Manifolds of mappings and shapes

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Abstract

In his Habilitationsvortrag, Riemann described infinite dimensional manifolds parameterizing functions and shapes of solids. This is taken as an excuse to describe convenient calculus in infinite dimensions which allows for short and transparent proofs of the main facts of the theory of manifolds of smooth mappings. Smooth manifolds of immersions, diffeomorphisms, and shapes, and weak Riemannian metrics on them are treated, culminating in the surprising fact, that geodesic distance can vanish completely for them.

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1 Introduction

At the very birthplace of the notion of manifolds, in the Habilitationsschrift [60, end of section I], Riemann mentioned infinite dimensional manifolds explicitly:

> "Es giebt indess auch Mannigfaltigkeiten, in welchen die Ortsbestimmung nicht eine endliche Zahl, sondern entweder eine unendliche Reihe oder eine stetige Mannigfaltigkeit von Grössenbestimmungen erfordert. Solche Mannigfaltigkeiten bilden z.B. die möglichen Bestimmungen einer Function für ein gegebenes Gebiet, die möglichen Gestalten einer räumlichen Figur u.s.w."

The translation into English in [61] reads as follows:

> "There are manifoldnesses in which the determination of position requires not a finite number, but either an endless series or a continuous manifoldness of determinations of quantity. Such manifoldnesses are, for example, the possible determinations of a function for a given region, the possible shapes of a solid figure, &c."

If one reads this with a lot of good will one can interpret it as follows: When Riemann sketched the general notion of a manifold, he also forsaw the notion of an infinite dimensional manifold of mappings between manifolds, and of a manifold of shapes. He then went on to describe the notion of Riemannian metric and to talk about curvature. I will take this as an excuse to describe the theory of manifolds of mappings, of diffeomorphisms, and of shapes, and of some striking results about weak Riemannian geometry on these spaces. See [7] for an overview article which is much more comprehensive for the aspect of shape spaces.

An explicit construction of manifolds of smooth mappings modeled on Fréchet spaces was described by [20]. Differential calculus beyond the realm of Banach spaces has some inherent difficulties even in its definition; see section 2. Smoothness of composition and inversion was first treated on the group of all smooth diffeomorphisms of a compact manifold in [41]; however, there was a gap in the proof, which was first filled by [31]. Manifolds of C^k-mappings and/or mappings of Sobolev classes were treated by [22], [19], Smale-Abraham [1], and [59]. Since these are modeled on Banach spaces, they allow solution methods for equations and have found a lot of applications. See in particular [18].

2 A short review of convenient calculus in infinite dimensions

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. Namely, if for a locally convex vector space E and its dual E' the evaluation mapping $\mathrm{ev} : E \times E' \to \mathbb{R}$ is jointly continuous, then there are open neighborhoods of zero $U \subset E$ and $U' \subset E'$ with $\mathrm{ev}(U \times U') \subset [-1, 1]$.

But then U' is contained in the polar of an open set, and thus is bounded. So E' is normable, and a fortiori E is normable.

For locally convex spaces which are more general than Banach spaces, we sketch here the convenient approach as explained in [29] and [33].

The name *convenient calculus* mimicks the paper [63] whose results (but not the name 'convenient') was predated by [12], [13], [14]. They discussed compactly generated spaces as a cartesian closed category for algebraic topology. Historical remarks on only those developments of calculus beyond Banach spaces that led to convenient calculus are given in [33, end of chapter I, p. 73ff].

2.1 The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth curves. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not entirely depend on the locally convex topology of E, only on its associated bornology (system of bounded sets); see [33, 2.11]. The final topologies with respect to the following sets of mappings into E coincide; see [33, 2.13]:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since addition is no longer jointly continuous. Namely, even $c^\infty(\mathcal{D} \times \mathcal{D}) \neq c^\infty\mathcal{D} \times c^\infty\mathcal{D}$.

The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

2.2 Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following equivalent conditions holds (called c^∞-completeness); see [33, 2.14]:

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.

3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all continuous linear functionals on E.

   - Equivalently, for all $\lambda \in E'$, the dual of all bounded linear functionals.
   - Equivalently, for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.

   We call this *scalarwise* C^∞.

4. Any Mackey-Cauchy-sequence (i.e., $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

5. If B is bounded closed absolutely convex, then E_B is a Banach space.

6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.

7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

2.3 Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. See [33, 3.11].

If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings; see below. Beyond Fréchet spaces, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C_c^∞.

2.4 Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial; see [11] or [33, 3.4].

2. Multilinear mappings are smooth iff they are bounded; see [33, 5.5].

3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets; see [33, 3.18].

4. The chain rule holds; see [33, 3.18].

5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the injection

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R}, U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

and where $C^\infty(\mathbb{R},\mathbb{R})$ carries the topology of compact convergence in each derivative separately; see [33, 3.11 and 3.7].

6. The exponential law holds; see [33, 3.12].: For c^∞-open $V \subset F$,

$$C^\infty(U, C^\infty(V,G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces. *Note that this (for $U = \mathbb{R}$) is the main assumption of variational calculus. Here it is a theorem.*

7. A linear mapping $f : E \to C^\infty(V,G)$ is smooth (by (2) equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V,G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. (*Smooth uniform boundedness theorem*; see [33, theorem 5.26].)

7. A mapping $f : U \to L(F,G)$ is smooth if and only if $U \xrightarrow{f} L(F,G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in F$, because then it is scalarwise smooth by the classical uniform boundedness theorem.

8. The following canonical mappings are smooth. This follows from the exponential law by simple categorical reasoning; see [33, 3.13].

$$\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x) \\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y) \\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G) \\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G)) \\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G) \\
&C^\infty(\quad,\quad) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to \\
&\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1)) \\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g) \\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

This ends our review of the standard results of convenient calculus. Just for the curious reader and to give a flavor of the arguments, we enclose a lemma that is used many times in the proofs of the results above.

Lemma. (Special curve lemma, [33, 2.8]) *Let E be a locally convex vector space. Let x_n be a sequence which converges fast to x in E; i.e., for each $k \in \mathbb{N}$ the sequence $n^k(x_n - x)$ is bounded. Then the infinite polygon through the x_n can be parameterized as a smooth curve $c : \mathbb{R} \to E$ such that $c(\frac{1}{n}) = x_n$ and $c(0) = x$.*

2.5 Remark

Convenient calculus (i.e., having properties 6 and 7) exists also for:

- Real analytic mappings; see [32] or [33, Chapter II].
- Holomorphic mappings; see [40] or [33, Chapter II] (using the notion of [25, 26]).

- Many classes of Denjoy Carleman ultradifferentiable functions, both of Beurling type and of Roumieu-type, see [35, 36, 37, 39].
- With some adaptations, Lip^k; see [29].
- With more adaptations, even $C^{k,\alpha}$ (the k-th derivative is Hölder-continuous with index α); see [28], [27].

The following result is very useful if one wants to apply convenient calculus to spaces which are not tied to its categorical origin, like the Schwartz spaces $\mathcal{S}$, $\mathcal{D}$, or Sobolev spaces; for its uses see [50] and [38].

Theorem 2.1. [29, theorem 4.1.19] *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{V} \subset E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed sets. Then the following are equivalent:*

1. *c is smooth*
2. *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\ell \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\ell \circ c)^{(k)} = \ell \circ c^k$, for each $\ell \in \mathcal{V}$.*

If E is reflexive, then for any point separating subset $\mathcal{V} \subset E'$ the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed subsets, by [29, 4.1.23].

3 Manifolds of mappings and regular Lie groups

In this section I hope to demonstrate how convenient calculus allows for very short and transparent proofs of the core results in the theory of manifolds of smooth mappings.

3.1 The manifold structure on $C^\infty(M, N)$

Let M be a compact (for simplicity's sake) finite dimensional manifold and N a manifold. We use an auxiliary Riemann metric $\bar g$ on N. Then

$$\begin{tikzcd}
& 0_N \arrow[dl, bend right, "\text{zero section}"'] \arrow[d, hook] & N \arrow[d, hook] \arrow[dr, bend left, "\text{diagonal}"] & \\
TN & V^N \arrow[l, hook', "\text{open}"] \arrow[r, "{(\pi_N,\exp^{\bar g})}", "\cong"'] & V^{N\times N} \arrow[r, hook, "\text{open}"'] & N \times N
\end{tikzcd}$$

$C^\infty(M, N)$, the space of smooth mappings $M \to N$, has the following manifold structure. A chart, centered at $f \in C^\infty(M, N)$, is:

$$C^\infty(M, N) \supset U_f = \{g : (f, g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde U_f \subset \Gamma(f^*TN)$$
$$u_f(g) = (\pi_N, \exp^{\bar g})^{-1} \circ (f, g), \quad u_f(g)(x) = (\exp^{\bar g}_{f(x)})^{-1}(g(x))$$
$$(u_f)^{-1}(s) = \exp^{\bar g}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar g}_{f(x)}(s(x))$$

Lemma 3.1. $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; {\mathrm{pr}_2}^* f^*TN)$

This follows by cartesian closedness after trivializing the bundle f^*TN.

Lemma 3.2. *The chart changes are smooth (C^∞)*

$$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$$

Since they map smooth curves to smooth curves.

