Peter Michor

BANACH MODULES AND FUNCTORS ON CATEGORIES OF BANACH SPACES

PURE AND APPLIED MATHEMATICS

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BANACH MODULES AND FUNCTORS ON CATEGORIES OF BANACH SPACES

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Preface

This book is the final outgrowth of a sequence of seminars about functors on categories of Banach spaces (held in the years 1971 - 1975) and several doctoral dissertations prepared during this period under the supervision of J. Cigler. In the summer term of 1974, a lecture course was given by J. Cigler (CIGLER [11]), the notes of which were the basis for this book. It has been written for readers with a general background in functional analysis. The requirements for category theory are modest and the necessary concepts are developed in the course of the book as the need for them arises. Some familiarity with categorical thinking, however, would be useful. A convenient reference is MAC LANE [49].

We would like to specify the main authors of each chapter: I: Cigler, Michor; II: Losert; III: Cigler; IV: Michor; V: Losert, Michor; VI: Losert, Michor.

The first three chapters are an exposition of more or less well-known material in categorical setting; the first chapter is devoted to identify basic categorical notions in the category Ban of Banach spaces and to translate some useful information of functional analysis into categorical terms. The second section contains an exposition of the Waelbroeck-Buchwalter theory, which describes the dual category of Ban. The second chapter is devoted to various tensor products of Banach spaces and to the approximation property. The third chapter treats Banach modules over a Banach algebra with approximate identities in such a way as to exhibit the analogy with the theory of functors on categories of Banach spaces. Chapter IV gives first some basic facts and constructions for functors on categories of Banach spaces (which in most cases have no direct counterpart in general category theory) and then a discussion of the tensor product of functors. The next chapter is devoted to duality of functors: first all possible duality theories (satisfying certain axioms) are characterized, then the duality notion of MITIAGIN-SHVARTS [57] is systematically treated and finally the conjecture [57] that any dual functor be reflexive is decided in the negative. In the sixth chapter various Kan extensions over subcategories of Ban are treated: this is a generalization of the theory of GROTHENDIECK [33], and here some intimate connections with the geometric theory of Banach spaces are brought to light. There are several loose ends leading to the theory of operator ideals (as in chapters IV and V too). A certain attempt to bridge this gap has been made in MICHOR [55].

We had planned to write a section on the historical development, but when the manuscript was finished we were too exhausted to do this. Hence, citations in the text give convenient references only, but rarely give credit to the original source.

Thanks are due to the participants of the forementioned seminars, to the audience of the lecture course, to James B. Cooper and M. Grosser who read the whole manuscript and to Brigitte Mühlegger and Silvia Aschan for the beautiful typescript.

Vienna, February 1977 J.C.

V.L.

P.M.

Contents

Introduction

Banach modules and functors on categories of Banach spaces may be interpreted as generalizations of Banach spaces, where the field of scalars R or C has been replaced by a Banach algebra or a category of Banach spaces. This book is an outgrowth of the attempt to carry over Banach space theory to this more general setting. In order to motivate our theory let us give a short survey of the basic ideas without going into exact definitions and details.

In the last decades, category theory has been developed into a powerful tool for comparing different theories and studying their structure. From this point of view the most useful constructions with Banach spaces are limits and colimits (including sums, products, quotients and subspaces) together with the (projective) tensor product $X \hat{\otimes} Y$ and the space of all bounded linear maps $H(X,Y)$. A central role is played by the remarkable formula

$$H(X \hat{\otimes} Y,Z) = H(Y,H(X,Z))$$

which is called the exponential law because it takes the form $(Z^X)^Y = Z^{X \hat{\otimes} Y}$ if we set $H(X,Y) = Y^X$.

All this may be generalized to the case of Banach modules ([70]) if some care is taken in distinguishing between left, right and bimodules if the Banach algebra A is non-commutative. Suppose W is a right A-module and V is a left A-module.

