Euler's equation of fluids and $\text{Diff}_{H^\infty}(\mathbb{R}^n)$

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EPDiff

On the space of vector fields $\mathfrak{X}_{H^\infty}(\mathbb{R}^n) = H^\infty(\mathbb{R}^n)^n$ we consider a weak inner product of the form

$$\|v\|_L^2 = \int_{\mathbb{R}^n} \langle Lv, v\rangle \, dx$$

where L is a positive L^2-symmetric (pseudo-) differential operator. This gives rise to a right invariant metric on $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ whose geodesic equation was discussed a lot already.

Condsider the *momentum* $m = L(v)$ of a vector, so that $\langle v, w\rangle_L = \int \langle m, w\rangle dx$. Then the geodesic equation is of the form:

$$\partial_t m = -(v \cdot \nabla)m - \mathrm{div}(v)m - m \cdot (Dv)^t$$

$$\partial_t m_i = -\sum_j (v_j \partial_{x_j} m_i + \partial_{x_j} v_j \cdot m_i + m_j \partial_{x_i} v_j)$$

$$v = K * m, \qquad \text{where } K \text{ is the matrix-valued Green function of } L.$$

Suppose, the time dependent vector field v integrates to a flow φ via

$$\partial_t \varphi(x,t) = v(\varphi(x,t),t)$$

and we describe the momentum by a *measure-valued 1-form*

$$\widetilde{m} = \sum_i m_i dx_i \otimes (dx_1 \wedge \cdots \wedge dx_n)$$

so that $\|v\|_L^2 = \int (v, \widetilde{m})$ makes intrinsic sense. Then the geodesic equation is equivalent to: $\widetilde{m}$ is *invariant* under the flow φ, that is,

$$\widetilde{m}(\cdot,t) = \varphi(\cdot,t)_* \widetilde{m}(\cdot,0),$$

whose infinitesimal version is the following, using the Lie derivative:

$$\partial_t \widetilde{m}(\cdot,t) = -\mathcal{L}_{v(\cdot,t)} \widetilde{m}(\cdot,t).$$

Because of this invariance, if a geodesic begins with momentum of compact support, it will always have compact support; and if it begins with momentum which, along with all its derivatives, has 'rapid' decay at infinity, that is it is in $O(\|x\|^{-n})$ for every n, this too will persist. This comes from the lemma:

Lemma: [1. lecture] *If $\varphi \in \mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ and T is any smooth tensor on $\mathbb{R}^n$ with rapid decay at infinity, then $\varphi_*(T)$ is again smooth with rapid decay at infinity.*

Moreover this invariance gives us a Lagrangian form of EPDiff:

$$
\begin{aligned}
\partial_t \varphi(x,t) &= \int K^{\varphi(\cdot,t)}(x,y)(\varphi(y,t)_* \widetilde{m}(y,0)) \\
&= K^{\varphi(\cdot,t)} * (\varphi(\cdot,t)_* \widetilde{m}(\cdot,0)) \\
&\qquad \text{where } K^{\varphi}(x,y) = K(\varphi(x),\varphi(y))
\end{aligned}
$$

Aim: Solutions of Euler's equation are limits of solutions of equations in the EPDiff class with the operator:

$$L_{\varepsilon,\eta} = (I - \tfrac{\eta^2}{p}\triangle)^p \circ (I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div}), \qquad \text{for any } \varepsilon > 0, \eta \geq 0.$$

All solutions of Euler's equation are limits of solutions of these much more regular EPDiff equations and *give a bound on their rate of convergence*. In fact, so long as $p > n/2 + 1$, these EPDiff equations have a well-posed initial value problem with unique solutions for all time. Moreover, although $L_{0,\eta}$ does not make sense, the analog of its Green's function $K_{0,\eta}$ does make sense as do the geodesic equations in momentum form. These are, in fact, geodesic equations on the group of volume preserving diffeomorphisms SDiff and become Euler's equation for $\eta = 0$. An important point is that so long as $\eta > 0$, the equations have *soliton* solutions (called *vortons*) in which the momentum is a sum of delta functions.

