Differential Geometry and its Applications 3 (1993) 323–329
North-Holland

CHARACTERISTIC CLASSES FOR G-STRUCTURES

D. V. Alekseevsky
Peter W. Michor

Center 'Sophus Lie', Moscow
Institut für Mathematik, Universität Wien, Austria

April 10, 2008

Abstract. Let $G \subset GL(V)$ be a linear Lie group with Lie algebra $\mathfrak{g}$ and let $A(\mathfrak{g})^G$ be the subalgebra of G-invariant elements of the associative supercommutative algebra $A(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(V^*)$. To any G-structure $\pi : P \to M$ with a connection ω we associate a homomorphism $\mu_\omega : A(\mathfrak{g})^G \to \Omega(M)$. The differential forms $\mu_\omega(f)$ for $f \in A(\mathfrak{g})^G$ which are associated to the G-structure π can be used to construct Lagrangians. If ω has no torsion the differential forms $\mu_\omega(f)$ are closed and define characteristic classes of a G-structure. The induced homomorphism $\mu'_\omega : A(\mathfrak{g})^G \to H^*(M)$ does not depend on the choice of the torsionfree connection ω and it is the natural generalization of the Chern Weil homomorphism.

1. G-structures

1.1. G-structures. By a G-structure on a smooth finite dimensional manifold M we mean a principal fiber bundle $\pi : P \to M$ together with a representation $\rho : G \to GL(V)$ of the structure group in a real vector space V of dimension $\dim M$ and a 1-form σ (called the *soldering form*) on M with values in the associated bundle $P[V,\rho] = P \times_G V$ which is fiber wise an isomorphism and identifies T_xM with $P[V]_x$ for each $x \in M$. Then σ corresponds uniquely to a G-equivariant 1-form $\theta \in \Omega^1_{\text{hor}}(P;V)^G$ which is strongly horizontal in the sense that its kernel is exactly the vertical bundle VP. The form θ is called the *displacement form* of the G-structure. A G-structure is called *1-integrable* if it admits torsionfree connections, see 1.4 below.

We fix this setting $((P,p,M,G),(V,\rho),\theta)$ from now on.

1.2. Invariant forms. We consider a multilinear form $f \in \bigotimes^k V^* = L^k(V)$ which is invariant in the sense that $f \circ (\bigotimes^k \rho(g)) = f$ for each $g \in G$. Let us denote by

1991 *Mathematics Subject Classification.* 53C10, 57R20.

Key words and phrases. G-structures, characteristic classes.

Supported by Project P 7724 PHY of 'Fonds zur Förderung der wissenschaftlichen Forschung'.

$L^k(V)^G$ the space of all these invariant forms. For each $f \in L^k(V)^G$ we have for any $X \in \mathfrak{g}$, the Lie algebra of G,

$$\begin{aligned} 0 &= \tfrac{d}{dt}|_0 f(\rho(\exp(tX))v_1, \dots, \rho(\exp(tX))v_k), \\ &= \sum_{i=1}^k f(v_1, \dots, \rho'(X)v_i, \dots, v_k), \end{aligned}$$

where $\rho' = T_e\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is the differential of the representation ρ.

1.3 Products of differential forms. For $\varphi \in \Omega^p(P;\mathfrak{g})$ and $\Psi \in \Omega^q(P;V)$ let us define the form $\rho'_\wedge(\varphi)\Psi \in \Omega^{p+q}(P;V)$ by

$$\begin{aligned} &(\rho'_\wedge(\varphi)\Psi)(X_1, \dots, X_{p+q}) = \\ &\qquad = \frac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\rho'(\varphi(X_{\sigma 1}, \dots, X_{\sigma p}))\Psi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}). \end{aligned}$$