Then the tensor product $W \otimes_A V$ is defined but, in general, has only a Banach space structure.
In the same way, for two left A-modules V_1 and V_2 the set of all A-module homomorphisms $H_A(V_1,V_2)$ is in general only a Banach space, in fact, a closed subspace of $H(V_1,V_2)$.
For right modules W_1,W_2 we denote the corresponding space by $H^A(W_1,W_2)$.
The exponential law can be generalized in the following form:
Let A and B be Banach algebras, X a left B-module, V a left A-module and Z a left A- and right B-module, then $Z \hat{\otimes}_B X$ is a left A-module, $H_A(Z,V)$ a left B-module and

$$H_A(Z \otimes_B X,V) = H_B(X,H_A(Z,V)) \ .$$

Let us now adopt the point of view already mentioned of considering left A-modules as "generalized" Banach spaces with the field of scalars I replaced by the Banach algebra A. Our aim is to extend important constructions for Banach spaces to this more general setting. This is rather straightforward and uninteresting if A has a unit element e. The formulas $I \ddot{\otimes} X = X$ and $H(I,X) = X$ for Banach spaces take the form $A \hat{\otimes}_A V = V$ and $H_A(A,V) = V$.
The dual space $X' = H(X,I) = H(X,I')$ generalizes to $H_A(V,A') = H_A(V,H(A,I)) = H(A \hat{\otimes}_A V,I) = H(V,I) = V'$ which coincides with the Banach space dual. For Banach algebras without unit element (which occur in most of the interesting examples) things are not so easy.
For simplicity let us assume that A has an approximate identity, i.e. a net (e_ι) of elements $e_\iota \in A$ such that $\|e_\iota\| \leq 1$ and $\lim_\iota e_\iota a = \lim_\iota a e_\iota = a$ for all $a \in A$.

Assume further that all modules V are strong, i.e. they satisfy $\sup_{\|a\|\leq 1} \|av\| = \|v\|$ for all $v \in V$.

Then we have $A \hat{\otimes}_A V \subseteq V \subseteq H_A(A,V)$.

We call $V_e = A \hat{\otimes}_A V$ the essential part of V and $\overline{V} = H_A(A,V)$ the A-completion of V. It turns out that $(V_e)_e = V_e$, $\overline{(\overline{V})} = \overline{V}$, $\overline{(V_e)} = \overline{V}$, $(\overline{V})_e = V_e$. If $V = V_e$ we call V essential and if $V = \overline{V}$ we call it A-complete.

The dual $V^o = H_A(V,A')$ is an A-complete right A-module and coincides with $V_e{}'$.

There is a natural inclusion $i:V \to V^o{}_o$ given by $i(v)(v^o) = v^o(v)$ for all $v^o \in V^o$.

For special cases some aspects of this situation are well known: For $A = c_o$ and V a normed ideal in l^∞, the dual V^o reduces to the Köthe-dual. For A the algebra of compact operators in a Hilbert space H we get the theory of symmetrically-normed ideals of Schatten [75] and Gohberg-Krein [29].

The case of right modules can be reduced to left modules. Consider the opposite Banach algebra A^{op} which has the same elements as A but where multiplication $a \circ b$ is defined by $a \circ b = ba$. Then W is a right A module if and only if W is a left A^{op}-module.

For bimodules the situation is somewhat more complicated. To get a satisfactory theory we must suppose that the bimodule U satisfies $A \hat{\otimes}_A U = U \hat{\otimes}_A A$ and this space again is called the essential part of the bimodule.

We define the A-completion $\overline{U}$ of U as the set of all $(\psi,\varphi) \in H^A(A,U) \times H_A(A,U)$ satisfying $a\psi(b) = \varphi(a)b$ for all a,b. Then $\overline{U}$ is a bimodule containing U via the imbedding $u \to (\psi_u,\varphi_u)$ where $\psi_u(a) = ua$ and $\varphi_u(a) = au$ and we have $\overline{(U_e)} = \overline{U}$, $(\overline{U})_e = U_e$, $\overline{\overline{U}} = \overline{U}$. The dual bimodule is again defined by $U^o = H_A(U,A') = U_e{}' = H^A(U,A')$. U^o is an A-complete bimodule.