Relation to Euler's equ. Oseledetz 1988

We use the kernel

$$K_{ij}(x) = \delta_{ij}\delta_0(x) + \partial_{x_i}\partial_{x_j} H$$

where H is the Green's function of $-\Delta$. But K now has a rather substantial pole at the origin. If $V_n = \mathrm{Vol}(S^{n-1})$,

$$H(x) = \begin{cases} \frac{1}{(n-2)V_n}(1/|x|^{n-2}) & \text{if } n > 2, \\ \frac{1}{V_2}\log(1/|x|) & \text{if } n = 2 \end{cases}$$

so that, *as a function*

$$(M_0)_{ij}(x) := \partial_{x_i}\partial_{x_j} H(x) = \frac{1}{V_n} \cdot \frac{n x_i x_j - \delta_{ij}|x|^2}{|x|^{n+2}}, \quad \text{if } x \neq 0.$$

Convolution with any $(M_0)_{ij}$ is still a Calderon-Zygmund singular integral operator defined by the limit as $\varepsilon \to 0$ of its value outside an ε-ball, so it is reasonably well behaved. *As a distribution* there is another term:

$$\partial_{x_i}\partial_{x_j} H \overset{\text{distribution}}{=} (M_0)_{ij} - \frac{1}{n}\delta_{ij}\delta_0$$

$$P_{\mathrm{div}=0} : m \mapsto v = \left(m + \partial^2(H)_{\mathrm{distr}}\right) = \left(\tfrac{n-1}{n} \cdot m + M_0 * m\right)$$

is the *orthogonal projection* of the space of vector fields m onto the subspace of divergence free vector fields v, orthogonal in each Sobolev space H^p, $p \in \mathbb{Z}_{\geq 0}$. (Hodge alias Helmholtz projection).

The matrix $M_0(x)$ has $\mathbb{R}x$ as an eigenspace with eigenvalue $(n-1)/V_n|x|^n$ and $\mathbb{R}x^\perp$ as an eigenspace with eigenvalue $-1/V_n|x|^n$. Let $P_{\mathbb{R}x}$ and $P_{\mathbb{R}x^\perp}$ be the orthonormal projections onto the eigenspaces, then

$$P_{\mathrm{div}=0}(m)(x) = \tfrac{n-1}{n} \cdot m(x) +$$
$$+ \frac{1}{V_n} \cdot \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{1}{|y|^n} \left((n-1) P_{\mathbb{R}y}(m(x-y)) - P_{\mathbb{R}y^\perp}(m(x-y)\right) dy.$$

With this K, EPDiff in the variables (v, m) is the Euler equation in v with pressure a function of (v, m). *Oseledets's form for Euler*:

$$\boxed{\begin{aligned} v &= P_{\mathrm{div}=0}(m) \\ \partial_t m &= -(v \cdot \nabla)m - m \cdot (Dv)^t \end{aligned}}$$

Let $\widetilde{m} = \sum_i m_i dx_i$ be the 1-form associated to m. Since $\operatorname{div} v = 0$, we can use $\widetilde{m}$ instead of $\sum_i m_i dx_i \otimes dx_1 \wedge \ldots dx_n$. Integrated form:

$$\boxed{\begin{aligned} \partial_t \varphi &= P_{\mathrm{div}=0}(m) \circ \varphi \\ \widetilde{m}(\cdot, t) &= \varphi(\cdot, t)_* \widetilde{m}(\cdot, 0) \end{aligned}}$$

This uses the variables v, m instead of v and pressure.
Advantage: m, like vorticity, is constant when transported by the flow. m determines the vorticity the 2-form $\omega = d(\sum_i v_i dx_i)$, because v and m differ by a gradient, so $\omega = d\widetilde{m}$ also. Thus: vorticity is constant along flows follows from the same fact for momentum 1-form $\widetilde{m}$.

However, these equ. are not part of the true EPDiff framework because the operator $K = P_{\text{div}=0}$ is not invertible and there is no corresponding differential operator L.

In fact, v does not determine m as we have rewritten Euler's equation using extra non-unique variables m, albeit ones which obey a conservation law so they may be viewed simply as extra parameters.

Approximationg Euler by EPDiff

Replace the Green's function H of $-\Delta$ by the Green's function H_ε of the positive $\varepsilon^2 I - \triangle$ for $\varepsilon > 0$ (whose dimension is length^{-1}). The Green's function is be given explicitly using the 'K' Bessel function via the formula

$$H_\varepsilon(x) = c_n \varepsilon^{n-2} |\varepsilon x|^{1-n/2} K_{n/2-1}(|\varepsilon x|)$$

for a suitable constant c_n independent of ε. Then we get the modified kernel

$$(K_\varepsilon)_{ij} = \delta_{ij}\delta_0 + (\partial_{x_i}\partial_{x_j} H_\varepsilon)_{\mathrm{distr}}$$

This has exactly the same highest order pole at the origin as K did and the second derivative is again a Calderon-Zygmund singular integral operator minus the same delta function. The main difference is that this kernel has exponential decay at infinity, not polynomial decay. By weakening the requirement that the velocity be divergence free, the resulting integro-differential equation behaves much more locally, more like a hyperbolic equation rather than a parabolic one.

