Convenient Calculus and Differential Geometry in Infinite Dimensions

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Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.

Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.

For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous.

The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
    - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
    - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.
8. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called smooth or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C_c^∞.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection
$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R}, U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$
where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

   $$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces.
   Note that this is the main assumption of variational calculus. Here it is a theorem.
7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if
   $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$.
   (*Smooth uniform boundedness theorem*, see [KM97], theorem 5.26).

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

$$
\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G)\\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to\\
&\qquad\qquad\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}
$$

This ends our review of the standard results of convenient calculus.

Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]

Manifolds of mappings

Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar g$ on N. Then

$$\begin{tikzcd}
& 0_N \arrow[dl, bend right, "\text{zero section}"'] \arrow[d, hook] & & N \arrow[d, hook] \arrow[dr, bend left, "\text{diagonal}"] & \\
TN & V^N \arrow[l, hook', "\text{open}"] \arrow[rr, "(\pi_N, \exp^{\bar g})", "\cong"'] & & V^{N\times N} \arrow[r, hook, "\text{open}"'] & N \times N
\end{tikzcd}$$

$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde U_f \subset \Gamma(f^* TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar g})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar g}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar g}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar g}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; \mathrm{pr}_2{}^* f^*TN)$
By Cartesian Closedness (I am lying a little).

Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.

Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By Cartesian closedness.

Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves

Corollary (of the chart structure):
$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N).$$

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.

A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\mathrm{Evol}^X_t := g(t) = \mathrm{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*

Proof: $\mathrm{Diff}(M) \xrightarrow{open} C^\infty(M,M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t,\quad)$ is a smooth curve in $\mathrm{Diff}(M)$, then $f(t,\quad)^{-1}$ satisfies the implicit equation $f(t, f(t,\quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t,x) \mapsto f(t,\quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.
Let $X(t,x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\mathrm{Fl}_s^{\partial_t \times X}(t,x) = (t+s, \mathrm{Evol}^X(t,x))$ satisfies the ODE $\quad \partial_t \mathrm{Evol}(t,x) = X(t, \mathrm{Evol}(t,x))$. If $X(s,t,x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism.
QED.

The principal bundle of embeddings

For finite dimensional manifolds M, N with M compact, $\mathrm{Emb}(M,N)$, the space of embeddings of M into N, is open in $C^\infty(M,N)$, so it is a smooth manifold. $\mathrm{Diff}(M)$ acts freely and smoothly from the right on $\mathrm{Emb}(M,N)$.

Theorem: $\mathrm{Emb}(M,N) \to \mathrm{Emb}(M,N)/\mathrm{Diff}(M)$ *is a principal fiber bundle with structure group* $\mathrm{Diff}(M)$.

Proof: Auxiliary Riem. metric $\bar{g}$ on N. Given $f \in \mathrm{Emb}(M,N)$, view $f(M)$ as submanifold of N. $TN|_{f(M)} = \mathrm{Nor}(f(M)) \oplus Tf(M)$.
$\mathrm{Nor}(f(M)) : \xrightarrow[\cong]{\exp^{\bar{g}}} W_{f(M)} \xrightarrow{p_{f(M)}} f(M)$ tubular nbhd of $f(M)$.
If $g : M \to N$ is C^1-near to f, then
$\varphi(g) := f^{-1} \circ p_{f(M)} \circ g \in \mathrm{Diff}(M)$ and
$g \circ \varphi(g)^{-1} \in \Gamma(f^* W_{f(M)}) \subset \Gamma(f^* \mathrm{Nor}(f(M)))$.
This is the required local splitting. QED

The orbifold bundle of immersions

$\mathrm{Imm}(M, N)$, the space of immersions $M \to N$, is open in $C^\infty(M, N)$, and is thus a smooth manifold. The regular Lie group $\mathrm{Diff}(M)$ acts smoothly from the right, but no longer freely.