In the case U = A the A-completion of A is well-known in the literature under the name "double centralizer algebra" $\Delta(A)$. The algebra structure in $\Delta(A)$ is defined by $(g_1,f_1)(g_2,f_2) = (g_1g_2,f_2f_1)$. Then $\Delta(A)$ is a Banach algebra with unit element $(1_A,1_A)$. It contains A as twosided ideal and is the "largest" algebra with this property.

Whereas most results on Banach modules we have mentioned are well known in the literature - albeit not from this point of view which makes them appear quite trivial - their analogues for functors on Banach spaces seem to have escaped attention so far. What sort of analogy exists between Banach modules and functors? It is of course a purely formal one: let F be a (covariant) functor from some full subcategory $\underline{K}$ of Ban into the category Ban of all Banach spaces. Then for every $v \in F(X)$ and every $a : X \to Y$, av defined as F(a)v satisfies $1_Xv = F(1_X)v = v$, $\|av\| = \|F(a)v\| \leq \|a\|\|v\|$ and $b(av) = F(b)(F(a)v) = F(ba)v = (ba)v$ for $b : Y \to Z$.

Now let F_1 and F_2 be two covariant functors from $\underline{K}$ to Ban and $\varphi : F_1 \to F_2$ a natural transformation. This means that $\varphi_Y(F_1(a)v) = F_2(a)\varphi_X(v)$ for $v \in F_1(X)$ and $a : X \to Y$. In the above notation, i.e. without indices and functor symbol, this reads as $\varphi(av) = a\varphi(v)$ which looks like the condition for a generalized module homomorphism. The underlying idea is to consider the Banach algebra as an algebra of endomorphisms of some Banach space, say A itself, and then "split" this space into several components, thus getting a category.

In the same way contravariant functors correspond to right modules and co-contra-variant functors to bimodules in the sense introduced above.

The interesting fact - which shows that this is a good analogy - is that all we have done for Banach modules can be done also for functors: For every contravariant functor $G : \underline{K} \to \text{Ban}$ and every covariant functor $F: \underline{K} \to \text{Ban}$ a Banach space $G \hat{\otimes}_{\underline{K}} F$ is defined which is called their tensor product, for two functors F_1, F_2 of the same variance the Banach space $\text{Nat}(F_1, F_2)$ of all natural transformations between them is defined and the exponential law holds in the following form: Let $\underline{K}$ and $\underline{L}$ be subcategories of Ban, $M : \underline{L} \times \underline{K} \to \text{Ban}$ a contra-covariant bifunctor, $F_1 : \underline{L} \to \text{Ban}$ and $F_2 : \underline{K} \to \text{Ban}$ covariant functors, then the equation $\text{Nat}_{\underline{K}}(M \hat{\otimes}_{\underline{L}} F_1, F_2) = \text{Nat}_{\underline{L}}(F_1, \text{Nat}_{\underline{K}}(M, F_2))$ holds.

If H denotes the bifunctor on $\underline{K} \times \underline{K}$ which assigns to each pair of spaces in $\underline{K}$ the space $H(X,Y)$ of all bounded linear maps then we have $\text{Nat}(H,F) = F$ and $H \hat{\otimes} F = F$ as expected. These equations are well known under the name "Yoneda-lemma" in category theory. One can define a dual functor by the formula $\text{Nat}(F,H')$ but this reduces to a triviality because $\text{Nat}(F,H') = (H \hat{\otimes} F)' = F'$.

The concept of essential functor and complete functor, however, does not carry over immediately - there are several concepts of essential functor, the simplest being the functor of type Σ, which is just a generalization of the reasonable tensor norms of Schatten and Grothendieck; but there are other notions connected with left Kan extensions from certain subcategories of Ban (in the case of the category of all finite-dimensional Banach spaces one gets the notion of $\otimes$-norm of Grothendieck, here called "computable functor"). Likewise any concept of complete functor is connected with a right Kan extension from a certain subcategory.

