PROJECT APPLICATION: LIE THEORY AND APPLICATIONS II

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This project is planned as a continuation of the project 14195MAT which is finishing in the first half of 2004, and which was very successful.

The research in the new project will follow different lines which are explained below. The project should be located at the Erwin Schrödinger Institute and at the Institute for Mathematics of the University of Vienna jointly.

Papers cited as [Mxy] can be found (fulltext) via the homepage of Peter Michor
`http://www.mat.univie.ac.at/~michor/listpubl.html`
under the number [xy], and those cited as [Hxy] via the homepage of Stefan Haller
`http://www.mat.univie.ac.at/~stefan/pubs.html`
under the number [xy].

1. Invariant Theory. Collaboration of Peter Michor with Dmitri Alekseevsky from Hull (UK), Mark Losik from Saratov (RU) and Andreas Kriegl from Vienna.

In [M65] we investigated the following problem: Let

$$P(t) = x^n - \sigma_1(t)x^{n-1} + \cdots + (-1)^n\sigma_n(t)$$

be a polynomial with all roots real, smoothly parameterized by t near 0 in $\mathbb{R}$. Can we find n smooth functions $x_1(t), \dots, x_n(t)$ of the parameter t defined near 0, which are the roots of $P(t)$ for each t? We showed that this is possible under quite general conditions: real analyticity or no two roots should meet of infinite order. Some applications to perturbations of unbounded operators in Hilbert space are also given, the first progress in this direction since the result of Rellich in 1940. A counterexample in [M65] turned out to be wrong, by a computational mistake. Alerted by that we could even prove the contrary. So in [M91] we showed that one can always choose the roots twice differentiable if the the coefficients are C^{3n}, where n is the degree of the polynomial. In [M89] this has been used to strengthen our result on the differentiability of of the eigenvalues of a smooth curve of unbounded operators with compact resolvent on Hilbert space.

The problem from [M65] can be reformulated in the following way: Let the symmetric group S_n act in $\mathbb{R}^n$ by permuting the coordinates (the roots), and consider the polynomial mapping $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{R}^n \to \mathbb{R}^n$ whose components are the elementary symmetric polynomials (the coefficients). Given a smooth curve $c : \mathbb{R} \to \sigma(\mathbb{R}^n) \subset \mathbb{R}^n$, is it possible to find a smooth lift $\bar{c} : \mathbb{R} \to \mathbb{R}^n$ with $\sigma \circ \bar{c} = c$?

In [M73] we tackled the following generalization of this problem. Consider an orthogonal representation of a compact Lie group G on a real vector space V. Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. Then the mapping

$$\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$$

defines a bijection of the orbit space V/G to the semialgebraic set $\sigma(V) \subseteq \mathbb{R}^n$. A curve

$$c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$$

in the orbit space V/G is called smooth if it is smooth as a curve in $\mathbb{R}^n$. This is well defined, i.e. does not depend on the choice of generators.

Problem. Given a smooth curve $c : \mathbb{R} \to V/G$ in the orbit space, does there exist a smooth lift to V, i.e. a smooth curve $\bar{c} : \mathbb{R} \to V$ with $c = \sigma \circ \bar{c}$?

We gave satisfactory answers under similar conditions as in the paper [M65].

In the last project, in [M99] we could show that each suitably often differentiable curve can be lifted to a once differentiable curve over the invariant mapping (also past of a Dr. thesis). The hope is to to show that one can even lift it twice differentiable, but this seems to be very difficult.

In this project we want to investigate these two questions, but we also want to replace $\mathbb{R}$ by higher dimensional manifolds. The paper [M94] which is nearly finished is a first step in this direction.

Collaboration is also foreseen with the physicist Guiseppe Marmo from Napoli and with Janusz Grabowski from Warsaw on the question of finding a useful criterion for entanglement in quantum mechanics (for the purpose of quantum computing) which seems to have an invariant theoretic flavor.

2. Representation theory. Collaboration of Peter Michor with Bert Kostant, and Mona Linkmann. During the last project the paper [M87] was written which constructs the generalized Cayley transform from a Lie group to its Lie algebra, induced by certain representation. Many properties were deduced, mostly of algebraic geometric type. For the spin representation this equals the classical Cayley transform $A \mapsto (A - \mathrm{Id})(A + \mathrm{Id})^{-1}$ for matrices, multiplied by a rational function which kills the polar divisor of the classical Cayley transform. A PhD thesis is in development (Mona Linkmann) where the setup of the generalized Cayley transform is investigated in the setting of finite groups of Lie type: here it is the first connection between the group and its (finite) Lie algebra (the exponential mapping does not make sense). Research in this direction will go on.

