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The Division Theorem on Banach Spaces

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The following is a detailed exposition of the proof of the division theorem for smooth functions, following Nirenberg [7] up to his extension lemma and Mather's proof [5] of the latter. This proof is developped in the context of Banach spaces—the necessary modifications are minor.

1. The Division Theorem: Let E, F be real Banach spaces and let $d: R \times E \to R$ be a smooth function, defined near 0, such that $d(t, 0) = \bar{d}(t)\, t^k$ for some $k \geq 0$, where $\bar{d}(0) \neq 0$, $\bar{d}: R \to R$ is smooth, defined near 0.

Then given any smooth function near 0 $f: R \times E \to F$ there are smooth functions near 0 $q: R \times E \to F$, $r_i: E \to F$, $i = 0, 1, \ldots, k-1$, such that

$$f(t, x) = q(t, x)\, d(t, x) + \sum_{i=0}^{k-1} r_i(x)\, t^i.$$

2. Notation: Let $P_k: R \times R^k \to R$ be the Polynomial

$$P_k(t, \lambda) = t^k + \sum_{i=0}^{k-1} \lambda_i t^i, \quad \lambda = (\lambda_0, \ldots, \lambda_{k-1}).$$

3. The Polynomial Division Theorem: Let $f(t,x)$ be a smooth function $R \times E \to C \otimes F$.

Then there are smooth functions, defined near 0 in R^k, $q: R \times E \times R^k \to C \otimes F$, $r_i: E \times R^k \to C \otimes F$, $i = 0, \ldots, k-1$, such that $f(t,x) = q(t, x, \lambda)\, P_k(t, \lambda) + \sum_{i=0}^{k-1} r_i(x, \lambda)\, t^i$. If f is realvalued in $C \otimes F$ (i.e. takes its values in the real subspace $1 \otimes F$), then q and r_i may be chosen realvalued too.

4. Remarks: a) $C \otimes F = F \oplus iF$ is just the canonical complexification of the Banach space E, with some suitable norm.

b) The last assertion is trivial: just apply the projection $C \otimes F \to 1 \otimes F$ to q and r_i.

c) If f is in E defined near 0 only, then the polynomial division theorem remains valid for q and r_i defined near 0. Nothing in the proof to follow has to be changed. But the global version does not imply the local one in general, since there need not exist smooth partitions of unity on E (on $C([0,1])$ e.g. there is no smooth function with bounded support, cf. Bonic and Frampton [1]).

d) Without loss of generality we may assume that $f(., x)$ has compact support in R for each $x \in E$ (or near 0). For suppose the theorem is valid in this case and f is arbitrary, let $g_j(t)$, $h_j(t)$, $j \in N$ be two locally finite families of smooth functions with compact support such that $(g_j(t)\, h_j(t))_j$ is a partition of unity. Then for each j we may write

$$h_j(t) f(t,x) = q_j(t,x,\lambda)\; P_k(t,\lambda) + \sum_{i=0}^{k-1} r_{ij}(x,\lambda)\, t^i,$$

but then clearly

$$f(t,x) = q(t,x,\lambda)\, P_k(t,\lambda) + \sum_{i=1}^{k-1} r_i(x,\lambda)\, t^i$$

for

$$q(t,x,\lambda) = \sum_j g_j(t)\, q_j(t,x,\lambda),\quad r_i(x,\lambda) = \sum_j g_j(t)\, r_{ij}(x,\lambda).$$

5. Proof of the division theorem using the local form of the polynomial division theorem:

Given d as in 1. there are smooth functions defined near 0 $q\colon R \times E \times R^k \to R$, $r_i\colon E \times R^k \to R$, $i = 0, \ldots, k-1$, so that

$$d(t,x) = q(t,x,\lambda)\,P_k(t,\lambda) + \sum_{i=0}^{k-1} r_i(x,\lambda)\,t^i. \tag{6}$$

We claim that

$$q(0,0,0) \neq 0, r_i(0,0) = 0, \quad \frac{\partial r_j}{\partial \lambda_j}(0,0) \neq 0 \quad \text{for all } j,$$

$$\frac{\partial r_j}{\partial \lambda_i}(0,0) = 0 \quad \text{for } j < i. \tag{7}$$

By (6) we have

$$t^k \bar{d}(t) = d(t,0) = q(t,0,0)\,t^k + \sum_{i=0}^{k-1} r_i(0,0)\,t^i.$$

Looking at the Taylor expansions at 0 of both sides of this equation we see that $r_i(0,0) = 0$ for all i and $q(0,0,0) = \bar{d}(0) \neq 0$. Now differentiate (6) at $x = 0$, $\lambda = 0$ with respect to λ_i, to obtain

$$0 = q(t,0,0)\,t^i + \frac{\partial q}{\partial \lambda_i}(t,0,0)\,t^k + \sum_{j=0}^{k-1} \frac{\partial r_j}{\partial \lambda_i}(0,0)\,t^j.$$