The corresponding inverse is the differential operator

$$L_\varepsilon = I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}$$

$$v = K_\varepsilon * m, \quad m = L_\varepsilon(v)$$

$$\|v\|^2_{L_\varepsilon} = \int \langle v, v\rangle + \mathrm{div}(v).\,\mathrm{div}(v) dx$$

Geodesic equation:

$$\partial_t(v_i) = (K_\varepsilon)_{ij} * \partial_t(m_j)$$

$$= -(K_\varepsilon)_{ij} * (v_k v_{j,k}) - v_i\,\mathrm{div}(v) - \tfrac{1}{2}(K_\varepsilon)_{ij} * \left(|v(x)|^2 + (\tfrac{\mathrm{div}(v)}{\varepsilon})^2\right)_{,j}$$

Curiously though, the parameter ε can be scaled away. That is, if $v(x,t), m(x,t)$ is a solution of EPDiff for the kernel K_1, then $v(\varepsilon x, \varepsilon t), m(\varepsilon x, \varepsilon t)$ is a solution of EPDiff for K_ε.

Regularizing more

Compose L_ε with a scaled version of the standard regularizing kernel $(I - \triangle)^p$ to get

$$L_{\varepsilon,\eta} = (I - \tfrac{\eta^2}{p}\triangle)^p \circ (I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div})$$
$$K_{\varepsilon,\eta} := {L_{\varepsilon,\eta}}^{-1} = G_\eta^{(p)} * K_\varepsilon$$

where $G_\eta^{(p)}$ is the Green's function of $(I - \frac{\eta^2}{p}\triangle)^p$ and is again given explicitly by a 'K'-Bessel function $d_{p,n}\eta^{-n}|x|^{p-n/2}K_{p-n/2}(|x|/\eta)$. For $p \gg 0$, the kernel converges to a Gaussian with variance depending only on η, namely $(2\sqrt{\pi}\eta)^{-n}e^{-|x|^2/4\eta^2}$. This follows because the Fourier transform takes $G_\eta^{(p)}$ to $\left(1 + \frac{\eta^2|\xi|^2}{p}\right)^{-p}$, whose limit, as $p \to \infty$, is $e^{-\eta^2|\xi|^2}$. These approximately Gaussian kernels lie in C^q if $q \le p - (n+1)/2$.

So long as the kernel is in C^1, it is known that EPDiff has solutions for all time, as noted first by A.Trouve and L.Younes.

Theorem

Let $F(x) = f(|x|)$ be any integrable C^2 radial function on $\mathbb{R}^n$. Assume $n \geq 3$. Define:

$$H_F(x) = \int_{\mathbb{R}^n} \min\left(\tfrac{1}{|x|^{n-2}}, \tfrac{1}{|y|^{n-2}}\right) F(y)dy$$
$$= \frac{1}{|x|^{n-2}} \int_{|y|\leq|x|} F(y)dy + \int_{|y|\geq|x|} \frac{F(y)}{|y|^{n-2}} dy$$

Then H_F is the convolution of F with $\frac{1}{|x|^{n-2}}$, is in C^4 and:

$$\partial_i(H_F)(x) = -(n-2)\frac{x_i}{|x|^n} \int_{|y|\leq|x|} F(y)dy$$
$$\partial_i\partial_j(H_F)(x) = (n-2)\left(\frac{nx_ix_j - \delta_{ij}|x|^2}{|x|^{n+2}} \int_{|y|\leq|x|} F(y)dy - V_n \frac{x_ix_j}{|x|^2} F(x)\right)$$

If $n = 2$, the same holds if you replace $1/|x|^{n-2}$ by $\log(1/|x|)$ and omit the factors $(n-2)$ in the derivatives.