Then $\rho'_\wedge(\varphi) : \Omega^*(P;V) \to \Omega^{*+p}(P;V)$ is a graded $\Omega(P)$-module homomorphism of degree p. Recall also that $\Omega(P;\mathfrak{g})$ is a graded Lie algebra with the bracket

$$\begin{aligned} &[\varphi, \psi]_\wedge(X_1, \dots, X_{p+q}) = \\ &\qquad = \frac{1}{p!\,q!} \sum_\sigma \operatorname{sign}\sigma \,[\varphi(X_{\sigma 1}, \dots, X_{\sigma p}), \psi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})]_{\mathfrak{g}}. \end{aligned}$$

One may easily check that for the graded commutator in $\operatorname{End}(\Omega(P;V))$ we have

$$\rho'_\wedge([\varphi,\psi]_\wedge) = [\rho'_\wedge(\varphi), \rho'_\wedge(\psi)] = \rho'_\wedge(\varphi) \circ \rho'_\wedge(\psi) - (-1)^{pq}\rho'_\wedge(\psi) \circ \rho'_\wedge(\varphi)$$

so that $\rho'_\wedge : \Omega^*(P;\mathfrak{g}) \to \operatorname{End}^*(\Omega(P;V))$ is a homomorphism of graded Lie algebras.

Let $\bigotimes V$ be the tensoralgebra generated by V. For $\Phi, \Psi \in \Omega(P;\bigotimes V)$ we will use the associative bigraded product

$$\begin{aligned} &(\Phi \otimes_\wedge \Psi)(X_1, \dots, X_{p+q}) = \\ &\qquad = \frac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\Phi(X_{\sigma 1}, \dots, X_{\sigma p}) \otimes \Psi(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) \end{aligned}$$

1.4. The covariant exterior derivative. Let $\omega \in \Omega^1(P;\mathfrak{g})^G$ be a principal connection on the principal bundle (P, p, M, G). Let $\chi : TP \to HP$ denote the corresponding projection onto the horizontal bundle $HP := \ker\omega$. The covariant exterior derivative $d_\omega : \Omega^k(P;V) \to \Omega^{k+1}_{\text{hor}}(P;V)$ is then given as usual by $d_\omega\Psi = \chi^* d\Psi = (d\Psi) \circ \Lambda^{k+1}(\chi)$.

Lemma. *For $\Psi \in \Omega_{\text{hor}}(P;V)^G$ the covariant exterior derivative is given by $d_\omega\Psi = d\Psi + \rho'_\wedge(\omega)\Psi$.*

Proof. If we insert one vertical vector field, say the fundamental vector field ζ_X for $X \in \mathfrak{g}$, into $d_\omega\Psi$, we get 0 by definition. For the right hand side we use $i_{\zeta_X}\Psi = 0$ and

$\mathcal{L}_{\zeta_X}\Psi = \frac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^{\zeta_X}_t)^*\Psi = \frac{\partial}{\partial t}\big|_0 \Psi\circ\Lambda^p(r^{\exp tX}) = \frac{\partial}{\partial t}\big|_0 \rho(\exp(-tX))\Psi = -\rho'(X)\Psi$ to get

$$\begin{aligned} i_{\zeta_X}(d\Psi+\rho'_\wedge(\omega)\Psi) &= i_{\zeta_X}d\Psi + di_{\zeta_X}\Psi + \rho'_\wedge(i_{\zeta_X}\omega)\Psi - \rho'_\wedge(\omega)i_{\zeta_X}\Psi \\ &= \mathcal{L}_{\zeta_X}\Psi + \rho'_\wedge(X)\Psi = 0. \end{aligned}$$

Let now all vector fields ξ_i be horizontal, then we get

$$\begin{aligned} (d_\omega\Psi)(\xi_0,\dots,\xi_k) &= (\chi^* d\Psi)(\xi_0,\dots,\xi_k) = d\Psi(\xi_0,\dots,\xi_k), \\ (d\Psi+\rho'_\wedge(\omega)\Psi)(\xi_0,\dots,\xi_k) &= d\Psi(\xi_0,\dots,\xi_k). \quad \square \end{aligned}$$

1.5. Definition. If $\theta\in\Omega^1_{\mathrm{hor}}(P;V)^G$ is the displacement form of a G-structure then the *torsion* of the connection ω with respect to the G-structure is $\tau := d_\omega\theta = d\theta + \rho'_\wedge(\omega)\theta$.