Again Taylor expansion at 0 tells us that for $j < i$ we have $\frac{\partial r_j}{\partial \lambda_i}(0,0) = 0$ and $\frac{\partial r_i}{\partial \lambda_i}(0,0) = -q(0,0,0) \neq 0$. So (7) follows. Let now $R = (r_0, \ldots, r_{k-1})\colon E \times R^k \to R^k$, then $D_2 R(0,0) = \left(\frac{\partial r_i}{\partial \lambda_j}(0,0)\right)\colon R^k \to R^k$ and this matrix is invertible by (7).

Now consider the mapping $(x,\lambda) \mapsto (x, R(x,\lambda))$ from $E \times R^k$ into itself, defined near 0. Its derivative at 0 has the form

$$\begin{pmatrix} Id_E & 0 \\ D_1 R(0,0) & D_2 R(0,0) \end{pmatrix}$$

and so is invertible too. By the inverse function theorem on Banach spaces (cf. S. Lang [4], I, §5, this mapping is locally invertible at $(0,0)$, its inverse (again fibered over E) being of the form $(x,\lambda)\mapsto(x,\, s(x,\lambda))$. Then of course $R(x,\, s(x,\lambda))=\lambda$.

Let now $\bar{P},\bar{q}\colon R\times E\to R$ be given by $\bar{P}(t,x)=P_k(t,\; s(x,0))$, $\bar{q}(t,x)=q(t,x,s(x,0))$. Using (6) again for $\lambda=s(x,0)$ we have

$$d(t,x)=q(t,x,s(x,0))\,P_k(t,s(x,0))+\sum_{i=1}^{k-1} r_i(x,s(x,0))\,t^i$$
$$=\bar{q}(t,x)\,\bar{P}(t,x).$$

$1/\bar{q}(t,x)$ exists and is smooth near 0 since $\bar{q}(0,0)=Q(0,0,0)\neq 0$, so $\bar{P}(t,x)=d(t,x)/\bar{q}(t,x)$ near 0.

Now if some f is given as in 1. then by 3. again there are functions $m\colon R\times E\times R^k\to F$, $n_i\colon E\times R^k\to F$, $i=0,\ \ldots,\ k-1$, defined near 0, such that (for $\lambda=s(x,0)$)

$$f(t,x)=m(t,x,s(x,0))\,P_k(t,s(x,0))+\sum_{i=0}^{k-1} n_i(x,s(x,0))\,t^i$$

$$=\frac{m(t,x,s(x,0))}{\bar{q}(t,x)}d(t,x)+\sum_{i=0}^{k-1} n_i(x,s(x,0))\,t^i$$

$$=\tilde{q}(t,x)\,d(t,x)+\sum_{i=0}^{k-1}\tilde{r}_i(x)\,t^i.$$

qed.

8. For the proof of 3. we will need two lemmas. Before proving the first one, some *notation*:

Let $f\colon C\to C$ be smooth as a real function. If $z=x+iy$,

then $df=\dfrac{\partial f}{\partial x}dx+\dfrac{\partial f}{\partial y}dy=\dfrac{\partial f}{\partial z}dz+\dfrac{\partial f}{\partial \bar{z}}d\bar{z}$,

where $\dfrac{\partial f}{\partial z}=\dfrac{1}{2}\left(\dfrac{\partial f}{\partial x}-i\dfrac{\partial f}{\partial y}\right)$, $\dfrac{\partial f}{\partial \bar{z}}=\dfrac{1}{2}\left(\dfrac{\partial f}{\partial x}+i\dfrac{\partial f}{\partial y}\right)$.

$$d(f\,dz)=df\wedge dz=0+\frac{\partial f}{\partial \bar{z}}d\bar{z}\wedge dz.$$

9. Lemma: Let $f: C \to C \otimes F$ be smooth. Let γ be a simple closed curve in C whose interiour is U. Then for $w \in U$ we have

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz + \frac{1}{2\pi i} \iint_{U} \frac{\partial f(z)}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - w}.$$

(If f is holomorphic $C \to C \otimes F$, i.e. $\frac{\partial f}{\partial \bar{z}} = 0$, this reduces to the Banach space valued Cauchy formula. The integrals in this lemma are meant to be Bochner integrals: Riemannian sums will converge in the Banach space $C \otimes F$. See Dunford-Schwartz I [3] for a discussion of vector valued integration.)