These two concepts can be beautifully formalized into notions that look like those in the theory of Banach modules if one introduces the notion of a Banach semicategory. It has the same relation to a category as a general Banach algebra to a Banach algebra with identity. It is useful to consider such Banach semicategories with approximate identities. A typical example is the semicategory whose objects are Hilbert spaces with morphisms all compact linear operators K(X,Y) between them. For infinite dimensional spaces K(X,X) does not contain the identity operator but there exist approximate identities, e.g. a net of finite dimensional orthoprojections converging strongly to the identity operator. Here $K \hat{\otimes} F = F_e$ where the essential part of the functor F coincides with the type Σ subfunctor in the sense of Mityagin-Shvarts, Nat(K,F) is a sort of completion of F and Nat(F,K') coincides essentially with the dual functor in the sense of Mityagin-Shvarts because $K(X,Y)' = X \hat{\otimes} Y'$.

For another example let $\underline{K}$ consist of one object, a Banach algebra A and choose as morphisms its elements, where composition is multiplication. A functor on this semicategory is nothing else than a left A-module.

We see therefore that the concept of semicategory allows - despite its seeming overgenerality - a useful theory which unifies Banach modules and functors.

We do not, however, develop the theory of Banach semicategories in this book in order to keep its size within bounds. A detailed exposition can be found in Michor [56].

Needless to say that the theory developed here gives some simplifications of existing theories and contains a number of concrete corollaries which may perhaps convince those who do not like such "general nonsense" that our theory belongs to mathematics and not to metaphysics.

CHAPTER I

The categories Ban and W

§ 1. The categories Ban_1 and Ban_∞

1.1. Let Ban denote the class of all Banach spaces over the one-dimensional space I (which may be $\mathbb{R}$ or $\mathbb{C}$). In order to avoid set-theoretical difficulties with some constructions we suppose that all Banach spaces are "small", i.e. belong to some given universe.

There are two important categories connected with Ban. One is the category Ban_∞ whose objects are the spaces in Ban and whose morphisms are the bounded linear maps between Banach spaces. This is an additive category but it has rather bad properties with respect to limits and colimits.
The other is the category Ban_1 with the same objects but where the morphisms consist only of all linear contractions (i.e. bounded linear maps φ satisfying $\|\varphi\| \leq 1$) between Banach spaces. It is this category we are mainly interested in. It has the advantage that in it all limits and colimits exist, a fact which is very important.

1.2. It would be rather tedious to state all results separately for Ban_1 and for Ban_∞. So we shall use the abbreviation "category Ban" to mean either Ban_1 or Ban_∞ if some statement holds for both categories. The set of all morphisms from X to Y in Ban_∞ coincides with the Banach space H(X,Y) of all bounded linear maps from X to Y, whereas the set of all morphisms Hom (X,Y)

in Ban_1 consists of the unit ball of $H(X,Y)$.
In particular, we have $\mathrm{Hom}\ (I,X) = OX$, where OX denotes the unit ball of X.
Our first task will be the translation of functional-analytic concepts into the language of category theory and vice versa. We shall always concentrate on typical aspects and not strive for maximum generality.

1.3. We recall that a morphism $u: X \to Y$ is called monomorphism (in short mono) if the equality $u \circ f = u \circ g$ for any morphisms $f,g: Z \to X$ is satisfied only if $f = g$.
Since $u \circ f = u \circ g$ is equivalent to $u \circ (\frac{f-g}{2}) = 0$ where $\frac{f-g}{2}$ is again a morphism, u is mono if and only if $u \circ f = 0$ for any morphism $f: Z \to X$ implies $f = 0$.

<u>Proposition:</u> A morphism $u: X \to Y$ in Ban is mono if and only if u is injective.

<u>Proof:</u> Every injective map u is mono since $u(f(z)) = 0$ for all $z \in Z$ implies $f(z) = 0$ for all $z \in Z$.
Now let u be mono and let $Z = \ker u = \{x \in X: u(x) = 0\}$ and $f: Z \to X$ be the inclusion map. Then $u \circ f = 0$ and therefore $f = 0$ which implies $Z = \ker u = (0)$, i.e. u is injective.

1.4. A morphism $u: X \to Y$ is epi if $f \circ u = 0$ for any $f: Y \to Z$ implies $f = 0$.

<u>Proposition:</u> A morphism $u: X \to Y$ in Ban is epi if and only if $u(X)$ is dense in Y.