Recall that a G-structure is called 1-integrable if it admits a connection without torsion. This notion has also been investigated in [Kolář, Vadovičová] where it was called prolongable.

1.6. Chern-Weil forms. For differential forms $\psi_i\in\Omega^{p_i}(P;V)$ and $f\in L^k(V) = (\bigotimes^k V)^*$ we can construct the following differential forms:

$$\begin{aligned} \psi_1\otimes_\wedge\dots\otimes_\wedge\psi_k &\in \Omega^{p_1+\dots+p_k}(P;V\otimes\dots\otimes V), \\ f^{\psi_1,\dots,\psi_k} := f\circ(\psi_1\otimes_\wedge\dots\otimes_\wedge\psi_k) &\in \Omega^{p_1+\dots+p_k}(P). \end{aligned}$$

The exterior derivative of the latter one is clearly given by

$$\begin{aligned} d(f\circ(\psi_1\otimes_\wedge\dots\otimes_\wedge\psi_k)) &= f\circ d(\psi_1\otimes_\wedge\dots\otimes_\wedge\psi_k) = \\ &= f\circ\Big(\textstyle\sum_{i=1}^k(-1)^{p_1+\dots+p_{i-1}}\psi_1\otimes_\wedge\dots\otimes_\wedge d\psi_i\otimes_\wedge\dots\otimes_\wedge\psi_k\Big). \end{aligned}$$

We also set $f^\psi := f^{\psi,\dots,\psi} = \mathrm{alt}\, f(\psi,\dots,\psi)$ for $\psi\in\Omega^p(P;V)$. Note that the form $f^{\psi_1,\dots,\psi_k}$ is G-invariant and horizontal if all $\psi_i\in\Omega^{p_i}_{\mathrm{hor}}(P;V)^G$ and $f\in L^k(V)^G$. It is then the pullback of a form on M.

1.7. Lemma. *Let $0\neq\psi\in\Omega^p(P;V)$ and $f\in L^k(V)$. Then we have:*

$$f^\psi\neq 0 \iff \begin{cases} \mathrm{alt}\, f\neq 0, & \text{if } p \text{ is odd,} \\ \mathrm{sym}\, f\neq 0, & \text{if } p \text{ is even,} \end{cases}$$

where alt *and* sym *are the natural projections onto $\Lambda(V^*)$ and $S(V^*)$, respectively.* $\square$

1.8. Lemma. *If $f\in L^k(V)^G$ is invariant then we have*

$$f\circ\Big(\textstyle\sum_{i=1}^k(-1)^{p_1+\dots+p_{i-1}}\psi_1\otimes_\wedge\dots\otimes_\wedge\rho'_\wedge(\omega)\psi_i\otimes_\wedge\dots\otimes_\wedge\psi_k\Big) = 0.$$

Proof. This follows from the infinitesimal condition of invariance for f given in 1.2 by applying the alternator. $\square$

2. Obstructions to 1-integrability of G-structures

2.1. Proposition. *Let $\pi : P \to M$ be a G-structure and let $f \in L^k(V)^G$ be an invariant tensor. For arbitrary G-equivariant horizontal V-valued forms $\psi_i \in \Omega^{p_i}_{\text{hor}}(P;V)^G$ we consider the $(p_1+\dots+p_k)$-form $f^{\psi_1,\dots,\psi_k}$ on M as above. If there is a connection ω for the G-structure π such that $d_\omega\psi_i = 0$ for all i, then the form $f^{\psi_1,\dots,\psi_k}$ is closed.*