Lemma 3.3. $C^\infty(\mathbb{R}, C^\infty(M,N)) \cong C^\infty(\mathbb{R} \times M, N)$.

By lemma 3.1.

Lemma 3.4. *Composition $C^\infty(P,M) \times C^\infty(M,N) \to C^\infty(P,N)$, $(f,g) \mapsto g \circ f$, is smooth*

Since it maps smooth curves to smooth curves.

Corollary 3.5. *The tangent bundle of $C^\infty(M,N)$ is given by*

$$TC^\infty(M,N) = C^\infty(M,TN) \xrightarrow{C^\infty(M,\pi_N)} C^\infty(M,N).$$

This follows from the chart structure.

3.2 Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_eG$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to [53, 54, 55, 56, 57, 58] for Fréchet Lie groups, was weakened and made more transparent by [51], and then carried over to convenient Lie groups in [34], see also [33, 38.4]. We shall write $\mu : G \times G \to G$ for the mutiplication with $x.y = \mu(x,y) = \mu_x(y) = \mu^y(x)$ for left and right translation.

A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.

- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\mathrm{Evol}^X_t := g(t) = \mathrm{evol}_G(tX)$.

Up to now, every Lie group modeled on convenient vector spaces is regular.

Theorem 3.6. *For each compact manifold M, the diffeomorphism group* $\mathrm{Diff}(M)$ *is a regular Lie group.*

Proof. $\operatorname{Diff}(M) \xrightarrow{open} C^\infty(M,M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\operatorname{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t,x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.

Let $X(t,x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\operatorname{Fl}_s^{\partial_t \times X}(t,x) = (t+s, \operatorname{Evol}^X(t,x))$ satisfies the ordinary differential equation

$$\partial_t \operatorname{Evol}(t,x) = X(t, \operatorname{Evol}(t,x)).$$

If $X(s,t,x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the equation depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism. □

Groups of smooth diffeomorphisms on $\mathbb{R}^n$

If we consider the group of all orientation preserving diffeomorphisms $\operatorname{Diff}(\mathbb{R}^n)$ of $\mathbb{R}^n$, it is not an open subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with the compact C^∞-topology. So it is not a smooth manifold in the usual sense, but we may consider it as a Lie group in the cartesian closed category of Frölicher spaces, see [33, Section 23], with the structure induced by the injection $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Or one can use the setting of 'manifolds' based on smooth curves instead of charts, with lots of extra structure (tangent bundle, parallel transport, geodesic structure), described in [45, 46]; this gives a category of smooth 'manifolds' where those which have Banach spaces as tangent fibes are exactly the usual smooth manifolds modeled on Banach spaces, which is cartesian closed: $C^\infty(M,N)$ and $\operatorname{Diff}(M)$ are always 'manifolds' for 'manifolds' M and N, and the exponential law holds.

We shall now describe regular Lie groups in $\operatorname{Diff}(\mathbb{R}^n)$ which are given by diffeomorphisms of the form $f = \operatorname{Id}_{\mathbb{R}} + g$ where g is in some specific convenient vector space of bounded functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Now we discuss these spaces on $\mathbb{R}^n$, we describe the smooth curves in them, and we describe the corresponding groups. These results are from [50] and from [38, 39] for the more exotic groups.

The group $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$

The space $\mathcal{B}(\mathbb{R}^n)$ (called $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ by [62]) consists of all smooth functions which have all derivatives (separately) bounded. It is a Fréchet space. By [64], the space $\mathcal{B}(\mathbb{R}^n)$ is linearly isomorphic to $\ell^\infty \hat{\otimes}\, \mathfrak{s}$ for any completed tensor-product between the projective one and the injective one, where $\mathfrak{s}$ is the nuclear Fréchet space of rapidly decreasing real sequences. Thus $\mathcal{B}(\mathbb{R}^n)$ is not reflexive, not nuclear, not smoothly paracompact.

The space $C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}^n))$ of smooth curves in $\mathcal{B}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ and each $t \in \mathbb{R}$ the expression $\partial_t^k \partial_x^\alpha c(t,x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally in t.*

To see this use Theorem 2.1 for the set $\{\mathrm{ev}_x : x \in \mathbb{R}\}$ of point evaluations in $\mathcal{B}(\mathbb{R}^n)$. Here $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ and $c^k(t) = \partial_t^k f(t, \quad)$.
$\mathrm{Diff}^+_{\mathcal{B}}(\mathbb{R}^n) = \big\{f = \mathrm{Id} + g : g \in \mathcal{B}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) \geq \varepsilon > 0\big\}$ *denotes the corresponding group*, see below.

The group $\mathrm{Diff}_{W^{\infty,p}}(\mathbb{R}^n)$

For $1 \leq p < \infty$, the space $W^{\infty,p}(\mathbb{R}^n) = \bigcap_{k\geq 1} L^p_k(\mathbb{R}^n)$ is the intersection of all L^p-Sobolev spaces, the space of all smooth functions such that each partial derivative is in L^p. It is a reflexive Fréchet space. It is called $\mathcal{D}_{L^p}(\mathbb{R}^n)$ in [62]. By [64], the space $W^{\infty,p}(\mathbb{R}^n)$ is linearly isomorphic to $\ell^p \hat{\otimes} \mathfrak{s}$. Thus it is not nuclear, not Schwartz, not Montel, and smoothly paracompact only if p is an even integer.
The space $C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n))$ of smooth curves in $W^{\infty,p}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ the expression $\|\partial_t^k \partial_x^\alpha f(t, \quad)\|_{L^p(\mathbb{R}^n)}$ is locally bounded near each $t \in \mathbb{R}$.*

The proof is literally the same as for $\mathcal{B}(\mathbb{R}^n)$, noting that the point evaluations are continuous on each Sobolev space L^p_k with $k > \frac{n}{p}$.
$\mathrm{Diff}^+_{W^{\infty,p}}(\mathbb{R}^n) = \big\{f = \mathrm{Id} + g : g \in W^{\infty,p}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\big\}$ *denotes the corresponding group.*

The group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

The algebra $\mathcal{S}(R^n)$ of rapidly decreasing functions is a reflexive nuclear Fréchet space.
The space $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ of smooth curves in $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\geq 0}$ and $\alpha \in \mathbb{N}^n_{\geq 0}$, the expression $(1 + |x|^2)^m \partial_t^k \partial_x^\alpha c(t, x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally uniformly bounded in $t \in \mathbb{R}$.*

$\mathrm{Diff}^+_{\mathcal{S}}(\mathbb{R}^n) = \big\{f = \mathrm{Id} + g : g \in \mathcal{S}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\big\}$ *is the corresponding group.*

The group $\mathrm{Diff}_c(\mathbb{R}^n)$

The algebra $C^\infty_c(\mathbb{R}^n)$ of all smooth functions with compact support is a nuclear (LF)-space.
The space $C^\infty(\mathbb{R}, C^\infty_c(\mathbb{R}^n))$ of smooth curves in $C^\infty_c(\mathbb{R}^n)$ consists of all functions $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For each compact interval $[a, b]$ in $\mathbb{R}$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $f(t, x) = 0$ for $(t, x) \in [a, b] \times (\mathbb{R}^n \setminus K)$.*

$\mathrm{Diff}_c(\mathbb{R}^n) = \big\{f = \mathrm{Id} + g : g \in C^\infty_c(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\big\}$ *is the correponding group.* The case $\mathrm{Diff}_c(\mathbb{R}^n)$ is well-known since 1980.

Ideal properties of function spaces

The function spaces discussed are boundedly mapped into each other as follows:

$$C_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow W^{\infty,p}(\mathbb{R}^n) \xrightarrow{p\leq q} W^{\infty,q}(\mathbb{R}^n) \longrightarrow \mathcal{B}(\mathbb{R}^n)$$

and each space is a bounded locally convex algebra and a bounded $\mathcal{B}(\mathbb{R}^n)$-module. Thus each space is an ideal in each larger space.

Theorem 3.7 ([50] and [38])**.** *The sets of diffeomorphisms* $\mathrm{Diff}_c(\mathbb{R}^n)$*,* $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$*,* $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$*, and* $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms*

$$\mathrm{Diff}_c(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{W^{\infty,p}}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n) \ .$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$*.*

The proof of this theorem relies on repeated use of the Faà di Bruno formula for higher derivatives of composed functions. This offers difficulties on non-compact manifolds, where one would need a non-commutative Faà di Bruno formula for iterated covariant derivatives. In the paper [38] many more similar groups are discussed, modeled on spaces of Denjoy-Carleman ultradifferentiable functions. It is also showm that for $p > 1$ the group $\mathrm{Diff}_{W^{\infty,p}\cap L^1}(\mathbb{R}^n)$ is only a topological group with smooth right translations — a property which is similar to the one of finite order Sobolev groups $\mathrm{Diff}_{W^{k,p}}(\mathbb{R}^n)$. Some of these groups were used extensively in [52].