Proof: First we reduce the lemma to the one dimensional case: The first integral exists in $C \otimes F$ since γ is compact and $f(z)/(z - w)$ is continuous on γ. The second one exists, since $\partial f / \partial \bar{z}$ is continuous on $\bar{U}$ and $\frac{dz \wedge d\bar{z}}{z - w}$ defines a finite Radon measure on $\bar{U}$.

Now we use duality. Take any continuous C—linear functional φ on $C \otimes F$. That commutes with integration (with the limits of Riemannian sums by continuity and with those sums by linearity) and with $\frac{\partial}{\partial \bar{z}}$ by the chain rule, since it is its own derivative. So we may compute:

$$\varphi\left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz + \frac{1}{2\pi i} \iint_{U} \frac{\partial}{\partial \bar{z}} f(z) \frac{dz \wedge dz}{z - w}\right)$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(f(z))}{z - w} dz + \frac{1}{2\pi i} \iint_{U} \frac{\partial}{\partial \bar{z}} \varphi(f(z)) \frac{dz \wedge dz}{z - w}$$

$= \varphi(f(w))$ by the one dimensional formula. So by the theorem of Hahn Banach the formula holds in $C \otimes F$.

Now we prove the one dimensional case. Let $w \in U$, choose $\varepsilon < \min\{|w-z| : z \in \gamma\}$. Let $U_\varepsilon = U \setminus$(disc of radius about w) and $\gamma_\varepsilon = \partial U_\varepsilon$.

We apply 8. and Stokes' theorem to the function $f(z)/(z-w)$, which is smooth on a neighbourhood of U_ε.

$$\iint_{U_\varepsilon} \frac{\partial f}{\partial \bar z} \frac{dz \wedge d\bar z}{z-w} = -\iint_{U_\varepsilon} d\left(\frac{f(z)}{z-w}\, dz\right) = -\int_{\partial U_\varepsilon} \frac{f(z)}{z-w}\, dz$$

$$= -\int_\gamma \frac{f(z)}{z-w}\, dz + \int_0^{2\pi} \frac{f(w+\varepsilon \exp(i\vartheta))}{\varepsilon \exp(i\vartheta)}\, i\varepsilon \exp(i\vartheta)\, d\vartheta.$$

As $\varepsilon \to 0$, the last integral converges to $2\pi i\, f(w)$ by uniform continuity of f, and the integral on the left-hand-side converges to $\iint_U \frac{\partial f}{\partial \bar z} \frac{dz \wedge d\bar z}{z-w}$, since $\frac{\partial f}{\partial \bar z}$ is bounded, $\frac{dz \wedge d\bar z}{z-w}$ induces a finite Radon measure which, applied to the difference set $U \setminus U_\varepsilon$, converges to 0.

qed.

10. The Nirenberg Extension Lemma: Let f: $R \times E \to C \otimes F$ be a smooth function with support contained in $K \times E$ for some compact K in R. Then there exists a smooth function $\tilde f$: $C \times E \times C^k \to C \otimes F$ such that

$$\tilde f(t, x, \lambda) = f(t, x) \text{ for } t \in R \text{ and all } \lambda \in C^k. \qquad (11)$$

$$\frac{\partial \tilde f}{\partial \bar z}(z, x, \lambda) \text{ vanishes to infinite order for } \{\operatorname{Im} z = 0\} \qquad (12)$$

and on $\{(z, \lambda) : P_k(z, \lambda) = 0\}$ for all $x \in E$.

13. Proof of the polynomial division theorem 3., using the Nirenberg extension lemma 10.

Let f be as in 10. and let $\tilde f$ be its extension. It suffices to prove the theorem for such an f, cf. 4.d).

Let γ be a smooth simple closed curve near 0 in C, U the interiour of γ, $0 \in U$.

For $P_k(z,\lambda) = z^k + \sum_{i=0}^{k-1} \lambda_i z^i$, $\lambda = (\lambda_0, \ldots, \lambda_{k-1}) \in C^k$, $z \in C$

we have

$$P_k(z,\lambda) - P_k(w,\lambda) = z^k - w^k + \sum_{i=1}^{k-1} \lambda_i (z^i - w^i)$$

$$= (z-w)(z^{k-1} + z^{k-2} w + \ldots + z w^{k-2} + w^{k-1} +$$

$$+ \sum_{i=1}^{k-1} \lambda_i (z^{i-1} + z^{i-2} w + \ldots + z w^{i-2} + w^{i-1}))$$

$$= (z-w) \sum_{i=0}^{k-1} p_i(z,\lambda) w^i \text{ for polynomials } p_i \text{ in } z, \lambda.$$

So $$\frac{P_k(z,\lambda)}{z-w} = \frac{P_k(w,\lambda)}{z-w} + \sum_{i=0}^{k-1} p_i(z,\lambda) w^i.$$