Proof. We use $d_\omega\psi_i = d\psi_i + \rho'_\wedge(\omega)\psi_i$ from lemma 1.4, and lemma 1.8, to obtain

$$df^{\psi_1,\dots,\psi_k} = f\circ\Big(\textstyle\sum_{i=1}^k(-1)^{p_1+\dots+p_{i-1}}\psi_1\otimes_\wedge\dots\otimes_\wedge d_\omega\psi_i\otimes_\wedge\dots\otimes_\wedge\psi_k\Big) = 0. \quad\square$$

2.2. Corollary. *1. For a G-structure $\pi : P \to M$ with displacement form θ we have a natural homomorphism of associative algebras*

$$\nu : \Lambda(V^*)^G \to \Omega(M),$$
$$f \mapsto f^\theta = f(\theta,\dots,\theta).$$

2. If the G-structure is 1-integrable then the image of ν consists of closed forms and we get an induced homomorphism

$$\nu^* : \Lambda(V^*)^G \to H^*(M).$$

If M and G are compact then ν^* is injective.

Proof. If the G-structure is 1-integrable then there is a connection ω with vanishing torsion $\tau = d_\omega\theta = 0$. Then the result follows from proposition 2.1.

If G is compact, any torsionfree connection ω for $\pi : P \to M$ is the Levi-Civita connection for some Riemannian metric. Any form f^θ, which is parallel with respect to to ω, is harmonic and can thus not be exact for compact M. So ν^* is injective. $\square$

Problem. Is the homomorphism ν^* injective for compact M but noncompact G?

2.3. Remark. Given a principal connection ω on P there is the induced covariant exterior derivative $\nabla : \Omega^p(M;P[V]) \to \Omega^{p+1}(M;P[V])$ on the associated vector bundle $P[V]$. The soldering form (see 1.1) $\sigma : TM \to P[V]$ is an isomorphism of vector bundles and we may consider the pull back covariand derivative $\sigma^*\nabla$ on TM. Next we consider the 'combined' covariant derivative $D^{\sigma^*\nabla,\nabla}$ on the vector bundle $L(TM,P[V])$ given by $D^{\sigma^*\nabla,\nabla}_X A = \nabla_X\circ A - A\circ(\sigma^*\nabla)_X$. Obviously we have $D^{\sigma^*\nabla,\nabla}\sigma = 0$. Consequently for any $f \in L^k(V)^G$ we have that $f^\theta \in \Omega^k(M)$ is parallel for the connection induced on Λ^kT^*M from $\sigma^*\nabla$ on TM.

3. The generalized Chern-Weil homomorphism for G-structures

3.1. The Chern-Weil homomorphism. Let ω be a connection for a G-structure $\pi : P \to M$ with curvature form $\Omega \in \Omega^2_{\text{hor}}(P,\mathfrak{g})$. Then the Bianchi identity $d_\omega\Omega = 0$ holds. If we apply proposition 2.1 to $\psi_i = \Omega$ we obtain a homomorphism

$$\gamma : S(\mathfrak{g}^*)^G \to \Omega(M),$$

given by $\gamma(f) = f^\Omega$. Since the image of γ consists of closed forms we have an induced homomorphism

$$\gamma' : S(\mathfrak{g}^*)^G \to H^*(M).$$

This is the well known Chern-Weil homomorphism.

3.2. The algebra $A(\mathfrak{g}, V)$. In oder to generalize the Chern Weil homomorphism we associate to a Lie algebra $\mathfrak{g}$ and a vector space V the associative graded commutative algebra

$$A(\mathfrak{g}, V) := S(\mathfrak{g}^*) \otimes \Lambda(V^*),$$

where the generators of the symmetric algebra $S(\mathfrak{g}^*)$ have degree 2. We may also consider $A(\mathfrak{g}, V)$ as a graded Lie algebra with the bracket

$$[a \otimes \varphi, b \otimes \psi] := \{a, b\} \otimes \varphi \wedge \psi, \quad a, b \in S(\mathfrak{g}^*), \varphi, \psi \in \Lambda(V^*),$$

where $\{a, b\}$ is the usual Poisson-Lie bracket in $S(\mathfrak{g}^*)$.