Corollary 3.8. $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *acts on* Γ_c*,* $\Gamma_{\mathcal{S}}$ *and* Γ_{H^∞} *of any tensorbundle over* $\mathbb{R}^n$ *by pullback. The infinitesimal action of the Lie algebra* $\mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ *on these spaces by the Lie derivative thus maps each of these spaces into itself. A fortiori,* $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ *acts on* $\Gamma_{\mathcal{S}}$ *of any tensor bundle by pullback.*

3.3 Manifolds of immersions and shape spaces

For finite dimensional manifolds M, N with M compact, $\mathrm{Emb}(M,N)$, the space of embeddings of M into N, is open in $C^\infty(M,N)$, so it is a smooth manifold. $\mathrm{Diff}(M)$ acts freely and smoothly from the right on $\mathrm{Emb}(M,N)$.

Theorem 3.9. $\mathrm{Emb}(M,N) \to \mathrm{Emb}(M,N)/\mathrm{Diff}(M) = B(M,N)$ *is a principal fiber bundle with structure group* $\mathrm{Diff}(M)$*.*

Note that $B(M,N)$ is the smooth manifold of all submanifolds of N which are of diffeomorphism type M. Therefore it is also called the *nonlinear Grassmannian* in [30], where this theorem is extended to the case when M has boundary. From another point of view, $B(M,N)$ is called the *differentiable Chow variety* in [43]. It is an example of a *shape space*.

Proof. We use an auxiliary Riemannian metric $\bar{g}$ on N. Given $f \in \operatorname{Emb}(M,N)$, we view $f(M)$ as a submanifold of N and we split the the tangent bundle of N along $f(M)$ as $TN|_{f(M)} = \operatorname{Nor}(f(M)) \oplus Tf(M)$. The exponential mapping describes a tubular neighborhood of $f(M)$ via

$$\operatorname{Nor}(f(M)) \xrightarrow[\cong]{\exp^{\bar{g}}} W_{f(M)} \xrightarrow{p_{f(M)}} f(M).$$

If $g : M \to N$ is C^1-near to f, then $\varphi(g) := f^{-1} \circ p_{f(M)} \circ g \in \operatorname{Diff}(M)$ and we may consider $g \circ \varphi(g)^{-1} \in \Gamma(f^* W_{f(M)}) \subset \Gamma(f^* \operatorname{Nor}(f(M)))$. This is the required local splitting. □

$\operatorname{Imm}(M,N)$, the space of immersions $M \to N$, is open in $C^\infty(M,N)$, and is thus a smooth manifold. The regular Lie group $\operatorname{Diff}(M)$ acts smoothly from the right, but no longer freely.

Theorem 3.10 ([17])**.** *For an immersion $f : M \to N$, the isotropy group*

$$\operatorname{Diff}(M)_f = \{\varphi \in \operatorname{Diff}(M) : f \circ \phi = f\}$$

is always a finite group, acting freely on M; so $M \xrightarrow{p} M/\operatorname{Diff}(M)_f$ is a covering of manifolds and f factors to $f = \bar{f} \circ p$.

Thus $\operatorname{Imm}(M,N) \to \operatorname{Imm}(M,N)/\operatorname{Diff}(M) =: B_i(M,N)$ *is a projection onto an infinite dimensional orbifold.*

The space $B_i(M,N)$ is another example of a *shape space*. It appeared in the form of $B_i(S^1,\mathbb{R}^2)$, the shape space of plane immersed curves, in [48] and [49].

4 Weak Riemannian manifolds

If an infinite dimensional manifold is not modeled on a Hilbert space, then a Riemannian metric cannot describe the topology on each tangent space. We have to deal with more complicated situations.

4.1 Manifolds, vector fields, differential forms

Let M be a smooth manifold modeled on convenient vector spaces. Tangent vectors to M are kinematic ones.

The reason for this is that eventually we want flows of vector fields, and that there are too many derivations in infinite dimensions, even on a Hilbert space H: Let $\alpha \in L(H,H)$ be a continuous linear functional which vanishes on the subspace of compact operators, thus also on $H \otimes H$. Then $f \mapsto \alpha(d^2 f(0))$ is a derivation at 0 on $C^\infty(H)$, since $\alpha(d^2(f.g)(0)) = \alpha\big(d^2 f(0).g(0) + df(0) \otimes dg(0) + dg(0) \otimes df(0) + f(0).d^2 g(0)\big)$ and α vanishes on the two middle terms. There are even non-zero derivations which differentiate 3 times, see [33, 28.4].

The (kinematic) tangent bundle TM is then a smooth vector bundle as usual. Differential forms of degree k are then smooth sections of the bundle $L^k_{\text{skew}}(TM;\mathbb{R})$ of skew symmetric k-linear functionals on the tangent bundle, since this is the only version which admits exterior derivative, Lie derivatives along vector field, and pullbacks along arbitray smooth mappings; see [33, 33.21]. The de Rham cohomology equals singular cohomology with real coefficients if the manifold is smoothly paracompact; see [33, Section 34]. If a vector field admits a flow, then each integral curve is uniquely given as a flow line; see [33, 32.14].

4.2 Weak Riemannian manifolds

Let (M, g) be a smooth manifold modeled on convenient locally convex vector spaces. A smooth Riemannian metric on M is called weak if $g_x : T_xM \to T_x^*M$ is only injective for each $x \in M$. The image $g(TM) \subset T^*M$ is called the *smooth cotangent bundle* associated to g. Then g^{-1} is the metric on the smooth cotangent bundle as well as the morphism $g(TM) \to TM$. We have a special class of 1-forms $\Omega^1_g(M) := \Gamma(g(TM))$ for which the musical mappings makes sense: $\alpha^\sharp = g^{-1}\alpha \in \mathfrak{X}(M)$ and $X^\flat = gX$. These 1-forms separate points on TM. The exterior derivative is defined by $d : \Omega^1_g(M) \to \Omega^2(M) = \Gamma(L^2_{\text{skew}}(TM; \mathbb{R}))$ since the embedding $g(TM) \subset T^*M$ is a smooth fiber linear mapping.

Existence of the Levi-Civita covariant derivative is equivalent to: *The metric itself admits symmetric gradients with respect to itself.* Locally this means: If M is c^∞-open in a convenient vector space V_M. Then:

$$D_{x,X}g_x(X, Y) = g_x(X, \operatorname{grad}_1 g(x)(X, Y)) = g_x(\operatorname{grad}_2 g(x)(X, X), Y)$$

where $D_{x,X}$ denote the directional derivative at x in the direction X, and where the mappings $\operatorname{grad}_1 g$ and $\operatorname{sym}\operatorname{grad}_2 g : M \times V_M \times V_M \to V_M$, given by $(x, X) \mapsto \operatorname{grad}_{1,2} g(x)(X, X)$, are smooth and quadratic in $X \in V_M$.

4.3 Weak Riemannian metrics on spaces of immersions

For a compact manifold M and a finite dimensional Riemannian manifold $(N, \bar g)$ we can consider the following weak Riemannian metrics on the manifold $\operatorname{Imm}(M, N)$ of smooth immersions $M \to N$:

$$\begin{aligned}
G^0_f(h,k) &= \int_M \bar g(h,k)\operatorname{vol}(f^*\bar g) && \text{the } L^2\text{-metric,}\\
G^s_f(h,k) &= \int_M \bar g((1+\Delta^{f^*\bar g})^s h,k)\operatorname{vol}(f^*\bar g) && \text{a Sobolev metric of order } s,\\
G^\varphi_f(h,g) &= \int_M \Phi(f)\bar g(h,k)\operatorname{vol}(f^*\bar g) && \text{an almost local metric.}
\end{aligned}$$

Here $\operatorname{vol}(f^*\bar g)$ is the volume density on M of the pullback metric $g = f^*\bar g$, where $\Delta^{f^*\bar g}$ is the (Bochner) Laplacion with respect to g and $\bar g$ acting on sections of f^*TN, and where $\Phi(f)$ is a positive function of the total volume $\operatorname{Vol}(f^*g) = \int_M \operatorname{vol}(f^*g)$, of the scalar curvature $\operatorname{Scal}(f^*\bar g)$, and of the mean curvature $\operatorname{Tr}(S^f)$, S^f being the second fundamental form. See [9], [10] for more information. All these metrics are invariant for the right action of the reparameterization group $\operatorname{Diff}(M)$, so they descend to metrics on shape space $B_i(M, N)$ (off the singularities) such that the projection $\operatorname{Imm}(M,N) \to B_i(M,N)$ is a Riemannian submersion of a benign type: the G-orthogonal component to the tangent space to the $\operatorname{Diff}(M)$-orbit consists always of smooth vector fields. So there is no need to use the notion of robust weak Riemannian metrics discussed below.

