE \to TM \times_M E$ is fiber linear over E and injective, so by reason of dimensions it is a fiber linear isomorphism: Its inverse is denoted by

$$C := ((Tp, \pi_E) \mid HE)^{-1} : TM \times_M E \to HE \hookrightarrow TE.$$

So $C : TM \times_M E \to TE$ is fiber linear over E and is a right inverse for (Tp, π_E). C is called the *horizontal lift* associated to the connection Φ.

Note the formula $\Phi(\xi_u) = \xi_u - C(Tp.\xi_u, u)$ for $\xi_u \in T_uE$. So we can equally well describe a connection Φ by specifying C. Then we call Φ the *vertical projection* and $\chi := \mathrm{id}_{TE} - \Phi = C \circ (Tp, \pi_E)$ will be called the *horizontal projection*.

9.4. Curvature. If $\Phi : TE \to VE$ is a connection on a fiber bundle (E, p, M, S), then as in 8.13 the curvature R of Φ is given by

$$2R = [\Phi, \Phi] = [Id - \Phi, Id - \Phi] = [\chi, \chi] \in \Omega^2(E; VE).$$

We have $R(X, Y) = \frac{1}{2}[\Phi, \Phi](X, Y) = \Phi[\chi X, \chi Y]$, so R is an obstruction against integrability of the horizontal subbundle. Since the vertical bundle VE is integrable, by 8.14 we have the *Bianchi identity* $[\Phi, R] = 0$.

9.5. Pullback. Let (E, p, M, S) be a fiber bundle and consider a smooth mapping $f : N \to M$. Since p is a submersion, f and p are transversal and thus the pullback $N \times_{(f,M,p)} E$ exists. It will be called the *pullback* of the fiber bundle E by F and we will denote it by f^*E. The following diagram sets up some further notation for it:

$$\begin{tikzcd} f^*E \arrow[r, "p^*f"] \arrow[d, "f^*"'] & E \arrow[d, "p"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

Proposition. *In the situation above we have:*

(1) *(f^*E, f^*p, N, S) is again a fiber bundle, and p^*f is a fiber wise diffeomorphism.*
(2) *If $\Phi \in \Omega^1(E; TE)$ is a connection on the bundle E, the vector valued form $f^*\Phi$, given by $(f^*\Phi)_u(X) := T_u(p^*f)^{-1}.\Phi.T_u(p^*f).X$ for $X \in T_uE$, is again a connection on the bundle f^*E. The forms $f^*\Phi$ and Φ are p^*f-related in the sense of 8.15.*
(3) *The curvatures of $f^*\Phi$ and Φ are also p^*f-related: $R(f^*\Phi) = f^*R(\Phi)$.*

Proof. (1). If (U_α, ψ_α) is a fiber bundle atlas of (E, p, M, S) in the sense of 9.1, then $(f^{-1}(U_\alpha), (f^*p, pr_2 \circ \psi_\alpha \circ p^*f))$ is visibly a fiber bundle atlas for (f^*E, f^*p, N, S), by the formal universal properties of a pullback. (2) is obvious. (3) follows from (2) and 8.15.6. $\square$

9.6. Flat connections. Let us suppose that a connection Φ on the bundle (E, p, M, S) has zero curvature. Then by 9.4 the horizontal bundle is integrable and gives rise to the *horizontal foliation*. Each point $u \in E$ lies on a unique leaf $L(u)$ such that $T_vL(u) = H_vE$ for each $v \in L(u)$. The restriction $p \mid L(u)$ is locally a diffeomorphism, but in general it is neither surjective nor is it a covering onto its image. This is seen by devising suitable horizontal foliations on the trivial bundle $pr_2 : \mathbf{R} \times S^1 \to S^1$.

9.7. Local description of connections. Let Φ be a connection on (E, p, M, S). Let us fix a fiber bundle atlas (U_α) with transition functions $(\psi_{\alpha\beta})$, and let us consider the connection $(\psi_\alpha)^{-1})^*\Phi \in \Omega^1(U_\alpha \times S; U_\alpha \times TS)$, which may be written in the form

$$((\psi_\alpha)^{-1})^*\Phi)(\xi_x, \eta_y) =: -\Gamma^\alpha(\xi_x, y) + \eta_y \text{ for } \xi_x \in T_xU_\alpha \text{ and } \eta_y \in T_yS,$$

since it reproduces vertical vectors. The Γ^α are given by

$$(0_x, \Gamma^\alpha(\xi_x, y)) := -T(\psi_\alpha).\Phi.T(\psi_\alpha)^{-1}.(\xi_x, 0_y).$$

We consider Γ^α as an element of the space $\Omega^1(U_\alpha; \mathfrak{X}(S))$, a 1-form on U^α with values in the infinite dimensional Lie algebra $\mathfrak{X}(S)$ of all vector fields on the standard fiber. The Γ^α are called the *Christoffel forms* of the connection Φ with respect to the bundle atlas (U_α, ψ_α).

Lemma. *The transformation law for the Christoffel forms is*

$$T_y(\psi_{\alpha\beta}(x, \quad)).\Gamma^\beta(\xi_x, y) = \Gamma^\alpha(\xi_x, \psi_{\alpha\beta}(x, y)) - T_x(\psi_{\alpha\beta}(\quad , y)).\xi_x.$$

The curvature R of Φ satisfies

$$(\psi_\alpha^{-1})^* R = d\Gamma^\alpha + [\Gamma^\alpha, \Gamma^\alpha]_{\mathfrak{X}(S)}.$$

The formula for the curvature is the *Maurer-Cartan formula* which in this general setting appears only in the level of local description.

Proof. From $\psi_\alpha \circ (\psi_\beta)^{-1}(x, y) = (x, \psi_{\alpha\beta}(x, y))$ we get that $T(\psi_\alpha \circ (\psi_\beta)^{-1}).(\xi_x, \eta_y) = (\xi_x, T_{(x,y)}(\psi_{\alpha\beta}).(\xi_x, \eta_y))$ and thus:

$$\begin{aligned}
T(\psi_\beta^{-1}).(0_x, \Gamma^\beta(\xi_x, y)) &= -\Phi(T(\psi_\beta^{-1})(\xi_x, 0_y)) = \\
&= -\Phi(T(\psi_\alpha^{-1}).T(\psi_\alpha \circ \psi_\beta^{-1}).(\xi_x, 0_y)) = \\
&= -\Phi(T(\psi_\alpha^{-1})(\xi_x, T_{(x,y)}(\psi_{\alpha\beta})(\xi_x, 0_y))) = \\
&= -\Phi(T(\psi_\alpha^{-1})(\xi_x, 0_y)) - \Phi(T(\psi_\alpha^{-1})(0_x, T_{(x,y)}\psi_{\alpha\beta}(\xi_x, 0_y)) = \\
&= T(\psi_\alpha^{-1}).(0_x, \Gamma^\alpha(\xi_x, \psi_{\alpha\beta}(x, y))) - T(\psi_\alpha^{-1})(0_x, T_x(\psi_{\alpha\beta}(\quad , y)).\xi_x).
\end{aligned}$$

This implies the transformation law.

For the curvature R of Φ we have by 9.4 and 9.5.3

$$\begin{aligned}
&(\psi_\alpha^{-1})^* R\,((\xi^1, \eta^1), (\xi^2, \eta^2)) = \\
&= (\psi_\alpha^{-1})^* R\,[(Id - (\psi_\alpha^{-1})^*\Phi)(\xi^1, \eta^1), (Id - (\psi_\alpha^{-1})^*\Phi)(\xi^2, \eta^2)] = \\
&= (\psi_\alpha^{-1})^*\Phi[(\xi^1, \Gamma^\alpha(\xi^1)), (\xi^2, \Gamma^\alpha(\xi^2))] = \\
&= (\psi_\alpha^{-1})^*\Phi\left([\xi^1, \xi^2], \xi^1\Gamma^\alpha(\xi^2) - \xi^2\Gamma^\alpha(\xi^1) + [\Gamma^\alpha(\xi^1), \Gamma^\alpha(\xi^2)]\right) = \\
&= -\Gamma^\alpha([\xi^1, \xi^2]) + \xi^1\Gamma^\alpha(\xi^2) - \xi^2\Gamma^\alpha(\xi^1) + [\Gamma^\alpha(\xi^1), \Gamma^\alpha(\xi^2)] = \\
&= d\Gamma^\alpha(\xi^1, \xi^2) + [\Gamma^\alpha(\xi^1), \Gamma^\alpha(\xi^2)]_{\mathfrak{X}(S)}. \quad \square
\end{aligned}$$

9.8. Theorem. Parallel transport. *Let Φ be a connection on a bundle (E, p, M, S) and let $c : (a, b) \to M$ be a smooth curve with $0 \in (a, b)$, $c(0) = x$.*

Then there is a neighborhood U of $E_x \times \{0\}$ in $E_x \times (a, b)$ and a smooth mapping $\operatorname{Pt}_c : U \to E$ such that:

(1) *$p(\operatorname{Pt}(c, t, u_x)) = c(t)$ if defined, and $\operatorname{Pt}(c, u_x, 0) = u_x$.*
(2) *$\Phi(\frac{d}{dt} \operatorname{Pt}(c, t, u_x)) = 0_{c(t)}$ if defined.*
(3) *Reparametrisation invariance: If $f : (a', b') \to (a, b)$ is smooth with $0 \in (a', b')$, then $\operatorname{Pt}(c, f(t), u_x) = \operatorname{Pt}(c \circ f, t, \operatorname{Pt}(c, u_x, f(0)))$ if defined.*
(4) *U is maximal for properties (1) and (2).*
(5) *The parallel transport is smooth as a mapping*

$$C^\infty(\mathbb{R}, M) \times_{(\mathrm{ev}_0, M, p)} E \times \mathbb{R} \supset U \xrightarrow{\operatorname{Pt}} E,$$

where U is its domain of definition.
(6) *If $X \in \mathfrak{X}(M)$ is a vector field on the base then the parallel transport along the flow lines of X is given by the flow of the horizontal lift CX of X:*

$$\operatorname{Pt}(\operatorname{Fl}^X(x), t, u_x) = \operatorname{Fl}^{CX}_t(u_x).$$

Proof. We first give three different proofs of assertions (1) – (4).

First proof. In local bundle coordinates $\Phi(\frac{d}{dt} \operatorname{Pt}(c, u_x, t)) = 0$ is an ordinary differential equation of first order, nonlinear, with initial condition $\operatorname{Pt}(c, u_x, 0) = u_x$. So there is a maximally defined local solution curve which is unique. All further properties are consequences of uniqueness.

Second proof. Consider the pullback bundle $(c^*E, c^*p, (a, b), S)$ and the pullback connection $c^*\Phi$ on it. It has zero curvature, since the horizontal bundle is 1-dimensional. By 9.6 the horizontal foliation exists and the parallel transport just follows a leaf and we may map it back to E, in detail: $\operatorname{Pt}(c, u_x, t) = p^*c((c^*p \mid L(u_x))^{-1}(t))$.

Third proof. Consider a fiber bundle atlas (U_α, ψ_α) as in 9.7. Then $\psi_\alpha(\operatorname{Pt}(c, \psi_\alpha^{-1}(x, y), t)) = (c(t), \gamma(y, t))$, where

$$0 = \left((\psi_\alpha^{-1})^*\Phi\right)\left(\tfrac{d}{dt}c(t), \tfrac{d}{dt}\gamma(y, t)\right) = -\Gamma^\alpha\left(\tfrac{d}{dt}c(t), \gamma(y, t)\right) + \tfrac{d}{dt}\gamma(y, t),$$

so $\gamma(y, t)$ is the integral curve (evolution line) through $y \in S$ of the time dependent vector field $\Gamma^\alpha\left(\frac{d}{dt}c(t)\right)$ on S. This vector field visibly depends smoothly on c. Clearly local solutions exist and all properties follow.

(5). By the considerations of section 6 it suffices to show that Pt maps a smooth curve

$$(c_1, c_2, c_3) : \mathbb{R} \to U \subset C^\infty(\mathbb{R}, M) \times_{(\mathrm{ev}_0, M, p)} E \times \mathbb{R}$$

to a smooth curve in E. We use the description of the smooth curves in $C^\infty(\mathbb{R}, M)$ from 6.2 and see that the ordinary differential equation considered in the first proof then just depends on some parameters more. By the theory of ordinary differential equations the solution is smooth in this parameters also, which we then set equal.

(6) is obvious from the definition. $\square$

9.9. Lemma. *Let Φ be a connection on a bundle (E, p, M, S) with curvature R and horizontal lift C. Let $X \in \mathfrak{X}(M)$ be a vector field on the base.*

Then for the horizontal lift $CX \in \mathfrak{X}(E)$ we have

$$\mathcal{L}_{CX}\Phi = \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}_t^{CX})^*\Phi = [CX, \Phi] = -\tfrac{1}{2} i_{CX} R.$$

Proof. From 8.16.(5) we get $\mathcal{L}_{CX}\Phi = \frac{\partial}{\partial t}\big|_0 (\mathrm{Fl}_t^{CX})^*\Phi = [CX, \Phi]$. From 8.11.(2) we have

$$\begin{aligned} i_{CX} R &= i_{CX}[\Phi, \Phi] \\ &= [i_{CX}\Phi, \Phi] - [\Phi, i_{CX}\Phi] + 2 i_{[\Phi, CX]}\Phi \\ &= -2\Phi[CX, \Phi] \end{aligned}$$

The vector field CX is p-related to X and $\Phi \in \Omega^1(E; TE)$ is p-related to $0 \in \Omega^1(M; TM)$, so by 8.15.(7) the form $[CX, \Phi] \in \Omega^1(E; TE)$ is also p-related to $0 = [X, 0] \in \Omega^1(M; TM)$. So $Tp.[CX, \Phi] = 0$, $[CX, \Phi]$ has vertical values, and $[CX, \Phi] = \Phi[CX, \Phi]$. $\square$

9.10. A connection Φ on (E, p, M, S) is called a *complete connection*, if the parallel transport Pt_c along any smooth curve $c : (a, b) \to M$ is defined on the whole of $E_{c(0)} \times (a, b)$. The third proof of theorem 9.8 shows that on a fiber bundle with compact standard fiber any connection is complete.

The following is a sufficient condition for a connection Φ to be complete:

> There exists a fiber bundle atlas (U_α, ψ_α) and complete Riemannian metrics g_α on the standard fiber S such that each Christoffel form $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(S))$ takes values in the linear subspace of g_α-bounded vector fields on S

For in the third proof of theorem 9.8 above the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t))$ on S is g_α-bounded for compact time intervals. So by continuation the solution exists over $c^{-1}(U_\alpha)$, and thus globally.

A complete connection is called an *Ehresmann connection* in [Greub - Halperin - Vanstone I, p 314], where it is also indicated how to prove the following result.

Theorem. *Each fiber bundle admits complete connections.*

Proof. Let $\dim M = m$. Let (U_α, ψ_α) be a fiber bundle atlas as in 9.1. By topological dimension theory [Nagata, 1965] the open cover (U_α) of M admits a refinement such that any $m+2$ members have empty intersection. Let $U_\alpha)$ itself have this property. Choose a smooth partition of unity (f_α) subordinated to (U_α). Then the sets $V_\alpha := \{\, x : f_\alpha(x) > \frac{1}{m+2} \,\} \subset U_\alpha$ form still an open cover of M since $\sum f_\alpha(x) = 1$ and at most $m+1$ of the $f_\alpha(x)$ can be nonzero. By renaming assume that each V_α is connected. Then we choose an open cover (W_α) of M such that $\overline{W_\alpha} \subset V_\alpha$.

Now let g_1 and g_2 be complete Riemannian metrics on M and S, respectively (see [Nomizu - Ozeki, 1961] or [Morrow, 1970]). For not connected Riemannian manifolds complete means that each connected component is complete. Then $g_1|U_\alpha \times g_2$ is a Riemannian metric on $U_\alpha \times S$ and we consider the metric $g := \sum f_\alpha \psi_\alpha^*(g_1|U_\alpha \times g_2)$ on E. Obviously $p : E \to M$ is a Riemannian submersion for the metrics g and g_1. We choose now the connection $\Phi : TE \to VE$ as the orthonormal projection with respect to the Riemannian metric g.

Claim: Φ is a complete connection on E.
Let $c : [0,1] \to M$ be a smooth curve. We choose a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that $c([t_i, t_{i+1}]) \subset V_{\alpha_i}$ for suitable α_i. It suffices to show that $\operatorname{Pt}(c(t_i + \quad), t, u_{c(t_i)})$ exists for all $0 \le t \le t_{i+1} - t_i$ and all $u_{c(t_i)} \in E_{c(t_i)}$, for all i — then we may piece them together. So we may assume that $c : [0,1] \to V_\alpha$ for some α. Let us now assume that for some $(x,y) \in V_\alpha \times S$ the parallel transport $\operatorname{Pt}(c, t, \psi_\alpha(x,y))$ is defined only for $t \in [0, t')$ for some $0 < t' < 1$. By the third proof of 9.8 we have $\operatorname{Pt}(c, t, \psi_\alpha(x,y)) = \psi_\alpha(c(t), \gamma(t))$, where $\gamma : [0, t') \to S$ is the maximally defined integral curve through $y \in S$ of the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t), \quad)$ on S. We put $g_\alpha := (\psi_\alpha^{-1})^* g$, then $(g_\alpha)_{(x,y)} = (g_1)_x \times (\sum_\beta f_\beta(x) \psi_{\beta\alpha}(x, \quad)^* g_2)_y$. Since $pr_1 : (V_\alpha \times S, g_\alpha) \to (V_\alpha, g_1|V_\alpha)$ is a Riemannian submersion and since the connection $(\psi_\alpha^{-1})^*\Phi$ is also given by orthonormal projection onto the vertical bundle, we get

$$\infty > g_1\text{-length}_0^{t'}(c) = g_\alpha\text{-length}(c, \gamma) = \int_0^{t'} |(c'(t), \tfrac{d}{dt}\gamma(t))|_{g_\alpha}\, dt =$$

$$= \int_0^{t'} \sqrt{|c'(t)|^2_{g_1} + \textstyle\sum_\beta f_\beta(c(t))(\psi_{\alpha\beta}(c(t),-)^* g_2)(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t))}\, dt \geq$$

$$\geq \int_0^{t'} \sqrt{f_\alpha(c(t))}\, |\tfrac{d}{dt}\gamma(t)|_{g_2}\, dt \geq \sqrt{m+2} \int_0^{t'} |\tfrac{d}{dt}\gamma(t)|_{g_2} dt.$$

So g_2-lenght(γ) is finite and since the Riemannian metric g_2 on S is complete, $lim_{t\to t'}\gamma(t) =: \gamma(t')$ exists in S and the integral curve γ can be continued. □

10. Principal Fiber Bundles and G-Bundles

10.1. Definition. Let G be a Lie group and let (E, p, M, S) be a fiber bundle as in 9.1. A *G-bundle structure* on the fiber bundle consists of the following data:

(1) A left action $\ell : G \times S \to S$ of the Lie group on the standard fiber.

(2) A fiber bundle atlas (U_α, ψ_α) whose transition functions $(\psi_{\alpha\beta})$ act on S via the G-action: There is a family of smooth mappings $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$ which satisfies the cocycle condition $\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x) = \varphi_{\alpha\gamma}(x)$ for $x \in U_{\alpha\beta\gamma}$ and $\varphi_{\alpha\alpha}(x) = e$, the unit in the group, such that $\psi_{\alpha\beta}(x, s) = \ell(\varphi_{\alpha\beta}(x), s) = \varphi_{\alpha\beta}(x).s$.

A fiber bundle with a G-bundle structure is called a *G-bundle*. A fiber bundle atlas as in (2) is called a *G-atlas* and the family $(\varphi_{\alpha\beta})$ is also called a cocycle of transition functions, but now for the G-bundle.

To be more precise, two G-atlases are said to be equivalent (to describe the same G-bundle), if their union is also a G-atlas. This translates as follows to the two cocycles of transition functions, where we assume that the two coverings of M are the same (by passing to the common refinement, if necessary): $(\varphi_{\alpha\beta})$ and $(\varphi'_{\alpha\beta})$ are called *cohomologous* if there is a family $(\tau_\alpha : U_\alpha \to G)$ such that $\varphi_{\alpha\beta}(x) = \tau_\alpha(x)^{-1}.\varphi'_{\alpha\beta}(x).\tau_\beta(x)$ holds for all $x \in U_{\alpha\beta}$.

In (2) one should specify only an equivalence class of G-bundle structures or only a cohomology class of cocycles of G-valued transition functions. From any open cover (U_α) of M, some cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$ for it, and a left G-action on a manifold S, we may construct a G-bundle by gluing, which depends only on the cohomology class of the cocycle. By some abuse of notation we write (E, p, M, S, G) for a fiber bundle with specified G-bundle structure.

Examples: The tangent bundle of a manifold M is a fiber bundle with structure group $GL(n)$. More general a vector bundle (E, p, M, V) is a fiber bundle with standard fiber the vector space V and with $GL(V)$-structure.

10.2. Definition. A *principal (fiber) bundle* (P, p, M, G) is a G-bundle with typical fiber a Lie group G, where the left action of G on G is just the left translation.

So by 10.1 we are given a bundle atlas $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ such that we have $\varphi_\alpha\varphi_\beta^{-1}(x, a) = (x, \varphi_{\alpha\beta}(x).a)$ for the cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$. This is now called a *principal bundle atlas*. Clearly the principal bundle is uniquely specified by the cohomology class of its cocycle of transition functions.

Each principal bundle admits a unique right action $r : P \times G \to P$, called the *principal right action*, given by $\varphi_\alpha(r(\varphi_\alpha^{-1}(x,a),g)) = (x,ag)$. Since left and right translation on G commute, this is well defined. As in 1.3 we write $r(u,g) = u.g$ when the meaning is clear. The principal right action is visibly free and for any $u_x \in P_x$ the partial mapping $r_{u_x} = r(u_x, \quad) : G \to P_x$ is a diffeomorphism onto the fiber through u_x, whose inverse is denoted by $\tau_{u_x} : P_x \to G$. These inverses together give a smooth mapping $\tau : P \times_M P \to G$, whose local expression is $\tau(\varphi_\alpha^{-1}(x,a), \varphi_\alpha^{-1}(x,b)) = a^{-1}.b$. This mapping is also uniquely determined by the implicit equation $r(u_x, \tau(u_x, v_x)) = v_x$, thus we also have $\tau(u_x.g, u'_x.g') = g^{-1}.\tau(u_x, u'_x).g'$ and $\tau(u_x, u_x) = e$.

10.3. Lemma. *Let $p : P \to M$ be a surjective submersion (a fibred manifold), and let G be a Lie group which acts freely on P such that the orbits of the action are exactly the fibers $p^{-1}(x)$ of p. Then (P,p,M,G) is a principal fiber bundle.*

Proof. Let the action be a right one by using the group inversion if necessary. Let $s_\alpha : U_\alpha \to P$ be local sections (right inverses) for $p : P \to M$ such that (U_α) is an open cover of M. Let $\varphi_\alpha^{-1} : U_\alpha \times G \to P|U_\alpha$ be given by $\varphi_\alpha^{-1}(x,a) = s_\alpha(x).a$, which is obviously injective with invertible tangent mapping, so its inverse $\varphi_\alpha : P|U_\alpha \to U_\alpha \times G$ is a fiber respecting diffeomorphism. So $(U_\alpha, \varphi_\alpha)$ is already a fiber bundle atlas. Let $\tau : P \times_M P \to G$ be given by the implicit equation $r(u_x, \tau(u_x, u'_x)) = u'_x$, where r is the right G-action. τ is smooth by the implicit function theorem and clearly $\tau(u_x, u'_x.g) = \tau(u_x, u'_x).g$. Thus we have $\varphi_\alpha \varphi_\beta^{-1}(x,g) = \varphi_\alpha(s_\beta(x).g) = (x, \tau(s_\alpha(x), s_\beta(x).g)) = (x, \tau(s_\alpha(x), s_\beta(x)).g)$ and $(U_\alpha, \varphi_\alpha)$ is a principal bundle atlas. $\square$

10.4. Remarks. In the proof of Lemma 10.3 we have seen, that a principal bundle atlas of a principal fiber bundle (P,p,M,G) is already determined if we specify a family of smooth sections of P, whose domains of definition cover the base M.

Lemma 10.3 can serve as an equivalent definition for a principal bundle. But this is true only if an implicit function theorem is available, so in topology or in infinite dimensional differential geometry one should stick to our original definition.

From the Lemma itself it follows, that the pullback f^*P over a smooth mapping $f : M' \to M$ is again a principal fiber bundle.

10.5. Homogeneous spaces. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let K be a closed subgroup of G, then by a well known theorem K is a closed Lie subgroup whose Lie algebra will be denoted by $\mathcal{K}$. There is a unique structure of a smooth manifold on the quotient space G/K such that the projection $p : G \to G/K$ is a submersion, so by the implicit function theorem p admits local sections.

Theorem. $(G, p, G/K, K)$ *is a principal fiber bundle.*

Proof. The group multiplication of G restricts to a free right action $\mu : G \times K \to G$, whose orbits are exactly the fibers of p. By lemma 10.3 the result follows. $\square$

For the convenience of the reader we discuss now the best know homogeneous spaces.