3. Geodesics on infinite dimensional weak Riemannian manifolds. In [M98] a very exciting collaboration of Peter Michor with David Mumford was started: This paper in preparation arose from attempts to find a suitable topological metric on the space of 2-dimensional shapes. By a shape one could mean the boundary of compact region in the plane, so we can start by those with smooth boundaries. Thus the space of shapes for us is at the beginning the orbit space $B_e(S^1, \mathbb{R}^2) = \operatorname{Emb}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$ of the space of all embedded regular smooth (C^∞) simple closed curves in the plane, under the action by composition from the right by diffeomorphisms of the circle. David Mumford was led to study the space B_e from its relevance to computer vision. To understand an image of the world, one needs to identify the most salient objects present in this image. In addition to readily quantifiable properties like color and area, objects in the world and their projections depicted by 2D images possess a 'shape' which is readily used by human observers to distinguish, for example, cats from dogs, BMWs from Hondas, etc. In fact people are not puzzled by what it means to say two shapes are *similar* but rather find this a natural question. This suggests that we construct, on some crude level, a mental metric which can be used to recognize familiar objects by the similarity of their shapes and to cluster categories of related objects like cats. Incidentally, immersions also arise in vision when a 3D object partially occludes itself from some viewpoint, hence its full 2D contour has visible and invisible parts which, together, form an immersed curve in the image plane. In fact, most of the results carry over to orbit spaces of some spaces of immersions which we have to consider anyhow for shapes with hidden parts like an arm behind a body. In particular we investigate the metric for a constant $A > 0$:

$$G_c^A(h, k) := \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle |c'(\theta)|\, d\theta$$

where κ_c is the curvature of the curve c. For $A = 0$ this induces 0 as geodesic distance function on the orbit space, which by itself is a very surprising result. But for $A > 0$ it induces a point separating metric. We give some interesting estimates for the distance function, derive the geodesic equation and the sectional curvature, solve the geodesic equation with simple endpoints numerically, and pose some open questions. Such questions have also arisen in Teichmüller theory and string theory, where the so-called Weil-Peterssen metric on the space of shapes (also called the 'universal Teichmüller space') has been much studied. In a second part of this paper, we will compare our metric to this remarkable (homogeneous!) metric.

4. Actions of finite dimensional Lie groups and structures of orbit spaces. Collaboration of Peter Michor with D. Alekseevsky from Moscow, M. Losik from Saratov, Andreas Kriegl and Simon Hochgerner from Vienna.

The project is to continue the investigations in the direction of obtaining a better understanding of the geometry of the orbit space of an isometric Lie group action. Paper [M85] gives a good description of the orbit space and of the image of geodesics on it: Short normal geodesics map to distance minimizing curves, whereas other geodesics map in special examples to solutions of very interesting dynamical systems generalizing the Calogero-Moser system. The thesis of Simon Hochgerner is devoted to study the complete integrability of these generalizations. Paper [M97] develops reflection groups on Riemannian manifolds and shows how to reconstruct them from the orbit space (a generalized Weyl chamber). Research in this direction will go on.

5. Actions of Lie algebras on manifolds. Collaboration of Peter Michor with Franz Kamber from Illinois and Boris Khesin from Toronto.

In the paper [M56] we started to investigate the differential geometry of an action of a Lie algebra on a manifold, i.e. only an infinitesimal Lie group action. We want to study how this action can be extended to a Lie group action on an enlarged manifold. There exists always a universal solution to this problem, as shown in the papers [M83], [M92], and [M96]. In particular, in [M92] we have constructed the flow completion of Burgers' equation which we viewed as a dynamical system on an infinite dimensional space where the solution run into singularities in finite time. This research should go on to construct the (purely infinite dimensional) flow completion to a positive semigroup with infinitesimal generator with its universal property: this would allow also to investigate the viscosity solutions of Burgers' equation when the viscosity term goes to 0.

6. Poisson harmonic forms. (Stefan Haller) Consider a Poisson manifold (P,Λ). The Poisson structure gives rise to a codifferential $\delta:\Omega^*(P)\to\Omega^{*-1}(P)$. Brylinski used this codifferential to define a subspace of Poisson harmonic cohomology classes in the deRham cohomology $H^*(P)$ of P. As in [H9] one can us this codifferential to define a filtration $H^*_m(P,\Lambda)\subseteq H^*(P)$, where $H^*_0(P,\Lambda)$ is exactly Brylinski's space of Poisson harmonic classes. Moreover on the associated bigraded $\tilde H^*_m(P,\Lambda):=H^*_m(P,\Lambda)/H^*_{m-1}(P,\Lambda)$ we get mappings $F_m:\tilde H^*_m(P,\Lambda)\to H^{*-2}_m(P,\Lambda)$. Usually F_m is not a differential — this is not part of a spectral sequence. For a Poisson mapping $f:(P_1,\Lambda_1)\to(P_2,\Lambda_2)$ the induced map in cohomology $f^*:H^*(P_2)\to H^*(P_1)$ has to preserve this structure. Particularly we should get obstructions on the homotopy type of Poisson mappings.