<u>Proof:</u> If u has dense image in Y and $f \circ u = 0$, then f vanishes on the dense subset $u(X)$ of Y and, by continuity, $f \equiv 0$.

Now let $u\colon X \to Y$ be epi, let $\overline{u(X)}$ be the closure of $u(X)$ in Y and let $f\colon Y \to Y/\overline{u(X)}$ be the canonical projection. Then $f \circ u = 0$ and therefore $f = 0$, which means that $\overline{u(X)} = Y$.

1.5. A morphism $u\colon X \to Y$ is called invertible or an isomorphism if there exists a morphism $v\colon Y \to X$ such that $v \circ u = 1_X$ and $u \circ v = 1_Y$.

<u>Proposition</u>: The isomorphisms in Ban_∞ are the bounded linear maps which are bijective and the isomorphisms in Ban_1 are the surjective isometries.

<u>Proof</u>: This follows immediately from the open mapping theorem.

<u>Remark</u>: Every isomorphism in Ban is also a bimorphism, i.e. at the same time mono and epi. It is worth noting, that there are bimorphisms which are not isomorphisms, e.g. the canonical imbedding of l^1 into c_0.

In the following we shall sometimes identify two spaces X and Y if they are isomorphic, i.e. in Ban_1 if there exists an isometry between them, and in Ban_∞ if they can be mapped bijectively onto another by a morphism.

1.6. Each morphism $u\colon X \to Y$ in Ban_1 has a canonical decomposition

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ \pi \downarrow & & \uparrow \iota \\ X/u^{-1}(0) & \xrightarrow{\quad \tilde{u} \quad} & \overline{u(X)} \end{array}$$

Here $\pi = \operatorname{coim} u$ is the quotient map of X onto $X/u^{-1}(0)$ and $\iota = \operatorname{im} u$ is the isometric embedding of the subspace $\overline{u(X)}$ into Y.

Those morphisms u for which $\tilde{u}$ is an isomorphism are called strict morphisms.
Thus the strict monos are the isometric embeddings and the strict epis the quotient maps (modulo an isomorphism).

1.7. The strict monos and epis in Ban_1 can be characterized in categorical terms:

Definition: A morphism $u: X \to Y$ in Ban_1 is called an extreme monomorphism if $u = m \circ e$ with m mono and e epi implies that e is an isomorphism.

Proposition: A morphism u in Ban_1 is extremely mono if and only if it is strictly mono.

Proof: Let $u: X \to Y$ be strictly mono, i.e. an isometric embedding and let $u = m \circ e$. Then
$\|x\| = \|u(x)\| = \|m(e(x))\| \leq \|e(x)\| \leq \|x\|$ for all $x \in X$, i.e. e is an isometry with dense image and therefore an isomorphism.

On the other hand let u be extremely mono and $u = \iota \circ \tilde{u} \circ \pi$ the canonical decomposition.
Then $u = (\iota \circ \tilde{u}) \circ \pi$ and therefore π is an isometric isomorphism, i.e. the identity on X. Thus $u = \iota \circ \tilde{u}$, which implies that $\tilde{u}$ is an isomorphism, i.e. that u is strict.

A morphism $u: X \to Y$ in Ban_1 is called an extreme epimorphism, if $u = m \circ e$ with m mono and e epi implies that m is an isomorphism.

Proposition: A morphism u in Ban_1 is extremely epi if and only if it is strictly epi.

Proof: Let u: $X \to Y$ be strictly epi, i.e. a quotient map and let $u = m \circ e$. Let $Z = u^{-1}(0)$.

Then

$$\inf_{z \in Z} \|x+z\| = \|u(x)\| = \|m(e(x))\| \leq \|e(x)\| = \|e(x+z)\| \leq \|x+z\|$$

for all $z \in Z$.

Thus $\|m(e(x))\| = \|e(x)\|$ for all x, i.e. m is an isometry which is surjective since u is surjective.

On the other hand let u be extremely epi and $u = \iota \circ u \circ \pi$ the canonical decomposition.

Then as above ι is an isomorphism, i.e. $\overline{u(X)} = Y$.