no L	$K_{0,0} = P_{\mathrm{div}=0} = \delta_{ij}\delta_0 + (\partial_i\partial_j H)_{\mathrm{distr}}$
no L	$K_{0,\eta} = G_\eta^{(p)} * P_{\mathrm{div}=0}$ – see above
$L_{\varepsilon,0} = I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}$	$K_{\varepsilon,0} = \delta_{ij}\delta_0 + \partial_i\partial_j H_\varepsilon$
$L_{\varepsilon,\eta} = \left(I - \frac{\eta^2}{p}\triangle\right)^p \circ$ $\circ \left(I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}\right)$	$K_{\varepsilon,\eta} = \delta_{ij} G_\eta^{(p)} + \partial_i\partial_j (G_\eta^{(p)} * H_\varepsilon)$

Theorem: *Let $\varepsilon \geq 0, \eta > 0, p \geq (n+3)/2$ and $K = K_{\varepsilon,\eta}$ be the corresponding kernel. For any vector-valued distribution m_0 whose components are finite signed measures, consider the Lagrangian equation for a time varying C^1-diffeomorphism $\varphi(\cdot, t)$ with $\varphi(x,0) \equiv x$:*

$$\partial_t \varphi(x,t) = \int K(\varphi(x,t) - \varphi(y,t))(D\varphi(y,t))^{-1,\top} m_0(y) dy.$$

Here $D\varphi$ is the spatial derivative of φ. This equation has a unique solution for all time t.

Proof: The Eulerian velocity at φ is:

$$V_\varphi(x) = \int K(x - \varphi(y))(D\varphi(y))^{-1,\top} m_0(y) dy$$

and $W_\varphi(x) = V_\varphi(\varphi(x))$ is the velocity in 'material' coordinates. Note that because of our assumption on m_0, if φ is a C^1-diffeomorphism, then V_φ and W_φ are C^1 vector fields on $\mathbb{R}^n$; in fact, they are as differentiable as K is, for suitably decaying m. The equation can be viewed as a the flow equation for the vector field $\varphi \mapsto W_\varphi$ on the union of the open sets

$$U_c = \left\{\varphi \in C^1(\mathbb{R}^n)^n : \| \operatorname{Id} - \varphi\|_{C^1} < 1/c,\ \det(D\varphi) > c\right\},$$

where $c > 0$. The union of all U_c is the group $\operatorname{Diff}_{C^1_b}(\mathbb{R}^n)$ of all C^1-diffeomorphisms which, together with their inverses, differ from the identity by a function in $C^1(\mathbb{R}^n)^n$ with bounded C^1-norm. We claim this vector field is locally Lipschitz on each U_c:

$$\|W_{\varphi_1} - W_{\varphi_2}\|_{C^1} \leq C.\|\varphi_1 - \varphi_2\|_{C^1}$$

where C depends only on c: Use that K is uniformly continuous and use $\|D\varphi^{-1}\| \leq \|D\varphi\|^{n-1}/|\det(D\varphi)|$.

As a result we can integrate the vector field for short times in $\mathrm{Diff}_{C^1_b}(\mathbb{R}^n)$. But since $(D\varphi(y,t))^{-1,\top} m_0(y)$ is then again a signed finite $\mathbb{R}^n$-valued measure,

$$\int V_{\varphi(\cdot,t)}(x)(D\varphi(y,t))^{-1,\top} m_0(y)dx = \|V_{\varphi(\cdot,t)}\|_{L_{\varepsilon,\eta}}$$

is actually finite for each t. Using the fact that in EPDiff the $L_{\varepsilon,\eta}$-energy $\|V_{\varphi(\cdot,t)}\|_{L_{\varepsilon,\eta}}$ of the $L_{\varepsilon,\eta}$-geodesic is constant in t, we get a bound on the norm $\|V_{\varphi(\cdot,t)}\|_{H^p}$, depending of course on η but independent of t, hence a bound on $\|V_{\varphi(\cdot,t)}\|_{C^1}$. Thus $\|\varphi(\cdot,t)\|_{C^0}$ grows at most linearly in t. But $\partial_t D\varphi = DW_\varphi = DV_\varphi \cdot D\varphi$ which shows us that $D\varphi$ grows at most exponentially in t. Hence $\det D\varphi$ can shrink at worst exponentially towards zero, because $\partial_t \det(D\varphi) = \mathrm{Tr}(\mathrm{Adj}(D\varphi).\partial_t D\varphi)$. Thus for all finite t, the solution $\varphi(\cdot,t)$ stays in a bounded subset of our Banach space and the ODE can continue to be solved. QED.