Theorem: [Cervera,Mascaro,M,1991] *For an immersion $f : M \to N$, the isotropy group $\mathrm{Diff}(M)_f = \{\varphi \in \mathrm{Diff}(M) : f \circ \phi = f\}$ is always a finite group, acting freely on M; so $M \xrightarrow{p} M/\mathrm{Diff}(M)_f$ is a convering of manifold and f factors to $f = \bar{f} \circ p$.*

Thus $\mathrm{Imm}(M, N) \to \mathrm{Imm}(M, N)/Diff(M)$ *is a projection onto an honest infinite dimensional orbifold.*

A Zoo of diffeomorphism groups on $\mathbb{R}^n$

We will prove that the following groups of diffeomorphisms on $\mathbb{R}^n$ are regular Lie groups: [M,Mumford,2013], partly [B.Walter,2012]

- $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, the group of all diffeomorphisms which differ from the identity by a function which is bounded together with all derivatives separately.
- $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, the group of all diffeomorphisms which differ from the identity by a function in the intersection H^∞ of all Sobolev spaces H^k for $k \in \mathbb{N}_{\geq 0}$.
- $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, the group of all diffeomorphisms which fall rapidly to the identity.

Since we are giving a kind of uniform proof, we also mention the group $\mathrm{Diff}_c(\mathbb{R}^n)$ of all diffeomorphisms which differ from the identity only on a compact subset.
In particular, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is essential if one wants to prove that the geodesic equation of a right Riemannian invariant metric is well-posed with the use of Sobolov space techniques.

Theorem FK

We need more on convenient calculus.

[FK88], theorem 4.1.19.

Theorem

Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{V} \subset E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed sets. Then the following are equivalent:

1. *c is smooth*
2. *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\ell \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\ell \circ c)^{(k)} = \ell \circ c^k$, for each $\ell \in \mathcal{V}$.*

If E is reflexive, then for any point separating subset $\mathcal{V} \subset E'$ the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed subsets, by [FK88], 4.1.23.

Faá di Bruno formula

Let $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ and let $f \in C^\infty(\mathbb{R}^k, \mathbb{R}^l)$. Then the p-th deivative of $f \circ g$ looks as follows where sym_p denotes symmetrization of a p-linear mapping:

$$\frac{d^p(f \circ g)(x)}{p!} =$$
$$= \operatorname{sym}_p\Big(\sum_{j=1}^{p} \sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=p}} \frac{d^j f(g(x))}{j!}\Big(\frac{d^{\alpha_1}g(x)}{\alpha_1!}, \dots, \frac{d^{\alpha_j}g(x)}{\alpha_j!}\Big)\Big)$$

The one dimensional version is due to [Faá di Bruno 1855], the only beatified mathematician.

Groups of smooth diffeomorphisms in the zoo

If we consider the group of all orientation preserving diffeomorphisms $\mathrm{Diff}(\mathbb{R}^n)$ of $\mathbb{R}^n$, it is not an open subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with the compact C^∞-topology. So it is not a smooth manifold in the usual sense, but we may consider it as a Lie group in the cartesian closed category of Frölicher spaces, see [KM97], section 23, with the structure induced by the injection $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^)$.

We shall now describe regular Lie groups in $\mathrm{Diff}(\mathbb{R}^n)$ which are given by diffeomorphisms of the form $f = \mathrm{Id}_\mathbb{R} + g$ where g is in some specific convenient vector spaces of bounded functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Now we discuss these spaces on $\mathbb{R}^n$, we describe the smooth curves in them, and we describe the corresponding groups.

The group $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ in the zoo

The space $\mathcal{B}(\mathbb{R}^n)$ (called $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ by [L.Schwartz 1966]) consists of all smooth functions which have all derivatives (separately) bounded. It is a Fréchet space. By [Vogt 1983], the space $\mathcal{B}(\mathbb{R}^n)$ is linearly isomorphic to $\ell^\infty \hat{\otimes}\, \mathfrak{s}$ for any completed tensor-product between the projective one and the injective one, where $\mathfrak{s}$ is the nuclear Fréchet space of rapidly decreasing real sequences. Thus $\mathcal{B}(\mathbb{R}^n)$ is not reflexive, not nuclear, not smoothly paracompact.
The space $C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}^n))$ of smooth curves in $\mathcal{B}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ and each $t \in \mathbb{R}$ the expression $\partial_t^k \partial_x^\alpha c(t, x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally in t.*

To see this use thm FK for the set $\{\mathrm{ev}_x : x \in \mathbb{R}\}$ of point evaluations in $\mathcal{B}(\mathbb{R}^n)$. Here $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ and $c^k(t) = \partial_t^k f(t, \quad)$.
$\mathrm{Diff}^+_{\mathcal{B}}(\mathbb{R}^n) = \{f = \mathrm{Id} + g : g \in \mathcal{B}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) \geq \varepsilon > 0\}$ *denotes the corresponding group*, see below.