Therefore $u = \tilde{u} \circ \pi$, which implies that u is an isomorphism, which means that u is strict.

1.8. Theorem (Banach-Schauder): A morphism u: $X \to Y$ in Ban_1 is strictly epi if and only if $u(O\,X)$ is dense in $O\,Y$.

Proof: First we show that u is surjective if $u(O\,X)$ is dense in $O\,Y$.

For every $n = 1, 2, \ldots$ the set $u(\frac{1}{2^n}\,O\,X)$ is dense in $\frac{1}{2^n}\,O\,Y$. Thus for $y \in O\,Y$ there is an x_1 with $\|x_1\| < 1$ and $\|y - u(x_1)\| < \frac{1}{2}$. For $y - u(x_1)$ there is an x_2 with $\|x_2\| < \frac{1}{2}$ and $\|y - u(x_1) - u(x_2)\| < \frac{1}{2^2}$, etc.

Let $x = \sum_{n=1}^{\infty} x_n$. Then $\sum_1^{\infty} \|x_n\| < 2$ and so $\|x\| < 2$. Furthermore $u(x) = y$. Thus u is surjective.

Now let $u = \tilde{u} \circ \pi$ be the canonical decomposition. Since $\tilde{u}$ is surjective it has an inverse $\tilde{v}$ in Ban_{∞}.

Moreover

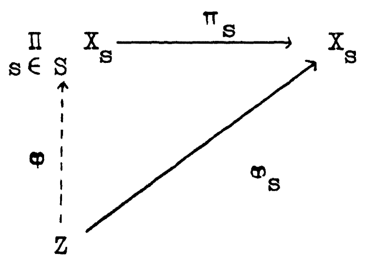
$$\|\tilde{v}\| = \sup_{\|y\| \leq 1} \|\tilde{v}(y)\| = \sup_{\|x\| < 1} \|\tilde{v}(u(x))\| = \sup_{\|x\| \leq 1} \|\pi(x)\| = 1.$$

This implies that $\tilde{u}$ is an isomorphism in Ban_1 and therefore u is strictly epi.

1.9. Products: Let $(X_s)_{s\in S}$ be a family of Banach spaces, where S is an arbitrary index set. By definition the product of this family in Ban_1 - if it exists - is a Banach space $\prod_{s\in S} X_s$ together with a family $(\pi_s)_{s\in S}$ of morphisms $\pi_s\colon \prod_{s\in S} X_s \to X_s$, such that for each Banach space Z and each family $(\varphi_s)_{s\in S}$ of morphisms $\varphi_s\colon Z \to X_s$, there is a unique morphism

$$\varphi = \prod_{s\in S} \varphi_s\colon Z \to \prod_{s\in S} X_s$$

such that the diagram



commutes.

Proposition: For each family $(X_s)_{s\in S}$ the product $\prod_{s\in S} X_s$ exists in Ban_1 and is (isometrically isomorphic to) the space of all elements $x = (x_s)_{s\in S}$, $x_s \in X_s$, such that

$$\|x\|_\infty = \sup_s \|x_s\|_{X_s} < \infty.$$

The morphisms π_s are the projections $\pi_s(x) = x_s$ onto the s'th coordinate. Each π_s is a quotient map.

<u>Proof</u>: Let $\varphi_s : Z \to X_s$ be a family of morphisms.

If there exists $\varphi : Z \to \Pi X_s$ with the stated properties, we must have $\pi_s(\varphi(z)) = \varphi_s(z)$, i.e. $\varphi(z) = (\varphi_s(z))$. But it is clear that if we define φ by this formula, then it satisfies all requirements. Moreover we have

$$\|\varphi\| = \sup_s \|\varphi_s\| .$$

For the special case $X_s \equiv X$ we get $\prod_{s \in S} X_s = l_S^{\infty}(X)$.