Theorem 4.1. *1. Geodesic distance on* $\operatorname{Imm}(M,N)$*, defined as the infimum of path-lenghts of smooth isotopies between two immersions, vanishes for the* L^2*-metric* G^0*.*

2. Geodesic distance is positive on $B_i(M,N)$ *for the almost local metric* G^Φ *if* $\Phi(f) \geq 1 + A\operatorname{Tr}(S^F)$*, or if* $\Phi(f) \geq A\operatorname{Vol}(f^*\bar g)$*, for some* $A > 0$*.*

3. Geodesic distance is positive on $B_i(M,N)$ *for the Sobolev metric* G^s *if* $s \geq 1$*.*

4. The geodesic equation is locally well-posed on $\operatorname{Imm}(M,N)$ *for the Sobolev metric* G^s *if* $s \geq 1$*, and globally well-posed (and thus geodesically complete) on* $\operatorname{Imm}(S^1,\mathbb{R}^n)$*, if* $s \geq 2$*.*

1 is due to [48] for $B_i(S^1,\mathbb{R}^2)$, to [47] for $B_i(M,N)$ and for $\operatorname{Diff}(M)$, which combines to the result for $\operatorname{Imm}(M,N)$ as noted in [3]. 2 is proved in [10]. For 3 see [9]. 4 is due to [16] and [15].

4.4 Weak right invariant Riemannian metrics on regular Lie groups

Let G be a regular convenient Lie group, with Lie algebra $\mathfrak{g}$. Let $\mu : G \times G \to G$ be the group multiplication, μ_x the left translation and μ^y the right translation, $\mu_x(y) = \mu^y(x) = xy = \mu(x,y)$. Let $L, R : \mathfrak{g} \to \mathfrak{X}(G)$ be the left- and right-invariant vector field mappings, given by $L_X(g) = T_e(\mu_g).X$ and $R_X = T_e(\mu^g).X$, respectively. They are related by $L_X(g) = R_{\operatorname{Ad}(g)X}(g)$. Their flows are given by

$$\begin{aligned}
\operatorname{Fl}^{L_X}_t(g) &= g.\exp(tX) = \mu^{\exp(tX)}(g),\\
\operatorname{Fl}^{R_X}_t(g) &= \exp(tX).g = \mu_{\exp(tX)}(g).
\end{aligned}$$

The right Maurer–Cartan form $\kappa = \kappa^r \in \Omega^1(G,\mathfrak{g})$ is given by $\kappa_x(\xi) := T_x(\mu^{x^{-1}})\cdot\xi$.
The left Maurer–Cartan form $\kappa^l \in \Omega^1(G,\mathfrak{g})$ is given by $\kappa_x(\xi) := T_x(\mu_{x^{-1}})\cdot\xi$.
κ^r *satisfies the left Maurer-Cartan equation* $d\kappa - \frac{1}{2}[\kappa,\kappa]^\wedge_{\mathfrak{g}} = 0$, where $[\quad,\quad]_\wedge$ denotes the wedge product of $\mathfrak{g}$-valued forms on G induced by the Lie bracket. Note that $\frac{1}{2}[\kappa,\kappa]_\wedge(\xi,\eta) = [\kappa(\xi),\kappa(\eta)]$.
κ^l *satisfies the right Maurer-Cartan equation* $d\kappa + \frac{1}{2}[\kappa,\kappa]^\wedge_{\mathfrak{g}} = 0$.

Namely, evaluate $d\kappa^r$ on right invariant vector fields R_X, R_Y for $X, Y \in \mathfrak{g}$.

$$\begin{aligned}
(d\kappa^r)(R_X,R_Y) &= R_X(\kappa^r(R_Y)) - R_Y(\kappa^r(R_X)) - \kappa^r([R_X,R_Y])\\
&= R_X(Y) - R_Y(X) + [X,Y] = 0 - 0 + [\kappa^r(R_X),\kappa^r(R_Y)].
\end{aligned}$$

The (exterior) derivative of the function $\operatorname{Ad} : G \to GL(\mathfrak{g})$ can be expressed by

$$d\operatorname{Ad} = \operatorname{Ad}.(\operatorname{ad}\circ\kappa^l) = (\operatorname{ad}\circ\kappa^r).\operatorname{Ad}$$

since we have $d\operatorname{Ad}(T\mu_g.X) = \frac{d}{dt}|_0 \operatorname{Ad}(g.\exp(tX)) = \operatorname{Ad}(g).\operatorname{ad}(\kappa^l(T\mu_g.X))$.

Let $\gamma = \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive-definite bounded (weak) inner product. Then

$$\gamma_x(\xi,\eta) = \gamma\big(T(\mu^{x^{-1}})\cdot\xi,\, T(\mu^{x^{-1}})\cdot\eta\big) = \gamma\big(\kappa(\xi),\,\kappa(\eta)\big)$$

is a right-invariant (weak) Riemannian metric on G and any (weak) right-invariant bounded Riemannian metric is of this form, for suitable γ. Denote by $\check\gamma : \mathfrak{g} \to \mathfrak{g}^*$ the mapping induced by γ, from the Lie algebra into its dual (of bounded linear functionals) and by $\langle \alpha, X\rangle_{\mathfrak{g}}$ the duality evaluation between $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

Let $g : [a,b] \to G$ be a smooth curve. The velocity field of g, viewed in the right trivializations, coincides with the right logarithmic derivative

$$\delta^r(g) = T(\mu^{g^{-1}})\cdot \partial_t g = \kappa(\partial_t g) = (g^*\kappa)(\partial_t).$$

The energy of the curve $g(t)$ is given by

$$E(g) = \frac{1}{2}\int_a^b \gamma_g(g',g')dt = \frac{1}{2}\int_a^b \gamma\big((g^*\kappa)(\partial_t),(g^*\kappa)(\partial_t)\big)\,dt.$$

For a variation $g(s,t)$ with fixed endpoints we then use that

$$d(g^*\kappa)(\partial_t,\partial_s) = \partial_t(g^*\kappa(\partial_s)) - \partial_s(g^*\kappa(\partial_t)) - 0,$$

partial integration, and the left Maurer–Cartan equation to obtain

$$\begin{aligned}
\partial_s E(g) &= \frac{1}{2}\int_a^b 2\gamma\big(\partial_s(g^*\kappa)(\partial_t),\,(g^*\kappa)(\partial_t)\big)\,dt\\
&= \int_a^b \gamma\big(\partial_t(g^*\kappa)(\partial_s) - d(g^*\kappa)(\partial_t,\partial_s),\,(g^*\kappa)(\partial_t)\big)\,dt\\
&= -\int_a^b \gamma\big((g^*\kappa)(\partial_s),\,\partial_t(g^*\kappa)(\partial_t)\big)\,dt\\
&\qquad - \int_a^b \gamma\big([(g^*\kappa)(\partial_t),(g^*\kappa)(\partial_s)],\,(g^*\kappa)(\partial_t)\big)\,dt\\
&= -\int_a^b \big\langle \check\gamma(\partial_t(g^*\kappa)(\partial_t)),\,(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak{g}}\,dt\\
&\qquad - \int_a^b \big\langle \check\gamma(((g^*\kappa)(\partial_t)),\,\mathrm{ad}_{(g^*\kappa)(\partial_t)}(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak{g}}\,dt\\
&= -\int_a^b \big\langle \check\gamma(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check\gamma(((g^*\kappa)(\partial_t)),\,(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak{g}}\,dt.
\end{aligned}$$

Thus the curve $g(0,t)$ is critical for the energy if and only if

$$\check\gamma(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check\gamma(((g^*\kappa)(\partial_t)) = 0.$$

In terms of the right logarithmic derivative $u : [a,b] \to \mathfrak{g}$ of $g : [a,b] \to G$, given by $u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}})\cdot g'(t)$, the *geodesic equation* has the expression

$$\boxed{\partial_t u = -\,\check\gamma^{-1}\,\mathrm{ad}(u)^*\,\check\gamma(u)}$$

Thus the geodesic equation exists in general if and only if $\mathrm{ad}(X)^*\check\gamma(X)$ is in the image of $\check\gamma : \mathfrak g \to \mathfrak g^*$, i.e.,

$$\mathrm{ad}(X)^*\check\gamma(X) \in \check\gamma(\mathfrak g)$$

for every $X \in \mathfrak X$. This condition then leads to the existence of the Christoffel symbols. Arnold in [2] asked for the more restrictive condition $\mathrm{ad}(X)^*\check\gamma(Y) \in \check\gamma(\mathfrak g)$ for all $X, Y \in \mathfrak g$. The geodesic equation for the *momentum* $p := \gamma(u)$ is

$$p_t = -\,\mathrm{ad}(\check\gamma^{-1}(p))^* p.$$

There are situations, see theorem 4.5 or [6], where only the more general condition is satisfied, but where the usual transpose $\mathrm{ad}^\top(X)$ of $\mathrm{ad}(X)$,

$$\mathrm{ad}^\top(X) := \check\gamma^{-1} \circ \mathrm{ad}_X^* \circ \check\gamma$$

does not exist for all X.