Lemma: *If $\eta \geq 0$ and $\varepsilon > 0$ are bounded above, then the norm*

$$\|v\|^2_{k,\varepsilon,\eta} = \sum_{|\alpha| \leq k} \int \langle D^\alpha L_{\varepsilon,\eta} v, D^\alpha v \rangle dx$$

is bounded above and below by the metric, with constants independent of ε and η:

$$\|v\|^2_{H^k} + \frac{1}{\varepsilon^2} \|\mathrm{div}(v)\|^2_{H^k} + \sum_{k+1 \leq |\alpha| \leq k+p} \eta^{2(|\alpha|-k)} \int |D^\alpha v|^2 + \frac{1}{\varepsilon^2} |D^\alpha \mathrm{div}(v)|^2$$

Main estimate: *Assume k is sufficiently large, for instance $k \geq (n + 2p + 4)$ works, then the velocity field of a solution satisfies:*

$$\boxed{|\partial_t \left(\|v\|^2_{k,\varepsilon,\eta} \right)| \leq C.\|v\|^3_{k,\varepsilon,\eta}}$$

where, so long ε and η are bounded above, the constant C is independent of ε and η.

Theorem: *Fix k, p, n with $p > n/2 + 1, k \geq n + 2p + 4$ and assume $(\varepsilon, \eta) \in [0, M]^2$ for some $M > 0$. Then there are constants t_0, C such that for all initial $v_0 \in H^{k+p+1}$, there is a unique solution $v_{\varepsilon,\eta}(x, t)$ of EPDiff (including the limiting Euler case) for $t \in [0, t_0]$. The solution $v_{\varepsilon,\eta}(\cdot, t) \in H^{k+p+1}$ depends continuously on $\varepsilon, \eta \in [0, M]^2$ and satisfies $\|v_{\varepsilon,\eta}(\cdot, t)\|_{k,\varepsilon,\eta} < C$ for all $t \in [0, t_0]$.*

Theorem: *Take any k and M and any smooth initial velocity $v(\cdot, 0)$. Then there are constants t_0, C such that Euler's equation and $(\varepsilon, 0)$-EPDiff have solutions v_0 and v_ε respectively for $t \in [0, t_0]$ and all $\varepsilon < M$ and these satisfy:*

$$\|v_0(\cdot, t) - v_\varepsilon(\cdot, t)\|_{H^k} \leq C\varepsilon.$$

Theorem: *Let $\varepsilon > 0$. Take any k and M and any smooth initial velocity $v(\cdot, 0)$. Then there are constants t_0, C such that $(\varepsilon, 0)$-EPDiff and (ε, η)-EPDiff have solutions v_0 and v_η respectively for $t \in [0, t_0]$ and all $\varepsilon, \eta < M$ and these satisfy:*

$$\|v_0(\cdot, t) - v_\eta(\cdot, t)\|_{H^k} \leq C\eta^2.$$

Vortons: Soliton-like solutions via landmark theory

We have a C^1 kernel, so we can consider solutions in which momentum m is supported in a finite set $\{P_1, \cdots, P_N\}$, so that the components of the momentum field are given by $m^i(x) = \sum_a m_{ai}\delta(x - P_a)$. The support is called the set of landmark points and in this case, EPDiff reduces to a set of Hamiltonian ODE's based on the kernel $K = K_{\varepsilon,\eta}, \varepsilon \geq 0, \eta > 0$:

$$\text{Energy } E = \sum_{a,b} m_{ai} K_{ij}(P_a - P_b) m_{jb}$$

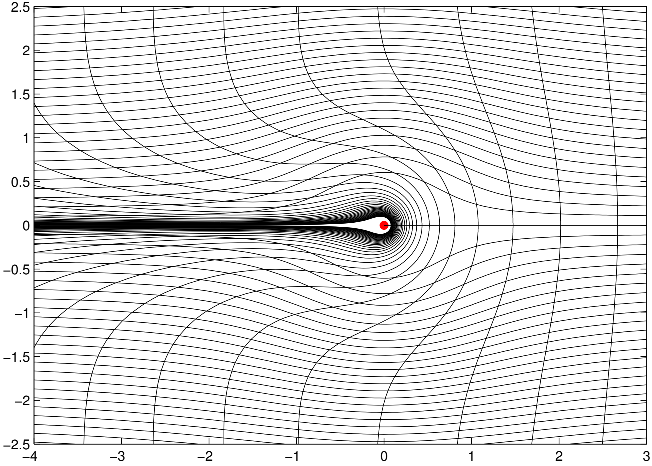
$$\frac{dP_{ai}}{dt} = \sum_{b,j} K_{ij}(P_a - P_b) m_{bj}$$

$$\frac{dm_{ai}}{dt} = -\sum_{b,j,k} \partial_{x_i} K_{jk}(P_a - P_b) m_{aj} m_{bk}$$