Now by 9. we can compute

$$\tilde{f}(w,x,\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(z,x,\lambda)}{z-w} dz + \frac{1}{2\pi i} \iint_U \frac{\partial \tilde{f}(z,x,\lambda)}{\partial z} \frac{dz \wedge d\bar{z}}{z-w}$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(z,x,\lambda)}{z-w} \frac{P_k(z,\lambda)}{P_k(z,\lambda)} dz +$$

$$+ \frac{1}{2\pi i} \iint_U \frac{\partial \tilde{f}(z,x,\lambda)}{\partial \bar{z}} \frac{P_k(z,\lambda)}{P_k(z,\lambda)} \frac{dz \wedge d\bar{z}}{z-w}$$

$$= \left[\frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(z,x,\lambda)}{(z-w)} \frac{1}{P_k(z,\lambda)} dz + \right.$$

$$+\frac{1}{2\pi i}\iint_U \frac{\partial\tilde{f}(z,x,\lambda)}{\partial\bar{z}}\frac{1}{P_k(z,\lambda)}\frac{dz\wedge d\bar{z}}{z-w}\Bigg]P_k(w,\lambda)+$$

$$+\sum_{i=0}^{k-1}\Bigg[\frac{1}{2\pi i}\int_\gamma \tilde{f}(z,x,\lambda)\frac{p_i(z,\lambda)}{P_k(z,\lambda)}dz+$$

$$+\frac{1}{2\pi i}\iint_U \frac{\partial\tilde{f}(z,x,\lambda)}{\partial\bar{z}}\frac{p_i(z,\lambda)}{P_k(z,\lambda)}dz\wedge d\bar{z}\Bigg]\cdot w^i$$

$$=q(w,x,\lambda)P_k(w,\lambda)+\sum_{i=0}^{k-1}r_i(x,\lambda)w^i, \text{ where}$$

$$q(w,x,\lambda)=\frac{1}{2\pi i}\int_\gamma \frac{\tilde{f}(z,x,\lambda)}{(z-w)}\frac{1}{P_k(z,\lambda)}dz+$$

$$+\frac{1}{2\pi i}\iint_U \frac{\partial\tilde{f}(z,x,\lambda)}{\partial\bar{z}}\frac{1}{P_k(z,\lambda)}\frac{dz\wedge d\bar{z}}{z-w}, \text{ and}$$

$$r_i(x,\lambda)=\frac{1}{2\pi i}\int_\gamma \tilde{f}(z,x,\lambda)\frac{p_i(z,\lambda)}{P_k(z,\lambda)}dz+$$

$$+\frac{1}{2\pi i}\iint_U \frac{\partial\tilde{f}(z,x,\lambda)}{\partial\bar{z}}\frac{p_i(z,\lambda)}{P_k(z,\lambda)}dz\wedge d\bar{z}.$$

All these integrals are Bochner integrals in $C\otimes F$. We have to check, that they are defined and yield smooth functions. The (formal) computation above is valid, since we used only linearity of the integral. Now a Bochner integral is defined, if the function is continuous and the domain (or its closure) is compact. The result is smooth in the remaining variables, if all derivatives of the integrand are continuous (we may interchange differentiation and integration).

The first integrals in the definition of both q and r_i are defined and smooth as long as the zeros of $P_k(z,\lambda)$ in z do not occur on the curve γ if λ is small enough.

Let us check this: Assume that for $z \in \gamma$ we have $0 < \eta_1 < |z| < \eta_2 < 1$, $\max |\lambda_i| < \varepsilon$. Then

$$|z^k + \sum_{i=0}^{k-1} \lambda_i z^i| \geq |z^k| - \sum_{i=0}^{k-1} |\lambda_i||z^i| > \eta_1^k - k\varepsilon\eta_2.$$

For ε small enough the last number will be positive.

The second integrals in the definitions exist and are smooth, since $\frac{\partial \tilde{f}}{\partial \bar{z}}$ vanishes to infinite order on the zeros of P_k and for real z (we need w real in the theorem): this takes care of $1/(z-w)$. qed.