The group $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ in the zoo

The space $H^\infty(\mathbb{R}^n) = \bigcap_{k\ge 1} H^k(\mathbb{R}^n)$ is the intersection of all Sobolev spaces which is a reflexive Fréchet space. It is called $\mathcal{D}_{L^2}(\mathbb{R}^n)$ in [L.Schwartz 1966]. By [Vogt 1983], the space $H^\infty(\mathbb{R}^n)$ is linearly isomorphic to $\ell^2 \hat{\otimes}\, \mathfrak{s}$. Thus it is not nuclear, not Schwartz, not Montel, but still smoothly paracompact.
The space $C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n))$ of smooth curves in $H^\infty(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\ge 0}$, $\alpha \in \mathbb{N}^n_{\ge 0}$ the expression $\|\partial_t^k \partial_x^\alpha f(t, \quad)\|_{L^2(\mathbb{R}^n)}$ is locally bounded near each $t \in \mathbb{R}$.*

The proof is literally the same as for $\mathcal{B}(\mathbb{R}^n)$, noting that the point evaluations are continuous on each Sobolev space H^k with $k > \frac{n}{2}$.
$\mathrm{Diff}^+_{H^\infty}(\mathbb{R}) = \big\{ f = \mathrm{Id} + g : g \in H^\infty(\mathbb{R}), \det(\mathbb{I}_n + dg) > 0 \big\}$ *denotes the correponding group.*

The group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ in the zoo

The algebra $\mathcal{S}(R^n)$ of rapidly decreasing functions is a reflexive nuclear Fréchet space.
The space $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ of smooth curves in $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\geq 0}$ and $\alpha \in \mathbb{N}^n_{\geq 0}$, the expression $(1 + |x|^2)^m \partial_t^k \partial_x^\alpha c(t, x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally uniformly bounded in $t \in \mathbb{R}$.*

$\mathrm{Diff}^+_{\mathcal{S}}(\mathbb{R}^n) = \left\{ f = \mathrm{Id} + g : g \in \mathcal{S}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0 \right\}$ *is the correponding group.*

The group $\mathrm{Diff}_c(\mathbb{R}^n)$ in the zoo

The algebra $C_c^\infty(\mathbb{R}^n)$ of all smooth functions with compact support is a nuclear (LF)-space.
The space $C^\infty(\mathbb{R}, C_c^\infty(\mathbb{R}^n))$ of smooth curves in $C_c^\infty(\mathbb{R}^n)$ consists of all functions $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For each compact interval $[a, b]$ in $\mathbb{R}$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $f(t, x) = 0$ for $(t, x) \in [a, b] \times (\mathbb{R}^n \setminus K)$.*

$\mathrm{Diff}_c(\mathbb{R}^n) = \big\{ f = \mathrm{Id} + g : g \in C_c^\infty(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0 \big\}$ *is the correponding group.*

Ideal properties of function spaces in the zoo

The function spaces are boundedly mapped into each other as follows:

$$C_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n) \longrightarrow \mathcal{B}(\mathbb{R}^n)$$

and each space is a bounded locally convex algebra and a bounded $\mathcal{B}(\mathbb{R}^n)$-module. Thus each space is an ideal in each larger space.

Main theorem in the Zoo

Theorem

The sets of diffeomorphisms $\mathrm{Diff}_c(\mathbb{R}^n)$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *and* $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms*

$$\mathrm{Diff}_c(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{H^\infty}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)\ .$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$*.*

Corollary

$\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *acts on* Γ_c, $\Gamma_{\mathcal{S}}$ *and* Γ_{H^∞} *of any tensorbundle over* $\mathbb{R}^n$ *by pullback. The infinitesimal action of the Lie algebra* $\mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ *on these spaces by the Lie derivative thus maps each of these spaces into itself. A fortiori,* $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ *acts on* $\Gamma_{\mathcal{S}}$ *of any tensor bundle by pullback.*

Proof of the main zoo theorem

Let $\mathcal{A}$ denote any of $\mathcal{B}$, H^∞, $\mathcal{S}$, or c, and let $\mathcal{A}(\mathbb{R}^n)$ denote the corresponding function space. Let $f(x) = x + g(x)$ for $g \in \mathcal{A}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + dg) > 0$ and for $x \in \mathbb{R}^n$.