We describe now the *covariant derivative* and the *curvature*. The right trivialization $(\pi_G, \kappa^r) : TG \to G \times \mathfrak g$ induces the isomorphism $R : C^\infty(G, \mathfrak g) \to \mathfrak X(G)$, given by $R(X)(x) := R_X(x) := T_e(\mu^x) \cdot X(x)$, for $X \in C^\infty(G, \mathfrak g)$ and $x \in G$. Here $\mathfrak X(G) := \Gamma(TG)$ denotes the Lie algebra of all vector fields. For the Lie bracket and the Riemannian metric we have

$$\begin{aligned}
[R_X, R_Y] &= R(-[X, Y]_{\mathfrak g} + dY \cdot R_X - dX \cdot R_Y),\\
R^{-1}[R_X, R_Y] &= -[X, Y]_{\mathfrak g} + R_X(Y) - R_Y(X),\\
\gamma_x(R_X(x), R_Y(x)) &= \gamma(X(x), Y(x))\,,\ x \in G.
\end{aligned}$$

In what follows, we shall perform all computations in $C^\infty(G, \mathfrak g)$ instead of $\mathfrak X(G)$. In particular, we shall use the convention

$$\nabla_X Y := R^{-1}(\nabla_{R_X} R_Y) \qquad \text{for } X, Y \in C^\infty(G, \mathfrak g)$$

to express the Levi-Civita covariant derivative.

Lemma 4.2. [6, 3.3] *Assume that for all $\xi \in \mathfrak g$ the element $\mathrm{ad}(\xi)^*\check\gamma(\xi) \in \mathfrak g^*$ is in the image of $\check\gamma : \mathfrak g \to \mathfrak g^*$ and that $\xi \mapsto \check\gamma^{-1}\,\mathrm{ad}(\xi)^*\check\gamma(\xi)$ is bounded quadratic (or, equivalently, smooth). Then the Levi-Civita covariant derivative of the metric γ exists and is given for any $X, Y \in C^\infty(G, \mathfrak g)$ in terms of the isomorphism R by*

$$\nabla_X Y = dY.R_X + \rho(X)Y - \frac{1}{2}\,\mathrm{ad}(X)Y,$$

where

$$\rho(\xi)\eta = \tfrac14 \check\gamma^{-1}\big(\mathrm{ad}^*_{\xi+\eta}\check\gamma(\xi+\eta) - \mathrm{ad}^*_{\xi-\eta}\check\gamma(\xi-\eta)\big) = \tfrac12 \check\gamma^{-1}\big(\mathrm{ad}^*_\xi\check\gamma(\eta) + \mathrm{ad}^*_\eta\check\gamma(\xi)\big)$$

is the polarized version. $\rho : \mathfrak g \to L(\mathfrak g, \mathfrak g)$ is bounded, and we have $\rho(\xi)\eta = \rho(\eta)\xi$. We also have

$$\begin{aligned}
\gamma\big(\rho(\xi)\eta, \zeta\big) &= \frac12\gamma(\xi, \mathrm{ad}(\eta)\zeta) + \frac12\gamma(\eta, \mathrm{ad}(\xi)\zeta),\\
\gamma(\rho(\xi)\eta, \zeta) &+ \gamma(\rho(\eta)\zeta, \xi) + \gamma(\rho(\zeta)\xi, \xi) = 0.
\end{aligned}$$

For $X, Y \in C^\infty(G, \mathfrak{g})$ we have

$$[R_X, \operatorname{ad}(Y)] = \operatorname{ad}(R_X(Y)) \quad \text{and} \quad [R_X, \rho(Y)] = \rho(R_X(Y)).$$

The *Riemannian curvature* is then computed by

$$\begin{aligned}
\mathcal{R}(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)} \\
&= [R_X + \rho_X - \tfrac{1}{2}\operatorname{ad}_X, R_Y + \rho_Y - \tfrac{1}{2}\operatorname{ad}_Y] \\
&\quad - R(-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) - \rho(-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) \\
&\quad + \frac{1}{2}\operatorname{ad}(-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) \\
&= [\rho_X, \rho_Y] + \rho_{[X,Y]_\mathfrak{g}} - \frac{1}{2}[\rho_X, \operatorname{ad}_Y] + \frac{1}{2}[\rho_Y, \operatorname{ad}_X] - \frac{1}{4}\operatorname{ad}_{[X,Y]_\mathfrak{g}}.
\end{aligned}$$

which is visibly a tensor field.

For the numerator of the sectional curvature we obtain

$$\begin{aligned}
\gamma\big(\mathcal{R}(X,Y)X, Y\big) &= \gamma(\rho_X\rho_Y X, Y) - \gamma(\rho_Y\rho_X X, Y) + \gamma(\rho_{[X,Y]}X, Y) \\
&\quad - \frac{1}{2}\gamma(\rho_X[Y,X], Y) + \frac{1}{2}\gamma([Y, \rho_X X], Y) \\
&\quad + 0 - \frac{1}{2}\gamma([X, \rho_Y X], Y) - \frac{1}{4}\gamma([[X,Y],X], Y) \\
&= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|^2_\gamma + \frac{3}{4}\|[X,Y]\|^2_\gamma \\
&\quad - \frac{1}{2}\gamma(X, [Y,[X,Y]]) + \frac{1}{2}\gamma(Y, [X,[X,Y]]) \\
&= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|^2_\gamma + \frac{3}{4}\|[X,Y]\|^2_\gamma \\
&\quad - \gamma(\rho_X Y, [X,Y]]) + \gamma(Y, [X,[X,Y]]).
\end{aligned}$$

If the adjoint $\operatorname{ad}(X)^\top : \mathfrak{g} \to \mathfrak{g}$ exists, this is easily seen to coincide with Arnold's original formula [2],

$$\begin{aligned}
\gamma(\mathcal{R}(X,Y)X, Y) = &-\frac{1}{4}\|\operatorname{ad}(X)^\top Y + \operatorname{ad}(Y)^\top X\|^2_\gamma + \gamma(\operatorname{ad}(X)^\top X, \operatorname{ad}(Y)^\top Y) \\
&+ \frac{1}{2}\gamma(\operatorname{ad}(X)^\top Y - \operatorname{ad}(Y)^\top X, \operatorname{ad}(X)Y) + \frac{3}{4}\|[X,Y]\|^2_\gamma.
\end{aligned}$$

4.5 Weak right invariant Riemannian metrics on diffeomorphism groups

Let N be a finite dimensional manifold. We consider the following regular Lie groups: $\operatorname{Diff}(N)$, the group of all diffeomorphisms of N if N is compact. $\operatorname{Diff}_c(N)$, the group of diffeomorphisms with compact support, if N is not compact. If $N = \mathbb{R}^n$, we also may consider one of the following: $\operatorname{Diff}_\mathcal{S}(\mathbb{R}^n)$, the group of all diffeomorphisms which fall rapidly to the identity. $\operatorname{Diff}_{W^{\infty,p}}(\mathbb{R}^n)$, the group of all diffeomorphisms which are modelled on the space $W^{\infty,p}(\mathbb{R}^n)^n$, the intersection of all $W^{k,p}$-Sobolev spaces of vector fields. The last type of groups works also for a *Riemannian manifold of bounded geometry* $(N,\bar{g})$; see [21] for Sobolev spaces on them. In the following we write $\mathrm{Diff}_{\mathcal{A}}(N)$ for any of these groups. The Lie algebras are the spaces $\mathfrak{X}_{\mathcal{A}}(N)$ of vector fields, where $\mathcal{A}\in\{C_c^\infty,\mathcal{S},W^{\infty,p}\}$, with the negative of the usual bracket as Lie bracket.

A right invariant weak inner product on $\mathrm{Diff}_{\mathcal{A}}(N)$ is given by a smooth positive definite inner product γ on the Lie algebra $\mathfrak{X}_{\mathcal{A}}(N)$ which is described by the operator $L=\check{\gamma}:\mathfrak{X}_{\mathcal{A}}(N)\to\mathfrak{X}_{\mathcal{A}}(N)'$ and we shall denote its inverse by $K=L^{-1}:L(\mathfrak{X}_{\mathcal{A}}(N))\to\mathfrak{X}_{\mathcal{A}}(N)$. Under suitable conditions on L (like an elliptic coercive (pseudo) differential operator of high enough order) the operator K turns out to be the reproducing kernel of a Hilbert space of vector fields which is contained in the space of either C_b^1 (bounded C^1 with respect to $\bar{g}$) or C_b^2 vector fields. See [65, Chapter 12], [44], and [52] for uses of the reproducing Hilbert space approach. The right invariant metric is then defined as in 4.4, where $\langle\ ,\ \rangle_{\mathfrak{X}_{\mathcal{A}}(N)}$ is the duality:

$$G^L_\varphi(X\circ\varphi,Y\circ\varphi)=G^L_{\mathrm{Id}}(X,Y)=\gamma(X,Y)=\langle L(X),Y\rangle_{\mathfrak{X}_{\mathcal{A}}(N)}.$$

Mis-using the notation for L we will often also write

$$G^L_{\mathrm{Id}}(X,Y)=\int_N\bar{g}(LX,Y)\,\mathrm{vol}(\bar{g}).$$

Examples of metrics are:

$$\begin{aligned}
G^0_{\mathrm{Id}}(X,Y)&=\int_N\bar{g}(X,Y)\,\mathrm{vol}(\bar{g}) &&\text{the } L^2 \text{ metric,}\\
G^s_{\mathrm{Id}}(X,Y)&=\int_N\bar{g}((1+\Delta^{\bar{g}})^sX,Y)\,\mathrm{vol}(\bar{g}) &&\text{a Sobolev metric of order } s,\\
G^{\dot{H}^1}_{\mathrm{Id}}(X,Y)&=\int_{\mathbb{R}}X'.Y'dx=-\int_{\mathbb{R}}X''Y\,dx &&\text{where } X,Y\in\mathfrak{X}_{\mathcal{A}}(\mathbb{R}).
\end{aligned}$$