14. To prove the Nirenberg extension lemma we need another lemma first. We denote

$$\delta(y, \lambda) = \inf\{|y - \operatorname{Im} z| : z \in C,\ P_k(z, \lambda) = 0\} \text{ for } y \in R \text{ and } \lambda \in C^k.$$

15. Lemma (Mather): There exists a continuous function $\rho: R \times C^k \times R \to [0, 1]$ such that

$$\rho(\xi, \lambda, y) = 0 \text{ in a neighbourhood of } y = 0. \tag{16}$$

$$\rho(\xi, \lambda, y) = 0 \text{ when } |\xi y| \geq 1. \tag{17}$$

$$\frac{\partial}{\partial y}\rho(\xi, \lambda, y) = 0 \text{ in a neighbourhood of } \delta(y, \lambda) = 0. \tag{18}$$

(19) The function $\rho(\xi, \lambda, y)$ is infinitely often differentiable with respect to λ, y, and its derivatives are continuous with respect to all variables and satisfy

$$\left| \frac{\partial^{|\alpha|}}{\partial \lambda^\alpha} \frac{\partial^{|\beta|}}{\partial \bar{\lambda}^\beta} \frac{\partial^\gamma}{\partial y^\gamma} \rho(\xi, \lambda, y) \right| \leq C(\alpha, \beta, \gamma, K)(1 + |\xi|^{1+|\alpha|+|\beta|+\gamma})$$

for all multiindices α, β and all $\gamma \in N$, and all $\lambda \in K$ where K is compact in C^k and C is a constant depending as indicated.

20. Proof of the Nirenberg extension lemma 10., using lemma 15.

Given a smooth function $f: R \times E \to C \otimes F$, in R compactly supported, we consider the Fouriertransform

$$\hat{f}(\xi, x) = \int_{-\infty}^{\infty} f(t, x)\, e^{-2\pi i t \xi}\, dt.$$

This integral exists in $C \otimes F$ since $f(., x)$ is compactly supported, and $\hat{f}(\xi, x)$ is smooth (compare the last arguments in 13.). Furthermore $\|\hat{f}(\xi, x)\| \leq \int_{-\infty}^{\infty} \|f(t, x)\|\, dt$, so is uniformly bounded in ξ for each $x \in E$. If $p(\xi)$ is a polynomial, then

$$p(\xi)\hat{f}(\xi, x) = \int_{-\infty}^{\infty} f(t, x)\, p(\xi)\, e^{-2\pi i t \xi}\, dt$$

$$= \int_{-\infty}^{\infty} f(t, x)\, p\left(-\frac{1}{2\pi i}\frac{\partial}{\partial t}\right)(e^{-2\pi i t \xi})\, dt$$

$$= \int_{-\infty}^{\infty} p\left(-\frac{1}{2\pi i}\frac{\partial}{\partial t}\right)(f(t, x))\, e^{-2\pi i t \xi}\, dt,$$

the last equation holds, since $-\frac{1}{2\pi i}\frac{\partial}{\partial t}$ is formally selfadjoint (use integration by parts, after reducing to $F = R$ by duality as in the proof of 9.).

So $|p(\xi)|\,\|\hat{f}(\xi, x)\|$ is uniformly bounded too and $\|\hat{f}(\xi, x)\|$ is rapidly decreasing in ξ, and each derivative of $\hat{f}$ has the same property (use the same argument for the derivative). We define now the extension $\tilde{f}$ of f. For $(z, x, \lambda) \in C \times E \times C^k$ we put

$$\tilde{f}(z, x, \lambda) = \int_{-\infty}^{\infty} \rho(\xi, \lambda, \operatorname{Im} z)\, e^{2\pi i \xi z} \hat{f}(\xi, x)\, d\xi,$$

where ρ is the function of lemma 15.

We claim that this integral is uniformly absolutely convergent in $C \otimes F$ and that we can differentiate under the integral sign, i.e. for any multiindices α, β and $\gamma, \delta \in N$ the following integral is again uniformly absolutely convergent:

$$\int_{-\infty}^{\infty} \frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha}} \frac{\partial^{|\beta|}}{\partial \bar{\lambda}^{\beta}} \frac{\partial^{\gamma}}{\partial z^{\gamma}} \frac{\partial^{\delta}}{\partial \bar{z}^{\delta}} (\rho(\xi, \lambda, \operatorname{Im} z)\, e^{2\pi i z \xi}) \hat{f}(\xi, x)\, d\xi.$$

For, by (17) and (19),

$$\left| \frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha}} \frac{\partial^{|\beta|}}{\partial \bar{\lambda}^{\beta}} \frac{\partial^{\gamma}}{\partial z^{\gamma}} \frac{\partial^{\delta}}{\partial \bar{z}^{\delta}} (\rho(\xi, \lambda, \operatorname{Im} z)\, e^{2\pi i z \xi}) \right|$$

is uniformly bounded by a polynomial in $|\xi|$, and $\|\hat{f}(\xi, x)\|$ is rapidly decreasing. So the integral exists in $C \otimes F$ (it does so on each compact in R, and if we piece together compacts in an appropriate manner the integrals of the norm of the function over these compacts will converge). An even simpler argument applies to each derivative of $\hat{f}$ with respect to x. So $\tilde{f}$ exists and is smooth.