Each such f is a diffeomorphism. By the inverse function theorem f is a locally a diffeomorphism everywhere. Thus the image of f is open in $\mathbb{R}^n$. We claim that it is also closed. So let $x_i \in \mathbb{R}^n$ with $f(x_i) = x_i + g(x_i) \to y_0$ in $\mathbb{R}^n$. Then $f(x_i)$ is a bounded sequence. Since $g \in \mathcal{A}(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$, the x_i also form a bounded sequence, thus contain a convergent subsequence. Without loss let $x_i \to x_0$ in $\mathbb{R}^n$. Then $f(x_i) \to f(x_0) = y_0$. Thus f is surjective. This also shows that f is a proper mapping (i.e., compact sets have compact inverse images under f). A proper surjective submersion is the projection of a smooth fiber bundle. In our case here f has discrete fibers, so f is a covering mapping and a diffeomorphism since $\mathbb{R}^n$ is simply connected.

$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)_0$ is closed under composition.

$$((\mathrm{Id}+f)\circ(\mathrm{Id}+g))(x) = x + g(x) + f(x+g(x))$$

We have to check that $x \mapsto f(x+g(x))$ is in $\mathcal{A}(\mathbb{R}^n)$ if $f,g \in \mathcal{A}(\mathbb{R}^n)^n$. For $\mathcal{A} = \mathcal{B}$ this follows by the Faà di Bruno formula. For $\mathcal{A} = \mathcal{S}$ or $\mathcal{S}_1$ we need furthermore: $(\partial_x^\alpha f)(x+g(x)) = O\Big(\frac{1}{(1+|x+g(x)|^2)^k}\Big) = O\Big(\frac{1}{(1+|x|^2)^k}\Big)$ which holds since $\frac{1+|x|^2}{1+|x+g(x)|^2}$ is globally bounded.

For $\mathcal{A} = H^\infty$ we also need that

$$\int_{\mathbb{R}^n} |(\partial_x^\alpha f)(x+g(x))|^2\,dx = \tag{3}$$

$$\int_{\mathbb{R}^n} |(\partial^\alpha f)(y)|^2 \,\frac{dy}{|\det(\mathbb{I}_n+dg)((\mathrm{Id}+g)^{-1}(y))|} \le C(g)\int_{\mathbb{R}^n} |(\partial^\alpha f)(y)|^2 dy;$$

this holds since the denominator is globally bounded away from 0 since g and dg vanish at ∞ by the lemma of Riemann-Lebesque.

The case $\mathcal{A}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ or $C_{c,1}^\infty(\mathbb{R}^n)$ is easy and well known.

Multiplication is smooth on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$.

Suppose that the curves $t \mapsto \mathrm{Id} + f(t, \quad)$ and $t \mapsto \mathrm{Id} + g(t, \quad)$ are in $C^\infty(\mathbb{R}, \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n))$ which means that the functions $f, g \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^n)$ satisfy condition $\mathcal{A}$.

We have to check that $f(t, x + g(t, x))$ also satisfies condition $\mathcal{A}$.

For this we reread the proof that composition preserves $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ and pay attention to the further parameter t.

The inverse $(\mathrm{Id}+g)^{-1}$ is again an element in $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$

For $g \in \mathcal{A}(\mathbb{R}^n)^n$ we write $(\mathrm{Id}+g)^{-1} = \mathrm{Id}+f$.
We have to check that $f \in \mathcal{A}(\mathbb{R}^n)^n$.

$$
\begin{aligned}
(\mathrm{Id}+f) \circ (\mathrm{Id}+g) = \mathrm{Id} \implies & \; x + g(x) + f(x+g(x)) = x \\
\implies & \; x \mapsto f(x+g(x)) = -g(x) \text{ is in } \mathcal{A}(\mathbb{R}^n)^n.
\end{aligned}
$$

First the case $\mathcal{A} = \mathcal{B}$. We know already that $\mathrm{Id}+g$ is a diffeomorphism. By definition, we have $\det(\mathbb{I}_n + dg(x)) \geq \varepsilon > 0$ for some ε. This implies that

$$
\|(\mathbb{I}_n + dg(x))^{-1}\|_{L(\mathbb{R}^n,\mathbb{R}^n)} \quad \text{is globally bounded,}
$$

using that $\|A^{-1}\| \leq \frac{\|A\|^{n-1}}{|\det(A)|}$ for any linear $A : \mathbb{R}^n \to \mathbb{R}^n$. Moreover,

$$
\begin{aligned}
(\mathbb{I}_n+df(x+g(x)))(\mathbb{I}_n+dg(x)) = \mathbb{I}_n \implies & \det(\mathbb{I}_n+df(x+g(x))) = \\
= \det(\mathbb{I}_n + dg(x))^{-1} \geq \|\mathbb{I}_n + dg(x)\|^{-n} \geq \eta > 0 & \text{ for all } x.
\end{aligned}
$$