The group $SO(n)$ acts transitively on $S^{n-1} \subset \mathbf{R}^n$. The isotropy group of the "north pole" $(0, \dots, 0, 1)$ is the subgroup

$$\begin{pmatrix} 1 & 0 \\ 0 & SO(n-1) \end{pmatrix}$$

which we identify with $SO(n-1)$. So $S^{n-1} = SO(n)/SO(n-1)$ and $(SO(n), p, S^{n-1}, SO(n-1))$ is a principal fiber bundle. Likewise
$(O(n), p, S^{n-1}, O(n-1))$,
$(SU(n), p, S^{2n-1}, SU(n-1))$,
$(U(n), p, S^{2n-1}, U(n-1))$, and
$(Sp(n), p, S^{4n-1}, Sp(n-1))$ are principal fiber bundles.

The *Grassmann manifold* $G(k, n; \mathbf{R})$ is the space of all k-planes containing 0 in $\mathbf{R}^n$. The group $O(n)$ acts transitively on it and the isotropy group of the k-plane $\mathbf{R}^k \times \{0\}$ is the subgroup

$$\begin{pmatrix} O(k) & 0 \\ 0 & O(n-k) \end{pmatrix},$$

therefore $G(k, n; \mathbf{R}) = O(n)/O(k) \times O(n-k)$ is a compact manifold and we get the principal fiber bundle $(O(n), p, G(k, n; \mathbf{R}), O(k) \times O(n-k))$. Likewise
$(SO(n), p, \tilde{G}(k, n, \mathbf{R}), SO(k) \times SO(n-k))$,
$(U(n), p, G(k, n, \mathbf{C}), U(k) \times U(n-k))$, and
$(Sp(n), p, G(k, n, \mathbf{H}), Sp(k) \times Sp(n-k))$ are principal fiber bundles.

The *Stiefel manifold* $V(k, n, \mathbf{R})$ is the space of all orthonormal k-frames in $\mathbf{R}^n$. Clearly the group $O(n)$ acts transitively on $V(k, n, \mathbf{R})$ and the isotropy subgroup of $(e_1, \dots, e_k)$ is $\mathbf{1} \times O(n-k)$, so $V(k, n, \mathbf{R}) = O(n)/O(n-k)$ is a compact manifold and we get a principal fiber bundle $(O(n), p, V(k, n, \mathbf{R}), O(n-k))$ But $O(k)$ also acts from the right on $V(k, n, \mathbf{R})$, its orbits are exactly the fibers of the projection $p : V(k, n, \mathbf{R}) \to G(k, n, \mathbf{R})$. So by lemma 10.3 we get a principal fiber bundle $(V(k,n,\mathbf{R}), p, G(k,n,\mathbf{R}), O(k))$. Indeed we have the following diagram where all arrows are projections of principal fiber bundles, and where the respective structure groups are written on the arrows:

(a)
$$\begin{tikzcd} O(n) \arrow[r, "O(n-k)"] \arrow[d, "O(k)"'] & V(k,n,\mathbf{R}) \arrow[d, "O(k)"'] \\ V(n-k,n,\mathbf{R}) \arrow[r, "O(n-k)"'] & G(k,n,\mathbf{R}), \end{tikzcd}$$

It is easy to see that $V(k,n)$ is also diffeomorphic to the space $\{ A \in L(\mathbf{R}^k, \mathbf{R}^n) : A^t.A = \mathbf{1}_k \}$, i.e. the space of all linear isometries $\mathbf{R}^k \to \mathbf{R}^n$. There are furthermore complex and quaternionic versions of the Stiefel manifolds.

More examples will be given in sections on jets below.

10.6. Homomorphisms. Let $\chi : (P, p, M, G) \to (P', p', M', G)$ be a *principal fiber bundle homomorphism*, i.e. a smooth G-equivariant mapping $\chi : P \to P'$. Then obviously the diagram

(a)
$$\begin{tikzcd} P \arrow[r, "\chi"] \arrow[d, "p"'] & P' \arrow[d, "p'"'] \\ M \arrow[r, "\bar\chi"'] & M' \end{tikzcd}$$

commutes for a uniquely determined smooth mapping $\bar\chi : M \to M'$. For each $x \in M$ the mapping $\chi_x := \chi | P_x : P_x \to P'_{\bar\chi(x)}$ is G-equivariant and therefore a diffeomorphism, so diagram (a) is a pullback diagram.

But the most general notion of a homomorphism of principal bundles is the following. Let $\Phi : G \to G'$ be a homomorphism of Lie groups. $\chi : (P, p, M, G) \to (P', p', M', G')$ is called a *homomorphism over* Φ *of principal bundles*, if $\chi : P \to P'$ is smooth and $\chi(u.g) = \chi(u).\Phi(g)$ holds in general. Then χ is fiber respecting, so diagram (a) makes again sense, but it is no longer a pullback diagram in general.

If χ covers the identity on the base, it is called a *reduction of the structure group* G' to G for the principal bundle (P', p', M', G') — the name comes from the case, when Φ is the embedding of a subgroup.

By the universal property of the pullback any general homomorphism χ of principal fiber bundles over a group homomorphism can be written as the composition of a reduction of structure groups and a pullback homomorphism as follows, where we also indicate the structure groups:

$$\text{(b)}\qquad \begin{tikzcd} (P,G) \arrow[r] \arrow[d, "p"'] & (\bar\chi^*P',G') \arrow[r] \arrow[d] & (P',G') \arrow[d, "p'"'] \\ M \arrow[r, equal] & M \arrow[r, "\bar\chi"'] & M' \end{tikzcd}$$

10.7. Associated bundles. Let (P,p,M,G) be a principal bundle and let $\ell : G\times S \to S$ be a left action of the structure group G on a manifold S. We consider the right action $R : (P\times S)\times G \to P\times S$, given by $R((u,s),g) = (u.g, g^{-1}.s)$.

Theorem. *In this situation we have:*

(1) *The space $P\times_G S$ of orbits of the action R carries a unique smooth manifold structure such that the quotient map $q : P\times S \to P\times_G S$ is a submersion.*
(2) *$(P\times_G S, \bar p, M, S, G)$ is a G-bundle, where $\bar p : P\times_G S \to M$ is given by*

$$\text{(a)}\qquad \begin{tikzcd} P\times S \arrow[r, "q"] \arrow[d, "pr_1"'] & P\times_G S \arrow[d, "\bar p"'] \\ P \arrow[r, "p"'] & M. \end{tikzcd}$$

In this diagram $q_u : \{u\}\times S \to (P\times_G S)_{p(u)}$ is a diffeomorphism for each $u\in P$.
(3) *$(P\times S, q, P\times_G S, G)$ is a principal fiber bundle with principal action R.*
(4) *If $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha\times G)$ is a principal bundle atlas with cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta}\to G)$, then together with the left action $\ell : G\times S\to S$ this cocycle is also one for the G-bundle $(P\times_G S, \bar p, M, S, G)$.*

NOTATION: $(P\times_G S,\bar p, M, S, G)$ is called the *associated bundle* for the action $\ell : G\times S\to S$. We will also denote it by $P[S,\ell]$ or simply $P[S]$ and we will write p for $\bar p$ if no confusion is possible. We also define the smooth mapping $\tau : P\times_M P[S,\ell] \to S$ by $\tau(u_x, v_x) := q_{u_x}^{-1}(v_x)$. It satisfies $\tau(u, q(u,s)) = s$, $q(u_x, \tau(u_x,v_x)) = v_x$, and $\tau(u_x.g, v_x) = g^{-1}.\tau(u_x,v_x)$. In the special situation, where $S = G$ and the action is left translation, so that $P[G] = P$, this mapping coincides with τ considered in 10.2.

Proof. In the setting of the diagram in (2) the mapping $p\circ pr_1$ is constant on the R-orbits, so $\bar p$ exists as a mapping. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha\times G)$

be a principal bundle atlas with transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$. We define $\psi_\alpha^{-1} : U_\alpha \times S \to \bar{p}^{-1}(U_\alpha) \subset P \times_G S$ by $\psi_\alpha^{-1}(x,s) = q(\varphi_\alpha^{-1}(x,e),s)$, which is fiber respecting. For each orbit in $\bar{p}^{-1}(x) \subset P \times_G S$ there is exactly one $s \in S$ such that this orbit passes through $(\varphi_\alpha^{-1}(x,e),s)$, since the principal right action is free. Thus $\psi_\alpha^{-1}(x.\quad) : S \to \bar{p}^{-1}(x)$ is bijective. Furthermore

$$\begin{aligned}
\psi_\beta^{-1}(x,s) = q(\varphi_\beta^{-1}(x,e),s) &= \\
&= q(\varphi_\alpha^{-1}(x,\varphi_{\alpha\beta}(x).e),s) = q(\varphi_\alpha^{-1}(x,e).\varphi_{\alpha\beta}(x),s) = \\
&\qquad = q(\varphi_\alpha^{-1}(x,e),\varphi_{\alpha\beta}(x).s) = \psi_\alpha^{-1}(x,\varphi_{\alpha\beta}(x).s),
\end{aligned}$$

so $\psi_\alpha\psi_\beta^{-1}(x,s) = (x,\varphi_{\alpha\beta}(x).s)$ So (U_α,ψ_α) is a G-atlas for $P \times_G S$ and makes it into a smooth manifold and a G-bundle. The defining equation for ψ_α shows that q is smooth and a submersion and consequently the smooth structure on $P \times_G S$ is uniquely defined, and $\bar{p}$ is smooth by the universal properties of a submersion.

By the definition of ψ_α the diagram

$$\text{(b)} \qquad \begin{array}{ccc}
p^{-1}(U_\alpha) \times S & \xrightarrow{\varphi_\alpha \times Id} & U_\alpha \times G \times S \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Id \times \ell} \\
\bar{p}^{-1}(U_\alpha) & = & U_\alpha \times S
\end{array}$$

commutes; since its lines are diffeomorphisms we conclude that $q_u : \{u\} \times S \to \bar{p}^{-1}(p(u))$ is a diffeomorphism. So (1), (2), and (4) are checked. (3) We rewrite the last diagram in the following form:

$$\text{(c)} \qquad \begin{array}{cccc}
p^{-1}(U_\alpha) \times S & = & q^{-1}(V_\alpha) \xrightarrow{\lambda_\alpha} & V_\alpha \times G \\
 & & {\scriptstyle q}\downarrow & \downarrow{\scriptstyle pr_1} \\
 & & \bar{p}^{-1}(U_\alpha) \quad = & V_\alpha
\end{array}$$

Here $V_\alpha := \bar{p}^{-1}(U_\alpha) \subset P \times_G S$ and the diffeomorphism λ_α is defined by $\lambda_\alpha^{-1}(\psi_\alpha^{-1}(x,s),g) := (\varphi_\alpha(x,g), g^{-1}.s)$. Then we have

$$\begin{aligned}
&\lambda_\beta^{-1}(\psi_\alpha^{-1}(x,s),g) = \lambda_\beta^{-1}(\psi_\beta^{-1}(x,,\varphi_{\beta\alpha}(x).s),g) = \\
&= (\varphi_\beta^{-1}(x,g), g^{-1}.\varphi_{\beta\alpha}(x).s) = (\varphi_\alpha^{-1}(x,\varphi_{\alpha\beta}(x).g), g^{-1}.\varphi_{\alpha\beta}(x)^{-1}.s) = \\
&\qquad\qquad = (\lambda_\alpha^{-1}(\psi_\alpha^{-1}(x,s),\varphi_{\alpha\beta}(x).g),
\end{aligned}$$

so $\lambda_\alpha\lambda_\beta^{-1}(\psi_\alpha^{-1}(x,s),g) = (\psi_\alpha^{-1}(x,s),\varphi_{\alpha\beta}(x).g)$ and we conclude that $(P \times S, q, P \times_G S, G)$ is a principal bundle with structure group G and the same cocycle $(\varphi_{\alpha\beta})$ we started with. $\square$

10.8. Corollary. *Let (E, p, M, S, G) be a G-bundle, specified by a cocycle of transition functions $(\varphi_{\alpha\beta})$ with values in G and a left action ℓ of G on S. Then from the cocycle of transition functions we may glue a unique principal bundle (P, p, M, G) such that $E = P[S, \ell]$.* $\square$

This is the usual way a differential geometer thinks of an associated bundle. He is given a bundle E, a principal bundle P, and the G-bundle structure then is described with the help of the mappings τ and q.

10.9. Equivariant mappings and associated bundles.

1. Let (P, p, M, G) be a principal fiber bundle and consider two left actions of G, $\ell : G \times S \to S$ and $\ell' : G \times S' \to S'$. Let furthermore $f : S \to S'$ be a G-equivariant smooth mapping, so $f(g.s) = g.f(s)$ or $f \circ \ell_g = \ell'_g \circ f$. Then $Id_P \times f : P \times S \to P \times S'$ is equivariant for the actions $R : (P \times S) \times G \to P \times S$ and $R' : (P \times S') \times G \to P \times S'$, so there is an induced mapping

(a)
$$\begin{array}{ccc} P \times S & \xrightarrow{Id \times f} & P \times S' \\ {\scriptstyle q}\downarrow & & {\scriptstyle q'}\downarrow \\ P \times_G S & \xrightarrow{Id \times_G f} & P \times_G S', \end{array}$$

which is fiber respecting over M, and a homomorphism of G-bundles in the sense of the definition 10.10 below.

2. Let $\chi : (P, p, M, G) \to (P', p', M', G)$ be a principal fiber bundle homomorphism as in 10.6. Furthermore we consider a smooth left action $\ell : G \times S \to S$. Then $\chi \times Id_S : P \times S \to P' \times S$ is G-equivariant and induces a mapping $\chi \times_G Id_S : P \times_G S \to P' \times_G S$, which is fiber respecting over M, fiber wise a diffeomorphism, and again a homomorphism of G-bundles in the sense of definition 10.10 below.

3. Now we consider the situation of 1 and 2 at the same time. We have two associated bundles $P[S, \ell]$ and $P'[S', \ell']$. Let $\chi : (P, p, M, G) \to (P', p', M', G)$ be a principal fiber bundle homomorphism and let $f : S \to S'$ be an G-equivariant mapping. Then $\chi \times f : P \times S \to P' \times S'$ is clearly G-equivariant and therefore induces a mapping $\chi \times_G f : P[S, \ell] \to P'[S', \ell']$ which again is a homomorphism of G-bundles.

4. Let S be a point. Then $P[S] = P \times_G S = M$. Furthermore let $y \in S'$ be a fixed point of the action $\ell' : G \times S' \to S'$, then the inclusion $i : \{y\} \hookrightarrow S'$ is G-equivariant, thus $Id_P \times i$ induces $Id_P \times_G i : M = P[\{y\}] \to P[S']$, which is a global section of the associated bundle $P[S']$.

If the action of G on S is trivial, so $g.s = s$ for all $s \in S$, then the associated bundle is trivial: $P[S] = M \times S$. For a trivial principal fiber bundle any associated bundle is trivial.

10.10. Definition. In the situation of 10.8, a smooth fiber respecting mapping $\gamma : P[S,\ell] \to P'[S',\ell']$ covering a smooth mapping $\bar\gamma : M \to M'$ of the bases is called a *homomorphism of G-bundles*, if the following conditions are satisfied: P is isomorphic to the pullback $\bar\gamma^* P'$, and the local representations of γ in pullback-related fiber bundle atlases belonging to the two G-bundles are fiber wise G-equivariant.

Let us describe this in more detail now. Let $(U'_\alpha, \psi'_\alpha)$ be a G-atlas for $P'[S',\ell']$ with cocycle of transition functions $(\varphi'_{\alpha\beta})$, belonging to the principal fiber bundle atlas $(U'_\alpha, \varphi'\alpha)$ of (P', p', M', G). Then the pullback-related principal fiber bundle atlas $(U_\alpha = \bar\gamma^{-1}(U'_\alpha), \gamma_\alpha)$ for $P = \bar\gamma^* P'$ as described in the proof of 9.5 has the cocycle of transition functions $(\varphi_{\alpha\beta} = \varphi'_{\alpha\beta} \circ \bar\gamma)$; it induces the G-atlas (U_α, ψ_α) for $P[S,\ell]$. Then $(\psi'_\alpha \circ\gamma\circ\psi_\alpha^{-1})(x,s) = (\bar\gamma(x), \gamma_\alpha(x,s))$ and $\gamma_\alpha(x, \quad) : S \to S'$ should be G-equivariant for all α and all $x \in U_\alpha$.

Lemma. *Let $\gamma : P[S,\ell] \to P'[S',\ell']$ be a homomorphism of associated bundles and G-bundles. Then there is a principal bundle homomorphism $\chi : (P,p,M,G) \to (P',p',M',G)$ and a G-equivariant mapping $f : S \to S'$ such that $\gamma = \chi \times_G f : P[S,\ell] \to P'[S',\ell']$.*

Proof. The homomorphism $\chi : (P,p,M,G) \to (P',p',M',G)$ of principal fiber bundles is already determined by the requirement that $P = \bar\gamma^* P'$, and we have $\bar\gamma = \bar\chi$. The G-equivariant mapping $f : S \to S'$ can be read off the following diagram

(a)
$$\begin{array}{ccc} P \times_M P[S] & \xrightarrow{\tau} & S \\ {\scriptstyle \chi\times_M\gamma}\downarrow & & {\scriptstyle f}\downarrow \\ P' \times_{M'} P'[S'] & \xrightarrow[\tau']{} & S', \end{array}$$

which by the assumptions is seen to be well defined in the right column. $\square$

So a homomorphism of associated bundles is described by the whole triple $(\chi : P \to P', f : S \to S'$ (G-equivariant) $, \gamma : E \to E')$, such that diagram (a) commutes.

10.11. Associated vector bundles. Let (P,p,M,G) be a principal fiber bundle, and consider a representation $\rho : G \to GL(V)$ of G on a finite dimensional vector space V. Then $P[V,\rho]$ is an associated fiber bundle with structure group G, but also with structure group $GL(V)$, for in the canonically associated fiber bundle atlas the transition functions have also values in $GL(V)$. So by section 6 $P[V,\rho]$ is a vector bundle.

If (E, p, M) is a vector bundle with n-dimensional fibers we may consider the open subset $GL(\mathbf{R}^n, E) \subset L(M \times \mathbf{R}^n, E)$, consisting of all invertible linear mappings, which is a fiber bundle over the base M. Composition from the right by elements of $GL(n)$ gives a free right action on $GL(\mathbf{R}^n, E)$ whose orbits are exactly the fibers, so by lemma 10.3 we have a principal fiber bundle $(GL(\mathbf{R}^n, E), p, M, GL(n))$. The associated bundle $GL(\mathbf{R}^n, E)[\mathbf{R}^n]$ for the banal representation of $GL(n)$ on $\mathbf{R}^n$ is isomorphic to the vector bundle (E, p, M) we started with, for the evaluation mapping $ev : GL(\mathbf{R}^n, E) \times \mathbf{R}^n \to E$ is invariant under the right action R of $GL(n)$, and locally in the image there are smooth sections to it, so it factors to a fiber linear diffeomorphism $GL(\mathbf{R}^n, E)[\mathbf{R}^n] = GL(\mathbf{R}^n, E) \times_{GL(n)} \mathbf{R}^n \to E$. The principal bundle $GL(\mathbf{R}^n, E)$ is called the *linear frame bundle of* E. Note that local sections of $GL(\mathbf{R}^n)$ are exactly the local frame fields of the vector bundle E.

To illustrate the notion of reduction of structure group, we consider now a vector bundle $(E, p, M, \mathbf{R}^n)$ equipped with a *Riemannian metric* g, that is a section $g \in C^\infty(S^2E^*)$ such that g_x is a positive definite inner product on E_x for each $x \in M$. Any vector bundle admits Riemannian metrics: local existence is clear and we may glue with the help of a partition of unity on M, since the positive definite sections form an open convex subset. Now let $s' = (s'_1, \dots, s'_n) \in C^\infty(GL(\mathbf{R}^n, E)|U)$ be a local frame field of the bundle E over $U \subset M$. Now we may apply the Gram-Schmidt orthonormalization procedure to the basis $(s_1(x), \dots, s_n(x))$ of E_x for each $x \in U$. Since this procedure is real analytic, we obtain a frame field $s = (s_1, \dots, s_n)$ of E over U which is orthonormal with respect to g. We call it an *orthonormal frame field.* Now let (U_α) be an open cover of M with orthonormal frame fields $s^\alpha = (s^\alpha_1, \dots, s^\alpha_n)$, where s^α is defined on U_α. We consider the vector bundle charts $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbf{R}^n)$ given the orthonormal frame fields: $\psi_\alpha^{-1}(x, v^1, \dots, v^n) = \sum s^\alpha_i(x).v^i =: s^\alpha(x).v$. For $x \in U_{\alpha\beta}$ we have $s^\alpha_i(x) = \sum s^\beta_j(x).g_{\beta\alpha}{}^j_i(x)$ for C^∞-functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbf{R}$. Since $s^\alpha(x)$ and $s^\beta(x)$ are both orthonormal bases of E_x, the matrix $g_{\alpha\beta}(x) = (g_{\alpha\beta}{}^j_i(x))$ is an element of $O(n, \mathbf{R})$. We write $s^\alpha = s^\beta.g_{\beta\alpha}$ for short. Then we have $\psi_\beta^{-1}(x, v) = s^\beta(x).v = s^\alpha(x).g_{\alpha\beta}(x).v = \psi_\alpha^{-1}(x, g_{\alpha\beta}(x).v)$ and consequently $\psi_\alpha\psi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x).v)$. So the $(g_{\alpha\beta} : U_{\alpha\beta} \to O(n, \mathbf{R}))$ are the cocycle of transition functions for the vector bundle atlas (U_α, ψ_α). So we have constructed an $O(n, \mathbf{R})$-structure on E. The corresponding principal fiber bundle will be denoted by $O(\mathbf{R}^n, (E, g))$; it is usually called the *orthonormal frame bundle* of E. It is derived from the linear frame bundle $GL(\mathbf{R}^n, E)$ by reduction of the structure group from $GL(n)$ to $O(n)$. The phenomenon discussed here plays a prominent role in the theory of *classifying spaces*.

10.12. Sections of associated bundles. Let (P, p, M, G) be a principal fiber bundle and $\ell : G \times S \to S$ a left action. Let $C^\infty(P, S)^G$ denote the space of all smooth mappings $f : P \to S$ which are G-equivariant in the sense that $f(u.g) = g^{-1}.f(u)$ holds for $g \in G$ and $u \in P$.

Theorem. *The sections of the associated bundle $P[S, \ell]$ correspond exactly to the G-equivariant mappings $P \to S$; we have a bijection*

$$C^\infty(P, S)^G \cong C^\infty(P[S]).$$

Proof. If $f \in C^\infty(P, S)^G$ we construct $s_f \in C^\infty(P[S])$ in the following way: $\operatorname{graph}(f) = (Id, f) : P \to P \times S$ is G-equivariant, since $(Id, f)(u.g) = (u.g, f(u.g)) = (u.g, g^{-1}.f(u)) = ((Id, f)(u)).g$. So it induces a smooth section $s_f \in C^\infty(P[S])$ as seen from

$$\text{(a)} \qquad \begin{array}{ccc} P & \xrightarrow{(Id,f)} & P \times S \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ M & \xrightarrow[f_s]{} & P[S] \end{array}$$

If conversely $s \in C^\infty(P[S])$ we define $f_s \in C^\infty(P, S)^G$ by $f_s := \tau \circ (Id_P \times_M f) : P = P \times_M M \to P \times_m P[S] \to S$. This is G-equivariant since $f_s(u_x.g) = \tau(u_x.g, f(x)) = g^{-1}.\tau(u_x, f(x)) = g^{-1}.f_s(u_x)$ by 10.4. The two constructions are inverse to each other since we have $f_{s(f)}(u) = \tau(u, s_f(p(u))) = \tau(u, q(u, f(u))) = f(u)$ and $s_{s(f)}(p(u)) = q(u, f_s(u)) = q(u, \tau(u, f(p(u))) = f(p(u))$. $\square$

10.13. Theorem. *Consider a principal fiber bundle (P, p, M, G) and a closed subgroup K of G. Then the reductions of structure group from G to K correspond bijectively to the global sections of the associated bundle $P[G/K, \bar\lambda]$, where $\bar\lambda : G \times G/K \to G/K$ is the left action on the homogeneous space.*

Proof. By theorem 10.12 the section $s \in C^\infty(P[G/K])$ corresponds to $f_s \in C^\infty(P, G/K)^G$, which is a surjective submersion since the action $\bar\lambda : G \times G/K \to G/K$ is transitive. Thus $P_s := f_s^{-1}(\bar e)$ is a submanifold of P which is stable under the right action of K on P. Furthermore the K-orbits are exactly the fibers of the mapping $p : P_s \to M$, so by lemma 10.3 we get a principal fiber bundle (P_s, p, M, K). The embedding $P_s \hookrightarrow P$ is then a reduction of structure groups as required.