By (16) and the Fourier inversion formula (which holds $C \otimes F$ too: use duality to reduce it to the case $F = R$ as in the proof of 9.) $\tilde{f}$ is an extension of f:

$$f(t, x) = \int_{-\infty}^{\infty} \hat{f}(\xi, x)\, e^{2\pi i \xi t}\, d\xi = \tilde{f}(t, x, \lambda), \quad t \in R.$$

So (11) holds.

By (16) again $\frac{\partial f}{\partial \bar{z}}$ vanishes to infinite order on $\{\mathrm{Im}\, z = 0\}$ and by (18) to infinite order on $\{(z,\lambda)\colon P_k(z,\lambda) = 0\}$, so (12) holds. qed.

21. Proof of lemma 15. Let

$$\sigma(\eta,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2 dx, \text{ so } \sigma\colon R\times C^k \to R. \tag{22}$$

We claim that (δ is defined in 14.)

$$1/2\,\delta(\eta,\lambda) \le \sigma(\eta,\lambda) \le k^2/2\,\delta(\eta,\lambda) \text{ if } \delta(\eta,\lambda) \neq 0. \tag{23}$$

To show this we integrate by residues.

Fix $\eta \in R$ and $\lambda \in C^k$, let $z_1, \ldots, z_k$ be the zeros of $z \mapsto P_k(z,\lambda)$. Then $P_k(z,\lambda) = \prod_i (z - z_i)$.

$$\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2 = \left|\frac{d}{dx}\log\prod_j (x+\eta i - z_j)\right|^2$$

$$= \left|\sum_{j=1}^{k} 1/(x+\eta i - z_j)\right|^2 = \left(\sum_j \frac{1}{x+\eta i - z_j}\right)\left(\sum_j \frac{1}{x-\eta i - \bar{z}_j}\right).$$

Let $Q(z) = \frac{1}{2\pi}\left(\sum_j \frac{1}{z+\eta i - z_j}\right)\left(\sum_j \frac{1}{z-\eta i-\bar{z}_j}\right)$, $z \in C$, so that for $x \in R$

$$Q(x) = \frac{1}{2\pi}\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2.$$

Clearly $Q(z)$ is meromorphic and $z^2\,Q(z)$ is bounded outside a suitable compact set. If $Q(x)$ has no real poles, i.e. if $\delta(\eta,\lambda) > 0$, then by the method of residues it follows that

$$\sigma(\eta,\lambda) = 2\pi \int_{-\infty}^{\infty} Q(x)\,dx = i \text{ (sum of all residues of } Q(z) \text{ in the upper half plane)}$$

$$= i\left(\sum_{j\in A}\sum_{l=1}^{k}\frac{1}{z_j - \bar{z}_l - 2\eta i} + \right.$$

$$\left. + \sum_{j\in B}\sum_{l=1}^{k}\frac{1}{\bar{z}_j - z_l + 2\eta i}\right),$$

where A denotes the set of all j such that $\operatorname{Im} z_j > \eta$ and B denotes the set of all j such that $\operatorname{Im} z_j < \eta$; we suppose furthermore that $z_j - \eta i \neq \bar{z}_k + \eta i$ for all j, k (this is a condition on λ), so the last equation holds.

Let now $b_{jk} = 1$ if $j, k \in A$, $b_{jk} = -1$ if $j, k \in B$, and $b_{jk} = 0$ otherwise. Then the above is equal to

$$\sum_{1\leq j,l\leq k} b_{jl}\frac{\operatorname{Im} z_j + \operatorname{Im} z_l - 2\eta}{|z_j - \bar{z}_l - 2\eta i|^2}.$$

This is the sum of k^2 nonnegative quantities each of which is $\leq 1/2\,\delta(\eta,\lambda)$ and at least one of them is $= 1/2\,\delta(\eta,\lambda)$. Hence (23) follows in case that $z_j - \eta i \neq \bar{z}_l + \eta i$ for all j, l. By continuity, (23) holds in general.

Now we want to estimate the partial derivatives of σ. We claim that

$$\left|\frac{\partial^{|\alpha|}}{\partial\lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial\bar{\lambda}^{\beta}}\frac{\partial^{\gamma}}{\partial\eta^{\gamma}}\sigma(\eta,\lambda)\right| \leq C(\alpha,\beta,\gamma,K)\,\left(1+\delta(\eta,\lambda)^{-2k(1+|\alpha|+|\beta|+\gamma)}\right) \tag{24}$$

for all multiindices α, β, all $\gamma \in N$ and all $\lambda \in K$, a compact subset of C^k, whenever $\delta(\eta,\lambda) \neq 0$.