For higher derivatives we write the Faa di Bruno formula as:

$$\frac{d^p(f \circ (\mathrm{Id} + g))(x)}{p!} = \mathrm{sym}_p \Big(\sum_{j=1}^{p} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1 + \cdots + \alpha_j = p}} \frac{d^j f(x + g(x))}{j!} \Big(\frac{d^{\alpha_1}(\mathrm{Id} + g)(x)}{\alpha_1 !}, \ldots, \frac{d^{\alpha_j}(\mathrm{Id} + g)(x)}{\alpha_j !} \Big) \Big)$$

$$= \frac{d^p f(x + g(x))}{p!} \Big(\mathrm{Id} + dg(x), \ldots, \mathrm{Id} + dg(x) \Big) \qquad (\textcolor{red}{\text{top extra}})$$

$$+ \, \mathrm{sym}_{p-1} \Big(\sum_{j=1}^{p-1} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1 + \cdots + \alpha_j = p \\ (h_{\alpha_1}, \ldots, h_{\alpha_j})}} \frac{d^j f(x + g(x))}{j!} \Big(\frac{d^{\alpha_1} h_{\alpha_1}(x)}{\alpha_1 !}, \ldots, \frac{d^{\alpha_j} h_{\alpha_j}(x)}{\alpha_j !} \Big) \Big)$$

where $h_{\alpha_i}(x)$ is $g(x)$ for $\alpha_i > 1$ (there is always such an i), and where $h_{\alpha_i}(x) = x$ or $g(x)$ if $\alpha_i = 1$.

Now we argue as follows:

The left hand side is globally bounded. We already know that $\mathrm{Id} + dg(x) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible with $\|(\mathbb{I}_n + dg(x))^{-1}\|_{L(\mathbb{R}^n,\mathbb{R}^n)}$ globally bounded.

Thus we can conclude by induction on p that $d^p f(x + g(x))$ is bounded uniformly in x, thus also uniformy in $y = x + g(x) \in \mathbb{R}$.

For general $\mathcal{A}$ we note that the left hand side is in $\mathcal{A}$. Since we already know that $f \in \mathcal{B}$, and since $\mathcal{A}$ is a $\mathcal{B}$-module, the last term is in $\mathcal{A}$. Thus also the first term is in $\mathcal{A}$, and any summand there containing just one $dg(x)$ is in $\mathcal{A}$, so the unique summand $d^p f(x, g(x))$ is also in $\mathcal{A}$. Thus inversion maps $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ into itself.

Inversion is smooth on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$.

We retrace the proof that inversion preserves $\mathrm{Diff}_{\mathcal{A}}$ assuming that $g(t,x)$ satisfies condition $\mathcal{A}$.

We see again that $f(t, x + g(t,x)) = -g(t,x)$ satisfies condition $\mathcal{A}$ as a function of t, x, and we claim that f then does the same.

We reread the proof paying attention to the parameter t and see that condition $\mathcal{A}$ is satisfied.

$\mathrm{Diff}_A(\mathbb{R}^n)$ is a regular Lie group

So let $t \mapsto X(t, \quad)$ be a smooth curve in the Lie algebra $\mathfrak{X}_{\mathcal{A}}(\mathbb{R}^n) = \mathcal{A}(\mathbb{R}^n)^n$, i.e., X satisfies condition $\mathcal{A}$.

The evolution of this time dependent vector field is the function given by the ODE

$$\begin{aligned} &\mathrm{Evol}(X)(t,x) = x + f(t,x), \\ &\begin{cases} \partial_t(x + f(t,x)) = f_t(t,x) = X(t, x + f(t,x)), \\ f(0,x) = 0. \end{cases} \end{aligned} \tag{7}$$

We have to show
first that $f(t, \quad) \in \mathcal{A}(\mathbb{R}^n)^n$ for each $t \in \mathbb{R}$,
second that it is smooth in t with values in $\mathcal{A}(\mathbb{R}^n)^n$, and
third that $X \mapsto f$ is also smooth.