If conversely we have a principal fiber bundle (P', p', M, K) and a reduction of structure groups $\chi : P' \to P$, then χ is an embedding covering the identity of M and is K-equivariant, so we may view P' as a sub fiber bundle of P which is stable under the right action of K. Now we consider the mapping $\tau : P \times_M P \to G$ from 10.2 and restrict it to $P \times_M P'$. Since we have $\tau(u_x, v_x.k) = \tau(u_x, v_x).k$ for $k \in K$ this restriction induces $f : P \to G/K$ by

$$\begin{array}{ccc} P \times_M P' & \xrightarrow{\tau} & G \\ \downarrow & & \downarrow p \\ P = P \times_M P'/K & \xrightarrow[f]{} & G/K; \end{array}$$

and from $\tau(u_x.g, v_x) = g^{-1}.\tau(u_x, v_x)$ it follows that f is G-equivariant as required. Finally $f^{-1}(\bar{e}) = \{u \in P : \tau(u, P'_{p(u)}) \subseteq K\} = P'$, so the two constructions are inverse to each other. $\square$

10.14. The bundle of gauges. If (P, p, M, G) is a principal fiber bundle we denote by $\mathrm{Aut}(P)$ the group of all G-equivariant diffeomorphisms $\chi : P \to P$. Then $p \circ \chi = \bar{\chi} \circ p$ for a unique diffeomorphism $\bar{\chi}$ of M, so there is a group homomorphism from $\mathrm{Aut}(P)$ into the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M. The kernel of this homomorphism is called $\mathrm{Gau}(P)$, the group of *gauge transformations*. So $\mathrm{Gau}(P)$ is the space of all $\chi : P \to P$ which satisfy $p \circ \chi = p$ and $\chi(u.g) = \chi(u).g$.

Theorem. *The group $Gau(P)$ of gauge transformations is equal to the space $C^\infty(P, (G, conj))^G \cong C^\infty(P[G, conj])$.*

Proof. We use again the mapping $\tau : P \times_M P \to G$ from 10.2. For $\chi \in \mathrm{Gau}(P)$ we define $f_\chi \in C^\infty(P, (G, conj))^G$ by $f_\chi := \tau \circ (Id, \chi)$. Then $f_\chi(u.g) = \tau(u.g, \chi(u.g)) = g^{-1}.\tau(u, \chi(u)).g = conj_{g^{-1}} f_\chi(u)$, so f_χ is indeed G-equivariant.

If conversely $f \in C^\infty(P, (G, conj))^G$ is given, we define $\chi_f : P \to P$ by $\chi_f(u) := u.f(u)$. It is easy to check that χ_f is indeed in $\mathrm{Gau}(P)$ and that the two constructions are inverse to each other. $\square$

10.15 The tangent bundles of homogeneous spaces. Let G be a Lie group and K a closed subgroup, with Lie algebras $\mathfrak{g}$ and $\mathcal{K}$, respectively. We recall the mapping $\mathrm{Ad}_G : G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ from 1.1 and put $\mathrm{Ad}_{G,K} := \mathrm{Ad}_G|K : K \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$. For $X \in \mathcal{K}$ and $k \in K$ we have $\mathrm{Ad}_{G,K}(k)X = \mathrm{Ad}_G(k)X = \mathrm{Ad}_K(k)X \in \mathcal{K}$, so $\mathcal{K}$ is an invariant subspace for the representation $\mathrm{Ad}_{G,K}$ of K in $\mathfrak{g}$,and we have the factor representation $\mathrm{Ad}^\perp : K \to GL(\mathfrak{g}/\mathcal{K}$. Then

$$\text{(a)} \qquad 0 \to \mathcal{K} \to \mathfrak{g} \to \mathfrak{g}/\mathcal{K} \to 0$$

is short exact and K-equivariant.

Now we consider the principal fiber bundle $(G, p, G/K, K)$ and the associated vector bundles $G[\mathfrak{g}/\mathcal{K}, \mathrm{Ad}^\perp]$ and $G[\mathcal{K}, \mathrm{Ad}_{G,K}]$.

Theorem. *In these circumstances we have*

$$T(G/K) = G[\mathfrak{g}/\mathcal{K}, Ad^\perp] = (G \times_K \mathcal{G}/\mathcal{K}, p, G/K, \mathfrak{g}/\mathcal{K}).$$

The left action $g \mapsto T(\bar\lambda_g)$ of G on $T(G/K)$ coincides with the left action of G on $G \times_K \mathfrak{g}/\mathcal{K}$. Furthermore $G[\mathfrak{g}/\mathcal{K}, Ad^\perp] \oplus G[\mathcal{K}, Ad_{G,K}]$ is a trivial vector bundle.

Proof. For $p : G \to G/K$ we consider the tangent mapping $T_e p : \mathfrak{g} \to T_{\bar e}(G/K)$ which is linear and surjective and induces a linear isomorphism $\overline{T_e p} : \mathfrak{g}/\mathcal{K} \to T_{\bar e}(G/K)$. For $k \in K$ we have $p \circ conj_k = p \circ \lambda_k \circ \rho_{k^{-1}} = \bar\lambda_k \circ p$ and consequently $T_e p \circ \mathrm{Ad}_{G,K}(k) = T_e p \circ T_e(conj_k) = T_{\bar e}\bar\lambda_k \circ T_e p$. Thus the isomorphism $\overline{T_e p} : \mathfrak{g}/\mathcal{K} \to T_{\bar e}(G/K)$ is K-equivariant for the representations $\mathrm{Ad}^\perp$ and $T_{\bar e}\bar\lambda : k \mapsto T_{\bar e}\bar\lambda_k$.

Now we consider the associated vector bundle $G[T_{\bar e}(G/K), T_{\bar e}\bar\lambda] = (G \times_K T_{\bar e}(G/K), p, G/K, T_{\bar e}(G/K))$, which is isomorphic to the vector bundle $G[\mathfrak{g}/\mathcal{K}, \mathrm{Ad}^\perp]$, since the representation spaces are isomorphic. The mapping $T\bar\lambda : G \times T_{\bar e}(G/K) \to T(G/K)$ is K-invariant and therefore induces a mapping ψ as in the following diagram:

$$\text{(b)} \qquad \begin{tikzcd}
G \times T_{\bar e}(G/K) \arrow[r, "T\bar\lambda"] \arrow[d, "q"'] & T(G/K) \arrow[d, equal] \\
G \times_K T_{\bar e}(G/K) \arrow[r, "\psi"'] \arrow[d, "p"'] & T(G/K) \arrow[d, "\pi_{G/K}"'] \\
G/K \arrow[r, equal] & G/K
\end{tikzcd}$$

This mapping ψ is an isomorphism of vector bundles.

It remains to show the last assertion. The short exact sequence (a) induces a sequence of vector bundles over G/K:

$$G/K \times 0 \to G[\mathcal{K}, \mathrm{Ad}_K] \to G[\mathcal{G}, \mathrm{Ad}_{G,K}] \to G[\mathfrak{g}/\mathcal{K}, \mathrm{Ad}^\perp] \to G/K \times 0$$

This sequence splits fiber wise thus also locally over G/K, so we obtain $G[\mathfrak{g}/\mathcal{K}, \mathrm{Ad}^\perp] \oplus G[\mathcal{K}, \mathrm{Ad}_{G,K}] \cong G[\mathfrak{g}, \mathrm{Ad}_{G,K}]$ and it remains to show that $G[\mathfrak{g}, \mathrm{Ad}_{G,K}]$ is a trivial vector bundle. Let $\varphi : G \times \mathfrak{g} \to G \times \mathfrak{g}$ be given by $\varphi(g,X) = (g, \mathrm{Ad}_G(g)X)$. Then for $k \in K$ we have $\varphi((g,X).k) = \varphi(gk, \mathrm{Ad}_{G,K}(g^{-1})X) = (ak, \mathrm{Ad}_G(g.k.k^{-1})X) = (gk, \mathrm{Ad}_G(g)X)$. So φ is K-equivariant from the "joint" K-action to the "on the left" K-action and therefore induces a mapping $\bar{\varphi}$ as in the diagram:

(c)
$$\begin{tikzcd}
G \times \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "q"'] & G \times \mathfrak{g} \arrow[d] \\
G \times_K \mathfrak{g} \arrow[r, "\bar{\varphi}"'] \arrow[d, "p"'] & G/K \times \mathfrak{g} \arrow[d, "pr_1"'] \\
G/K \arrow[r, equal] & G/K
\end{tikzcd}$$

The map $\bar{\varphi}$ is a vector bundle isomorphism. $\square$

10.16 Tangent bundles of Grassmann manifolds. From 10.5 we know that $(V(k,n) = O(n)/O(n-k), p, G(k,n), O(k))$ is a principal fiber bundle. Using the banal representation of $O(k)$ we consider the associated vector bundle $(E_k := V(k,n)[\mathbf{R}^k], p, G(k,n))$. It is called the *universal vector bundle* over $G(k,n)$ for reasons we will discuss below in chapter 11. Recall from 10.5 the description of $V(k,n)$ as the space of all linear isometries $\mathbf{R}^k \to \mathbf{R}^n$; we get from it the evaluation mapping $ev : V(k,n) \times \mathbf{R}^k \to \mathbf{R}^n$. The mapping (p, ev) in the diagram

(a)
$$\begin{tikzcd}
V(k,n) \times \mathbf{R}^k \arrow[r, "(p,ev)"] \arrow[d, "q"'] & G(kn) \times \mathbf{R}^n \arrow[d, equal] \\
V(k,n) \times_{O(k)} \mathbf{R}^k \arrow[r, "\psi"'] & G(kn) \times \mathbf{R}^n
\end{tikzcd}$$

is $O(k)$-invariant for the action R and factors therefore to an embedding of vector bundles $\psi : E_k \to G(k,n) \times \mathbf{R}^n$. So the fiber $(E_k)_W$ over the k-plane W in $\mathbf{R}^n$ is just the linear subspace W. Note finally that the fiber wise orthogonal complement ${E_k}^\perp$ of E_k in the trivial vector bundle $G(k,n) \times \mathbf{R}^n$ with its standard Riemannian metric is isomorphic to the universal vector bundle E_{n-k} over $G(n-k,n)$, where the isomorphism covers the diffeomorphism $G(k,n) \to G(n-k,n)$ given also by the orthogonal complement mapping.

Corollary. *The tangent bundle of the Grassmann manifold is*
$$TG(k,n) = L(E_k, {E_k}^\perp).$$

Proof. We have $G(k,n) = O(n)/(O(k) \times O(n-k))$, so by theorem 10.15 we get

$$TG(k,n) = O(n) \underset{O(k)\times O(n-k)}{\times} so(n)/(so(k) \times so(n-k)).$$

On the other hand we have $V(k,n) = O(n)/O(n-k)$ and the right action of $O(k)$ commutes with the right action of $O(n-k)$ on $O(n)$, therefore

$$V(k,n)[\mathbf{R}^k] = O(n)/O(n-k) \underset{O(k)}{\times} \mathbf{R}^k = O(n) \underset{O(k)\times O(n-k)}{\times} \mathbf{R}^k,$$

where $O(n-k)$ acts trivially on $\mathbf{R}^k$. Finally

$$\begin{aligned} L(E_k, {E_k}^{\perp}) &= L\left(O(n) \underset{O(k)\times O(n-k)}{\times} \mathbf{R}^k, O(n) \underset{O(k)\times O(n-k)}{\times} \mathbf{R}^{n-k} \right) \\ &= O(n) \underset{O(k)\times O(n-k)}{\times} L(\mathbf{R}^k, \mathbf{R}^{n-k}), \end{aligned}$$

where the left action of $O(k) \times O(n-k)$ on $L(\mathbf{R}^k, \mathbf{R}^{n-k})$ is given by $(A,B)(C) = B.C.A^{-1}$. Finally we have an $O(k) \times O(n-k)$ - equivariant linear isomorphism $L(\mathbf{R}^k, \mathbf{R}^{n-k}) \to so(n)/(so(k) \times so(n-k))$, as follows:

$$\begin{aligned} &so(n)/(so(k) \times so(n-k)) = \\ &\qquad \frac{(\text{skew})}{\begin{pmatrix} skew & 0 \\ 0 & skew \end{pmatrix}} = \left\{ \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} : \quad A \in L(\mathbf{R}^k, \mathbf{R}^{n-k}) \right\} \quad \square \end{aligned}$$

10.17. The tangent group of a Lie group. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We will use the notation from 1.1. First note that TG is also a Lie group with multiplication $T\mu$ and inversion $T\nu$, given by the expressions $T_{(a,b)}\mu.(\xi_a, \eta_b) = T_a(\rho_b).\xi_a + T_b(\lambda_a).\eta_b$ and $T_a\nu.\xi_a = -T_e(\lambda_{a^{-1}}).T_a(\rho_{a^{-1}}).\xi_a$.

Lemma. *Via the isomomorphism $T\rho : \mathfrak{g} \times G \to TG$, $T\rho.(X,g) = T_e(\rho_g).X$, the group structure on TG looks as follows: $(X,a).(Y,b) = (X + Ad(a)Y, a.b)$ and $(X,a)^{-1} = (-Ad(a^{-1})X, a^{-1})$. So TG is isomorphic to the semidirect product $\mathfrak{g}Ⓢ G$.*

Proof. $T_{(a,b)}\mu.(T\rho_a.X, T\rho_b.Y) = T\rho_b.T\rho_a.X + T\lambda_a.T\rho_b.Y =$
$\quad = T\rho_{ab}.X + T\rho_b.T\rho_a.T\rho_{a^{-1}}.T\lambda_a.Y = T\rho_{ab}(X + \mathrm{Ad}(a)Y)$.
$T_a\nu.T\rho_a.X = -T\rho_{a^{-1}}.T\lambda_{a^{-1}}.T\rho_a.X = -T\rho_{a^{-1}}.\mathrm{Ad}(a^{-1})X$. $\square$

Remark. In the left trivialization $T\lambda : G \times \mathfrak{g} \to TG$, $T\lambda.(g, X) = T_e(\lambda_g).X$, the semidirect product structure looks somewhat awkward: $(a, X).(b, Y) = (ab, \mathrm{Ad}(b^{-1})X + Y)$ and $(a, X)^{-1} = (a^{-1}, -\mathrm{Ad}(a)X)$.

10.18. Tangent and vertical bundles.
For a fiber bundle (E, p, M, S) the subbundle $VE = \{ \xi \in TE : \quad Tp.\xi = 0 \}$ of TE is called the *vertical bundle* and is denoted by (VE, π_E, E).

Theorem. *Let (P, p, M, G) be a principal fiber bundle with principal right action $r : P \times G \to P$. Let $\ell : G \times S \to S$ be a left action. Then the following assertions hold:*

(1) *(TP, Tp, TM, TG) is again a principal fiber bundle with principal right action $Tr : TP \times TG \to TP$.*
(2) *The vertical bundle $(VP, \pi, P, \mathfrak{g})$ of the principal bundle is trivial as a vector bundle over P: $VP = P \times \mathfrak{g}$.*
(3) *The vertical bundle of the principal bundle as bundle over M is again a principal bundle: $(VP, p \circ \pi, M, TG)$.*
(4) *The tangent bundle of the associated bundle $P[S, \ell]$ is given by $T(P[S, \ell]) = TP[TS, T\ell]$.*
(5) *The vertical bundle of the associated bundle $P[S, \ell]$ is given by $V(P[S, \ell]) = P[TS, T_2\ell] = P \times_G TS$.*

Proof. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ be a principal fiber bundle atlas with cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$. Since T is a functor which respects products, $(TU_\alpha, T\varphi_\alpha : TP|TU_\alpha \to TU_\alpha \times TG)$ is again a principal fiber bundle atlas with cocycle of transition functions $(T\varphi_{\alpha\beta} : TU_{\alpha\beta} \to TG)$, describing the principal fiber bundle (TP, Tp, TM, TG). The assertion about the principal action is obvious. So (1) follows. For completeness sake we include here the transition formula for this atlas in the right trivialization of TG:

$$\begin{aligned} &T(\varphi_\alpha \circ \varphi_\beta^{-1})(\xi_x, T_e(\rho_g).X) = \\ &\qquad = (\xi_x, T_e(\rho_{\varphi_{\alpha\beta}}(x).g).(\delta\varphi_{\alpha\beta}(\xi_x) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X)), \end{aligned}$$

where $\delta\varphi_{\alpha\beta} \in \Omega^1(U_{\alpha\beta}; \mathfrak{g})$ is the *right logarithmic derivative* of $\varphi_{\alpha\beta}$ which is given by $\delta\varphi_{\alpha\beta}(\xi_x) = T(\rho_{\varphi_{\alpha\beta}(x)^{-1}}).T(\varphi_{\alpha\beta}).\xi_x$.

(2) The mapping $(u, X) \mapsto T_e(r_u).X = T_{(u,e)}r.(0_u, X)$ is a vector bundle isomorphism $P \times \mathfrak{g} \to VP$ over P.

(3) Obviously $Tr : TP \times TG \to TP$ is a free right action which acts transitive on the fibers of $Tp : TP \to TM$. Since $VP = (Tp)^{-1}(0_M)$, the bundle $VP \to M$ is isomorphic to $TP|0_M$ and Tr restricts to a free right action, which is transitive on the fibers, so by lemma 10.3 the result follows.

(4) The transition functions of the fiber bundle $P[S,\ell]$ are given by the expression $\ell \circ (\varphi_{\alpha\beta} \times Id_S) : U_{\alpha\beta} \times S \to G \times S \to S$. Then the transition functions of $TP[S,\ell]$ are $T(\ell \circ (\varphi_{\alpha\beta} \times Id_S)) = T\ell \circ (T\varphi_{\alpha\beta} \times Id_{TS}) : TU_{\alpha\beta} \times TS \to TG \times TS \to TS$, from which the result follows.

(5) Vertical vectors in $TP[S,\ell]$ have local representations $(0_x, \eta_s) \in TU_{\alpha\beta} \times TS$. Under the transition functions of $TP[S,\ell]$ they transform as $T(\ell \circ (\varphi_{\alpha\beta} \times Id_S)).(0_x, \eta_s) = T\ell.(0_{\varphi_{\alpha\beta}(x)}, \eta_s) = T(\ell_{\varphi_{\alpha\beta}(x)}).\eta_s = T_2\ell.(x, \eta_s)$ and this implies the result $\square$

11. Principal and Induced Connections

11.1. Principal connections. Let (P, p, M, G) be a principal fiber bundle. Recall from 9.3 that a (general) connection on P is a fiber projection $\Phi : TP \to VP$, viewed as a 1-form in $\Omega^1(P; TP)$. Such a connection Φ is called a *principal connection* if it is G-equivariant for the principal right action $r : P \times G \to P$, so that $T(r^g).\Phi = \Phi.T(r^g)$ and Φ is r^g-related to itself, or $(r^g)^*\Phi = \Phi$ in the sense of 8.16, for all $g \in G$. By theorem 8.15.6 the curvature $R = \frac{1}{2}.[\Phi, \Phi]$ is then also r^g-related to itself for all $g \in G$.

Recall from 10.18.2 that the vertical bundle of P is trivialized as a vector bundle over P by the principal right action. So $\omega(X_u) := T_e(R_u)^{-1}.\Phi(X_u) \in \mathfrak{g}$ and in this way we get a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P; \mathfrak{g})$, which is called the *(Lie algebra valued) connection form* of the connection Φ. Recall from 1.3. the fundamental vector field mapping $\zeta : \mathcal{G} \to \mathfrak{X}(P)$ for the principal right action.

Lemma. *If $\Phi \in \Omega^1(P; VP)$ is a principal connection on the principal fiber bundle (P, p, M, G) then the connection form has the following two properties:*

(1) *ω reproduces the generators of fundamental vector fields, so we have $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$.*

(2) *ω is G-equivariant, so we have $((r^g)^*\omega)(X_u) = \omega(T_u(r^g).X_u) = Ad(g^{-1}).\omega(X_u)$ for all $g \in G$ and $X_u \in T_uP$. Consequently we have for the Lie derivative $\mathcal{L}_{\zeta_X}\omega = -ad(X).\omega$.*

Conversely a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying (1) *defines a connection Φ on P by $\Phi(X_u) = T_e(r_u).\omega(X_u)$, which is a principal connection if and only if* (2) *is satisfied.*

Proof. (1). $T_e(r_u).\omega(\zeta_X(u)) = \Phi(\zeta_X(u)) = \zeta_X(u) = T_e(r_u).X$. Since $T_e(r_u) : \mathcal{G} \to V_uP$ is an isomorphism, the result follows.

(2). From $T_e(r_{ug}).\omega(T_u(r^g).X_u) = \zeta_{\omega(T_u(r^g).X_u)}(ug) = \Phi(T_u(r^g).X_u)$ and $T_e(r_{ug}).Ad(g^{-1}).\omega(X_u) = \zeta_{Ad(g^{-1}).\omega(X_u)}(ug) = T_u(r^g).\zeta_{\omega(X_u)}(u) = T_u(r^g).\Phi(X_u)$ both directions follow. $\square$

11.2. Curvature. Let Φ be a principal connection on the principal fiber bundle (P, p, M, G) with connection form $\omega \in \Omega^1(P; \mathfrak{g})$. We already noted in 11.1 that the curvature $R = \frac{1}{2}[\Phi, \Phi]$ is then also G-invariant, $(r^g)^*R = R$ for all $g \in G$. Since R has vertical values we may again define a $\mathfrak{g}$-valued 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$ by $\Omega(X_u, Y_u) := -T_e(r_u)^{-1}.R(X_u, Y_u)$, which is called the *(Lie algebra-valued) curvature form* of the connection. We take the negative sign here to get the usual connection form as in [Kobayashi-Nomizu I, 1963].

We equip the space $\Omega(P;\mathfrak{g})$ of all $\mathfrak{g}$-valued forms on P in a canonical way with the structure of a graded Lie algebra by

$$\begin{aligned}&[\Psi,\Theta](X_1,\dots,X_{p+q}) =\\ &\quad= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}\sigma\,[\Psi(X_{\sigma 1},\dots,X_{\sigma p}),\Theta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]_{\mathfrak{g}}\end{aligned}$$

or equivalently by $[\psi\otimes X,\theta\otimes Y] := \psi\wedge\theta\otimes[X,Y]$. In particular for $\omega\in\Omega^1(P;\mathfrak{g})$ we have $[\omega,\omega](X,Y) = 2[\omega(X),\omega(Y)]$.

Theorem. *The curvature form Ω of a principal connection with connection form ω has the following properties:*

(1) *Ω is horizontal, i.e. it kills vertical vectors.*
(2) *Ω is G-equivariant in the following sense: $(r^g)^*\Omega = Ad(g^{-1}).\Omega$. Consequently $\mathcal{L}_{\zeta_X}\Omega = -ad(X).\Omega$.*
(3) *The Maurer-Cartan formula holds: $\Omega = d\omega + \frac{1}{2}[\omega,\omega]_{\mathfrak{g}}$.*

Proof. (1) is true for R by 9.4. For (2) we compute as follows:

$$\begin{aligned}&T_e(r_{ug}).((r^g)^*\Omega)(X_u,Y_u) = T_e(r_{ug}).\Omega(T_u(r^g).X_u,T_u(r^g).Y_u) =\\ &\quad= -R_{ug}(T_u(r^g).X_u,T_u(r^g).Y_u) = -T_u(r^g).((r^g)^*R)(X_u,Y_u) =\\ &\qquad= -T_u(r^g).R(X_u,Y_u) = T_u(r^g).\zeta_{\Omega(X_u,Y_u)}(u) =\\ &\qquad\quad= \zeta_{Ad(g^{-1}).\Omega(X_u,Y_u)}(ug) =\\ &\qquad\qquad= T_e(r_{ug}).Ad(g^{-1}).\Omega(X_u,Y_u), \quad \text{by 1.3.}\end{aligned}$$

(3). For $X\in\mathfrak{g}$ we have $i_{\zeta_X}R = 0$ by (1) and

$$\begin{aligned}i_{\zeta_X}(d\omega + \frac{1}{2}.[\omega,\omega]_{\mathfrak{g}}) &= i_{\zeta_X}d\omega + \frac{1}{2}[i_{\zeta_X}\omega,\omega] - \frac{1}{2}[\omega,i_{\zeta_X}\omega] =\\ &= \mathcal{L}_{\zeta_X}\omega + [X,\omega] = -ad(X)\omega + ad(X)\omega = 0\end{aligned}$$

So the formula holds for vertical vectors, and for horizontal vector fields $X,Y\in C^\infty(H(P))$ we have

$$R(X,Y) = \Phi[X-\Phi X, Y-\Phi Y] = \Phi[X,Y] = \zeta_{\omega([X,Y])}$$

$$(d\omega + \frac{1}{2}[\omega,\omega])(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = -\omega([X,Y]) \quad \square$$

11.3. Lemma. *Any principal fiber bundle (P, p, M, G) (with paracompact basis) admits principal connections.*

Proof. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)_\alpha$ be a principal fiber bundle atlas. Let us define $\gamma_\alpha(T\varphi_\alpha^{-1}(\xi_x, T_e\lambda_g.X)) := X$ for $\xi_x \in T_xU_\alpha$ and $X \in \mathfrak{g}$. An easy computation involving lemma 1.3 shows that $\gamma_\alpha \in \Omega^1(P|U_\alpha; \mathfrak{g})$ satisfies the requirements of lemma 11.1 and thus is a principal connection on $P|U_\alpha$. Now let (f_α) be a smooth partition of unity on M which is subordinated to the open cover (U_α), and let $\omega := \sum_\alpha (f_\alpha \circ p)\gamma_\alpha$. Since both requirements of lemma 11.1 are invariant under convex linear combinations, ω is a principal connection on P. □

11.4. Local descriptions of principal connections. We consider a principal fiber bundle (P, p, M, G) with some principal fiber bundle atlas $(U_\alpha, \psi_\alpha : P|U_\alpha \to U_\alpha \times G)$ and corresponding cocycle $(\varphi_{\alpha\beta} : U_{\alpha\beta} \to G)$ of transition functions. We consider the sections $s_\alpha \in C^\infty(P|U_\alpha)$ which are given by $\varphi_\alpha(s_\alpha(x)) = (x, e)$ and satisfy $s_\alpha.\varphi_{\alpha\beta} = s_\beta$.