The geodesic equation on $\mathrm{Diff}_{\mathcal{A}}(N)$. As explained in 4.4, the geodesic equation is given as follws: Let $\varphi:[a,b]\to\mathrm{Diff}_{\mathcal{A}}(N)$ be a smooth curve. In terms of its right logarithmic derivative $u:[a,b]\to\mathfrak{X}_{\mathcal{A}}(N)$, $u(t):=\varphi^*\kappa(\partial_t)=\varphi'(t)\circ\varphi(t)^{-1}$, the geodesic equation is

$$L(u_t)=L(\partial_t u)=-\operatorname{ad}(u)^*L(u).$$

The *condition for the existence of the geodesic equation* is as follows:

$$X\mapsto K(\operatorname{ad}(X)^*L(X))$$

is bounded quadratic $\mathfrak{X}_{\mathcal{A}}(N)\to\mathfrak{X}_{\mathcal{A}}(N)$. Using *Lie derivatives*, the computation of ad_X^* is especially simple. Namely, for any section ω of $T^*N\otimes\mathrm{vol}$ and vector fields $\xi,\eta\in\mathfrak{X}_{\mathcal{A}}(N)$, we have:

$$\int_N(\omega,[\xi,\eta])=\int_N(\omega,\mathcal{L}_\xi(\eta))=-\int_N(\mathcal{L}_\xi(\omega),\eta),$$

hence $\operatorname{ad}^*_\xi(\omega) = +\mathcal{L}_\xi(\omega)$. Thus the Hamiltonian version of the geodesic equation on the smooth dual $L(\mathfrak{X}_{\mathcal{A}}(N)) \subset \Gamma_{C^2_b}(T^*N \otimes \operatorname{vol})$ becomes

$$\partial_t \alpha = -\operatorname{ad}^*_{K(\alpha)} \alpha = -\mathcal{L}_{K(\alpha)}\alpha,$$

or, keeping track of everything,

$$\partial_t\varphi = u \circ \varphi, \qquad \partial_t\alpha = -\mathcal{L}_u\alpha \qquad u = K(\alpha) = \alpha^\sharp, \qquad \alpha = L(u) = u^\flat.$$

Theorem 4.3. *Geodesic distance vanishes on* $\operatorname{Diff}_{\mathcal{A}}(N)$ *for any Sobolev metric of order* $s < \frac{1}{2}$*. If* $N = S^1 \times C$ *with* C *compact, then geodesic distance vanishes also for* $s = \frac{1}{2}$*. It also vanises for the* L^2*-metric on the Virasoro group* $\mathbb{R} \rtimes \operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$*.*

Geodesic distance is positive on $\operatorname{Diff}_{\mathcal{A}}(N)$ *for any Sobolev metric of order* $s \geq 1$*. If* $\dim(N) = 1$ *then geodesic distance is also positive for* $s > \frac{1}{2}$*.*

This proved in [4], [5], and [3]. Note that low order Sobolev metrics have geodesic equations corresponding to well-known non-linear PDEs: On $\operatorname{Diff}(S^1)$ or $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ the L^2-geodesic equation is Burgers' equation, on the Virasoro group it is the KdV equation, and the (standard) H^1-geodesic is (in both cases a variant of) the Camassa-Holm equation; see [7, 7.2] for a more comprehensive overview. All these are completely integrable infinite dimensional Hamiltonian systems.

Theorem 4.4. *Let* $(N, \bar{g})$ *be a compact Riemannian manifold. Then the geodesic equation is locally well-posed on* $\operatorname{Diff}_{\mathcal{A}}(N)$ *and the geodesic exponential mapping is a local diffeomorphism for a Sobolev metric of integer order* $s \geq 1$*. For a Sobolev metric of integer order* $s > \frac{\dim(N)+3}{2}$ *the geodesic equation is even globally well-posed, so that* $(\operatorname{Diff}_{\mathcal{A}}(N), G^s)$ *is geodesically complete. This is also true for non-integer order* s *if* $N = \mathbb{R}^n$*.*

For $N = S^1$*, the geodesic equation is locally wellposed even for* $s \geq \frac{1}{2}$*.*

For these results see [9], [24], [23], [8].

Theorem 4.5. [6] *For* $\mathcal{A} \in \{C^\infty_c, \mathcal{S}, W^{\infty,1}\}$ *let*

$$\mathcal{A}_1(\mathbb{R}) = \{f \in C^\infty(R) \,:\, f' \in \mathcal{A}(\mathbb{R})\,,\, f(-\infty) = 0\}$$

and let $\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R}) = \{\varphi = \operatorname{Id} + f \,:\, f \in \mathcal{A}_1(\mathbb{R})\,,\, f' > -1\}$*. These are all regular Lie groups. The right invariant weak Riemannian metric* $G^{\dot{H}^1}_{\operatorname{Id}}(X, Y) = \int_{\mathbb{R}} X'Y'\,dx$ *is positive definite both on* $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ *where it does not admit a geodesic equation (a non-robust weak Riemannian manifold), and on* $\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *where it admits a geodesic equation but not in the stronger sense of Arnold. On* $\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic equation is the Hunter-Saxton equation*

$$(\varphi_t) \circ \varphi^{-1} = u, \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^x (u_x(z))^2\,dz\,,$$

and the induced geodesic distance is positive. We define the R-map by:

$$R : \operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}), \quad R(\varphi) = 2\,\big((\varphi')^{1/2} - 1\big)\,.$$

The R-map is invertible with inverse

$$R^{-1} : \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \quad R^{-1}(\gamma)(x) = x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2 + 4\gamma\, dx\ .$$

The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$, i.e., $R^*\langle\cdot,\cdot\rangle_{L^2(dx)} = G^{\dot{H}^1}$. Thus the space $\big(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\big)$ is a flat space in the sense of Riemannian geometry. There are explicit formulas for geodesics, geodesic distance, and geodesic splines, even for more restrictive spaces $\mathcal{A}_1$ like Denjoy-Carleman ultradifferentiable function classes. There are also soliton-like solutions. $(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), G^{\dot{H}^1})$ is geodesically convex, but not geodesically complete; the geodesic completion is the smooth semigroup $\mathrm{Mon}_{\mathcal{A}_1} = \{\varphi = \mathrm{Id} + f\ :\ f \in \mathcal{A}_1(\mathbb{R})\ ,\ f' \geq -1\}$. Any geodesic can hit the subgroup $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \subset \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ at most twice.

5 Robust weak Riemannian manifolds and Riemannian submersions

Another problem arises if we want to consider Riemannian submersions, in particular shape spaces as orbits of diffeomorphism groups, as explained in [43].

5.1 Robust weak Riemannian manifolds

Some constructions may lead to vector fields whose values do not lie in T_xM, but in the Hilbert space completion $\overline{T_xM}$ with respect to the weak inner product g_x. We need that $\bigcup_{x\in M}\overline{T_xM}$ forms a smooth vector bundle over M. In a coordinate chart on open $U \subset M$, $TM|_U$ is a trivial bundle $U \times V$ and all the inner products $g_x, x \in U$ define inner products on the same topological vector space V. They all should be bounded with respect to each other, so that the completion $\overline{V}$ of V with respect to g_x does not depend on x and $\bigcup_{x\in U}\overline{T_xM} \cong U \times \overline{V}$. This means that $\bigcup_{x\in M}\overline{T_xM}$ forms a smooth vector bundle over M with trivialisations the linear extensions of the trivialisations of the bundle $TM \to M$. Chart changes should respect this. This is a compatibility property between the weak Riemannian metric and some smooth atlas of M.

Definition A convenient weak Riemannian manifold (M, g) will be called a *robust* Riemannian manifold if

- The Levi-Civita convariant derivative of the metric g exists: The symmetric gradients should exist and be smooth.
- The completions $\overline{T_xM}$ form a smooth vector bundle as above.