For $0 \leq t \leq C$ we consider

$$|f(t,x)| \leq \int_0^t |f_t(s,x)| ds = \int_0^t |X(s, x + f(s,x))| \, ds. \tag{8}$$

Since $\mathcal{A} \subseteq \mathcal{B}$, the vector field $X(t,y)$ is uniformly bounded in $y \in \mathbb{R}^n$, locally in t. So the same is true for $f(t,x)$ by (7).

Next consider

$$
\begin{aligned}
\partial_t d_x f(t,x) &= d_x(X(t, x_f(t,x))) \qquad (9)\\
&= (d_x X)(t, x+f(t,x)) + (d_x X)(t, x+f(t,x)).d_x f(t,x)\\
\|d_x f(t,x)\| &\leq \int_0^t \|(d_x X)(s, x+f(s,x))\| ds\\
&\quad + \int_0^t \|(d_x X)(s, x+f(s,x))\|.\|d_x f(s,x)\| ds\\
&=: \alpha(t,x) + \int_0^t \beta(s,x).\|d_x f(s,x)\| ds
\end{aligned}
$$

By the Bellman-Grönwall inequality,

$$
\|d_x f(t,x)\| \leq \alpha(t,x) + \int_0^t \alpha(s,x).\beta(s,x).e^{\int_s^t \beta(\sigma,x)\, d\sigma}\, ds,
$$

which is globally bounded in x, locally in t.

For higher derivatives in x (where $p > 1$) we use Faá di Bruno as

$$\partial_t d_x^p f(t,x) = d_x^p(X(t,x+f(t,x))) = \operatorname{sym}_p\Big(\sum_{j=1}^{p} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j \\ \alpha_1+\dots+\alpha_j=p}} \frac{(d_x^j X)(t,x+f(t,x))}{j!}\Big(\frac{d_x^{\alpha_1}(x+f(t,x))}{\alpha_1!},\dots,\frac{d_x^{\alpha_j}(x+f(t,x))}{\alpha_j!}\Big)\Big)$$

$$= (d_x X)(t,x+f(t,x))\big(d_x^p f(t,x)\big) + \qquad \text{(bottom extra)}$$

$$+ \operatorname{sym}_p\Big(\sum_{j=2}^{p} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j \\ \alpha_1+\dots+\alpha_j=p}} \frac{(d_x^j X)(t,x+f(t,x))}{j!}\Big(\frac{d_x^{\alpha_1}(x+f(t,x))}{\alpha_1!},\dots,\frac{d_x^{\alpha_j}(x+f(t,x))}{\alpha_j!}\Big)\Big)$$

We can assume recursively that $d_x^j f(t,x)$ is globally bounded in x, locally in t, for $j < p$. Then we have reproduced the situation of (9) (with values in the space of symmetric p-linear mappings $(\mathbb{R}^n)^p \to \mathbb{R}^n$) and we can repeat the argument above involving the Bellman-Grönwall inequality to conclude that $d_x^p f(t,x)$ is globally bounded in x, locally in t.

To conclude the same for $\partial_t^m d_x^p f(t,x)$ we just repeat the last arguments for $\partial_t^m f(t,x)$. So we have now proved that $f \in C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n))$.

To prove that $C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)) \ni X \mapsto \mathrm{Evol}(X)(1, \quad) \in \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ is smooth, we consider a smooth curve X in $C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n))$; thus $X(t_1, t_2, x)$ is smooth on $\mathbb{R}^2 \times \mathbb{R}^n$, globally bounded in x in each derivative separately, locally in $t = (t_1, t_2)$ in each derivative. Or, we assume that t is 2-dimensional in the argument above. But then it suffices to show that $(t_1, t_2) \mapsto X(t_1, t_2, \quad) \in \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ is smooth along smooth curves in $\mathbb{R}^2$, and we are again in the situation we have just treated.

Thus $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ is a regular Lie group.

If $\mathcal{A} = \mathcal{S}$, we already know that $f(s,x)$ is globally bounded in x, locally in t. Thus may insert $X(s, x + f(s,x)) = O(\frac{1}{(1+|x+f(s,x)|^2)^k}) = O(\frac{1}{(1+|x|^2)^k})$ into (8) and can conclude that $f(t,x) = O(\frac{1}{(1+|x|^2)^k})$ globally in x, locally in t, for each k.

Using this argument, we can repeat the proof for the case $\mathcal{A} = \mathcal{B}$ from above.

Thus $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is a regular Lie group.