Let now $\mathfrak{g}$ be the Lie algebra of the Lie group G and let $\rho : G \to GL(V)$ be a representation. Then G acts naturally on $A(\mathfrak{g}, V)$, and we denote $A(\mathfrak{g}, V)^G$ the subalgebra of G-invariant elements in $A(\mathfrak{g}, V)$.

3.3. Remark. The associative algebra $A(\mathfrak{g}, V)^G$ contains the subalgebra $S(\mathfrak{g}^*)^G \otimes \Lambda(V^*)^G$, in general as a proper subalgebra. Actually, let $G \subset GL(V)$ be the isotropy group of an irreducible Riemannian symmetric space M. Then the curvature tensor of M defines an element of $(\mathfrak{g}^* \otimes \Lambda^2 V^*)^G \subset A(\mathfrak{g}, V)^G$ that does not belong to $(\mathfrak{g}^*)^G \otimes (\Lambda^2 V^*)^G = 0$

3.4. The generalized Chern-Weil homomorphism. Now we are in a position to combine the constructions 2.2 and 3.1.

Theorem. *Let $\pi : P \to M$ be a G-structure on M with displacement form θ. Any connection ω in π defines a homomorphism of associative algebras*

$$\mu : A(\mathfrak{g}, V)^G \to \Omega(M)$$
$$(S^p(\mathfrak{g}^*) \otimes \Lambda^q V^*)^G \ni f \mapsto f^{\Omega,\theta} = f(\underbrace{\Omega, \dots, \Omega}_{p}, \underbrace{\theta, \dots, \theta}_{q})$$

If the connection ω has no torsion then the image of μ consists of closed forms and μ induces a homomorphism

$$\mu' : A(\mathfrak{g}, V)^G \to H^*(M),$$

which is independent of the choice of the torsionfree connection.

In other words, any G-invariant tensor $f \in S^p(\mathfrak{g}^*) \otimes \Lambda^q(V^*)$ defines a cohomology class $[f^{\Omega,\theta}] \in H^{2p+q}(M)$ which is an invariant of the 1-integrable G-structure. We call it a *characteristic class* of the 1-integrable G-structure π.

Proof. It just remains to show that the cohomology class $[f^{\Omega,\theta}]$ does not depend on the choice of the torsionfree connection for the G-structure $\pi : P \to M$.

So let ω_0, ω_1 be two torsionfree connections for the G-structure, let $\varphi = \omega_1 - \omega_0$, and denote by $\Omega_t = d_{\omega_t}\Omega_t$ the curvature form of the torsionfree connection $\omega_t = \omega_0 + t\varphi = (1-t)\omega_0 + t\omega_1$. We claim that for $f \in (S^p(\mathfrak{g}^*) \otimes \Lambda^q(V^*))^G$ we have

$$f^{\Omega_1,\theta} - f^{\Omega_0,\theta} = d(Tf), \quad \text{where} \tag{1}$$
$$Tf = p \int_0^1 f(\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\, dt$$

is the transgression form of f on P. The assertion is immediate from (1). To prove it we compute $\partial_t f^{\Omega_t,\theta}$ using the identities $\partial_t\Omega_t = d_{\omega_t}\varphi$ (see [Kobayashi, Nomizu II, p. 296]), $d_{\omega_t}\Omega_t = 0$, and $d_{\omega_t}\theta = 0$.

$$\begin{aligned}\partial_t f^{\Omega_t,\theta} &= p\, f(\partial_t\Omega_t, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\\ &= p\, f(d_{\omega_t}\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\\ &= p\, d_{\omega_t} f(\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta)\\ &= p\, d\, f(\varphi, \Omega_t, \dots, \Omega_t, \theta, \dots, \theta). \quad \square\end{aligned}$$

3.5. Remarks about secondary characteristic classes. If the characteristic forms $f^{\Omega_1,\theta}$ and $f^{\Omega_0,\theta}$ associated with two torsionfree connections ω_1 and ω_0 vanish we obtain a secondary characteristic class $[Tf]$. It is a natural generalization of the classical Chern-Simons characteristic class, see [Chern, Simons], [Kobayashi, Ochiai].