Theorem 5.1. *If a right invariant weak Riemannian metric on a regular Lie group admits the Levi-Civita covariant derivative, then it is already robust.*

Proof. By right invariance, each right translation $T\mu^g$ extends to an isometric isomorphims $\overline{T_xG} \to \overline{T_{xg}G}$. By the uniform boundedness theorem these isomorphisms depend smoothly on $g \in G$. $\square$

5.2 Covariant curvature and O'Neill's formula

In [43, 2.2] one finds the following formula for the numerator of sectional curvature, which is valid for *closed smooth* 1-forms $\alpha, \beta \in \Omega^1_g(M)$ on a weak Riemannianian manifold (M, g). Recall that we view $g : TM \to T^*M$ and so g^{-1} is the dual inner product on $g(TM)$ and $\alpha^\sharp = g^{-1}(\alpha)$.

$$
\begin{aligned}
&g\big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\big) = \\
&-\tfrac12\alpha^\sharp\alpha^\sharp(\|\beta\|^2_{g^{-1}}) - \tfrac12\beta^\sharp\beta^\sharp(\|\alpha\|^2_{g^{-1}}) + \tfrac12(\alpha^\sharp\beta^\sharp+\beta^\sharp\alpha^\sharp)g^{-1}(\alpha,\beta)\\
&\qquad\qquad\big(\text{last line } = -\alpha^\sharp\beta([\alpha^\sharp,\beta^\sharp]) + \beta^\sharp\alpha([\alpha^\sharp,\beta^\sharp]])\big)\\
&-\tfrac14\|d(g^{-1}(\alpha,\beta))\|^2_{g^{-1}} + \tfrac14 g^{-1}\big(d(\|\alpha\|^2_{g^{-1}}), d(\|\beta\|^2_{g^{-1}})\big)\\
&+\tfrac34\big\|[\alpha^\sharp,\beta^\sharp]\big\|^2_g
\end{aligned}
$$

This is called Mario's formula since Mario Micheli derived the coordinate version in his 2008 thesis. Each term depends only on g^{-1} with the exception of the last term. The role of the last term (which we call the O'Neill term) will become clear in the next result. Let $p : (E, g_E) \to (B, g_B)$ be a Riemannian submersion between infinite dimensional robust Riemann manifolds; i.e., for each $b \in B$ and $x \in E_b := p^{-1}(b)$ the tangent mapping $T_xp : (T_xE, g_E) \to (T_bB, g_B)$ is a surjective metric quotient map so that

$$\|\xi_b\|_{g_B} := \inf\big\{\|X_x\|_{g_E} \ :\ X_x \in T_xE, T_xp.X_x = \xi_b\big\}.$$

The infinimum need not be attained in T_xE but will be in the completion $\overline{T_xE}$. The orthogonal subspace $\{Y_x : g_E(Y_x, T_x(E_b)) = 0\}$ will therefore be taken in $\overline{T_x(E_b)}$ in T_xE. If $\alpha_b = g_B(\alpha_b^\sharp,\quad) \in g_B(T_bB) \subset T_b^*B$ is an element in the g_B-smooth dual, then $p^*\alpha_b := (T_xp)^*(\alpha_b) = g_B(\alpha_b^\sharp, T_xp\quad) : T_xE \to \mathbb{R}$ is in T_x^*E but in general it is not an element in the smooth dual $g_E(T_xE)$. It is, however, an element of the Hilbert space completion $\overline{g_E(T_xE)}$ of the g_E-smooth dual $g_E(T_xE)$ with respect to the norm $\|\quad\|_{g_E^{-1}}$, and the element $g_E^{-1}(p^*\alpha_b) =: (p^*\alpha_b)^\sharp$ is in the $\|\quad\|_{g_E}$-completion $\overline{T_xE}$ of T_xE. We can call $g_E^{-1}(p^*\alpha_b) =: (p^*\alpha_b)^\sharp$ the *horizontal lift* of $\alpha_b^\sharp = g_B^{-1}(\alpha_b) \in T_bB$.

Theorem 5.2. [43, 2.6] *Let $p : (E, g_E) \to (B, g_B)$ be a Riemann submersion between infinite dimensional robust Riemann manifolds. Then for* **closed** *1-forms $\alpha, \beta \in \Omega^1_{g_B}(B)$ O'Neill's formula holds in the form:*

$$
\begin{aligned}
g_B\big(R^B(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big) &= g_E\big(R^E((p^*\alpha)^\sharp,(p^*\beta)^\sharp)(p^*\beta)^\sharp,(p^*\alpha)^\sharp\big)\\
&\quad + \tfrac34\|[(p^*\alpha)^\sharp,(p^*\beta)^\sharp]^{ver}\|^2_{g_E}
\end{aligned}
$$

Proof. The last (O'Neill) term is the difference between curvature on E and the pullback of the curvature on B. □

5.3 Semilocal version of Mario's formula, force, and stress

In all interesting examples of orbits of diffeomorphisms groups through a template shape, Mario's covariant curvature formula leads to complicated and impenetrable formulas. Efforts to break this down to comprehensible pieces led to the concepts of symmetrized force and (shape-) stress explained below. Since acceleration sits in the second tangent bundle, one either needs a covariant derivative to map it down to the tangent bundle, or at least rudiments of local charts. In [43] we managed the local version. Interpretations in mechanics or elasticity theory are still lacking.

Let (M, g) be a robust Riemannian manifold, $x \in M$, $\alpha, \beta \in g_x(T_xM)$. Assume we are given local smooth vector fields X_α and X_β such that:

1. $X_\alpha(x) = \alpha^\sharp(x), \quad X_\beta(x) = \beta^\sharp(x)$,
2. Then $\alpha^\sharp - X_\alpha$ is zero at x hence has a well defined derivative $D_x(\alpha^\sharp - X_\alpha)$ lying in $\operatorname{Hom}(T_xM, T_xM)$. For a vector field Y we have $D_x(\alpha^\sharp - X_\alpha).Y_x = [Y, \alpha^\sharp - X_\alpha](x) = \mathcal{L}_Y(\alpha^\sharp - X_\alpha)|_x$. The same holds for β.
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$,
4. $[X_\alpha, X_\beta] = 0$.

Locally constant 1-forms and vector fields will do. We then define:

$$\begin{aligned}
\mathcal{F}(\alpha,\beta) &:= \tfrac{1}{2} d(g^{-1}(\alpha,\beta)), \qquad \text{a 1-form on } M \text{ called the } \textit{force},\\
\mathcal{D}(\alpha,\beta)(x) &:= D_x(\beta^\sharp - X_\beta).\alpha^\sharp(x)\\
&= d(\beta^\sharp - X_\beta).\alpha^\sharp(x), \quad \in T_xM \text{ called the } \textit{stress.}\\
&\Longrightarrow \mathcal{D}(\alpha,\beta)(x) - \mathcal{D}(\beta,\alpha)(x) = [\alpha^\sharp, \beta^\sharp](x)
\end{aligned}$$

Then in terms of force and stress the numerator of sectional curvature looks as follows:

$$\begin{aligned}
&g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)(x) = R_{11} + R_{12} + R_2 + R_3\,, \qquad \text{where}\\
&\quad R_{11} = \tfrac{1}{2}\big(\mathcal{L}^2_{X_\alpha}(g^{-1})(\beta,\beta) - 2\mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}(g^{-1})(\alpha,\beta) + \mathcal{L}^2_{X_\beta}(g^{-1})(\alpha,\alpha)\big)(x)\,,\\
&\quad R_{12} = \langle \mathcal{F}(\alpha,\alpha), \mathcal{D}(\beta,\beta)\rangle + \langle \mathcal{F}(\beta,\beta), \mathcal{D}(\alpha,\alpha)\rangle - \langle \mathcal{F}(\alpha,\beta), \mathcal{D}(\alpha,\beta) + \mathcal{D}(\beta,\alpha)\rangle\,,\\
&\quad R_2 = \big(\|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} - \big\langle \mathcal{F}(\alpha,\alpha)), \mathcal{F}(\beta,\beta)\big\rangle_{g^{-1}}\big)(x)\,,\\
&\quad R_3 = -\tfrac{3}{4}\|\mathcal{D}(\alpha,\beta) - \mathcal{D}(\beta,\alpha)\|^2_{g_x}\,.
\end{aligned}$$

5.4 Landmark space as homogeneus space of solitons

This subsection is based on [42]; the method explained here has many applications in computational anatomy and elswhere, under the name LDDMM (large diffeomorphic deformation metric matching).

A *landmark* $q = (q_1, \dots, q_N)$ is an N-tuple of distinct points in $\mathbb{R}^n$; landmark space $\operatorname{Land}^N(\mathbb{R}^n) \subset (\mathbb{R}^n)^N$ is open. Let $q^0 = (q^0_1, \dots, q^0_N)$ be a fixed standard template landmark. Then we have the surjective mapping

$$\begin{aligned}
&\operatorname{ev}_{q^0} : \operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n) \to \operatorname{Land}^N(\mathbb{R}^n),\\
&\varphi \mapsto \operatorname{ev}_{q^0}(\varphi) = \varphi(q^0) = (\varphi(q^0_1), \dots, \varphi(q^0_N)).
\end{aligned}$$

Given a Sobolev metric of order $s > \frac{n}{2} + 2$ on $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n)$, we want to induce a Riemannian metric on $\operatorname{Land}^N(\mathbb{R}^n)$ such that ev_{q_0} becomes a Riemannian submersion.