If $\mathcal{A} = H^\infty$ we first consider the differential of (8),

$$\|d_x f(s,x)\| = \Big\| \int_0^t d_x(X(s, \quad))(x + f(s,x)).(\mathbb{I}_n + df(s, \quad)(x))\, ds \Big\|$$
$$\leq \int_0^t \big\| d_x(X(s, \quad))(x + f(s,x)) \big\|.C, ds \qquad (10)$$

since $d_x f(s,x)$ is globally bounded in x, locally in s, by the case $\mathcal{A} = \mathcal{B}$. The same holds for $f(s,x)$. Moreover, $X(s, \quad)$ vanishes near infinity by the lemma of Riemann-Lebesque, so that the same holds for $f(s, \quad)$ by (10).

Now we consider

$$\int_{\mathbb{R}^n} \|(d_x^p f)(t,x)\|^2\,dx = \int_{\mathbb{R}^n} \Big\| \int_0^t d_x^p\big(X(s, \mathrm{Id} + f(s, \quad))\big)(x)\,ds \Big\|^2\,dx.$$

We apply Faá di Bruno in the form (top extra) to the integrand, remember that we already know that each $d^{\alpha_i}(\mathrm{Id} + f(s, \quad))(x)$ is globally bounded, locally in s, thus the last term is

$$\leq \int_{\mathbb{R}^n} \Big(\int_0^t \sum_{j=1}^p \|(d_x^j X)(s, x + f(s,x))\|.C_j\,ds \Big)^2\,dx$$

$$= \int_{\mathbb{R}^n} \Big(\int_0^t \sum_{j=1}^p \|(d_x^j X)(s, y)\|.C_j\,ds \Big)^2 \frac{dy}{|\det(\mathbb{I}_n + df(s, \quad))((\mathbb{I}_n + f(s, \quad))^{-1}(y)|}$$

which is finite since $X(s, \quad) \in H^\infty$ and since the determinand in the denominator is bounded away from zero – we just checked that $d_x f(s, \quad)$ vanishes at infinity. We repeat this for $\partial_t^m d_x^p f(t,x)$. This shows that $\mathrm{Evol}(X)(t, \quad) \in \mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ for each t.

Choosing t two-dimensional (as in the case $\mathcal{A} = \mathcal{B}$) we can conclude that $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a regular Lie group.

$\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

So let $g \in \mathcal{B}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + dg(x)) \geq \varepsilon > 0$ for all x, and $s \in \mathcal{S}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + ds(x)) > 0$ for all x. We consider

$$
\begin{aligned}
(\mathrm{Id} + g)^{-1}(x) = x + f(x) \quad &\text{for } f \in \mathcal{B}(\mathbb{R}^n)^n \\
\iff\ & f(x + g(x)) = -g(x)
\end{aligned}
$$

$$
\begin{aligned}
\big((\mathrm{Id} + g)^{-1} \circ (\mathrm{Id} + s) \circ (\mathrm{Id} + g)\big)(x) &= \big((\mathrm{Id} + f) \circ (\mathrm{Id} + s) \circ (\mathrm{Id} + g)\big)(x) \\
&= x + g(x) + s(x + g(x)) + f\big(x + g(x) + s(x + g(x))\big) \\
&= x + s(x + g(x)) - f(x + g(x)) + f\big(x + g(x) + s(x + g(x))\big).
\end{aligned}
$$

Since $g(x)$ is globally bounded we get $s(x + g(x)) = O((1 + |x + g(x)|^{-k}) = O((1 + |x|)^{-k})$ for each k. For $d_x^p(s \circ (\mathrm{Id} + g))(x)$ this follows from Faá di Bruno in the form of (top extra).

Moreover we have

$$
\begin{aligned}
&f\big(x+g(x)+s(x+g(x))\big)-f(x+g(x))= \\
&\qquad\qquad =\int_0^1 df\big(x+g(x)+ts(x+g(x))\big)(s(x+g(x)))\,dt
\end{aligned}
$$

which is in $\mathcal{S}(\mathbb{R}^n)^n$ as a function of x since df is in $\mathcal{B}$ and $s(x+g(x))$ is in $\mathcal{S}$.

$\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

We redo the last proof under the assumption that $s \in H^\infty(\mathbb{R}^n)^n$.

By the argument in (3) we see that $s(x + g(x))$ is in H^∞ as a function of x.

The rest is as above.

This finishes the proof of the main theorem.

Thank you!