(1) Let $\Theta \in \Omega^1(G, \mathfrak{g})$ be the left logarithmic derivative of the identity, i.e. $\Theta(\eta_g) := T_g(\lambda_{g^{-1}}).\eta_g$. We will use the forms $\Theta_{\alpha\beta} := {\varphi_{\alpha\beta}}^*\Theta \in \Omega^1(U_{\alpha\beta}; \mathfrak{g})$.

Let $\Phi = \zeta \circ \omega \in \Omega^1(P; VP)$ be a principal connection, $\omega \in \Omega^1(P; \mathfrak{g})$. We may associate the following local data to the connection:

(2) $\omega_\alpha := {s_\alpha}^*\omega \in \Omega^1(U_\alpha; \mathfrak{g})$, the physicists version of the connection.
(3) The Christoffel forms $\Gamma^\alpha \in \Omega^1(U_\alpha; \mathfrak{X}(G))$ from 9.7, which are given by $(0_x, \Gamma^\alpha(\xi_x, g)) = -T(\varphi_\alpha).\Phi.T(\varphi_\alpha)^{-1}(\xi_x, 0_g)$.
(4) $\gamma_\alpha := (\varphi_\alpha^{-1})^*\omega \in \Omega^1(U_\alpha \times G; \mathfrak{g})$, the local expressions of ω.

Lemma. *These local data have the following properties and are related by the following formulas.*

(5) *The forms $\omega_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})$ satisfy the transition formulas*

$$\omega_\alpha = Ad(\varphi_{\beta\alpha}^{-1})\omega_\beta + \Theta_{\beta\alpha},$$

and any set of forms like that with this transition behavior determines a unique principal connection.

(6) $\gamma_\alpha(\xi_x, T\lambda_g.X) = \gamma_\alpha(\xi_x, 0_g) + X = Ad(g^{-1})\omega_\alpha(\xi_x) + X$.
(7) $\Gamma^\alpha(\xi_x, g) = -T_e(\lambda_g).\gamma_\alpha(\xi_x, 0_g) = -T_e(\lambda_g).Ad(g^{-1})\omega_\alpha(\xi_x)$.

Proof. From the definition of the Christoffel forms we have

$$\begin{aligned}
\Gamma^\alpha(\xi_x, g) &= -T(\varphi_\alpha).\Phi.T(\varphi_\alpha)^{-1}(\xi_x, 0_g) \\
&= -T(\varphi_\alpha).T_e(r_{\varphi_\alpha(x,g)})\omega.T(\varphi_\alpha)^{-1}(\xi_x, 0_g) \\
&= -T_e(\varphi_\alpha \circ r_{\varphi_\alpha(x,g)})\omega.T(\varphi_\alpha)^{-1}(\xi_x, 0_g) \\
&= -T_e(\lambda_g)\omega.T(\varphi_\alpha)^{-1}(\xi_x, 0_g) = -T_e(\lambda_g)\gamma_\alpha(\xi_x, 0_g).
\end{aligned}$$

This is the first part of (7). The second part follows from (6).

$$\begin{aligned}\gamma_\alpha(\xi_x, T\lambda_g.X) &= \omega(T(\varphi_\alpha)^{-1}(\xi_x, 0_g)) = \\ &= \gamma_\alpha(\xi_x, 0_g) + \omega(\zeta_X(\varphi_\alpha^{-1}(x,g))) = \gamma_\alpha(\xi_x, 0_g) + X.\end{aligned}$$

So the first part of (6) holds. The second part is seen from

$$\begin{aligned}\gamma_\alpha(\xi_x, 0_g) &= \gamma_\alpha(\xi_x, T_e(\rho_g)0_e) = (\omega \circ T(\varphi_\alpha)^{-1} \circ T(\rho_g))(\xi_x, 0_e) = \\ &= (\omega \circ T(r^g \circ \varphi_\alpha^{-1}))(\xi_x, 0_e) = Ad(g^{-1})\omega(T(\varphi_\alpha^{-1})(\xi_x, 0_e)) \\ &= Ad(g^{-1})(s_\alpha{}^*\omega)(\xi_x) = Ad(g^{-1})\omega_\alpha(\xi_x).\end{aligned}$$

Via (7) the transition formulas for the ω_α are easily seen to be equivalent to the transition formulas for the Christoffel forms in lemma 9.7. $\square$

11.5. The covariant derivative. Let (P,p,M,G) be a principal fiber bundle with principal connection $\Phi = \zeta \circ \omega$. We consider the horizontal projection $\chi = Id_{TP} - \Phi : TP \to HP$, cf. 9.3, which satisfies $\chi \circ \chi = \chi$, $\operatorname{im}\chi = HP$, $\ker\chi = VP$, and $\chi \circ T(r^g) = T(r^g \circ \chi)$ for all $g \in G$.

If W is a finite dimensional vector space, we consider the mapping $\chi^* : \Omega(P;W) \to \Omega(P;W)$ which is given by

$$(\chi^*\varphi)_u(X_1, \dots, X_k) = \varphi_u(\chi(X_1), \dots, \chi(X_k)).$$

The mapping χ^* is a projection onto the subspace of *horizontal differential forms*, i.e. the space $\Omega_{hor}(P;W) := \{\psi \in \Omega(P;W) : i_X\psi = 0 \text{ for } X \in VP\}$. The notion of horizontal form is independent of the choice of a connection.

The projection χ^* has the following properties: $\chi^*(\varphi\wedge\psi) = \chi^*\varphi\wedge\chi^*\psi$, if one of the two forms has real values; $\chi^*\circ\chi^* = \chi^*$; $\chi^*\circ(r^g)^* = (r^g)^*\circ\chi^*$ for all $g \in G$; $\chi^*\omega = 0$; and $\chi^* \circ \mathcal{L}(\zeta_X) = \mathcal{L}(\zeta_X) \circ \chi^*$. They follow easily from the corresponding properties of χ, the last property uses that $\mathrm{Fl}_t^{\zeta(X)} = r^{\exp tX}$.

Now we define the *covariant exterior derivative* $d_\omega : \Omega^k(P;W) \to \Omega^{k+1}(P;W)$ by the prescription $d_\omega := \chi^* \circ d$.

Theorem. *The covariant exterior derivative d_ω has the following properties.*

1. $d_\omega(\varphi \wedge \psi) = d_\omega(\varphi) \wedge \chi^*\psi + (-1)^{\deg\varphi}\chi^*\varphi \wedge d_\omega(\psi)$.
2. $\mathcal{L}(\zeta_X) \circ d_\omega = d_\omega \circ \mathcal{L}(\zeta_X)$ *for each* $X \in \mathfrak{g}$.
3. $(r^g)^* \circ d_\omega = d_\omega \circ (r^g)^*$ *for each* $g \in G$.
4. $d_\omega \circ p^* = d \circ p^* = p^* \circ d : \Omega(M;W) \to \Omega_{hor}(P;W)$.
5. $d_\omega\omega = \Omega$, *the curvature form.*
6. $d_\omega\Omega = 0$, *the Bianchi identity.*
7. $d_\omega \circ \chi^* - d_\omega = \chi^* \circ i(R) \circ d$, *where R is the curvature.*
8. $d_\omega \circ d_\omega = \chi^* \circ i(R) \circ d$.

Proof. (1) through (4) follow from the properties of χ^*.

(5). We have

$$\begin{aligned}(d_\omega\omega)(\xi,\eta) &= (\chi^* d\omega)(\xi,\eta) = d\omega(\chi\xi,\chi\eta)\\ &= (\chi\xi)\omega(\chi\eta) - (\chi\eta)\omega(ch\xi) - \omega([\chi\xi,\chi\eta])\\ &= -\omega([\chi\xi,\chi\eta]) \text{ and}\\ R(\xi,\eta) &= -\zeta(\Omega(\xi,\eta)) = \Phi[\chi\xi,\chi\eta].\end{aligned}$$

(6). We have

$$\begin{aligned}d_\omega\Omega &= d_\omega(d\omega + \tfrac{1}{2}[\omega,\omega])\\ &= \chi^* dd\omega + \tfrac{1}{2}\chi^* d[\omega,\omega]\\ &= \tfrac{1}{2}\chi^*([d\omega,\omega] - [\omega,d\omega]) = \chi^*[d\omega,\omega]\\ &= [\chi^* d\omega, \chi^*\omega] = 0, \text{ since } \chi^*\omega = 0.\end{aligned}$$

(7). For $\varphi \in \Omega(P;W)$ we have

$$\begin{aligned}&(d_\omega\chi^*\varphi)(X_0,\dots,X_k) = (d\chi^*\varphi)(\chi(X_0),\dots,\chi(X_k))\\ &\quad= \sum_{0\le i\le k}(-1)^i\chi(X_i)((\chi^*\varphi)(\chi(X_0),\dots,\widehat{\chi(X_i)},\dots,\chi(X_k)))\\ &\qquad+\sum_{i<j}(-1)^{i+j}(\chi^*\varphi)([\chi(X_i),\chi(X_j)],\chi(X_0),\dots\\ &\qquad\qquad\dots,\widehat{\chi(X_i)},\dots,\widehat{\chi(X_j)},\dots)\\ &\quad= \sum_{0\le i\le k}(-1)^i\chi(X_i)(\varphi(\chi(X_0),\dots,\widehat{\chi(X_i)},\dots,\chi(X_k)))\\ &\qquad+\sum_{i<j}(-1)^{i+j}\varphi([\chi(X_i),\chi(X_j)] - \Phi[\chi(X_i),\chi(X_j)],\chi(X_0),\dots\\ &\qquad\qquad\dots,\widehat{\chi(X_i)},\dots,\widehat{\chi(X_j)},\dots)\\ &\quad= (d\varphi)(\chi(X_0),\dots,\chi(X_k)) + (i_R\varphi)(\chi(X_0),\dots,\chi(X_k))\\ &\quad= (d_\omega + \chi^* iR)(\varphi)(X_0,\dots,X_k).\end{aligned}$$

(8).

$$\begin{aligned}d_\omega d_\omega &= \chi^* d\chi^* d = (\chi^* i_R + \chi^* d)d \qquad \text{by (7)}\\ &= \chi^* i_R. \quad \square\end{aligned}$$

11.6 Theorem. *Let (P, p, M, G) be a principal fiber bundle with principal connection ω. Then the parallel transport for the principal connection is globally defined and g-equivariant.*

In detail: For each smooth $c : \mathbb{R} \to M$ there is a smooth mapping $\mathrm{Pt}_c : \mathbb{R} \times P_{c(0)} \to P$ such that the following holds:

(1) $\mathrm{Pt}(c, t, u) \in P_{c(t)}$, $\mathrm{Pt}(c, 0) = Id_{P_{c(0)}}$, *and* $\omega(\frac{d}{dt} \mathrm{Pt}(c, t, u)) = 0$.
(2) $\mathrm{Pt}(c, t) : P_{c(0)} \to P_{c(t)}$ *is G-equivariant, i.e.* $\mathrm{Pt}(c, t, u.g) = \mathrm{Pt}(c, t, u).g$.
(3) *For smooth $f : \mathbb{R} \to \mathbb{R}$ we have* $\mathrm{Pt}(c, f(t), u) = \mathrm{Pt}(c \circ f, t, \mathrm{Pt}(c, f(0), u))$.

Proof. The Christoffel forms $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(G))$ of the connection ω with respect to a principal fiber bundle atlas $(U_\alpha, \varphi_\alpha)$ take values in the Lie subalgebra $\mathfrak{X}_L(G)$ of all left invariant vector fields on G, which are bounded with respect to any left invariant Riemannian metric on G. Each left invariant metric on a Lie group is complete. So the connection is complete by the remark in 9.10.

Properties (1) and (3) follow from theorem 9.8, and (2) is seen as follows: $\omega(\frac{d}{dt} \mathrm{Pt}(c, t, u).g) = Ad(g^{-1})\omega(\frac{d}{dt} \mathrm{Pt}(c, t, u)) = 0$ implies that $\mathrm{Pt}(c, t, u).g = \mathrm{Pt}(c, t, u.g)$. $\square$

11.7. Holonomy groups. Let (P, p, M, G) be a principal fiber bundle with principal connection $\Phi = \zeta \circ \omega$. We assume that M is connected and we fix $x_0 \in M$.

In view of developments which we make later in section 12 we define the holonomy group $\mathrm{Hol}(\Phi, x_0) \subset \mathrm{Diff}(P_{x_0})$ as the group of all $\mathrm{Pt}(c, 1) : P_{x_0} \to P_{x_0}$ for c any piecewise smooth closed loop through x_0. (Reparametrizing c by a function which is flat at each corner of c we may assume that any c is smooth.) If we consider only those curves c which are nullhomotopic, we obtain the restricted holonomy group $\mathrm{Hol}_0(\Phi, x_0)$.

Now let us fix $u_0 \in P_{x_0}$. The elements $\tau(u_0, \mathrm{Pt}(c, t, u_0)) \in G$ form a subgroup of the structure group G which is isomorphic to $\mathrm{Hol}(\Phi, x_0)$; we denote it by $\mathrm{Hol}(\omega, u_0)$ and we call it also the *holonomy group* of the connection. Considering only nullhomotopic curves we get the *restricted holonomy group* $\mathrm{Hol}_0(\omega, u_0)$ a normal subgroup $\mathrm{Hol}(\omega, u_0)$.

Theorem. *1. We have* $\mathrm{Hol}(\omega, u_0.g) = Ad(g^{-1})\,\mathrm{Hol}(\omega, u_0)$ *and* $\mathrm{Hol}_0(\omega, u_0.g) = Ad(g^{-1})\,\mathrm{Hol}_0(\omega, u_0)$.

2. For each curve c in M we have $\mathrm{Hol}(\omega, \mathrm{Pt}(c, t, u_0)) = \mathrm{Hol}(\omega, u_0)$ *and* $\mathrm{Hol}_0(\omega, \mathrm{Pt}(c, t, u_0)) = \mathrm{Hol}_0(\omega, u_0)$.

3. $\mathrm{Hol}_0(\omega, u_0)$ *is a connected Lie subgroup of G and the quotient group* $\mathrm{Hol}(\omega, u_0)/\mathrm{Hol}_0(\omega, u_0)$ *is at most countable, so* $\mathrm{Hol}(\omega, u_0)$ *is also a Lie subgroup of G.*

4. The Lie algebra $\operatorname{hol}(\omega, u_0) \subset \mathfrak{g}$ *of* $\operatorname{Hol}(\omega, u_0)$ *is linearly generated by* $\{\Omega(X_u, Y_u) : X_u, Y_u \in T_u P\}$. *It is isomorphic to the Lie algebra* $\operatorname{hol}(\Phi, x_0)$ *we considered in 9.9.*

5. For $u_0 \in P_{x_0}$ *let* $P(\omega, u_0)$ *be the set of all* $\operatorname{Pt}(c, t, u_0)$ *for* c *any (piecewise) smooth curve in* M *with* $c(0) = x_0$ *and for* $t \in \mathbb{R}$. *Then* $P(\omega, u_0)$ *is a sub fiber bundle of* P *which is invariant under the right action of* $\operatorname{Hol}(\omega, u_0)$*; so it is itself a principal fiber bundle over* M *with structure group* $\operatorname{Hol}(\omega, u_0)$ *and we have a reduction of structure group, cf. 10.6 and 10.13. The pullback of* ω *to* $P(\omega, u_0)$ *is then again a principal connection form* $i^*\omega \in \Omega^1(P(\omega, u_0); \operatorname{hol}(\omega, u_0))$.

6. P *is foliated by the leaves* $P(\omega, u)$, $u \in P_{x_0}$.

7. If the curvature $\Omega = 0$ *then* $\operatorname{Hol}_0(\omega, u_0) = \{e\}$ *and each* $P(\omega, u)$ *is a covering of* M.

8. If one uses piecewise C^k*-curves for* $1 \le k < \infty$ *in the definition, one gets the same holonomy groups.*

In view of assertion 5 a principal connection ω is called *irreducible* if $\operatorname{Hol}(\omega, u_0)$ equals the structure group G for some (equivalently any) $u_0 \in P_{x_0}$.

Proof. 1. This follows from the properties of the mapping τ from 10.2 and from the from the G-equivariancy of the parallel transport.

The rest of this theorem is a compilation of well known results, and we refer to [Kobayashi-Nomizu I, 1963, p. 83ff] for proofs. □

11.8. Inducing principal connections on associated bundles. Let (P, p, M, G) be a principal bundle with principal right action $r : P \times G \to P$ and let $\ell : G \times S \to S$ be a left action of the structure group G on some manifold S. Then we consider the associated bundle $P[S] = P[S, \ell] = P \times_G S$, constructed in 10.7. Recall from 10.18 that its tangent and vertical bundle are given by $T(P[S, \ell]) = TP[TS, T\ell] = TP \times_{TG} TS$ and $V(P[S, \ell]) = P[TS, T_2\ell] = P \times_G TS$.

Let $\Phi = \zeta \circ \omega \in \Omega^1(P; TP)$ be a principal connection on the principal bundle P. We construct the *induced connection* $\bar{\Phi} \in \Omega^1(P[S], T(P[S]))$ by the following diagram:

$$\begin{tikzcd}
TP \times TS \arrow[r, "\Phi \times Id"] \arrow[d, "Tq = q'"'] & TP \times TS \arrow[r, equal] \arrow[d, "q'"'] & T(P \times S) \arrow[d, "Tq"'] \\
TP \times_{TG} TS \arrow[r, "\bar{\Phi}"'] & TP \times_{TG} TS \arrow[r, equal] & T(P \times_G S).
\end{tikzcd}$$

Let us first check that the top mapping $\Phi \times Id$ is TG-equivariant. For $g \in G$ and $X \in \mathfrak{g}$ the inverse of $T_e(\lambda_g)X$ in the Lie group TG is denoted by $(T_e(\lambda_g)X)^{-1}$, see lemma 10.17. Furthermore by 1.3 we have

$$\begin{aligned} Tr(\xi_u, T_e(\lambda_g)X) &= T_u(r^g)\xi_u + Tr((O_P \times L_X)(u,g)) \\ &= T_u(r^g)\xi_u + T_g(r_u)(T_e(\lambda_g)X) \\ &= T_u(r^g)\xi_u + \zeta_X(ug). \end{aligned}$$

We may compute

$$\begin{aligned} &(\Phi \times Id)(Tr(\xi_u, T_e(\lambda_g)X), T\ell((T_e(\lambda_g)X)^{-1}, \eta_s)) \\ &\quad = (\Phi(T_u(r^g)\xi_u + \zeta_X(ug)), T\ell((T_e(\lambda_g)X)^{-1}, \eta_s)) \\ &\quad = (\Phi(T_u(r^g)\xi_u) + \Phi(\zeta_X(ug)), T\ell((T_e(\lambda_g)X)^{-1}, \eta_s)) \\ &\quad = (\Phi(T_u(r^g)\xi_u) + \zeta_X(ug), T\ell((T_e(\lambda_g)X)^{-1}, \eta_s)) \\ &\quad = (Tr((\Phi \times Id)\xi_u, T_e(\lambda_g)X), T\ell((T_e(\lambda_g)X)^{-1}, \eta_s)) \end{aligned}$$

So the mapping $\Phi \times Id$ factors to $\bar\Phi$ as indicated in the diagram, and we have $\bar\Phi \circ \bar\Phi = \bar\Phi$ from $(\Phi \times Id) \circ (\Phi \times Id) = \Phi \times Id$. The mapping $\bar\Phi$ is fiberwise linear, since $\Phi \times Id$ and $q' = Tq$ are. The image of $\bar\Phi$ is

$$\begin{aligned} q'(VP \times TS) &= q'(\ker(Tp : TP \times TS \to TM)) \\ &= \ker(Tp : TP \times_{TG} TS \to TM) = V(P[S,\ell]). \end{aligned}$$

Thus $\bar\Phi$ is a connection on the associated bundle $P[S]$. We call it the *induced connection*.

From the diagram it also follows, that the vector valued forms $\Phi \times Id \in \Omega^1(P \times S; TP \times TS)$ and $\bar\Phi \in \Omega^1(P[S]; T(P[S]))$ are $(q : P \times S \to P[S])$-related. So by 8.15 we have for the curvatures

$$\begin{aligned} &R_{\Phi\times Id} = \tfrac{1}{2}[\Phi \times Id, \Phi \times Id] = \tfrac{1}{2}[\Phi, \Phi] \times 0 = R_\Phi \times 0, \\ &R_{\bar\Phi} = \tfrac{1}{2}[\bar\Phi, \bar\Phi], \end{aligned}$$

that they are also q-related, i.e. $Tq \circ (R_\Phi \times 0) = R_{\bar\Phi} \circ (Tq \times_M Tq)$.

By uniqueness of the solutions of the defining differential equation we also get that

$$\mathrm{Pt}_{\bar\Phi}(c, t, q(u, s)) = q(\mathrm{Pt}_\Phi(c, t, u), s).$$

11.9. Recognizing induced connections. We consider again a principal fiber bundle (P, p, M, G) and a left action $\ell : G \times S \to S$. Suppose that $\Psi \in \Omega^1(P[S]; T(P[S]))$ is a connection on the associated bundle $P[S] = P[S, \ell]$. Then the following question arises: When is the connection Ψ induced from a principal connection on P? If this is the case, we say that Ψ is compatible with the G-structure on $P[S]$. The answer is given in the following

Theorem. *Let Ψ be a (general) connection on the associated bundle $P[S]$. Let us suppose that the action ℓ is infinitesimally effective, i.e. the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(S)$ is injective.*

Then the connection Ψ is induced from a principal connection ω on P if and only if the following condition is satisfied:

> *In some (equivalently any) fiber bundle atlas (U_α, ψ_α) of $P[S]$ belonging to the G-structure of the associated bundle the Christoffel forms $\Gamma^\alpha \in \Omega^1(U_\alpha; \mathfrak{X}(S))$ have values in the sub Lie algebra $\mathfrak{X}_{fund}(S)$ of fundamental vector fields for the action ℓ.*

Proof. Let $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$ be a principal fiber bundle atlas for P. Then by the proof of theorem 10.7 the induced fiber bundle atlas $(U_\alpha, \psi_\alpha : P[S]|U_\alpha \to U_\alpha \times S)$ is given by

$$\psi_\alpha^{-1}(x, s) = q(\varphi_\alpha^{-1}(x, e), s), \tag{1}$$

$$(\psi_\alpha \circ q)(\varphi_\alpha^{-1}(x, g), s) = (x, g.s). \tag{2}$$

Let $\Phi = \zeta \circ \omega$ be a principal connection on P and let $\bar{\Phi}$ be the induced connection on the associated bundle $P[S]$. By 9.7 its Christoffel symbols are given by

$$\begin{aligned}
\Gamma^\alpha_{\bar{\Phi}}(\xi_x, s) &= -(T(\psi_\alpha) \circ \bar{\Phi} \circ T(\psi_\alpha^{-1}))(\xi_x, 0_s) \\
&= -(T(\psi_\alpha) \circ \bar{\Phi} \circ Tq \circ (T(\varphi_\alpha^{-1}) \times Id))(\xi_x, 0_e, 0_s) && \text{by (1)} \\
&= -(T(\psi_\alpha) \circ Tq \circ (\Phi \times Id))(T(\varphi_\alpha^{-1})(\xi_x, 0_e), 0_s) && \text{by 11.8} \\
&= -(T(\psi_\alpha) \circ Tq)(\Phi(T(\varphi_\alpha^{-1})(\xi_x, 0_e)), 0_s) \\
&= (T(\psi_\alpha) \circ Tq)(T(\varphi_\alpha^{-1})(\Gamma^\alpha_\Phi(\xi_x, e)), 0_s) && \text{by 11.4,(3)} \\
&= -T(\psi_\alpha \circ q \circ (\varphi_\alpha^{-1} \times Id))(0_x, \omega_\alpha(\xi_x), 0_s) && \text{by 11.4,(7)} \\
&= -T_e(\ell^s)\omega_\alpha(\xi_x) && \text{by (2)} \\
&= -\zeta_{\omega_\alpha(\xi_x)}(s).
\end{aligned}$$

So the condition is necessary. Now let us conversely suppose that a connection Ψ on $P[S]$ is given such that the Christoffel forms Γ^α_Ψ with respect to a fiber bundle atlas of the G-structure have values in $\mathfrak{X}_{fund}(S)$. Then unique $\mathfrak{g}$-valued forms $\omega_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})$ are given by the equation

$$\Gamma^\alpha_\Psi(\xi_x) = \zeta(\omega_\alpha(\xi_x)),$$

since the action is infinitesimally effective. From the transition formulas 9.7 for the Γ^α_Ψ follow the transition formulas 11.4.(5) for the ω^α, so that they give a unique principal connection on P, which by the first part of the proof induces the given connection Ψ on $P[S]$. $\square$

11.10. Inducing principal connections on associated vector bundles. Let (P,p,M,G) be a principal fiber bundle and let $\rho : G \to GL(W)$ be a representation of the structure group G on a finite dimensional vector space W. We consider the associated vector bundle $(E := P[W,\rho], p, M, W)$, which was treated in some detail in 10.11.