We have

$$\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2 = R(x,\eta,\lambda)/\left|P_k(x+\eta i,\lambda)\right|^2, \qquad (25)$$

where $R(x,\eta,\lambda)$ is a polynomial in $(x,\lambda,\eta)\in R\times C^k\times R$ of degree $2k-2$ in x.

Any first partial derivative of (25) is of the form $R_1(x,\eta,\lambda)/|P_k(x+\eta i,\lambda)|^4$, where R_1 is a polynomial of degree at most $2k-2+2k=4k-2$ in x.

Any j-th partial derivative of (25) is of the form $R_j(x,\eta,\lambda)/|P_k(x+\eta i,\lambda)|^{2(1+j)}$, where R_j is a polynomial of degree at most $2k(1+j-1)-2+2(1+j-1)k=2kj-2$ in x by induction. $|P_k(x+\eta i,\lambda)|^{2(1+j)}$ is a polynomial in $x\in R$, $\eta\in R$, $\lambda\in C^k$, with leading coefficient 1 in x, of degree $2k(1+j)$ in x, this is $2k+2$ higher than the degree of R_j in x. The same argument applies to η, if $\delta(\eta,\lambda)\neq 0$, i.e. if there are no poles on the line $x+\eta i$, $x\in R$. So the dominating factor is the distance to the next pole, in the appropriate power, and $\delta(\eta,\lambda)$ measures the "vertical" distance to the next pole. So we obtain the following: For any compact subset K of C^k there exists a constant $C(K,j)$ such that

$$\frac{R_j(x,\eta,\lambda)}{P_k(x+\eta i,\lambda)^{2(i+j)}} \le C(K,j)\frac{1}{(1+|x|^{2k+2})(1+|\eta|^{2k+2})}\cdot$$

$$\cdot\left(1+\delta(\eta,\lambda)^{-2k(1+|\alpha|+|\beta|+\gamma)}\right).$$

$1/(1+|x|^{2k+2})$ is integrable along R, so we may integrate the above estimate with respect to x to obtain (24).

We will construct the function ρ.

Let g be a smooth function $[0,\infty)\to R$ satisfying

$$\begin{aligned} &g(t)=1 && \text{if } 0\le t\le 4k^3,\\ &0\le g(t)\le 1 && \text{if } 4k^3\le t\le 8k^3,\\ &g(t)=0 && \text{if } 8k^3\le t. \end{aligned} \qquad (26)$$

Let h be a second smooth function $[0, \infty) \to R$ satisfying

$$\begin{aligned} h(t) &= 0 && \text{if } t \leq \varepsilon \text{ for some } 0 < \varepsilon < 1/2, \\ 0 \leq h(t) &\leq 1 && \text{if } \varepsilon \leq t \leq 1 - \varepsilon, \\ h(t) &= 1 && \text{if } 1 - \varepsilon \leq t. \end{aligned} \tag{27}$$

Then define $\rho(\xi, \lambda, y)$ for $(\xi, \lambda, y) \in R \times C^k \times R$ as follows:

$$\begin{aligned} \rho(\xi, \lambda, y) &= 0 && \text{if } 1/(1+|\xi|) \leq |y|, \\ &= 1 && \text{if } |y| \leq 1/2(1+|\xi|), \\ &= h\left(4(1+|\xi|)\int_y^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta\right) && \text{if } 1/2(1+|\xi|) \leq y \leq 1/(1+|\xi|), \\ &= h\left(4(1+|\xi|)\int_{-\frac{1}{1+|\xi|}}^{y} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta\right) && \text{if } -1/(1+|\xi|) \leq y \leq -1/2(1+|\xi|). \end{aligned}$$

First we claim that

$$\int_{-\frac{1}{2(1+|\xi|)}}^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta \geq 1/4(1+|\xi|) \tag{29}$$

$$\int_{-\frac{1}{1+|\xi|}}^{\frac{1}{2(1+|\xi|)}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta \geq 1/4(1+|\xi|). \tag{30}$$

For that remember the definition of δ (14) and (23).

Let m be Lebesque measure on R and $I(\xi) = [1/2(1+|\xi|), 1/(1+|\xi|)]$. Then a simple geometrical argument (there are at most k different zeros of $P_k(., \lambda)$ for fixed λ) gives

$$m(\{\eta \in I(\xi) : \delta(\eta,\lambda) \geq r\}) = m(I(\xi)) - m(\{\eta \in I(\xi) : \delta(\eta,\lambda) < r\})$$
$$\geq 1/2(1+|\xi|) - 2rk.$$

We are interested in the set of those η, for which $g(\sigma(\eta,\lambda)/(1+|\xi|)) = 1$. Sufficient for that is $\sigma(\eta,\lambda)/(1+|\xi|) \leq 4k^3$ by (26). By (23) $\sigma(\eta,\lambda) \leq k^2/2\,\delta(\eta,\lambda)$, so we obtain the sufficient condition $\delta(\eta,\lambda) \geq 1/8k(1+|\xi|)$. But now $m(\{\eta \in I(\xi) : \delta(\eta,\lambda) \geq 1/8k(1+|\xi|)\}) \geq$ $\geq 1/4(1+|\xi|)$ as we computed above. (29) follows since on this set $g = 1$.