In view of this we would like to compute the filtration and the F_m for nice Poisson manifolds. In [H9] the special case when (P,Λ) is a symplectic manifold was treated. It turned out that all the structure is determined by the cohomology class $[\omega]\in H^2(P)$ of the symplectic structure ω, and is effectively computable in terms of the cohomology ring $H^*(P)$. Also if (P,Λ) is the product of a symplectic manifold with another manifold B the situation is simple and computable. As next step we would like to study the filtration and the mappings F_m in the case when the symplectic leaves of (P,Λ) define a Hamiltonian fibration $M\to P\to B$. At least if the structure group of this fibration is a compact subgroup G of the Hamiltonian group we have $H^*(P)=H^*(M)\otimes H^*(B)$, additively, according to a theorem of Atiyah–Bott. The deRham model for the equivariant cohomology $H^*_G(M)$ should permit one to get a hold on the filtration $H^*_m(P)$ and the mappings F_m.

The second issue we would like to study is the following: If (P,Λ) is symplectic, Poincaré duality maps the filtration $H^*_m(P,\Lambda)$ onto its dual and the maps F_m to their duals. Examples show that Poincaré duality ceases to have this property if (P,Λ) has singularities, e.g. leaves of different dimensions. This failure can be used to define invariants associated to (P,Λ). We would like to show that in the case when the symplectic leaves of (P,Λ) define a fibration Poincaré duality still has the nice property and these invariants vanish. In this case the invariants obtained from Poincaré duality could be considered as very condensed geometric information about the singularities of (P,Λ).

7. Dynamics versus spectral geometry. Joint work of Stefan Haller with Dan

Burghelea, Ohio State University, USA.

Consider a closed manifold M, a closed one form ω and a Riemannian metric g. The metric and the closed one form give rise to a vector field X. Suppose X has only non-degenerate zeros and is Morse–Smale. Counting the instantons one obtains incident numbers which give rise to a Morse–Novikov complex. In [H6] we showed how one can use spectral geometry to recover all Novikov incident numbers. We would like to pursue this and study the Morse–Bott case. So we would like to drop the condition that X has only non-degenerate zeros and replace it with the Morse–Bott condition, that is the zeros of X are submanifolds and one has a non-degeneracy condition in the normal direction.

In contrast to the Morse case the dynamics of a closed one form has periodic orbits in general. Assuming all periodic orbits are non-generate one can count them. In [H12] we show how one can recover the number of periodic orbits with the help of spectral geometry. We would like to extend this to the Bott situation. In this context this means we will drop the condition that the closed orbits are non-degenerate and replace it with an appropriate Bott–like condition.

8. Vortex filament equation. Joint work of Stefan Haller with Cornelia Vizman, West University of Timisoara, Romania.

In [H11] we introduced a generalization of the vortex filament equation. On an oriented Riemannian manifold the space of oriented codimension 2 submanifolds carries a natural symplectic structure. The Hamiltonian vector field of the function which assigns to such a submanifold its volume is JH. Here H is the mean curvature of the submanifold and J is the complex structure on its normal bundle induced from the metric g and the orientations. Thus the Hamilton equation formally reads

$$\frac{\partial \Sigma_t}{\partial t} = JH(\Sigma_t),$$

Σ_t being the submanifold at time t. We would like to study this evolution equation, first establishing short time existence and uniqueness of solutions.

Personnel and budget.

Name	Position	per year
Stefan Haller	1/2 Senior Post Doc for 3 years	27.015,00
Armin Rainer	1/2 Post Doc for 3 years	22.860,00
Mona Linkmann	1/2 Post Doc for 3 years	22.860,00
Simon Hochgerner	1/2 Post Doc for 3 years	22.860,00
N.N. Dissertant	2 x 1/2 Dissertant for 3 years	27.620,00
Invitations	Guests, 75 E/day	15.000,00
	Sum per year	138.215,00
	Sum for 3 years, in EURO	414.645,00

The invitation money is for inviting guests to Vienna, among them the scientists mentioned in the report. Stefan Haller will apply for Habilitation shortly. Rainer, Linkmann, and Hochgerner are now working on their Dr. theses and will be finished with them when the program is starting.

Start of the program: September 2004.

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- ESI preprint server:
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