The tangent bundle $T(E) = TP \times_{TG} TW$ has two vector bundle structures with the projections

$$\begin{aligned} \pi_E : T(E) = TP \times_{TG} TW \to P \times_G W = E, \\ Tp \circ pr_1 : T(E) = TP \times_{TG} TW \to TM; \end{aligned}$$

the first one is the vector bundle structure of the tangent bundle, the second one is the derivative of the vector bundle structure on E.

Now let $\Phi = \zeta \circ \omega \in \Omega^1(P;TP)$ be a principal connection on P. We consider the induced connection $\bar{\Phi} \in \Omega^1(E;T(E))$ from 11.8. Inserting the projections of both vector bundle structures on $T(E)$ into the diagram in 11.8 one easily sees that the induced connection is linear in both vector bundle structures: the new aspect is that it is a linear endomorphism of the vector bundle (TE, Tp, TM). We say that it is a *linear connection* on the associated bundle.

Recall now from 11.8 the vertical lift $vl_E : E \times_M E \to VE$, which is an isomorphism, pr_1–π_E–fiberwise linear and also p–Tp–fiberwise linear.

Now we define the *connector* K of the linear connection $\bar{\Phi}$ by

$$K := pr_2 \circ (vl_E)^{-1} \circ \bar{\Phi} : TE \to VE \to E \times_M E \to E.$$

Lemma. *The connector $K : TE \to E$ is π_E–p–fiberwise linear and Tp–p–fiberwise linear and satisfies $K \circ vl_E = pr_2 : E \times_M E \to TE \to E$.*

Proof. This follows from the fiberwise linearity of the composants of K and from its definition. $\square$

11.11. Linear connections. If (E,p,M) is a vector bundle, a connection $\Psi \in \Omega^1(E;TE)$ such that $\Psi : TE \to VE \to TE$ is also Tp–Tp–fiberwise linear is called a *linear connection.* An easy check with 11.9 or a direct construction shows that Ψ is then induced from a unique principal connection on the linear frame bundle $GL(\mathbb{R}^n, E)$ of E (where n is the fiber dimension of E).

Equivalently a linear connection may be specified by a connector $K : TE \to E$ with the three properties of lemma 11.10. For then $HE := \{\xi_u : K(\xi_u) = 0_{p(u)}\}$ is a complement to VE in TE which is Tp–fiberwise linearly chosen.

11.12. Covariant derivative on vector bundles. Let (E, p, M) be a vector bundle with a linear connection, given by a connector $K : TE \to E$ with the properties in lemma 11.10.

For any manifold N, smooth mapping $s : N \to E$, and vector field $X \in \mathfrak{X}(N)$ we define the *covariant derivative* of s along X by

$$\nabla_X s := K \circ Ts \circ X : N \to TN \to TE \to E. \tag{1}$$

If $f : N \to M$ is a fixed smooth mapping, let us denote by $C^\infty_f(N, E)$ the vector space of all smooth mappings $s : N \to E$ with $p \circ s = f$ – they are called sections of E along f. From the universal property of the pullback it follows that the vector space $C^\infty_f(N, E)$ is canonically linearly isomorphic to the space $C^\infty(f^*E)$ of sections of the pullback bundle. Then the covariant derivative may be viewed as a bilinear mapping

$$\nabla : \mathfrak{X}(N) \times C^\infty_f(N, E) \to C^\infty_f(N, E). \tag{2}$$

Lemma. *This covariant derivative has the following properties:*

(3) $\nabla_X s$ *is* $C^\infty(N, \mathbb{R})$*-linear in* $X \in \mathfrak{X}(N)$.

(4) $\nabla_X s$ *is* $\mathbb{R}$*-linear in* $s \in C^\infty_f(N, E)$. *So for a tangent vector* $X_x \in T_xN$ *the mapping* $\nabla_{X_x} : C^\infty_f(N, E) \to E_{f(x)}$ *makes sense and* $(\nabla_X s)(x) = \nabla_{X(x)} s$.

(5) $\nabla_X(h.s) = dh(X).s + h.\nabla_X s$ *for* $h \in C^\infty(N, \mathbb{R})$*, the derivation property of* ∇_X.

(6) *For any manifold* Q *and smooth mapping* $g : Q \to N$ *and* $Y_y \in T_yQ$ *we have* $\nabla_{Tg.Y_y} s = \nabla_{Y_y}(s \circ g)$. *If* $Y \in \mathfrak{X}(Q)$ *and* $X \in \mathfrak{X}(N)$ *are* g*-related, then we have* $\nabla_Y(s \circ g) = (\nabla_X s) \circ g$.

Proof. All these properties follow easily from the definition (1). $\square$

Remark. Property (6) is not well understood in some differential geometric literature. See e.g. the clumsy and unclear treatment of it in [Eells-Lemaire, 1983].

For vector fields X, $Y \in \mathfrak{X}(M)$ and a section $s \in C^\infty(E)$ an easy computation shows that

$$\begin{aligned} R^E(X, Y)s :&= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ &= ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})s \end{aligned}$$

is $C^\infty(M, \mathbb{R})$-linear in X, Y, and s. By the method of 7.4 it follows that R^E is a 2-form on M with values in the vector bundle $L(E, E)$, i.e. $R^E \in \Omega^2(M; L(E,E))$. It is called the *curvature* of the covariant derivative.

For $f : N \to M$, vector fields X, $Y \in \mathfrak{X}(N)$ and a section $s \in C^\infty_f(N,E)$ along f we obtain

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = (f^* R^E)(X,Y)s.$$

11.13. Covariant exterior derivative. Let (E,p,M) be a vector bundle with a linear connection, given by a connector $K : TE \to E$.

For a smooth mapping $f : N \to M$ let $\Omega(N; f^*E)$ be the vector space of all forms on N with values in the vector bundle f^*E. We can also view them as forms on N with values along f in E, but we do not introduce an extra notation for this.

The graded space $\Omega(N; f^*E)$ is a graded $\Omega(N)$-module via

$$\begin{aligned}(\varphi \wedge \Phi)(X_1, \dots, X_{p+q}) &= \\ &= \tfrac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\ \varphi(X_{\sigma 1}, \dots, X_{\sigma p}) \Phi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}).\end{aligned}$$

It can easily be shown that the graded module homomorphisms $H : \Omega(N; f^*E) \to \Omega(N; f^*E)$ (so that $H(\varphi \wedge \Phi) = (-1)^{\deg H.\deg \varphi} \varphi \wedge H(\Phi))$ are exactly the mappings $\mu(K)$ for $K \in \Omega^q(N; f^*L(E,E))$, which are given by

$$\begin{aligned}(\mu(K)\Phi)(X_1, \dots, X_{p+q}) &= \\ &= \tfrac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\ K(X_{\sigma 1}, \dots, X_{\sigma p})(\Phi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})).\end{aligned}$$

The *covariant exterior derivative* $d_\nabla : \Omega^p(N : f^*E) \to \Omega^{p+1}(N; f^*E)$ is defined by (where the X_i are vector fields on N)

$$\begin{aligned}(d_\nabla \Phi)(X_0, \dots, X_p) &= \sum_{i=0}^p (-1)^i \nabla_{X_i} \Phi(X_0, \dots, \widehat{X_i}, \dots, X_p) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \Phi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p).\end{aligned}$$

Lemma. *The covariant exterior derivative is well defined and has the following properties.*

(1) *For $s \in C^\infty(f^*E) = \Omega^0(N; f^*E)$ we have $(d_\nabla s)(X) = \nabla_X s$.*
(2) $d_\nabla(\varphi \wedge \Phi) = d\varphi \wedge \Phi + (-1)^{\deg \varphi} \varphi \wedge d_\nabla \Phi$.
(3) *For smooth $g : Q \to N$ and $\Phi \in \Omega(N; f^*E)$ we have $d_\nabla(g^*\Phi) = g^*(d_\nabla \Phi)$.*
(4) $d_\nabla d_\nabla \Phi = \mu(f^* R^E)\Phi$.

Proof. It suffices to investigate decomposable forms $\Phi = \varphi \otimes s$ for $\varphi \in \Omega^p(N)$ and $s \in C^\infty(f^*E)$. Then from the definition we have $d_\nabla(\varphi \otimes s) = d\varphi \otimes s + (-1)^p \varphi \wedge d_\nabla s$. Since by 11.12,(3) $d_\nabla s \in \Omega^1(N; f^*E)$, the mapping d_∇ is well defined. This formula also implies (2) immediately. (3) follows from 11.12,(6). (4) is checked as follows:

$$\begin{aligned} d_\nabla d_\nabla(\varphi \otimes s) &= d_\nabla(d\varphi \otimes s + (-1)^p \varphi \wedge d_\nabla s) \text{ by (2)} \\ &= 0 + (-1)^{2p} \varphi \wedge d_\nabla d_\nabla s \\ &= \varphi \wedge \mu(f * R^E)s \text{ by the definition of } R^E \\ &= \mu(f^* R^E)(\varphi \wedge s). \quad \square \end{aligned}$$

11.14. Let (P, p, M, G) be a principal fiber bundle and let $\rho : G \to GL(W)$ be a representation of the structure group G on a finite dimensional vector space W.

Theorem. *There is a canonical isomorphism from the space of $P[W,\rho]$-valued differential forms on M onto the space of horizontal G-equivariant W-valued differential forms on P:*

$$\begin{aligned} q^\sharp : \Omega(M; P[W,\rho]) \to \Omega_{hor}(P; W)^G = \{\varphi \in \Omega(P; W) : i_X \varphi = 0 \\ \textit{for all } X \in VP, (r^g)^* \varphi = \rho(g^{-1}) \circ \varphi \textit{ for all } g \in G\}. \end{aligned}$$

In particular for $W = \mathbb{R}$ with trivial representation we see that

$$p^* : \Omega(M) \to \Omega_{hor}(P)^G = \{\varphi \in \Omega_{hor}(P) : (r^g)^* \varphi = \varphi\}$$

is also an isomorphism. The isomorphism

$$q^\sharp : \Omega^0(M; P[W]) = C^\infty(P[W]) \to \Omega^0_{hor}(P; W)^G = C^\infty(P, W)^G$$

is a special case of the one from 10.12.

Proof. Recall the smooth mapping $\tau : P \times_M P \to G$ from 10.2, which satisfies $r(u_x, \tau(u_x, v_x)) = v_x$, $\tau(u_x.g, u'_x.g') = g^{-1}.\tau(u_x, u'_x).g'$, and $\tau(u_x, u_x) = e$.

Let $\varphi \in \Omega^k_{hor}(P; W)^G$, $X_1, \dots, X_k \in T_u P$, and $X'_1, \dots, X'_k \in T_{u'} P$ such that $T_u p.X_i = T_{u'} p.X'_i$ for each i. Then we have for $g = \tau(u, u')$, so that $ug = u'$:

$$\begin{aligned} q(u \varphi_u(X_1, \dots, X_k)) &= q(ug, \rho(g^{-1}) \varphi_u(X_1, \dots, X_k)) \\ &= q(u', ((r^g)^* \varphi)_u(X_1, \dots, X_k)) \\ &= q(u', \varphi_{ug}(T_u(r^g).X_1, \dots, T_u(r^g).X_k)) \\ &= q(u', \varphi_{u'}(X'_1, \dots, X'_k)), \text{ since } T_u(r^g)X_i - X'_i \in V_{u'} P. \end{aligned}$$

By this prescription a vector bundle valued form $\Phi \in \Omega^k(M; P[W])$ is uniquely determined.

For the converse recall the smooth mapping $\bar{\tau} : P \times_M P[W, \rho] \to W$ from 10.7, which satisfies $\bar{\tau}(u, q(u, w)) = w$, $q(u_x, \bar{\tau}(u_x, v_x)) = v_x$, and $\bar{\tau}(u_x g, v_x) = \rho(g^{-1})\bar{\tau}(u_x, v_x)$.

For $\Phi \in \Omega^k(M; P[W])$ we define $q^\sharp \in \Omega^k(P; W)$ as follows. For $X_i \in T_uP$ we put

$$(q^\sharp\Phi)(X_1, \dots , X_k) := \bar{\tau}(u, \Phi_{p(u)}(T_up.X_1, \dots , T_up.X_k)).$$

Then $q^\sharp\Phi$ is smooth and horizontal. For $g \in G$ we have

$$\begin{aligned}
((r^g)^*(q^\sharp\Phi))_u(X_1, \dots , X_k) &= (q^\sharp\Phi)_{ug}(T_u(r^g).X_1, \dots , T_u(r^g).X_k) \\
&= \bar{\tau}(ug, \Phi_{p(ug)}(T_{ug}p.T_u(r^g).X_1, \dots , T_{ug}p.T_u(r^g).X_k)) \\
&= \rho(g^{-1})\bar{\tau}(u, \Phi_{p(u)}(T_up.X_1, \dots , T_up.X_k)) \\
&= \rho(g^{-1})(q^\sharp\Phi)_u(X_1, \dots , X_k).
\end{aligned}$$

Clearly the two constructions are inverse to each other. □

11.15. Let (P, p, M, G) be a principal fiber bundle with a principal connection $\Phi = \zeta \circ \omega$, and let $\rho : G \to GL(W)$ be a representation of the structure group G on a finite dimensional vector space W. We consider the associated vector bundle $(E := P[W, \rho], p, M, W)$, the induced connection $\bar{\Phi}$ on it and the corresponding covariant derivative.

Theorem. *The covariant exterior derivative d_ω from 11.5 on P and the covariant exterior derivative for $P[W]$-valued forms on M are connected by the mapping $q^\sharp$ from 11.14, as follows:*

$$q^\sharp \circ d_\nabla = d_\omega \circ q^\sharp : \Omega(M; P[W]) \to \Omega_{hor}(P; W)^G.$$

Proof. Let first $f \in \Omega^0_{hor}(P; W)^G = C^\infty(P, W)^G$, then $f = q^\sharp s$ for $s \in C^\infty(P[W])$ and we have $f(u) = \tau(u, s(p(u)))$ and $s(p(u)) = q(u, f(u))$ by 11.14 and 10.12. Therefore $Ts.Tp.X_u = Tq(X_u, Tf.X_u)$, where $Tf.X_u = (f(u), df(X_u)) \in TW = W \times W$. If $\chi : TP \to HP$ is the horizontal projection as in 11.5, we have $Ts.Tp.X_u = Ts.Tp.\chi.X_u = Tq(\chi.X_u, Tf.\chi.X_u)$. So we get

$$\begin{aligned}
&(q^\sharp d_\nabla s)(X_u) = \tau(u, (d_\nabla s)(Tp.X_u)) \\
&= \tau(u, \nabla_{Tp.X_u} s) && \text{by 11.13,(1)} \\
&= \tau(u, K.Ts.Tp.X_u) && \text{by 11.12.(1)}
\end{aligned}$$

$$
\begin{aligned}
&= \tau(u, K.Tq(\chi.X_u, Tf.\chi.X_u)) && \text{from above}\\
&= \tau(u, pr_2.vl^{-1}_{P[W]}.\bar{\Phi}.Tq(\chi.X_u, Tf.\chi.X_u)) && \text{by 11.10}\\
&= \tau(u, pr_2.vl^{-1}_{P[W]}.Tq.(\Phi\times Id)(\chi.X_u, Tf.\chi.X_u))) && \text{by 11.8}\\
&= \tau(u, pr_2.vl^{-1}_{P[W]}.Tq(0_u, Tf.\chi.X_u))) && \text{since } \Phi.\chi = 0\\
&= \tau(u, q.pr_2.vl^{-1}_{P\times W}.Tq(0_u, Tf.\chi.X_u))) && \text{since } q \text{ is fiber linear}\\
&= \tau(u, q(u, df.\chi.X_u)) = (\chi^* df)(X_u)\\
&= (d_\omega q^\sharp s)(X_u).
\end{aligned}
$$

Now we turn to the general case. It suffices to check the formula for a decomposable $P[W]$-valued form $\Psi = \psi\otimes s \in \Omega^k(M, P[W])$, where $\psi \in \Omega^k(M)$ and $s \in C^\infty(P[W])$. Then we have

$$
\begin{aligned}
d_\omega q^\sharp(\psi\otimes s) &= d_\omega(p^*\psi\cdot q^\sharp s)\\
&= d_\omega(p^*\psi)\cdot q^\sharp s + (-1)^k \chi^* p^*\psi\wedge d_\omega q^\sharp s && \text{by 11.5,(1)}\\
&= \chi^* p^* d\psi\cdot q^\sharp s + (-1)^k p^*\psi \wedge q^\sharp d_\nabla s && \text{from above}\\
&= p^* d\psi\cdot q^\sharp s + (-1)^k p^*\psi\wedge q^\sharp d_\nabla s\\
&= q^\sharp(d\psi\otimes s + (-1)^k\psi\wedge d_\nabla s)\\
&= q^\sharp d_\nabla(\psi\otimes s). \quad \square
\end{aligned}
$$

11.16. Corollary. *In the situation of theorem 11.15 above we have for the curvature form $\Omega \in \Omega^2_{hor}(P;\mathfrak{g})$ and the curvature $R^{P[W]} \in \Omega^2(M; L(P[W], P[W]))$ the relation*

$$q^\sharp_{L(P[W],P[W])} R^{P[W]} = \rho' \circ \Omega,$$

where $\rho' = T_e\rho : \mathfrak{g} \to L(W,W)$ is the derivative of the representation ρ.

Proof. We use the notation of the proof of theorem 11.15. By this theorem we have for X, $Y \in T_uP$

$$
\begin{aligned}
(d_\omega d_\omega q^\sharp_{P[W]} s)_u(X,Y) &= (q^\sharp d_\nabla d\nabla s)_u(X,Y)\\
&= (q^\sharp R^{P[W]} s)_u(X,Y)\\
&= \tau(u, R^{P[W]}(T_up.X, T_up.Y)s(p(u)))\\
&= (q^\sharp_{L(P[W],P[W])} R^{P[W]})_u(X,Y)(q^\sharp_{P[W]} s)(u).
\end{aligned}
$$

On the other hand we have by theorem 11.5.(8)

$$\begin{aligned}
(d_\omega d_\omega q^\sharp s)_u(X,Y) &= (\chi^* i_R dq^\sharp s)_u(X,Y)\\
&= (dq^\sharp s)_u(R(X,Y)) \quad \text{since } R \text{ is horizontal}\\
&= (dq^\sharp s)(-\zeta_{\Omega(X,Y)}(u)) \quad \text{by 11.2}\\
&= \tfrac{\partial}{\partial t}\big|_0 (q^\sharp s)(\mathrm{Fl}^{\zeta_{\Omega(X,Y)}}_{-t}(u))\\
&= \tfrac{\partial}{\partial t}\big|_0 \tau(u.\exp(-t\Omega(X,Y)), s(p(u.\exp(-t\Omega(X,Y)))))\\
&= \tfrac{\partial}{\partial t}\big|_0 \tau(u.\exp(-t\Omega(X,Y)), s(p(u)))\\
&= \tfrac{\partial}{\partial t}\big|_0 \rho(\exp t\Omega(X,Y))\tau(u, s(p(u))) \quad \text{by 11.7}\\
&= \rho'(\Omega(X,Y))(q^\sharp s)(u). \quad \square
\end{aligned}$$

12. Holonomy

12.1. Holonomy groups. Let (E,p,M,S) be a fiber bundle with a complete connection Φ, and let us assume that M is connected. We choose a fixed base point $x_0 \in M$ and we identify E_{x_0} with the standard fiber S. For each closed piecewise smooth curve $c : [0,1] \to M$ through x_0 the parallel transport $\operatorname{Pt}(c, \quad, 1) =: \operatorname{Pt}(c, 1)$ (pieced together over the smooth parts of c) is a diffeomorphism of S. All these diffeomorphisms form together the group $\operatorname{Hol}(\Phi, x_0)$, the *holonomy group* of Φ at x_0, a subgroup of the diffeomorphism group $\operatorname{Diff}(S)$. If we consider only those piecewise smooth curves which are homotopic to zero, we get a subgroup $Hol_0(\Phi, x_0)$, called the *restricted holonomy group* of the connection Φ at x_0.

12.2. Holonomy Lie algebra. In the setting of 12.1 let $C : TM \times_M E \to TE$ be the horizontal lifting as in 9.3, and let R be the curvature (9.4) of the connection Φ. For any $x \in M$ and $X_x \in T_xM$ the horizontal lift $C(X_x) := C(X_x, \quad) : E_x \to TE$ is a vector field along E_x. For X_x and $Y_x \in T_xM$ we consider $R(CX_x, CY_x) \in \mathfrak{X}(E_x)$. Now we choose any piecewise smooth curve c from x_0 to x and consider the diffeomorphism $\operatorname{Pt}(c,t) : S = E_{x_0} \to E_x$ and the pullback $\operatorname{Pt}(c,1)^*R(CX_x, CY_x) \in \mathfrak{X}(S)$. Let us denote by $\operatorname{hol}(\Phi, x_0)$ the closed linear subspace, generated by all these vector fields (for all $x \in M$, X_x, $Y_x \in T_xM$ and curves c from x_0 to x) in $\mathfrak{X}(S)$ with respect to the compact C^∞-topology, and let us call it the *holonomy Lie algebra* of Φ at x_0.

12.3. Lemma. $\operatorname{hol}(\Phi, x_0)$ *is a Lie subalgebra of* $\mathfrak{X}(S)$.

Proof. For $X \in \mathfrak{X}(M)$ we consider the local flow Fl_t^{CX} of the horizontal lift of X. It restricts to parallel transport along any of the flow lines of X in M. Then for vector fields on M the expression

$$\begin{aligned}
&\tfrac{d}{dt}|_0 (\operatorname{Fl}_s^{CX})^* (\operatorname{Fl}_t^{CY})^* (\operatorname{Fl}_{-s}^{CX})^* (\operatorname{Fl}_z^{CZ})^* R(CU, CV) \restriction E_{x_0} \\
&\qquad = (\operatorname{Fl}_s^{CX})^* [CY, (\operatorname{Fl}_{-s}^{CX})^* (\operatorname{Fl}_z^{CZ})^* R(CU, CV)] \restriction E_{x_0} \\
&\qquad = [(\operatorname{Fl}_s^{CX})^* CY, (\operatorname{Fl}_z^{CZ})^* R(CU, CV)] \restriction E_{x_0}
\end{aligned}$$

is in $\operatorname{hol}(\Phi, x_0)$, since it is closed in the compact C^∞-topology and the derivative can be written as a limit. Thus

$$[(\operatorname{Fl}_s^{CX})^* [CY_1, CY_2], (\operatorname{Fl}_z^{CZ})^* R(CU, CV)] \restriction E_{x_0} \in \operatorname{hol}(\Phi, x_0)$$

by the Jacobi identity and

$$[(\operatorname{Fl}_s^{CX})^* C[Y_1, Y_2], (\operatorname{Fl}_z^{CZ})^* R(CU, CV)] \restriction E_{x_0} \in \operatorname{hol}(\Phi, x_0),$$

so also their difference

$$[(\mathrm{Fl}^{CX}_s)^* R(CY_1, CY_2), (\mathrm{Fl}^{CZ}_z)^* R(CU, CV)] \upharpoonright E_{x_0}$$

is in $\mathrm{hol}(\Phi, x_0)$. $\square$

12.4. The following theorem is a generalization of the theorem of Ambrose and Singer on principal connections.

Theorem. *Let Φ be a connection on the fiber bundle (E, p, M, S) and let M be connected. Suppose that for some (hence any) $x_0 \in M$ the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$ is finite dimensional and consists of complete vector fields on the fiber E_{x_0}*

Then there is a principal bundle (P, p, M, G) with finite dimensional structure group G, an irreducible connection ω on it and a smooth action of G on S such that the Lie algebra $\mathfrak{g}$ of G equals the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$, the fiber bundle E is isomorphic to the associated bundle $P[S]$, and Φ is the connection induced by ω. The structure group G equals the holonomy group $\mathrm{Hol}(\Phi, x_0)$. P and ω are unique up to isomorphism.

By a theorem of [Palais, 1957] a finite dimensional Lie subalgebra of $\mathfrak{X}(E_{x_0})$ like $\mathrm{Hol}(\Phi, x_0)$ consists of complete vector fields if and only if it is generated by complete vector fields as a Lie algebra.

Proof. Let us again identify E_{x_0} and S. Then $\mathfrak{g} := \mathrm{hol}(\Phi, x_0)$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(S)$, and since each vector field in it is complete, there is a finite dimensional connected Lie group G_0 of diffeomorphisms of S with Lie algebra $\mathfrak{g}$, see [Palais, 1957].