The fiber of ev_{q^0} over a landmark $q = \varphi_0(q^0)$ is

$$\begin{aligned}\{\varphi \in \operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n) : \varphi(q^0) = q\} &= \varphi_0 \circ \{\varphi \in \operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n) : \varphi(q^0) = q^0\} \\ &= \{\varphi \in \operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n) : \varphi(q) = q\} \circ \varphi_0 \,.\end{aligned}$$

The tangent space to the fiber is

$$\{X \circ \varphi_0 : X \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n), X(q_i) = 0 \text{ for all } i\}.$$

A tangent vector $Y \circ \varphi_0 \in T_{\varphi_0} \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is $G^L_{\varphi_0}$-perpendicular to the fiber over q if and only if

$$\int_{\mathbb{R}^n} \langle LY, X \rangle \, dx = 0 \quad \forall X \text{ with } X(q) = 0.$$

If we require Y to be smooth then $Y = 0$. So we assume that $LY = \sum_i P_i.\delta_{q_i}$, a distributional vector field with support in q. Here $P_i \in T_{q_i}\mathbb{R}^n$. But then

$$\begin{aligned}Y(x) = L^{-1}\Big(\sum_i P_i.\delta_{q_i}\Big) &= \int_{\mathbb{R}^n} K(x-y)\sum_i P_i.\delta_{q_i}(y)\,dy = \sum_i K(x-q_i).P_i \,, \\ T_{\varphi_0}(\operatorname{ev}_{q^0}).(Y\circ\varphi_0) = Y(q_k)_k &= \sum_i (K(q_k - q_i).P_i)_k \,.\end{aligned}$$

Now let us consider a tangent vector $P = (P_k) \in T_q \operatorname{Land}^N(\mathbb{R}^n)$. Its horizontal lift with footpoint φ_0 is $P^{\text{hor}} \circ \varphi_0$ where the vector field P^{hor} on $\mathbb{R}^n$ is given as follows: Let $K^{-1}(q)_{ki}$ be the inverse of the $(N \times N)$-matrix $K(q)_{ij} = K(q_i - q_j)$. Then

$$\begin{aligned}P^{\text{hor}}(x) &= \sum_{i,j} K(x-q_i)K^{-1}(q)_{ij}P_j \,, \\ L(P^{\text{hor}}(x)) &= \sum_{i,j} \delta(x-q_i)K^{-1}(q)_{ij}P_j \,.\end{aligned}$$

Note that P^{hor} is a vector field of class H^{2l-1}.

The Riemannian metric on Land^N induced by the g^L-metric on $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is

$$\begin{aligned}g^L_q(P,Q) = G^L_{\varphi_0}(P^{\text{hor}}, Q^{\text{hor}}) &= \int_{\mathbb{R}^n} \langle L(P^{\text{hor}}), Q^{\text{hor}} \rangle \, dx \\ &= \int_{\mathbb{R}^n} \Big\langle \sum_{i,j} \delta(x-q_i)K^{-1}(q)_{ij}P_j, \sum_{k,l} K(x-q_k)K^{-1}(q)_{kl}Q_l \Big\rangle \, dx \\ &= \sum_{i,j,k,l} K^{-1}(q)_{ij}K(q_i-q_k)K^{-1}(q)_{kl}\langle P_j, Q_l\rangle \\ g^L_q(P,Q) &= \sum_{k,l} K^{-1}(q)_{kl}\langle P_k, Q_l\rangle.\end{aligned}$$

The *geodesic equation* in vector form is:

$$\ddot{q}_n = -\frac{1}{2}\sum_{k,i,j,l} K^{-1}(q)_{ki}\operatorname{grad} K(q_i-q_j)(K(q)_{in}-K(q)_{jn})K^{-1}(q)_{jl}\langle \dot{q}_k,\dot{q}_l\rangle$$
$$+\sum_{k,i} K^{-1}(q)_{ki}\Big\langle \operatorname{grad} K(q_i-q_n), \dot{q}_i-\dot{q}_n\Big\rangle \dot{q}_k\,.$$

The cotangent bundle $T^*\mathrm{Land}^N(\mathbb{R}^n) = \mathrm{Land}^N(\mathbb{R}^n)\times((\mathbb{R}^n)^N)^* \ni (q,\alpha)$. We shall treat $\mathbb{R}^n$ like scalars; $\langle\ \ ,\ \ \rangle$ is always the standard inner product on $\mathbb{R}^n$.
The metric looks like

$$(g^L)^{-1}_q(\alpha,\beta) = \sum_{i,j} K(q)_{ij}\langle \alpha_i,\beta_j\rangle, \qquad K(q)_{ij} = K(q_i-q_j).$$

The energy function is

$$E(q,\alpha) = \tfrac{1}{2}(g^L)^{-1}_q(\alpha,\alpha) = \tfrac{1}{2}\sum_{i,j} K(q)_{ij}\langle \alpha_i,\alpha_j\rangle$$

and its Hamiltonian vector field (using $\mathbb{R}^n$-valued derivatives to save notation) is

$$H_E(q,\alpha) = \sum_{i,k=1}^{N}\Big(K(q_k-q_i)\alpha_i\frac{\partial}{\partial q_k} + \operatorname{grad} K(q_i-q_k)\langle \alpha_i,\alpha_k\rangle\frac{\partial}{\partial \alpha_k}\Big).$$

So the *Hamitonian version of the geodesic equation* is the flow of this vector field:

$$\begin{cases} \dot{q}_k &= \sum_i K(q_i-q_k)\alpha_i \\ \dot{\alpha}_k &= -\sum_i \operatorname{grad} K(q_i-q_k)\langle \alpha_i,\alpha_k\rangle \end{cases}$$

We shall use *stress and force* to express the geodesic equation and curvature:

$$\alpha_k^\sharp = \sum_i K(q_k-q_i)\alpha_i, \quad \alpha^\sharp = \sum_{i,k} K(q_k-q_i)\langle \alpha_i, \tfrac{\partial}{\partial q^k}\rangle$$
$$\mathcal{D}(\alpha,\beta) := \sum_{i,j} dK(q_i-q_j)(\alpha_i^\sharp - \alpha_j^\sharp)\Big\langle \beta_j, \frac{\partial}{\partial q_i}\Big\rangle, \quad \text{the } \textit{stress}.$$
$$\mathcal{D}(\alpha,\beta)-\mathcal{D}(\beta,\alpha) = (D_{\alpha^\sharp}\beta^\sharp) - D_{\beta^\sharp}\alpha^\sharp = [\alpha^\sharp,\beta^\sharp], \quad \textit{Lie bracket.}$$
$$\mathcal{F}_i(\alpha,\beta) = \frac{1}{2}\sum_k \operatorname{grad} K(q_i-q_k)(\langle \alpha_i,\beta_k\rangle + \langle \beta_i,\alpha_k\rangle)$$
$$\mathcal{F}(\alpha,\beta) := \sum_i \langle \mathcal{F}_i(\alpha,\beta), dq_i\rangle = \frac{1}{2}\,d\,g^{-1}(\alpha,\beta) \quad \text{the } \textit{force}.$$

The geodesic equation on $T^*\,\mathrm{Land}^N(\mathbb{R}^n)$ then becomes

$$\begin{cases} \dot{q} &= \alpha^\sharp \\ \dot{\alpha} &= -\mathcal{F}(\alpha,\alpha)\,. \end{cases}$$

Next we shall compute *curvature via the cotangent bundle.* From the semilocal version of Mario's formula for the numerator of the sectional curvature for constant 1-forms α, β on landmark space, where $\alpha^\sharp_k = \sum_i K(q_k - q_i)\alpha_i$, we get directly:

$$
\begin{aligned}
&g^L\big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\big) = \\
&= \big\langle \mathcal{D}(\alpha,\beta) + \mathcal{D}(\beta,\alpha), \mathcal{F}(\alpha,\beta)\big\rangle \\
&\quad - \big\langle \mathcal{D}(\alpha,\alpha), \mathcal{F}(\beta,\beta)\big\rangle - \big\langle \mathcal{D}(\beta,\beta), \mathcal{F}(\alpha,\alpha)\big\rangle \\
&\quad - \tfrac{1}{2}\sum_{i,j}\Big(d^2K(q_i-q_j)(\beta^\sharp_i-\beta^\sharp_j, \beta^\sharp_i-\beta^\sharp_j)\langle\alpha_i,\alpha_j\rangle \\
&\qquad\qquad - 2d^2K(q_i-q_j)(\beta^\sharp_i-\beta^\sharp_j, \alpha^\sharp_i-\alpha^\sharp_j)\langle\beta_i,\alpha_j\rangle \\
&\qquad\qquad + d^2K(q_i-q_j)(\alpha^\sharp_i-\alpha^\sharp_j, \alpha^\sharp_i-\alpha^\sharp_j)\langle\beta_i,\beta_j\rangle\Big) \\
&\quad - \|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} + g^{-1}\big(\mathcal{F}(\alpha,\alpha),\mathcal{F}(\beta,\beta)\big). \\
&\quad + \tfrac{3}{4}\|[\alpha^\sharp,\beta^\sharp]\|^2_g
\end{aligned}
$$

5.5 Shape spaces of submanifolds as homogeneous spaces for the diffeomorphism group

Let M be a compact manifold and $(N,\bar g)$ a Riemannian manifold of bounded geometry as in subsection 4.5. The diffeomorphism group $\operatorname{Diff}_{\mathcal{A}}(N)$ acts also from the left on the manifold of $\operatorname{Emb}(M,N)$ embeddings and also on the *non-linear Grassmannian* or *differentiable Chow variety* $B(M,N) = \operatorname{Emb}(M,N)/\operatorname{Diff}(M)$. For a Sobolev metric of order $s > \frac{\dim(N)}{2} + 2$ one can then again induce a Riemannian metric on each $\operatorname{Diff}_{\mathcal{A}}(N)$-orbit, as we did above for landmark spaces. This is done in [43], where the geodesic equation is computed and where curvature is described in terms of stress and force.

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