Problem: study conditions when the secondary characteristic class $[Tf]$ does not depend on the choice of the torsionfree connections ω_1 and ω_0.

3.6. Examples of characteristic classes. Assume that a linear group $G \subset GL(V)$ preserves some pseudo Euclidean metric in $V = \mathbb{R}^n$. Then we may identify the Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$ with a subspace $\mathfrak{g} \subset \Lambda^2 V$. Suppose that there exists a G-invariant supplement $\mathfrak{d}$ to $\mathfrak{g}$ in $\Lambda^2 V$. Then the G-equivariant projection $\Lambda^2 V \to \mathfrak{g}$ along $\mathfrak{d}$ determines a G-invariant element $q \in \mathfrak{g} \otimes \Lambda^2 V^* \cong \mathfrak{g}^* \otimes \Lambda^2 V^*$. The element q defines a 4-form $q^{\Omega,\theta}$ on the base of any G-structure $\pi : P \to M$ with a connection ω and curvature Ω. It may be written as

$$q^{\Omega,\theta} = q(\Omega, \theta, \theta) = q^a_{bcd} R^b_{aef} \theta^c \wedge \theta^d \wedge \theta^e \wedge \theta^f,$$

where (q^a_{bcd}) is the coordinate expression of q in the standard basis (e_a) of $V = \mathbb{R}^n$, $\theta = e_a \otimes \theta^a$, and $\Omega = R^a_{bef}\theta^e \wedge \theta^f$.

If ω is torsionfree the 4-form $q^{\Omega,\theta}$ is closed and it defines a cohomology class $[q^{\Omega,\theta}] \in H^4(M)$ independently of the choice of ω.

3.7. Remarks about the classification of characteristic classes. The classification of characterictic classes for $G-$ *structures* with a given Lie group G reduces to the construction of generators of the associative algebra $A(\mathfrak{g}, V)^G = (S(\mathfrak{g}^*) \otimes \Lambda(V^*))^G$. We may also use the bracket to multiply characteristic classes. It suffices to solve this problem for those Lie groups G which appear as holonomy groups of torsionfree connection. Only for such groups G there exist 1-integrable non-flat G-structures. Under the additional hypothesis of irreducibility, all such groups were classified by [Berger], up to some gaps which were filled by [Bryant] and [Alekseevsky, Graev].

References

Alekseevsky, D. V.; Graev M. M., *Twistors and G-structures*, Preprint, University of Roma, (1991).

Berger, M., *Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniénnes*, Bull. Soc. Math. France 83 **155** (1955), 279–330.

Bryant, R., *Holonomy groups*, Lecture at the conference: Global Differential Geometry and Global Analysis, Berlin 1990.

Chern, S. S.; Simons, J., *Some cohomology classes in principal fiber bundles and their applications to Riemannian geometry*, Proc. Nat. Acad. Sci. (USA) **68** (1971), 791–794.

Chern, S. S.; Simons, J., *Charateristic forms and geometrical invariants*, Annals of Math. **99** (1974), 48–69.

Dupont, Johan L., *Curvature and characteristic classes*, Lecture Notes in Mathematics 640, Springer-Verlag, Berlin, 1978.

Kobayashi, S.; Nomizu, K., *Foundations of Differential Geometry. Vol. II.*, J. Wiley-Interscience, 1969.

Kobayashi, Sh.; Ochiai, T., *G-structures of order two and transgression operators*, J. Diff. Geo. **6** (1971), 213–230.

Kolář, I; Vadovičová, I, *On the structure functions of a G-structure*, Math. Slovaca **35** (1985), 277–282.

D. V. Alekseevsky: Center 'Sophus Lie', Krasnokazarmennaya 6, 111250 Moscow, USSR

P. W. Michor: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: MICHOR@AWIRAP.BITNET