Claim 1. G_0 equals $\mathrm{Hol}_0(\Phi, x_0)$, the restricted holonomy group.
Let $f \in \mathrm{Hol}_0(\Phi, x_0)$, then $f = \mathrm{Pt}(c, 1)$ for a piecewise smooth closed curve c through x_0, which is nullhomotopic. Since the parallel transport is essentially invariant under reparametrization, 9.8, we can replace c by $c \circ g$, where g is smooth and flat at each corner of c. So we may assume that c itself is smooth. Since c is homotopic to zero, by approximation we may assume that there is a smooth homotopy $H : \mathbb{R}^2 \to M$ with $H_1|[0, 1] = c$ and $H_0|[0, 1] = x_0$. Then $f_t := \mathrm{Pt}(H_t, 1)$ is a curve in $\mathrm{Hol}_0(\Phi, x_0)$ which is smooth as a mapping $\mathbb{R} \times S \to S$.

Claim 2. $(\frac{d}{dt} f_t) \circ f_t^{-1} =: Z_t$ is in $\mathfrak{g}$ for all t.
To prove claim 2 we consider the pullback bundle $H^* E \to \mathbb{R}^2$ with the induced connection H^Φ. It is sufficient to prove claim 2 there. Let $X = \frac{d}{ds}$ and $Y = \frac{d}{dt}$ be the constant vector fields on $\mathbb{R}^2$, so $[X, Y] = 0$. Then $\mathrm{Pt}(c, s) = \mathrm{Fl}^{CX}_s |S$ and so on. We put

$$f_{t,s} = \mathrm{Fl}^{CX}_{-s} \circ \mathrm{Fl}^{CY}_{-t} \circ \mathrm{Fl}^{CX}_s \circ \mathrm{Fl}^{CY}_t : S \to S,$$

so $f_{t,1} = f_t$. Then we have in the vector space $\mathfrak{X}(S)$

$$
\begin{aligned}
(\tfrac{d}{dt} f_{t,s}) \circ f_{t,s}^{-1} &= -(\mathrm{Fl}_s^{CX})^* CY + (\mathrm{Fl}_s^{CX})^* (\mathrm{Fl}_t^{CY})^* (\mathrm{Fl}_{-s}^{CX})^* CY, \\
(\tfrac{d}{dt} f_{t,s}) \circ f_{t,s}^{-1} &= \int_0^1 \tfrac{d}{ds} \left((\tfrac{d}{dt} f_{t,s}) \circ f_{t,s}^{-1} \right) ds \\
&= \int_0^1 \Big(-(\mathrm{Fl}_s^{CX})^* [CX, CY] + (\mathrm{Fl}_s^{CX})^* [CX, (\mathrm{Fl}_t^{CY})^* (\mathrm{Fl}_{-s}^{CX})^* CY] \\
&\qquad - (\mathrm{Fl}_s^{CX})^* (\mathrm{Fl}_t^{CY})^* (\mathrm{Fl}_{-s}^{CX})^* [CX, CY] \Big)\ ds.
\end{aligned}
$$

Since $[X, Y] = 0$ we have $[CX, CY] = \Phi[CX, CY] = R(CX, CY)$ and

$$
\begin{aligned}
(\mathrm{Fl}_t^{CX})^* CY &= C\left((\mathrm{Fl}_t^{X})^* Y \right) + \Phi\left((\mathrm{Fl}_t^{CX})^* CY \right) \\
&= CY + \int_0^1 \tfrac{d}{dt} \Phi (\mathrm{Fl}_t^{CX})^* CY\ dt = CY + \int_0^1 \Phi (\mathrm{Fl}_t^{CX})^* [CX, CY]\ dt \\
&= CY + \int_0^1 \Phi (\mathrm{Fl}_t^{CX})^* R(CX, CY)\ dt.
\end{aligned}
$$

Thus all parts of the integrand above are in $\mathfrak{g}$ and so $(\frac{d}{dt} f_{t,s}) \circ f_{t,s}^{-1}$ is in $\mathfrak{g}$ for all t and claim 2 follows.

Now claim 1 can be shown as follows. There is a unique smooth curve $g(t)$ in G_0 satisfying $T_e(\rho_{g(t)}) Z_t = Z_t.g(t) = \frac{d}{dt} g(t)$ and $g(0) = e$; via the action of G_0 on S the curve $g(t)$ is a curve of diffeomorphisms on S, generated by the time dependent vector field Z_t, so $g(t) = f_t$ and $f = f_1$ is in G_0. So we get $\mathrm{Hol}_0(\Phi, x_0) \subseteq G_0$.

Claim 3. $\mathrm{Hol}_0(\Phi, x_0)$ equals G_0.
In the proof of claim 1 we have seen that $\mathrm{Hol}_0(\Phi, x_0)$ is a smoothly arcwise connected subgroup of G_0, so it is a connected Lie subgroup by the results cited in 5.6. It suffices thus to show that the Lie algebra $\mathfrak{g}$ of G_0 is contained in the Lie algebra of $\mathrm{Hol}_0(\Phi, x_0)$, and for that it is enough to show, that for each $\xi \in \mathfrak{g}$ there is a smooth mapping $f : [-1, 1] \times S \to S$ such that the associated curve $\check{f}$ lies in $\mathrm{Hol}_0(\Phi, x_0)$ with $\check{f}'(0) = 0$ and $\check{f}''(0) = \xi$.

By definition we may assume $\xi = \mathrm{Pt}(c, 1)^* R(CX_x, CY_x)$ for $X_x, Y_x \in T_x M$ and a smooth curve c in M from x_0 to x. We extend X_x and Y_x to vector fields X and $Y \in \mathfrak{X}(M)$ with $[X, Y] = 0$ near x. We may also suppose that $Z \in \mathfrak{X}(M)$ is a vector field which extends $c'(t)$ along $c(t)$: make c simple piecewise smooth by deleting any loop, reparametrize it in such a way that it is again smooth (see the beginning of this proof) and approximate it by an embedding. The vector fields ξ thus obtained still generate the finite dimensional vector space $\mathfrak{g}$. So we have

$$\begin{aligned}
\xi &= (\mathrm{Fl}_1^{CZ})^* R(CX, CY) = (\mathrm{Fl}_1^{CZ})^*[CX, CY] \quad \text{since } [X,Y](x) = 0 \\
&= (\mathrm{Fl}_1^{CZ})^* \tfrac{1}{2} \tfrac{d^2}{dt^2}|_{t=0} (\mathrm{Fl}_{-t}^{CY} \circ \mathrm{Fl}_{-t}^{CX} \circ \mathrm{Fl}_t^{CY} \circ \mathrm{Fl}_t^{CX}) \\
&= \tfrac{1}{2} \tfrac{d^2}{dt^2}|_{t=0} (\mathrm{Fl}_{-1}^{CZ} \mathrm{Fl}_{-t}^{CY} \circ \mathrm{Fl}_{-t}^{CX} \circ \mathrm{Fl}_t^{CY} \circ \mathrm{Fl}_t^{CX}) \mathrm{Fl}_1^{CZ}),
\end{aligned}$$

where we used the well known formula expressing the Lie bracket of two vector fields as the second derivative of the (group theoretic) commutator of the flows. The parallel transport in the last equation first follows c from x_0 to x, then follows a small closed parallelogram near x in M (since $[X, Y] = 0$ near x) and then follows c back to x_0. This curve is clearly nullhomotopic, so claim 3 follows.

Step 4. Now we make $\mathrm{Hol}(\Phi, x_0)$ into a Lie group which we call G, by taking $\mathrm{Hol}_0(\Phi, x_0) = G_0$ as its connected component of the identity. Then $\mathrm{Hol}(\Phi, x_0)/\mathrm{Hol}_0(\Phi, x_0)$ is a countable group, since the fundamental group $\pi_1(M)$ is countable (by Morse Theory M is homotopy equivalent to a countable CW-complex).

Step 5. Construction of a cocycle of transition functions with values in G. Let $(U_\alpha, u_\alpha : U_\alpha \to \mathbb{R}^m)$ be a locally finite smooth atlas for M such that each $u_\alpha : U_\alpha \to \mathbb{R}^m)$ is surjective. Put $x_\alpha := u_\alpha^{-1}(0)$ and choose smooth curves $c_\alpha : [0,1] \to M$ with $c_\alpha(0) = x_0$ and $c_\alpha(1) = x_\alpha$. For each $x \in U_\alpha$ let $c_\alpha^x : [0,1] \to M$ be the smooth curve $t \mapsto u_\alpha^{-1}(t.u_\alpha(x))$, then c_α^x connects x_α and x and the mapping $(x,t) \mapsto c_\alpha^x(t)$ is smooth $U_\alpha \times [0,1] \to M$. Now we define a fiber bundle atlas $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times S)$ by $\psi_\alpha^{-1}(x,s) = \mathrm{Pt}(c_\alpha^x, 1)\,\mathrm{Pt}(c_\alpha, 1)\, s$. Then ψ_α is smooth since $\mathrm{Pt}(c_\alpha^x, 1) = \mathrm{Fl}_1^{CX_x}$ for a local vector field X_x depending smoothly on x. Let us investigate the transition functions.

$$\begin{aligned}
\psi_\beta \psi_\alpha^{-1}(x,s) &= \big(x, \mathrm{Pt}(c_\alpha, 1)^{-1}\, \mathrm{Pt}(c_\alpha^x, 1)^{-1}\, \mathrm{Pt}(c_\beta^x, 1)\, \mathrm{Pt}(c_\beta, 1)\, s\big) \\
&= \big(x, \mathrm{Pt}(c_\beta.C_\beta^x.(c_\alpha^x)^{-1}.(c_\alpha)^{-1}, 4)\, s\big) \\
&=: (x, \psi_{\beta\alpha}(x)\, s), \text{ where } \psi_{\beta\alpha} : U_{\beta\alpha} \to G.
\end{aligned}$$

Clearly $\psi_{\beta\alpha} : U_{\beta\alpha} \times S \to S$ is smooth which implies that $\psi_{\beta\alpha} : U_{\beta\alpha} \to G$ is also smooth. $(\psi_{\alpha\beta})$ is a cocycle of transition functions and we use it to glue a principal bundle with structure group G over M which we call (P, p, M, G). From its construction it is clear that the associated bundle $P[S] = P \times_G S$ equals (E, p, M, S).

Step 6. Lifting the connection Φ to P.
For this we have to compute the Christoffel symbols of Φ with respect

to the atlas of step 5. To do this directly is quite difficult since we have to differentiate the parallel transport with respect to the curve. Fortunately there is another way. Let $c:[0,1]\to U_\alpha$ be a smooth curve. Then we have

$$\begin{aligned}
&\psi_\alpha(\operatorname{Pt}(c,t)\psi_\alpha^{-1}(c(0),s)) = \\
&\quad = \Big(c(t), \operatorname{Pt}(c_\alpha^{-1},1)\operatorname{Pt}((c_\alpha^{c(t)})^{-1},1)\operatorname{Pt}(c,t)\operatorname{Pt}(c_\alpha^{c(t)},1)\operatorname{Pt}(c_\alpha,1)s\Big)\\
&\quad = (c(t),\gamma(t).s),
\end{aligned}$$

where $\gamma(t)$ is a smooth curve in the holonomy group G. Now let $\Gamma^\alpha\in\Omega^1(U_\alpha,\mathfrak{X}(S))$ be the Christoffel symbol of the connection Φ with respect to the chart (U_α,ψ_α). From the third proof of theorem 9.8 we have

$$\psi_\alpha(\operatorname{Pt}(c,t)\psi_\alpha^{-1}(c(0),s)) = (c(t),\bar\gamma(t,s)),$$

where $\bar\gamma(t,s)$ is the integral curve through s of the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t))$ on S. But then we get

$$\Gamma^\alpha(\tfrac{d}{dt}c(t))(s) = \tfrac{d}{dt}\bar\gamma(t,s) = \tfrac{d}{dt}(\gamma(t).s) = (\tfrac{d}{dt}\gamma(t)).s,$$

where $\frac{d}{dt}\gamma(t)\in\mathfrak{g}$. So Γ^α takes values in the Lie sub algebra of fundamental vector fields for the action of G on S. By theorem 11.9 the connection Φ is thus induced by a principal connection ω on P. Since by 11.8 the principal connection ω has the "same" holonomy group as Φ and since this is also the structure group of P, the principal connection ω is irreducible, see 11.7. $\square$

13. The nonlinear frame bundle of a fiber bundle

13.1. Let now (E, p, M, S) be a fiber bundle and let us fix a fiber bundle atlas (U_α) with transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \times S \to S$. By 6.8 we have $C^\infty(U_{\alpha\beta}, C^\infty(S, S)) \subseteq C^\infty(U_{\alpha\beta} \times S, S)$ with equality if and only if S is compact. Let us therefore assume from now on that S **is compact**. Then we assume that the transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \to \operatorname{Diff}(S, S)$.

Now we define the nonlinear frame bundle of (E, p, M, S) as follows. We consider the set $\operatorname{Diff}\{S, E\} := \bigcup_{x\in M} \operatorname{Diff}(S, E_x)$ and give it the infinite dimensional differentiable structure which one gets by applying the functor $\operatorname{Diff}(S, \quad)$ to the cocycle of transition functions $(\psi_{\alpha\beta})$. Then the resulting cocycle of transition functions for $\operatorname{Diff}\{S, E\}$ gives it the structure of a smooth principal bundle over M with structure group $\operatorname{Diff}(M)$. The principal action is just composition from the right.

We can consider now the smooth action $\mathrm{ev} : \operatorname{Diff}(S) \times S \to S$ and the associated bundle $\operatorname{Diff}\{S, E\}[S, \mathrm{ev}] = \frac{\operatorname{Diff}\{S,E\}\times S}{\operatorname{Diff}(S)}$. The mapping $\mathrm{ev} : \operatorname{Diff}\{S, E\} \times S \to E$ is invariant under the $\operatorname{Diff}(S)$-action and factors therefore to a smooth mapping $\operatorname{Diff}\{S, E\}[S, \mathrm{ev}] \to E$ as in the following diagram:

$$\begin{array}{ccc}
\operatorname{Diff}\{S, E\} \times S & \xrightarrow{pr} & \frac{\operatorname{Diff}\{S,E\}\times S}{\operatorname{Diff}(S)} \\
{\scriptstyle \mathrm{ev}}\big\downarrow & & \big\| \\
E & \longleftarrow & \operatorname{Diff}\{S, E\}[S, \mathrm{ev}].
\end{array}$$

The bottom mapping is easily seen to be a diffeomorphism. Thus the bundle $\operatorname{Diff}\{S, E\}$ is in full right the (nonlinear) *frame bundle* of E.

13.2. Lemma. *In the setting of 13.1 the infinite dimensional smooth manifold* $\operatorname{Diff}\{S, E\}$ *is a splitting smooth submanifold of* $\operatorname{Emb}(S, E)$*, with the obvious embedding.*

13.3. Let now $\Phi \in \Omega^1(E; TE)$ be a connection on E. We want to lift Φ to a principal connection on $\operatorname{Diff}\{S, E\}$, and for this we need a good description of the tangent space $T \operatorname{Diff}\{S, E\}$. With the methods of [Michor, 1980] one can show that

$$\begin{aligned}
T \operatorname{Diff}\{S, E\} = \bigcup_{x\in M} \{f \in C^\infty(S, TE|E_x) : Tp \circ f = \text{ one point}& \\
\text{in } T_xM \text{ and } \pi_E \circ f \in \operatorname{Diff}(S, E_x)\}&.
\end{aligned}$$

Starting from the connection Φ we can then consider $\omega(f) := T(\pi_E \circ f)^{-1} \circ \Phi \circ f : S \to TE \to VE \to TS$ for $f \in T \operatorname{Diff}\{S, E\}$. Then $\omega(f)$ is a vector field on S and we have

Lemma. *$\omega \in \Omega^1(\operatorname{Diff}\{S,E\};\mathfrak{X}(S))$ is a principal connection and the induced connection on $E = \operatorname{Diff}\{S,E\}[S,\mathrm{ev}]$ coincides with Φ.*

Proof. This follows directly from 11.9. But we also give a direct proof.

The fundamental vector field ζ_X on $\operatorname{Diff}\{S,E\}$ for $X \in \mathfrak{X}(S)$ is given by $\zeta_X(g) = Tg \circ X$. Then $\omega(\zeta_X(g)) = Tg^{-1} \circ \Phi \circ Tg \circ X = X$ since $Tg \circ X$ has vertical values. So ω reproduces fundamental vector fields.

Now let $h \in \operatorname{Diff}(S)$ and denote by r^h the principal right action. Then we have

$$\begin{aligned}((r^h)^*\omega)(f) &= \omega(T(r^h)f) = \omega(f \circ h) = T(\pi_E \circ f \circ h)^{-1} \circ \Phi \dot{f} \circ h \\ &= Th^{-1} \circ \omega(f) \circ h = \operatorname{Ad}_{\operatorname{Diff}(S)}(h^{-1})\omega(f). \quad \square\end{aligned}$$

13.4 Theorem. *Let (E,p,M,S) be a fiber bundle with compact standard fiber S. Then connections on E and principal connections on $\operatorname{Diff}\{S,E\}$ correspond to each other bijectively, and their curvatures are related as in 11.8. Each principal connection on $\operatorname{Diff}\{S,E\}$ admits a global parallel transport. The holonomy groups and the restricted holonomy groups are equal as subgroups of $\operatorname{Diff}(S)$.*

Proof. This follows directly from 11.8 and 11.9. Each connection on E is complete since S is compact, and the lift to $\operatorname{Diff}\{S,E\}$ of its parallel transport is the global parallel transport of the lift of the connection, so the two last assertions follow. $\square$

13.5. Remark on the holonomy Lie algebra. Let M be connected, let $\rho = -d\omega - \frac{1}{2}[\omega,\omega]_{\mathfrak{X}(S)}$ be the usual $\mathfrak{X}(S)$-valued curvature of the lifted connection ω on $\operatorname{Diff}\{S,E\}$. Then we consider the $\mathbb{R}$-linear span of all elements $\rho(\xi_f,\eta_f)$ in $\mathfrak{X}(S)$, where ξ_f, $\eta_f \in T_f \operatorname{Diff}\{S,E\}$ are arbitrary (horizontal) tangent vectors, and we call this span $\operatorname{hol}(\omega)$. Then by the $\operatorname{Diff}(S)$-equivariance of ρ the vector space $\operatorname{hol}(\omega)$ is an ideal in the Lie algebra $\mathfrak{X}(S)$.

13.6. Lemma. *Let $f : S \to E_{x_0}$ be a diffeomorphism in $\operatorname{Diff}\{S,E\}_{x_0}$. Then $f_* : \mathfrak{X}(S) \to \mathfrak{X}(E_{x_0})$ induces an isomorphism between $\operatorname{hol}(\omega)$ and the $\mathbb{R}$-linear span of all $g^*R(CX,CY)$, X, $Y \in T_xM$, and $g : E_{x_)} \to E_x$ any diffeomorphism.*

The proof is obvious. Note that the closure of $f_*(\operatorname{hol}(\omega))$ is (a priori) larger than $\operatorname{hol}(\Phi,x_0)$ of 12.2. Which one is the right holonomy Lie algebra?

14. Gauge theory for fiber bundles

We fix the setting of section 13. In particular the standard fiber S is supposed to be compact.

14.1. We consider the bundle $\operatorname{Diff}\{E,E\} := \bigcup_{x\in M} \operatorname{Diff}(E_x,E_x)$ which bears the smooth structure described by the cocycle of transition functions $\operatorname{Diff}(\psi_{\alpha\beta}^{-1},\psi_{\alpha\beta}) = (\psi_{\alpha\beta})_*(\psi_{\beta\alpha})^*$, where $(\psi_{\alpha\beta})$ is a cocycle of transition functions for the bundle (E,p,M,S).

14.2. Lemma. *The associated bundle* $\operatorname{Diff}\{S,E\}[\operatorname{Diff}(S),conj]$ *is isomorphic to the fiber bundle* $\operatorname{Diff}\{E,E\}$.

Proof. The mapping $A : \operatorname{Diff}\{S,E\}\times\operatorname{Diff}(S)\to\operatorname{Diff}\{E,E\}$, given by $A(f,g) := f\circ g\circ f^{-1} : E_x\to S\to S\to E_x$ for $f\in\operatorname{Diff}(S,E_x)$, is $\operatorname{Diff}(S)$-invariant, so it factors to a smooth mapping $\operatorname{Diff}\{S,E\}[\operatorname{Diff}(S)]\to\operatorname{Diff}\{E,E\}$. It is bijective ans admits locally over M smooth inverses, so it is a fiber respecting diffeomorphism. $\square$

14.3. The *gauge group* $\operatorname{Gau}(E)$ of the bundle (E,p,M,S) is by definition the group of all principal bundle automorphisms of the $\operatorname{Diff}(S)$-bundle $(\operatorname{Diff}\{S,E\}$ which cover the identity of M. The usual reasoning gives that $\operatorname{Gau}(E)$ equals the space of all smooth sections of the associated bundle $\operatorname{Diff}\{S,E\}[\operatorname{Diff}(S),conj]$ which by 14.2 equals the space of sections of the bundle $\operatorname{Diff}\{E,E\}\to M$. We equip it with the topology and differentiable structure as a space of smooth sections, see 6.1. Since only the image space is infinite dimensional but admits exponential mappings (induced from the finite dimensional ones) this makes no difficulties. Since the fiber S is compact we see from a local (on M) application of the exponential law 6.8 that $C^\infty(\operatorname{Diff}\{E,E\}\to M)\hookrightarrow\operatorname{Diff}(E)$ is an embedding of a splitting closed submanifold.

14.4. Theorem. *The gauge group* $\operatorname{Gau}(E) = C^\infty(\operatorname{Diff}\{E,E\})$ *is a splitting closed subgroup of* $\operatorname{Diff}(E)$, *if S is compact. It has an exponential mapping which is not surjective on any neighborhood of the identity. Its Lie algebra consists of all vertical vector fields with compact support on E, with the negative of the usual Lie bracket.*

Proof. The first statement has already been shown before the theorem. A curve through the identity of principal bundle automorphisms of $\operatorname{Diff}\{S,E\}\to M$ is a smooth curve through the identity in $\operatorname{Diff}(E)$ consisting of fiber respecting mappings. the derivative of such a curve is thus an arbitrary vertical vector field with compact support. The space of all these is therefore the Lie algebra of the gauge group, with the negative of the usual Lie bracket.

The exponential mapping is given by the flow operator of such vector fields. Since on each fiber it is just isomorphic the exponential mapping of $\mathrm{Diff}(S)$, it has all the properties of the latter. $\square$

14.5. Remark. If S is not compact we may circumvent the nonlinear frame bundle, and we may define the gauge group $\mathrm{Gau}(E)$ directly as the splitting closed subgroup of $\mathrm{Diff}(E)$ which consists of all fiber respecting diffeomorphisms which cover the identity of M. The Lei algebra of $\mathrm{Gau}(E)$ consists then of all vertical vector fields on E with compact support on E.

14.6 The space of connections. Let $J^1(E) \to E$ be the affine bundle of 1-jets of sections of $E \to M$. We have $J^1(E) = \{\ell \in L(T_xM, T_uE) : Tp \circ \ell = Id_{T_xM}, u \in E, p(u) = x\}$. Then a section of $J^1(E) \to E$ is just a horizontal lift mapping $TM \times_M E \to TE$ which is fiber linear over E, so it describes a connection as treated in 9.2 and we may view the space of sections $C^\infty(J^1(E) \to E)$ as the space of all connections.

14.7. Theorem. *The action of the gauge group* $\mathrm{Gau}(E)$ *on the space of connections* $C^\infty(J^1(E))$ *is smooth.*

Proof. This follows from 6.6 $\square$

14.8. We will now give a different description of the action. We view now a connection Φ again as a linear fiber wise projection $TE \to VE$, So the space of connections is now $\mathrm{Conn}(E) := \{\Phi \in \Omega^1(E; TE) : \Phi \circ \Phi = \Phi, \Phi(TE) = VE\}$. Since S is compact the canonical isomorphism $\mathrm{Conn}(E) \to C^\infty(J^1(E))$ is even a diffeomorphism. Then the action of $f \in \mathrm{Gau}(E) \subset \mathrm{Diff}(E)$ on $\Phi \in \mathrm{Conn}(E)$ is given by $f_*\Phi = (f^{-1})^*\Phi = Tf \circ \Phi \circ Tf^{-1}$. Now it is very easy to describe the infinitesimal action. Let X be a vertical vector field with compact support on E and consider its global flow Fl_t^X.

Then we have $\frac{d}{dt}|_0(\mathrm{Fl}_t^X)^*\Phi = \mathcal{L}_X\Phi = [X, \Phi]$, the Frölicher Nijenhuis bracket, by 8.16(5). The tangent space of $\mathrm{Conn}(E)$ at Φ is the space $T_\Phi \mathrm{Conn}(E) = \{\Psi \in \Omega^1(E; TE) : \Psi|VE = 0\}$. The "infinitesimal orbit" at Φ in $T_\Phi \mathrm{Conn}(E)$ is $\{[X, \Phi] : X \in C_c^\infty(VE)\}$.

The isotropy subgroup of a connection Φ is $\{f \in \mathrm{Gau}(E) : f^*\Phi = \Phi\}$. Clearly this just the group of all those f which respect the horizontal bundle $HE = \ker \Phi$. The most interesting object is of course the orbit space $\mathrm{Conn}(E)/\mathrm{Gau}(E)$. If the base manifold M is compact then one can show that the orbit space is stratified into smooth manifolds, each one corresponding to a conjugacy class of holonomy groups in $\mathrm{Diff}(S)$. Those strata whose holonomy group groups are up to conjugacy contained in a fixed compact group (like $SU(2)$) acting on S are diffeomorphic to the strata of the usual finite dimensional gauge theory. The proof of this assertions is quite complicated and will be dealt with in another paper, see [Gil-Medrano Michor, 1989] for the starting idea.