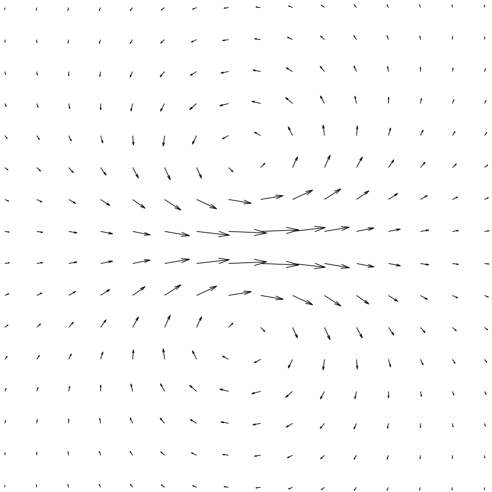
where a, b enumerate the points and i, j, k the dimensions in $\mathbb{R}^n$. These are essentially Roberts' equations from 1972.

One landmark point

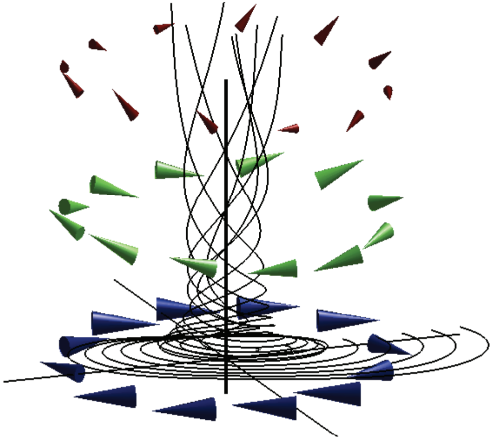
Its momentum must be constant hence so is its velocity. Therefore the momentum moves uniformly in a straight line ℓ from $-\infty$ to $+\infty$.



Momentum is tranformed to vortex-like velocity field by kernel $K_{0,\varepsilon}$

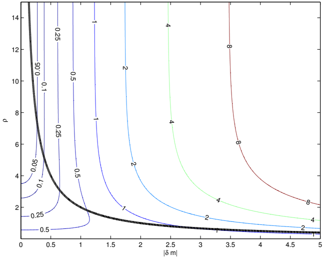


The dipole given by the kernel $K_{0,\eta}$ in dimension 2.



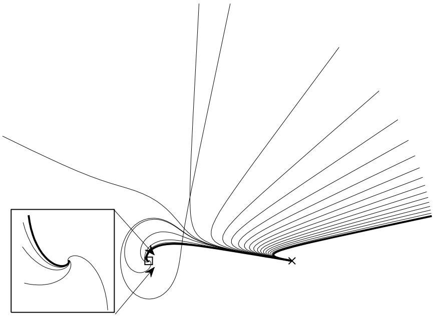
Streamlines and MatLab's 'coneplot' to visualize the vector field given by the x_1-derivative of the kernel $K_{0,1}$ times the vector $(1, 2, 0)$.

Two landmark points



Level sets of energy for the collision of two vortons with $\overline{m} = 0$, $\eta = 1$, $\omega = 1$. The coordinates are $\rho = |\delta P|$ and $|\delta m|$, and the state space is the double cover of the area above and right of the heavy black line, the two sheets being distinguished by the sign of $\langle \delta m, \delta P \rangle$. The heavy black line which is the curve $\rho \cdot |\delta m| = \omega$ where $\langle \delta m, \delta P \rangle = 0$. Each level set is a geodesic. If they hit the black line, they flip to the other sheet and retrace their path. Otherwise ρ goes to zero at one end of the geodesic.

Geodesics in the δP plane all starting at the point marked by an X but with $\overline{m} = m_1 + m_2 =$const. along the y-axis varying from 0 to 10. Here $\eta = 1$, the initial point is $(5, 0)$ and the initial momentum is $(-3, .5)$. Note how the two vortons repel each other on some geodesics and attract on others. A blow up shows the spiraling behavior as they collapse towards each other.



Thank you for listening.

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