A similar argument proves (30).

Using (29) and (30) we see that the definitions of $\rho(\xi,\lambda,y)$ coincide on overlapping intervalls, so $\rho(\xi,\lambda,y)$ is smooth in λ and y for fixed ξ.

Let us check now wether the conditions (16)-(19) of lemma 15 are satisfied:

(16) On a neighbourhood of $y = 0$, exactly for $|y| \leq 1/2(1+|\xi|)$, we have $\rho(\xi,\lambda,y) = 1$ by definition.

(17) If $|\xi y| \geq 1$, then $|y| \geq 1/|\xi| > 1/(1+|\xi|)$, so $\rho(\xi,\lambda,y) = 0$ by definition.

(18) We want that $\frac{\partial}{\partial y}\rho(\xi,\lambda,y) = 0$ in a neighbourhood of $\delta(y,\lambda) = 0$.

$$\frac{\partial}{\partial y}\rho(\xi,\lambda,y) = h'\left(4(1+|\xi|)\int_y^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta\right)\cdot$$
$$\cdot(-4(1+|\xi|)\,g(\sigma(y,\lambda)/(1+|\xi|))), \text{ if } y \in I(\xi).$$

If y is so near at $\{\eta : \delta(\eta,\lambda) = 0\}$ that

$\delta(y,\lambda) \leq 1/16k^3(1+|\xi|)$, then $\sigma(y,\lambda)/(1+|\xi|) \geq 1/2\delta(y,\lambda)(1+|\xi|) \geq 8k^3$,

using (23), so $g(\sigma(y,\lambda)/(1+|\xi|)) = 0$ by (26). If $\delta(y,\lambda) = 0$, then (23) does not hold but the conclusion holds by continuity. So $\frac{\partial}{\partial y}\rho(\xi, \lambda, y) = 0$. Exactly the same argument applies, if $y \in -I(\xi)$.

(19) We already know that $\rho(\xi, \lambda, y)$ is smooth with respect to λ, y. So we have only to estimate

$$\left|\frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{\lambda}^{\beta}}\frac{\partial^{\gamma}}{\partial y^{\gamma}}\rho(\xi, \lambda, y)\right|.$$

For fixed ξ we know that $\rho(\xi, \lambda, y)$ is constant outside $I(\xi) \cup (-I(\xi))$, in particular for $|y| \geq 1/(1 + |\xi|)$.

Let now $K \subseteq C^k$ be compact and consider $\lambda \in C^k$. We want to estimate for $\lambda \in K$ the expression

$$\frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{\lambda}^{\beta}}\frac{\partial^{\gamma}}{\partial y^{\gamma}} h\left(4(1 + |\xi|)\int_y^{\frac{1}{1+|\xi|}} g(\sigma(\eta, \lambda)/(1 + |\xi|))\, d\eta\right) \tag{31}$$

and the similar expression for $y \in (-I(\xi))$.

The partial derivative (31) is a polynominal in the partial derivatives of h and g (which are uniformly bounded since both are constant outside a compact set) and in $1 + |\xi|$, $1/(1 + |\xi|)$ and the parial derivatives of σ.

The latter are bounded by an expression

$$C(\alpha, \beta, \gamma, K)(1 + \delta(y, \lambda)^{-2k(1 + |\alpha| + |\beta| + \gamma)}),$$

using (24). Recall that $\delta(y, \lambda)$ is the "vertical" distance from y to the next zero of $P_k(., \lambda)$. If λ remains in K then the set of all these zeros is bounded, so this expression above becomes big only in a compact set, where we can bound it uniformly. So we can disregard all partial derivaties and of course $1/(1 + |\xi|)$ in (31). So (31) is bounded by a polynomial in $1 + |\xi|$, of order $|\alpha| + |\beta| + \gamma$, i.e. just the order partial derivative (31). So finally we obtain a bound of the form $C(\alpha, \beta, \gamma, K)\,(1 + |\xi|^{1 + |\alpha| + |\beta| + \gamma})$. qed.

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Added in proof: The Division Theorem has been used in R.J. Magnus, Universal unfoldings in Banach spaces: reduction and stability. Math. Proc. Cambridge Phil. Soc. **86** (1979), 41—55.

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