15. A classifying space for the diffeomorphism group

15.1. Let ℓ^2 be the Hilbert space of square summable sequences and let S be a compact manifold. By a slight generalization of theorem 7.3 (we use a Hilbert space instead of a Riemannian manifold N) the space $\operatorname{Emb}(S,\ell^2)$ of all smooth embeddings is an open submanifold of $C^\infty(S,\ell^2)$ and it is also the total space of a smooth principal bundle with structure group $\operatorname{Diff}(S)$ acting from the right by composition. The base space $B(S,\ell^2) := \operatorname{Emb}(S,\ell^2)/\operatorname{Diff}(S)$ is a smooth manifold modeled on Fréchet spaces which are projective limits of Hilbert spaces. $B(S,\ell^2)$ is a Lindelöf space in the quotient topology and the model spaces admit bump functions, thus $B(S,\ell^2)$ admits smooth partitions of unity. We may view $B(S,\ell^2)$ as the space of all submanifolds of ℓ^2 which are diffeomorphic to S, a nonlinear analogy of the infinite dimensional Grassmanian.

15.2. Lemma. *The total space* $\operatorname{Emb}(S,\ell^2)$ *is contractible.*

So by the general theory of classifying spaces the base space $B(S,\ell^2)$ is a classifying space of $\operatorname{Diff}(S)$. I will give a detailed description of the classifying process in 15.4 below.

Proof. We consider the continuous homotopy $A : \ell^2\times[0,1]\to\ell^2$ through isometries which is given by $A_0 = Id$ and by

$$\begin{aligned}A_t(a_0,a_1,a_2,\dots) = (a_0,\dots,a_{n-2},a_{n-1}\cos\theta_n(t),a_{n-1}\sin\theta_n(t),\\ a_n\cos\theta_n(t),a_n\sin\theta_n(t),a_{n+1}\cos\theta_n(t),a_{n+1}\sin\theta_n(t),\dots)\end{aligned}$$

for $\frac{1}{n+1}\le t\le\frac{1}{n}$, where $\theta_n(t)=\varphi(n((n+1)t-1))\frac{\pi}{2}$ for a fixed smooth function $\varphi:\mathbb{R}\to\mathbb{R}$ which is 0 on $(-\infty,0]$, grows monotonically to 1 in $[0,1]$, and equals 1 on $[1,\infty)]$.

Then $A_{1/2}(a_0,a_1,a_2,\dots)=(a_0,0,a_1,0,a_2,0,\dots)$ is in ℓ^2_{even} and on the other hand $A_1(a_0,a_1,a_2,\dots)=(0,a_0,0,a_1,0,a_2,0,\dots)$ is in ℓ^2_{odd}. The same homotopy makes sense as a mapping $A:\mathbb{R}^\infty\times\mathbb{R}\to\mathbb{R}^\infty$, and here it is easily seen to be smooth: a smooth curve in $\mathbb{R}^\infty$ is locally bounded and thus takes locally values in a finite dimensional subspace $\mathbb{R}^N\subset\mathbb{R}^\infty$. The image under A then has values in $\mathbb{R}^{2N}\subset\mathbb{R}^\infty$ and the expression is clearly smooth as a mapping into $\mathbb{R}^{2N}$. This is a variant of a homotopy constructed by [Ramadas, 1982].

Given two embeddings e_1 and $e_2\in\operatorname{Emb}(S,\ell^2)$ we first deform e_1 through embeddings to $e_1'\in\operatorname{Emb}(S,\ell^2_{\text{even}})$, and e_2 to $e_2'\in\operatorname{Emb}(S,\ell^2_{\text{odd}})$; then we connect them by $te_1'+(1-t)e_2'$ which is a smooth embedding for all t since the values are always orthogonal. $\square$

15.3. We consider the smooth action $\mathrm{ev} : \mathrm{Diff}(S) \times S \to S$ and the associated bundle $\mathrm{Emb}(S, \ell^2)[S, \mathrm{ev}] = \mathrm{Emb}(S, \ell^2) \times_{\mathrm{Diff}(S)} S$ which we call $E(S, \ell^2)$, smooth fiber bundle over $B(S, \ell^2)$ with standard fiber S. In view of the interpretation of $B(S, \ell^2)$ as the nonlinear Grassmannian we may visualize $E(S, \ell^2)$ as the "universal S-bundle" as follows: $E(S, \ell^2) = \{(N, x) \in B(S, \ell^2) \times \ell^2 : x \in N\}$ with the differentiable structure from the embedding into $B(S, \ell^2) \times \ell^2$.

The tangent bundle $TE(S, \ell^2)$ is then the space of all (N, x, ξ, v) where $N \in B(S, \ell^2)$, $x \in N$, ξ is a vector field along and normal to N in ℓ^2, and $v \in T_x\ell^2$ such that the part of v normal to T_xN equals $\xi(x)$. This follows from the description of the principal fiber bundle $\mathrm{Emb}(S, \ell^2) \to B(S, \ell^2)$ given in 7.3 combined with 6.7. Obviously the vertical bundle $VE(S, \ell^2)$ consists of all (N, x, v) with $x \in N$ and $v \in T_xN$. The orthonormal projection $p_{(N,x)} : \ell^2 \to T_xN$ defines a connection $\Phi^{\mathrm{class}} : TE(S, \ell^2) \to VE(S, \ell^2)$ which is given by $\Phi^{\mathrm{class}}(N, x, \xi, v) = (N, x, p_{(N,x)}v)$. It will be called the *classifying connection* for reasons to be explained in the next theorem.

15.4. Theorem. Classifying space for $\mathrm{Diff}(S)$**.** *The fiber bundle $(E(S, \ell^2), pr, B(S, \ell^2), S)$ is classifying for S-bundles and Φ^{class} is a classifying connection:*

For each finite dimensional bundle (E, p, M, S) and each connection Φ on E there is a smooth (classifying) mapping $f : M \to B(S, \ell^2)$ such that (E, Φ) is isomorphic to $(f^*E(S, \ell^2), f^*\Phi^{\mathrm{class}})$. Homotopic maps pull back isomorphic S-bundles and conversely (the homotopy can be chosen smooth). The pulled back connection is invariant under a homotopy H if and only if $i(C^{\mathrm{class}}T_{(x,t)}H.(0_x, \frac{d}{dt}))R^{\mathrm{class}} = 0$ where R^{class} is the curvature of Φ^{class}.

Since S is compact the classifying connection Φ^{class} is complete and its parallel transport $\mathrm{Pt}^{\mathrm{class}}$ has the following classifying property:

$$\tilde{f} \circ \mathrm{Pt}^{f^*\Phi^{\mathrm{class}}}(c, t) = \mathrm{Pt}^{\mathrm{class}}(f \circ c, t) \circ \tilde{f},$$

where $\tilde{f} : E \cong F^*E(S, \ell^2) \to E(S, \ell^2)$ is the fiberwise diffeomorphic which covers the classifying mapping $f : M \to B(S, \ell^2)$.

Proof. We choose a Riemannian metric g_1 on the vector bundle $VE \to E$ and a Riemannian metric g_2 on the manifold M. We can combine these two into the Riemannian metric $g := (Tp|\ker\Phi)^*g_2 \oplus g_1$ on the manifold E, for which the horizontal and vertical spaces are orthogonal. By the theorem of Nash there is an isometric imbedding $h : E \to \mathbb{R}^N$ for N large enough. We then embed $\mathbb{R}^N$ into the Hilbert space ℓ^2 and consider

$f : M \to B(S, \ell^2)$, given by $f(x) = h(E_x)$. Then

$$\begin{array}{ccc} E & \xrightarrow{\tilde f = (f,h)} & E(S,\ell^2) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr} \\ M & \xrightarrow[f]{} & B(S,\ell^2) \end{array}$$

is fiberwise a diffeomorphism, so this diagram is a pullback and we have $f^*E(S,\ell^2) = E$. Since $T(f,h)$ maps horizontal and vertical vectors to orthogonal ones, $(f,h)^*\Phi^{\text{class}} = \Phi$. If Pt denotes the parallel transport of the connection Φ and $c : [0,1] \to M$ is a (piecewise) smooth curve we have for $u \in E_{c(0))}$

$$\begin{aligned} \Phi^{\text{class}} \ \tfrac{\partial}{\partial t}\big|_0 \tilde f(\operatorname{Pt}(c,t,u)) &= \Phi^{\text{class}}.T\tilde f.\ \tfrac{\partial}{\partial t}\big|_0 \operatorname{Pt}(c,t,u) \\ &= T\tilde f.\Phi.\ \tfrac{\partial}{\partial t}\big|_0 \operatorname{Pt}(c,t,u) = 0, \quad \text{so} \\ \tilde f(\operatorname{Pt}(c,t,u)) &= \operatorname{Pt}^{\text{class}}(f\circ c, t, \tilde f(u)). \end{aligned}$$

Now let H be a continuous homotopy $M \times I \to B(S,\ell^2)$. Then we may approximate H by smooth mappings with the same H_0 and H_1, if they are smooth, see [Bröcker-Jänich, 1973], where the infnite dimensionality of $B(S,\ell^2)$ does not disturb. Then we consider the bundle $H^*E(S,\ell^2) \to M \times I$, equipped with the connection $H^*\Phi^{\text{class}}$, whose curvature is H^*R^{class}. Let ∂_t be the vector field tangential to all $\{x\}\times I$ on $M \times I$. Parallel transport along the lines $t \mapsto (x,t)$ with respect $H^*\Phi^{\text{class}}$ is given by the flow of the horizontal lift $(H^*C^{\text{class}})(\partial_t)$ of ∂_t. Let us compute its action on the connection $H^*\Phi^{\text{class}}$ whose curvature is H^*R^{class} by 9.5.(3). By lemma 9.9 we have

$$\begin{aligned} \tfrac{\partial}{\partial t}\left(\operatorname{Fl}_t^{(H^*C^{\text{class}})(\partial_t)}\right)^* H^*\Phi^{\text{class}} &= -\tfrac12 i_{(H^*C^{\text{class}})(\partial_t)}(H^*R^{\text{class}}) \\ &= -\frac12 H^*\left(i(C^{\text{class}}T_{(x,t)}H.(0_x,\tfrac{d}{dt}))R^{\text{class}}\right) \end{aligned}$$

which implies the result. $\square$

15.5. Theorem. [Ebin-Marsden, 1970] *Let S be a compact orientable manifold, let μ_0 be a positive volume form on S with total mass 1. Then* $\operatorname{Diff}(S)$ *splits smoothly into* $\operatorname{Diff}(S) = \operatorname{Diff}_{\mu_0}(S)\times\operatorname{Vol}(S)$ *where* $\operatorname{Diff}_{\mu_0}(S)$ *is the Lie group of all μ_0-preserving diffeomorphisms and* $\operatorname{Vol}(S)$ *is the space of all volume forms of total mass 1.*

Proof. We show first that there exists a smooth mapping $\tau : \operatorname{Vol}(S) \to \operatorname{Diff}(S)$ such that $\tau(\mu)^*\mu_0 = \mu$.

We put $\mu_t = \mu_0 + t(\mu - \mu_0)$. We want a smooth curve $t \mapsto f_t \in \mathrm{Diff}(S)$ with $f_t^* \mu_t = \mu_0$. We have $\frac{\partial}{\partial t} f_t = X_t \circ f_t$ for a time dependent vector field X_t on S. Then $0 = \frac{\partial}{\partial t} f_t^* \mu_t = f_t^* \mathcal{L}_{X_t} \mu_t + f_t^* \frac{\partial}{\partial t} \mu_t = f_t^*(\mathcal{L}_{X_t} \mu_t + (\mu = \mu_0))$, so $\mathcal{L}_{X_t} \mu_t = \mu_0 - \mu$ and consequently $\int_S \mathcal{L}_{X_t} \mu_t = \int_S (\mu_0 - \mu) = 0$. So the cohomology class of $\mathcal{L}_{X_t} \mu_t$ in $H^{\dim S}(S)$ is zero, and we have $\mathcal{L}_{X_t} \mu_t = d i_{X_t} \mu_t + i_{X_t} 0 = d\omega$ for some $\omega \in \Omega^{\dim S - 1}(S)$. Now we choose ω such that $d\omega = \mu_0 - \mu$ and we choose it smoothly depending on μ by the theorem of Hodge. Then the time dependent vector field X_t is uniquely determined by $i_{X_t} \mu_t = \omega$ since μ_t is nowhere 0. Let f_t be the evolution operator of X_t and put $\tau(\mu) = f_1^{-1}$.

Now we may prove the theorem proper. We define a mapping Ψ : $\mathrm{Diff}(S) \to \mathrm{Diff}_{\mu_0}(S) \times \mathrm{Vol}(S)$ by $\Psi(f) := (f \circ \tau(f^* \mu_0)^{-1}, f^* \mu_0)$, which is smooth by sections 6 and 7. An easy computation shows that the inverse is given by the smooth mapping $\Psi^{-1}(g, \mu) = g \circ \tau(\mu)$. $\square$

15.6. A consequence of theorem 15.5 is that the classifying spaces of $\mathrm{Diff}(S)$ and $\mathrm{Diff}_{\mu_0}(S)$ are homotopy equivalent. So their classifying spaces are also homotopy equivalent.

We now sketch a smooth classifying space for Diff_{μ_0}. Consider the space $B_1(S, \ell^2)$ of all submanifolds of ℓ^2 of type S and total volume 1 in the volume form induced from the inner product on ℓ^2. It is a closed splitting submanifold of codimension 1 of $B(S, \ell^2)$ by the Nash-Moser inverse function theorem, see [Hamilton, 1982]. This theorem is applicable if we use ℓ^2 as image space, because the modeling spaces are then tame Fréchet spaces in his sense. It is not applicable directly for $\mathbb{R}^\infty$ as image space.

15.7. Theorem. Classifying space for $\mathrm{Diff}^\omega(S)$**.** *Let S be a compact real analytic manifold. Then the space $\mathrm{Emb}^\omega(S, \ell^2)$ of real analytic embeddings of S into the Hilbert space ℓ^2 is the total space of a real analytic principal fiber bundle with structure group $\mathrm{Diff}^\omega(S)$ and real analytic base manifold $B^\omega(S, \ell^2)$, which is a classifying space for the Lie group $\mathrm{Diff}^\omega(S)$. It carries a universal $\mathrm{Diff}^\omega(S)$-connection.*

In other words:

$$(\mathrm{Emb}^\omega(S, N) \times_{\mathrm{Diff}^\omega(S)} S \to B^\omega(S, \ell^2)$$

classifies fiber bundles with typical fiber S and carries a universal (generalized) connection.

The proof is similar to that of 15.4 with the appropriate changes to C^ω.

16. A characteristic class

16.1. Bundles with fiber volumes. Let (E, p, M, S) be a fiber bundle and let μ be a smooth section of the line bundle $\Lambda^n V^*E \to E$ where $n = \dim S$, such that for each $x \in M$ the n-form $\mu_x := \mu|E_x$ is a positive volume form of total mass 1. Such a form will be called a *fiber volume* on E. Such forms exist on any oriented bundle with compact oriented fiber type. We may plug n vertical vector fields $X_i \in C^\infty(VE)$ into μ to get a function $\mu(X_1, \dots, X_n)$ on E, but we cannot treat μ as a differential form on E.

16.2. Lemma. *Let μ_0 be a fixed volume form on S and let μ be a fiber volume on the bundle E. Then there is a bundle atlas $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times S)$ such that $(\psi_\alpha|E_x)^*\mu_0 = \mu_x$ for all $x \in U_\alpha$.*

Proof. We start with any bundle atlas $(U_\alpha, \psi'_\alpha : E|U_\alpha \to U_\alpha \times S)$. By theorem 15.5 there is a diffeomorphism $\varphi_{\alpha,x} \in \operatorname{Diff}(S)$ such that $(\varphi_{\alpha,x})^*((\psi'_\alpha)^{-1}|\{x\} \times S)^*\mu_x = \mu_0$, and $\varphi_{\alpha,x}$ depends smoothly on x. Then $\psi_\alpha := (Id_{U_\alpha} \times \varphi_{\alpha,x}^{-1}) \circ \psi'_\alpha : E|U_\alpha \to U_\alpha \times S$ is the desired bundle chart. $\square$

16.3. Let now Φ be a connection on (E, p, M, S) and let $C : TM \times_M E \to TE$ be its horizontal lifting. Then for $X \in \mathfrak{X}(M)$ the flow Fl_t^{CX} of its horizontal lift CX respects fibers, so for the fiber volume μ the pullback $(\operatorname{Fl}_t^{CX})^*\mu$ makes sense and is again a fiber volume on E. We can now define a kind of covariant derivative by

$$\text{(a)} \qquad \nabla_X^\Phi \mu := \tfrac{d}{dt}|_0 (\operatorname{Fl}_t^{CX})^*\mu.$$

In fact ∇^Φ is a covariant derivative on the vector bundle $\Omega^n(p) := \Omega^n_{\text{ver}}(E) := \bigcup_{x\in M} \Omega^n(E_x)$ with its obvious smooth vector bundle structure, whose standard fiber is the nuclear Fréchet space $\Omega^n(S)$.

We shall need a description of the convariant derivative ∇^Φ in terms of Lie derivatives. So let $j : VE \to TE$ be the embedding of the vertical bundle and let us consider $j^* : \Omega^k(E) \to C^\infty(\Lambda^k V^*E)$. Likewise we consider $\Phi^* : C^\infty(\Lambda^k V^*E) \to \Omega^k(E)$. These are algebraic mappings, algebra homomorphisms for the wedge product, and satisfy $j^* \circ \Phi^* = Id_{C^\infty(\Lambda^k V^*E)}$. The other composition $\Phi^* \circ j^* : \Omega^k(E) \to \Omega^k(E)$ is a projection onto the image of Φ^*.

If $X \in \mathfrak{X}(M)$ is a vector field on M, then the horizontal lift $CX \in \mathfrak{X}(E)$ and X are p-related, i.e. $Tp \circ CX = X \circ p$. Thus the flows are p-conjugated: $p \circ \operatorname{Fl}_t^{CX} = \operatorname{Fl}_t^X \circ p$. So Fl_t^{CX} respects the vertical bundle

VE and we have $j \circ Fl_t^{CX} = \mathrm{Fl}_t^{CX} \circ j$. Therefore we have

(b)
$$\begin{aligned}\nabla_X^\Phi \mu &= \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}_t^{CX})^* \mu = \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}_t^{CX})^* j^* \Phi^* \mu \\ &= j^* \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}_t^{CX})^* \Phi^* \mu = j^* \mathcal{L}_{CX} \Phi^* \mu = 0 + j^* i_{CX} d\Phi^* \mu,\end{aligned}$$

where $\mathcal{L}_{CX}$ is the usual Lie derivative. This formula also shows that $\nabla_X^\Phi \mu$ is $C^\infty(M, \mathbb{R})$-linear in X, as asserted.

Now let (U_α, ψ_α) be a fiber bundle atlas for (E, p, M, S). Put $\Phi^\alpha := (\psi_\alpha^{-1})^* \Phi$, a connection on $U_\alpha \times S \to U_\alpha$. Its horizontal lift is given by $C^\alpha(X) = (X, \Gamma^\alpha(X))$. Let μ be a fiber k-form on $U_\alpha \times S$, so $\mu : U_\alpha \to \Omega^k(S)$ is just a smooth mapping. Thus we have for $x \in U_\alpha$

(c)
$$\begin{aligned}(\nabla_X^{\Phi^\alpha} \mu) &= \tfrac{\partial}{\partial t}\big|_0 ((\mathrm{Fl}^{\Gamma^\alpha(X)})^* (\mu \circ \mathrm{Fl}_t^X)) \\ &= d^{U_\alpha} \mu(X) + \mathcal{L}^S_{\Gamma^\alpha(X)} \mu \\ &= X(\mu) + \mathcal{L}^S_{\Gamma^\alpha(X)} \mu\end{aligned}$$

where d^{U_α} is the exterior derivative of $\Omega^k(S)$-valued forms on U_α, and where $\mathcal{L}^S$ is the Lie derivative on S.

16.4. Lemma. *Let Φ be a connection on the bundle (E, p, M, S) with curvature $R \in \Omega^2(E; VE)$. Then the curvature of the linear covariant derivative ∇^Φ on the vector bundle $(\Omega^k(p), M, \Omega^k(S))$ is given by $R(\nabla^\Phi)(X, Y)\mu = \mathcal{L}^v_{R(CX,CY)} \mu$, where $\mathcal{L}^v$ denotes the vertical Lie derivative, on each fiber separately.*

Proof. The easiest proof is local. Let (U_α, ψ_α) be a fiber bundle atlas for (E, p, M, S). Put again $\Phi^\alpha := (\psi_\alpha^{-1})^* \Phi$, a connection on $U_\alpha \times S \to U_\alpha$ with horizontal lift is given by $C^\alpha(X) = (X, \Gamma(X))$. Let μ be a fiber k-form on $U_\alpha \times S$, so $\mu : U_\alpha \to \Omega^k(S)$ is just a smooth mapping. Thus we have for X, $Y \in \mathfrak{X}(U_\alpha)$, where we write ∇^α for ∇^{Φ^α},

$$\begin{aligned}&R(\nabla^\alpha)(X, Y)\mu = (\nabla^\alpha_X \nabla^\alpha_Y - \nabla^\alpha_Y \nabla^\alpha_X - \nabla^\alpha_{[X,Y]})\mu \\ &= \nabla^\alpha_X (Y(\mu) + \mathcal{L}^S_{\Gamma(Y)}\mu) - \nabla^\alpha_Y (X(\mu) + \mathcal{L}^S_{\Gamma(Y)}\mu) - [X, Y](\mu) - \mathcal{L}^S_{\Gamma[X,Y]}\mu \\ &= X(Y(\mu)) + \mathcal{L}^S_{X\Gamma(Y)}\mu + \mathcal{L}^S_{\Gamma(Y)} X(\mu) + \mathcal{L}^S_{\Gamma(X)} Y(\mu) + \mathcal{L}^S_{\Gamma(X)} \mathcal{L}^S_{\Gamma(Y)}\mu \\ &\quad - Y(X(\mu)) - \mathcal{L}^S_{Y\Gamma(X)}\mu - \mathcal{L}^S_{\Gamma(X)} Y(\mu) - \mathcal{L}^S_{\Gamma(Y)} X(\mu) - \mathcal{L}^S_{\Gamma(Y)} \mathcal{L}^S_{\Gamma(X)}\mu \\ &\quad - [X, Y](\mu) - \mathcal{L}^S_{\Gamma[X,Y]}\mu \\ &= \mathcal{L}^S \left(X\Gamma(Y) - Y\Gamma(X) - \Gamma([X, Y]) \right) \mu + [\mathcal{L}^S_{\Gamma(X)}, \mathcal{L}^S_{\Gamma(Y)}]\mu \\ &= \mathcal{L}^S \left(d^{U_\alpha}\Gamma(X, Y) + [\Gamma(X), \Gamma(Y)] \right) \mu \\ &= \mathcal{L}^S(((\psi_\alpha^{-1}) * R)(X, Y))\mu.\end{aligned}$$

In this computation we used that $\mathcal{L}^S : \mathfrak{X}(S)\times\Omega^k(S) \to \Omega^k(S)$ is bilinear and continuous so that the product rule of differential calculus applies. The last formula comes from lemma 9.7. $\square$

16.5. Definition. The connection Φ on E is called *μ-respecting connection* if $\nabla^\Phi_X \mu = 0$ for all $X \in \mathfrak{X}(M)$.

Lemma.

(1) *A connection Φ is μ-respecting if and only if for some (any) bundle atlas (U_α, ψ_α) as in lemma 16.2 the corresponding Christoffel forms Γ^α take values in the Lie algebra $\mathfrak{X}_{\mu_0}(S)$ of divergence free vector fields on S (i.e. $\mathcal{L}_X\mu_0 = 0$).*

(2) *There exist many μ-respecting connections.*

(3) *If Φ is μ-respecting and if R is its curvature, then for X_x and $Y_x \in T_xM$ the form $i(R(CX_x, CY_x))\mu$ is closed in $\omega^{n-1}(E_x)$.*

Proof. 1. Put $\Phi^\alpha := (\psi_\alpha^{-1})^*\Phi$, a connection on $U_\alpha \times S \to U_\alpha$. Its horizontal lift is given by $C^\alpha(X) = (X, \Gamma^\alpha(X))$. Thus we have $\nabla^{\Phi^\alpha}_X \mu_0 = \mathcal{L}_{\Gamma^\alpha(X)}\mu_0$, and this is zero for all $X \in \mathfrak{X}(U_\alpha)$ if and only if $\Gamma^\alpha \in \Omega^1(U_\alpha; \mathfrak{X}_{\mu_0}(S))$.

2. By the first part μ-respecting connections exist locally and since ∇^Φ_X is $C^\infty(M,\mathbb{R})$-linear in $X \in \mathfrak{X}(M)$ we may glue them via a partition of unity on M.

3. We compute locally in a chart (U_α, ψ_α) belonging to a fiber bundle atlas satisfying lemma 16.2. Then by 1. we have $\Gamma^\alpha \in \Omega^1(U_\alpha; \mathfrak{X}_{\mu_0}(S))$, so by lemma 9.7 the local expression of the curvature $R^\alpha = (\psi_\alpha^{-1})^*R = d^{U_\alpha}\Gamma^\alpha + [\Gamma^\alpha, \Gamma^\alpha]$ is an element of $\Omega^2(U_\alpha; \mathfrak{X}_{\mu_0}(S))$. But then

$$d^S i^S_{R^\alpha(X,Y)}\mu_0 = \mathcal{L}^S_{R^\alpha(X,Y)}\mu_0 = 0. \quad \square$$

16.6. We consider the cohomology class $[i(R(CX, CY))\mu] \in H^{n-1}(E_x)$ for X, $Y \in T_xM$ and the finite dimensional vector bundle $H^{n-1}(p) := \bigcup_{x\in M} H^{n-1}(E_x) \to M$. It is described by a cocycle of transition functions $U_{\alpha\beta} \to GL(H^{n-1}(S))$, $x \mapsto H^{n-1}(\psi_{\beta\alpha}(x, \quad))$, which are locally constant by the homotopy invariance of cohomology. So the vector bundle $H^{n-1}(p) \to M$ admits a unique flat linear connection ∇ respecting the resulting discrete structure group, and the induced covariant exterior derivative $d_\nabla : \Omega^(M; H^{n-1}(p)) \to \Omega^{k+1}(M; H^{n-1}(p))$ satisfies $d_\nabla^2 = 0$ and defines the *De Rham cohomology of M with twisted coefficient domain $H^{n-1}(p)$.*

We may view $[i_R\mu] : X_x, Y_x \mapsto [i(R(CX_x, CY_x))\mu]$ as an element of $\Omega^2(M; H^{n-1}(p))$.

16.7. Lemma.

(1) *If* $\gamma \in C^\infty(\Lambda^k V^* E \to E)$ *induces a section* $[\gamma] : M \to H^k(p)$ *then we have* $\nabla_X[\gamma] = [\nabla^\Phi_X \gamma]$, *for any connection* Φ *on* E.

(2) *If* Φ *is a* μ*-respecting connection on* E *then* $d_\nabla[i_R\mu] = 0$.

Proof. (1). For any bundle atlas $(U_\alpha, \psi\alpha)$ the parallel sections of the vector bundle $H^k(p)$ for the unique flat connection respecting the discrete structure group are exactly those which in each local bundle chart $U_\alpha \times H^k(S)$ are given by the (locally) constant mappings: $U_\alpha \to H^k(S)$.

Obviously $\nabla^\Phi_X[\gamma] := [\nabla^\Phi_X\gamma]$ defines a connection and in the local bundle chart $U_\alpha \to H^k(S)$ we have

$$\begin{aligned}\nabla^{\Phi^\alpha}_X \gamma^\alpha &= \frac{d}{dt}|_0 (\mathrm{Fl}^{(X,\Gamma^\alpha(X))}_t)^* \gamma^\alpha \\ &= d^{U_\alpha}\gamma^\alpha(X) + \mathcal{L}^S_{\Gamma^\alpha(X)}\gamma^\alpha,\end{aligned}$$

where d^{U_α} is the exterior derivative on U_α of Ω^k-valued mappings. But if $[\gamma]$ is parallel for the flat connection respecting the discrete structure group, so $[\gamma^\alpha]$ is locally constant, then $d^{U_\alpha}\gamma^\alpha$ takes values in the space $B^k(S)$ of exact forms. And since $\gamma^\alpha(x)$ is closed for all $x \in U_\alpha$ the second summand $\mathcal{L}^S_{\Gamma^\alpha(X)}\gamma^\alpha = d^S i_{\Gamma^\alpha(X)}\gamma + i_{\Gamma^\alpha(X)}0$ takes values in the space of exact forms anyhow. So the two connections have the same local parallel sections, so they coincide.

(2). Let X_0, X_1, $X_2 \in \mathfrak{X}(M)$. Then we have

$$\begin{aligned}
&d_\nabla[i_R\mu](X_0, X_1, X_2) = \\
&\quad= \sum_{\text{cyclic}} (\nabla_{X_0}([i_R\mu](X_1, X_2)) - [i_R\mu]([X_0, X_1], X_2)) \\
&\quad= \sum_{\text{cyclic}} \big[\nabla_{X_0}(i_{R(CX_1,CX_2)}\mu) - i_{R(C[X_0,X_1],CX_2)}\mu\big] \\
&\quad= \sum_{\text{cyclic}} \big[j^*\mathcal{L}_{CX_0} i_{R(CX_1,CX_2)}\Phi^*\mu - i_{R(C[X_0,X_1],CX_2)}\mu\big] \\
&\quad= \sum \big[j^* i_{[CX_0,R(CX_1,CX_2)]}\Phi^*\mu + \\
&\qquad\qquad + j^* i_{R(CX_1,CX_2)}\mathcal{L}_{CX_0}\Phi^*\mu - i_{R(C[X_0,X_1],CX_2)}\mu\big] \\
&\quad= \big[\sum_{\text{cyclic}} j^* i\big([CX_0, R(CX_1, CX_2)] + 0 - R(C[X_0, X_1], CX_2)\big)\Phi^*\mu\big],
\end{aligned}$$

where we used $[i_{CX}, \mathcal{L}_Z] = i_{[CX,Z]}$, that $j^*\Phi^* = Id$, and that R has vertical values, which implies

$$\begin{aligned}j^* i_{R(CX_1,CX_2)}\mathcal{L}_{CX_0}\Phi^*\mu &= i_{R(CX_1,CX_2)} j^*\mathcal{L}_{CX_0}\Phi^*\mu \\ &= i_{R(CX_1,CX_2)}\nabla^\Phi_{X_0}\mu = 0.\end{aligned}$$

Now we need the Bianchi identity $[R,\Phi]=0$ from 9.4. Writing out the global formula 8.9 for the horizontal vector fields CX_i we get

$$\begin{aligned}0&=[R,\Phi](CX_0,CX_1,CX_2)\\&=\sum_{\text{cyclic}}(-\Phi[CX_0,R(CX_1,CX_2)]-R(C[X_0,X_1],CX_2)).\end{aligned}$$

From this it follows that $d_\nabla[i_R\mu]=0$. $\square$

16.8. Definition. So we may define a characteristic class $k(E,\mu)$ of the bundle (E,p,M,S) with fiber volume μ as the class

$$k(E,\mu):=[[i_R\mu]_{H^{n-1}(p)}]_{H^2(M;H^{n-1}(p))}.$$

16.9. Lemma. *The class $k(E,\mu)$ is independent of the choice of the μ-respecting connection.*

Proof. Let Φ and Φ' be two μ preserving connections on (E,p,M), and let C and C' be their horizontal liftings. We put $D=C-C':TM\times_M E\to VE$, which is fiber linear over E, then $C_t=C+tD=(1-t)C+tC'$ is a curve of horizontal lifts which preserve μ. In fact we have $\mathcal{L}_{D(X_x)}\mu|E_x=0$ for each $X_x\in T_xM$. Let Φ_t be the connection corresponding to C_t and let R_t be its curvature. Then we have

$$\begin{aligned}&R_t(C_tX,C_tY)=\Phi_t[C_tX,C_tY]=[C_tX,C_tY]-C_t[X,Y]\\&\quad=R(CX,CY)+t[CX,DY]+t[DX,CY]+t^2[DX,DY]-tD[X,Y].\\&\tfrac{\partial}{\partial t}\big|_0 R_t(C_tX,C_tY)=[CX,DY]+[DX,CY]+2t[DX,DY]-D[X,Y].\\&\tfrac{\partial}{\partial t}\big|_0 i(R_t(C_tX,C_tY))\mu=j^*\left(\tfrac{\partial}{\partial t}\big|_0 i(R_t(C_tX,C_tY))\right)\Phi^*\mu\\&\quad=j^*\left(i_{[CX,DY]}-i_{[CY,DX]}+2ti_{[DX,DY]}-i_{D[X,Y]}\right)\Phi^*\mu\\&\quad=j^*\left([\mathcal{L}_{CX},i_{DY}]-[\mathcal{L}_{CY},i_{DX}]+2t[\mathcal{L}_{DX},i_{DY}]-i_{D[X,Y]}\right)\Phi^*\mu.\\&\quad=j^*\left(\mathcal{L}_{CX}i_{DY}-0-\mathcal{L}_{CY}i_{DX}-i_{D[X,Y]}+2t(di_{DX}i_{DY}-0)\right)\Phi^*\mu.\end{aligned}$$

Here we used that D has vertical values which implies $j^*i_{DY}\mathcal{L}_{CX}\Phi^*\mu=i_{DY}j^*\mathcal{L}_{CX}\Phi^*\mu=i_{DY}\nabla^\Phi_X\mu=0$.

Next we use lemma 16.7.1 to compute the differential of the of the 1-form $[i_D\mu]\in\Omega^1(M;H^{n-1}(p))$.

$$\begin{aligned}d_\nabla[i_D\mu](X,Y)&=\nabla_X[i_{D(Y)}\mu]-\nabla_Y[i_{D(X)}\mu]-[i_{D([X,Y])}\mu]\\&=[\nabla^\Phi_X i_{D(Y)}\mu-\nabla^\Phi_Y i_{D(X)}\mu-i_{D([X,Y])}\mu]\\&=[j^*(\mathcal{L}_{CX}i_{D(Y)}-\mathcal{L}_{CY}i_{D(X)}-i_{D([X,Y])})\Phi^*\mu].\end{aligned}$$

So we get $\frac{\partial}{\partial t}\big|_0[i_{R_t}\mu]=d_\nabla[i_D\mu]$ in the space $\Omega^2(M;H^{n-1}(p))$. $\square$

16.10. Remarks. The idea of this class is originally due to [Kainz, 1985].

From the point of view of algebraic topology it is not very interesting since it coincides with the second differential in the Serre spectral sequence of the fibration $E \to M$ of the class corresponding to μ.

16.11. Review of linear connections on vector bundles. Let $(V, p, M, \mathbb{R}^n)$ be a vector bundle equipped with a fiber metric g. Let ∇ be a linear covariant derivative on E which respects g. Let $s = (s_i)$ be a local frame field over $U \subset M$ of E: so $s_i \in C^\infty(E|U)$ and the $s_i(x)$ are a linear basis of E_x for each $x \in U$.

Then s defines a vector bundle chart $\psi : E|U \to U \times \mathbb{R}^n$ by the prescription $\psi(\sum s_i u^i) = (u^i)$ or $\psi(s.u) = u$. If s' is another frame field, we have $s'_i = \sum_j s_j h^j_i$ or $s' = s.h$ for a function $h \in C^\infty(U \cap U', GL(n))$. For the vector bundle charts we have then $\psi' \psi^{-1}(x, u) = (x, h(x).u)$. We also have $T\psi : T(E|U) \to TU \times \mathbb{R}^n \times \mathbb{R}^n$, and for the chart change we have

$$\begin{aligned} T(\psi' \circ \psi^{-1}) : T(U \cap U') \times \mathbb{R}^n \times \mathbb{R}^n &\to T(U \cap U') \times \mathbb{R}^n \times \mathbb{R}^n \\ T(\psi' \circ \psi^{-1})(X_x, v, w) &= (X_x, h(x).v, dh(X_x).v + h(x).w) \end{aligned}$$

The *connection form* is defined by $\nabla_X s_i = \sum_j s_j \omega^j_i(X)$ or $\nabla s = s.\omega$ for $\omega \in \Omega^1(U, \mathfrak{gl}(n))$ in general and $\omega \in \Omega^1(U, \mathfrak{o}(n))$ if ∇ respects g and the frame field s is orthonormal. If $\nabla s' = s'.\omega'$ is the connection form for another frame $s' = s.h$, we have the transition formula $h.\omega' = \omega.h + dh$. For a general section $s.u = \sum_j s_j u^j$ we have $\nabla(s.u) = s.\omega.u + s.du$.

Now we want to express the connection $\Phi = \Phi^\nabla \in \Omega^1(E; VE)$, the horizontal lift $C : TM \times_M E \to TE$ and the connector $K = vpr \circ \Phi : TE \to E$ locally in terms of the connection form (compare with 11.10 and following). Since for a local section $x \mapsto (x, u(x))$ of $U \times \mathbb{R}^n \to U$ we have

$$\begin{aligned} \nabla_X(s.u) &= s.(\omega(X).u + du(X)) \\ &= K \circ T(s.u) \circ X \qquad \text{by 11.12}, \\ (T(\psi) \circ T(s.u))(X_x) &= (X_x, u(x), du(X_x)) \end{aligned}$$

we see that

$$\begin{aligned} (\psi \circ K \circ T(\psi^{-1}))(X_x, v, w) &= (X_x, \omega(X_x).v + w) \\ (T\psi \circ \Phi \circ T(\psi^{-1}))(X_x, v, w) &= (0_x, v, w + \omega(X_x).v) \\ (T\psi \circ C \circ (Id \times \psi))(X_x, v) &= (X_x, v, -\omega(X_x).v). \end{aligned}$$

The Christoffel form Γ of Φ with respect to the bundle chart ψ is given by 9.7

$$\begin{aligned}(0_x, \Gamma(X_x, v)) &= -T(\psi)\Phi T(\psi^{-1})(X_x, v, 0)\\ &= (0_x, v, -\omega(X_x).v)\\ \Gamma(X_x, v) &= -\omega(X_x).v\end{aligned}$$

Now the curvature $R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. If we put $R^\nabla.s = s.\Omega$, then we have $\Omega \in \Omega^2(U, \mathfrak{gl}(n))$ and $\Omega \in \Omega^2(U, \mathfrak{o}(n))$ if ∇ respects g and if s is an orthonormal frame field. Then the *curvature form* Ω is given by $\Omega = d\omega + \omega \wedge \omega$.

The curvature $R = \frac{1}{2}[\Phi, \Phi] \in \Omega^2(E; VE)$ can be written in terms of the Christoffel forms by 9.7

$$\begin{aligned}(\psi^{-1})^* R(X, Y) &= d^U \Gamma(X, Y) + [\Gamma(X), \Gamma(Y)]_{\mathfrak{X}(\mathbb{R}^n)}\\ &= d(-\omega)(X, Y) - (-\omega(X)) \wedge (-\omega(Y))\\ &= -\Omega(X, Y),\end{aligned}$$

since $GL(n)$ acts from the left on $\mathbb{R}^n$, so the fundamental vector field mapping is a Lie algebra anti homomorphism.

Let us consider now the unit sphere bundle $SE = \{u \in E : g(u, u) = 1\}$ in the vector bundle E. If we use an orthonormal frame then $\psi : SE|U \to U \times S^{n-1}$ is a fiber bundle chart. We have

$$\begin{aligned}T(U \times S^{n-1}) = \{(X_x, v, w) \in TU \times \mathbb{R}^n \times \mathbb{R}^n : |v| = 1, v \perp w = 0\}\\ \Phi_{\text{local}}(X_x, v, w) = (0_x, v, w + \omega(X_x).v)\\ \langle v, w + \omega(X_x).v\rangle = \langle v, \omega(X_x).v\rangle = 0,\end{aligned}$$

since $\omega(X_x) \in \mathfrak{o}(n)$. So Φ induces a connection on SE with the same Christoffel forms (acting on a submanifold now). If we assume that E is fiber orientable and if we consider the fiber volume μ on SE induced by g, then these Christoffel forms take values in the Lie algebra of divergence free vector fields, so the induced connection Φ^{SE} is μ-respecting. The local expression of the curvature of Φ^{SE} is again $-\Omega$.

16.12. Lemma. *1. For a fiber orientable vector bundle $(E, p, M, \mathbb{R}^n)$ with fiber dimension $n > 2$ and a fiber Riemannian metric g we consider the induced fiber volume μ on the unit sphere bundle SE. Then the characteristic class $k(SE, \mu) = 0$.*

2. For a complex line bundle (E, p, M) with smooth hermitian metric g. We consider again the induced fiber volume on the unit sphere bundle (SE, p, S^1). Then for the characteristic class we have

$$k(SE, \mu) = -c_1(E),$$

the negative of the first Chern class of E.

Proof. 1. This follows from $H^{n-2}(S^{n-1}) = 0$.

2. This follows since $H^0(p) = M \times \mathbb{R}$, so from 16.8 the class $k(SE, \mu)$ is induced by the negative of the curvature form Ω, which also defines the first Chern class. $\square$

17. Self duality

17.1. Let again (E, p, M, S) be a fiber bundle with compact standard fiber S. By a *fiberwise symplectic form* ω on E we mean a smooth section of the vector bundle $\Lambda^2 V^*E \to E$ such that $\omega_x := \omega \restriction E_x \in \Omega^2(E_x)$ is a symplectic form on each fiber E_x. So as in 16.1 we may plug two vertical vector fields X and Y into ω to get a function $\omega(X, Y)$ on E.

If $\dim S = 2n$, let $\omega_x^n = \omega_x \wedge \dots \wedge \omega_x$ be the Liouville volume form of ω_x on E_x. We will suppose that each fiber E_x has total mass 1 for this fiber volume: we may multiply ω by a suitable positive smooth function on M.

I do not know whether lemma 16.2 remains true for fiberwise symplectic structures.

17.2. Example. For a compact Lie group G let $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ be the coadjoint action. Then the union of all orbits of principal type (maximal dimension) is the total space of a fiber bundle which bears a canonical fiberwise symplectic form.

17.3. Let now Φ be a connection on (E, p, M, S) and let ∇ be the associated covariant derivative on the (infinite dimensional) vector bundle $\Omega^2_{\mathrm{ver}}(E) := \bigcup_{x \in M} \Omega^2(E_x)$ over M which is given by

$$\nabla_X \omega := \tfrac{d}{dt}|_0 (\mathrm{Fl}_t^{CX})^* \omega$$

as in 16.3.

Definition. Let ω be a fiberwise symplectic form on E. The connection Φ on E is called *ω-respecting* if $\nabla_X \omega = 0$ for all $X \in \mathfrak{X}(M)$.

17.4. Lemma. *If $\dim H^2(S; \mathbb{R}) = 1$ then there are ω-respecting connections on E and they form a convex set in the space of all connections.*

Proof. Let us first investigate the trivial situation. So $e = M \times S$ and the fiberwise symplectic form is given by a smooth mapping $\omega : M \to Z^2(S) \subset \Omega^2(S)$ such that ω_x is a symplectic form for each x. Since $\int_S \omega_x^n = 1$ for each x and since $\alpha \mapsto \alpha^n$ is a covering mapping $H^2(S) \setminus 0 = \mathbb{R} \setminus 0 \to H^{2n} \setminus 0 = \mathbb{R} \setminus 0$, we see that the cohomology class $[\omega_x] \in H^2(S)$ is locally constant on M, so $\omega : M \to \omega_{x_0} + B^2(S)$ if M is connected. Thus $d^M \omega : TM \to B^2(M)$ is a 1-form on M with values in the nuclear vector space $B^2(S)$ of exact forms, since it is a closed subspace of $\Omega^2(S)$ by Hodge theory. Let $G : \Omega^k(S) \to \Omega^k(S)$ be the Green operator of Hodge theory for some Riemannian metric on S and define $\Gamma \in \Omega^1(M : \mathfrak{X}(S))$

by $i(\Gamma(X_x))\omega_x = -d^*G(d^M\omega)_x(X_x)$. Since ω_x is non degenerate, Γ is uniquely determined by this procedure and we have

$$\mathcal{L}_{\Gamma(X_x)}\omega_x = d^S i_{\Gamma(X_x)}\omega_x + 0 = -dd^*G(d^M\omega)_x(X_x) = -(d^M\omega)_x(X_x),$$

since it is in $B^2(S)$. For the covariant derivative ∇ induced by the connection with Christoffel form Γ we have

$$\nabla_X\omega = \tfrac{d}{dt}|_0(\mathrm{Fl}_t^{(X,\Gamma(X))})^*\omega = (d^M\omega)(X) + \mathcal{L}_{\Gamma(X)}\omega = 0,$$

so the Christoffel form Γ defines an ω-respecting connection.

For the general situation we know now that there are local solutions, and we may use a partition of unity on M to glue the horizontal lifts to get a global ω-respecting connection.

The second assertion is obvious. $\square$

17.5. Let now Φ be an ω-respecting connection on (E,p,M,S) where ω is a fiberwise symplectic form on E. Let $R = R(\Phi)$ be the curvature of Φ and let C be the horizontal lifting of Φ. Then $i(R(CX_x,CY_x))\omega_x \in \Omega^1(E_x)$ and $i(R(CX_x,CY_x))\omega_x^n \in \Omega^{2n-1}(E_x)$. As in lemma 16.5 we may prove that both are closed forms. So $[i(R(CX_x,CY_x))\omega_x)] \in H^1(E_x)$ and $[i(R(CX_x,CY_x))\omega_x^n] \in H^{2n-1}(E_x)$.

Let us also fix an auxiliary Riemannian metric on the base manifold M and let us consider the associated Hodge -$*$-operator: $\Omega^k(M) \to \Omega^{m-k}(M)$, where $m = \dim M$. Let us suppose that the dimension of M is 4 and let us consider the forms $[i_R\omega] \in \Omega^2(M;H^1(p))$ and $*[i_R\omega^n] \in \Omega^2(M;H^{2n-1}(p))$ with values in the flat vector bundles $H^i(p)$ from 16.6. Both are d_∇-closed forms (before applying $*$ to the second one) since the proof of 16.7 applies without changes. Let $D_x : H^{2n-1}(E_x) \to H^1(E_x)$ be the Poincaré duality operator for the compact oriented manifold E_x; these fit together to a smooth vector bundle homomorphism $D : H^{2n-1}(p) \to H^1(p)$.

Definition. An ω-respecting connection Φ is called *self dual* or *anti self dual* if

$$\begin{aligned} D\,*\,[i_R\omega^n] &= [i_R\omega] \quad \text{or} \\ &= -[i_R\omega] \quad \text{respectively.} \end{aligned}$$

This notion makes sense also for not ω -respecting connections.

17.6. For the notion of self duality of 17.5 there is also a Yang-Mills functional if M^4 is supposed to be compact. Let Φ be any connection. For $X_i \in T_xM$ we consider the 4-form

$$\varphi_x(\Phi)(X_1,\dots,X_4) := \tfrac{1}{4}\operatorname{Alt}\int_{E_x}(i_R\omega)_x(X_1,X_2)\wedge *(i_R\omega^n)_x(X_3,X_4),$$

where Alt is the alternator of the indices of the X_i. The *Yang-Mills functional* is then

$$F(\Phi) := \int_M \varphi(\Phi).$$

17.7. A typical example of a fiber bundle with compact symplectic fibers is the union of coadjoint orbits of a fixed type of a compact Lie group. Closely related to this are open subsets of Poisson manifolds. ω-respecting connections exist if the orbit type has second Betti number 1. The Yang-Mills functional and the self duality notion exist even without this requirement. It would be nice to investigate the characteristic class of section 16 for the fiberwise symplectic structure instead of a fiber volume form for examples of coadjoint actions.

17.8. Finally we sketch another notion of self duality. We consider a fiberwise contact structure α on the fiber bundle (E,p,M,S) with compact standard fiber S: So $\alpha_x \in \Omega^1(E_x)$ depends smoothly on x and is a contact structure on E_x, i. e. $\alpha_x \wedge (d\alpha_x)^n$ is a positive volume form on E_x, where $\dim S = 2n+1$ now. A connection Φ on the bundle E is now called *α-respecting* if in analogy with 16.2 we have

$$\nabla^\Phi_X \alpha := \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^{CX}_t)^*\alpha = 0$$

for each vector field X on M. If R is the curvature of such a connection then $iR(CX,CY)\alpha$ is fiberwise locally constant by the analogue of 16.5.(3). If S is connected then $(X,Z)\mapsto iR(CX,CY)\alpha$ defines a closed 2-form on M with real coefficients and it defines also a cohomology class in $H^2(M;\mathbb{R})$. But it is not clear that α-respecting connections exist.

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List of Symbols

$C^\infty(E)$, also $C^\infty(E \to M)$ the space of smooth sections of a fibre bundle

$C^\infty(M, \mathbf{R})$ the space of smooth functions on M

$d : \Omega^p(M) \to \Omega^{p+1}(M)$ usually the exterior derivative

(E, p, M, S), also simply E usually a fibre bundle with total space E, base M, and standard fibre S

Fl_t^X the flow of a vector field X

$\mathbb{I}_k$, short for the $k \times k$-identity matrix $Id_{\mathbb{R}^k}$.

$\mathcal{L}_X$ Lie derivative

G usually a general Lie group with multiplication $\mu : G \times G \to G$, left translation λ, and right translation ρ

$\ell : G \times S \to S$ usually a left action

M usually a (base) manifold

$\mathbb{N}$ natural numbers

$\mathbb{N}_0$ nonnegative integers

$\mathbb{R}$ real numbers

$r : P \times G \to P$ usually a right action, in particular the principal right action of a principal bundle

TM the tangent bundle of a manifold M with projection $\pi_M : TM \to M$

$\mathbb{Z}$ Integers

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