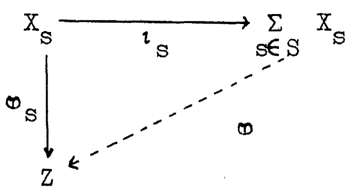
<u>Remark</u>: Ban_∞ has the undesirable property that infinite products do not exist. For suppose $(X, \pi_s)_{s \in S}$, S infinite, were a product of the family $(X_s)_{s \in S}$. Then for each family of morphisms $\varphi_s : Z \to X_s$ there would exist $\varphi : Z \to X$ such that $\varphi_s = \pi_s \circ \varphi$ and therefore $\|\varphi_s\| \leq \|\varphi\| \, \|\pi_s\|$. If the family φ_s contains a sequence φ_{s_n} such that $\|\varphi_{s_n}\| > n \, \|\pi_{s_n}\|$ this is clearly impossible.

1.10. <u>Coproducts</u> (or sums) are defined dually.

The coproduct in Ban_1 of a family $(X_s)_{s \in S}$ of Banach spaces X_s is a Banach space $\sum_{s \in S} X_s$ together with a family $(\iota_s)_{s \in S}$ of morphisms $\iota_s : X_s \to \sum_{s \in S} X_s$, such that for each family $(\varphi_s)_{s \in S}$ of morphisms

$\varphi_s : X_s \to Z$ there is a unique morphism

$\varphi : \sum_{s \in S} X_s \to Z$ such that the diagram



commutes.

Proposition: For each family $(X_s)_{s\in S}$ the coproduct $\sum_{s\in S} X_s$ exists in Ban_1 and is (isometrically isomorphic to) the space of all elements $x=(x_s)_{s\in S}$ such that

$$\|x\|_1 = \sum_{s\in S} \|x_s\|_{X_s} < \infty.$$

The morphisms ι_s are the injections $\iota_s(x_s) = (\delta_s^t x_s)_{t\in S}$ of X_s into X. Each ι_s is an isometric embedding.

Proof: Let $\varphi_s: X_s \to Z$ be a family of morphisms. If there exists $\varphi: \sum_{s\in S} X_s \to Z$ with the stated properties then it must satisfy $\varphi(\iota_s(x_s)) = \varphi_s(x_s)$, i.e. $\varphi((x_s)) = \sum_{s\in S} \varphi_s(x_s)$.

On the other hand if we define φ by this formula we have $\|\varphi((x_s))\| = \|\sum_{s\in S} \varphi_s(x_s)\| \le \sum_{s\in S} \|\varphi_s\| \, \|x_s\|_{X_s} \le \sum_{s\in S} \|x_s\|_{X_s} = \|x\|_1$ and the diagram commutes.

For the special case $X_s \equiv X$ we get $\sum_{s\in S} X_s = l_S^1(X)$.

Remark: If S is finite the products and sums in Ban_1 are also products and sums in Ban_∞ and are isomorphic in Ban_∞. In the category Ban_1 they are of course not isomorphic because they carry different norms.

The same reasoning as above shows that for an infinite index set S there are no sums in Ban_∞. If one prefers to work in Ban_∞ instead of in Ban_1 then one can define sums and products for bounded families of morphisms φ_s, i.e. for families satisfying $\sup_{s\in S} \|\varphi_s\| < \infty$. One then gets in fact only a reformulation of the result for Ban_1.

1.11. Proposition: Every Banach space X may be written as a quotient space of a space $l^1_S(I)$ and as a closed subspace of a space $l^\infty_T(I)$ for some index sets S and T.

Proof: Let $\{x_s\}$ be a dense subset of the unit ball OX of X and $\pi: l^1_S(I) \to X$ be defined by

$$\pi((\xi_s)) = \sum_{s \in S} \xi_s x_s \text{ for } (\xi_s) \in l^1_S(I).$$

Then $\Sigma \|\xi_s x_s\| \le \Sigma |\xi_s| = \|\xi\|_1$ and $\pi(O\, l^1_S(I))$ is dense in OX. The first assertion follows by the theorem of Banach-Schauder.

For the second assertion let T be a w*-dense subset of OX' and define

$$X \to l^\infty_T(I) \text{ by } x \to (\langle x,x'\rangle)_{x' \in T}.$$

Then

$$\|x\| = \sup_{x' \in T} |\langle x,x'\rangle|.$$

Remark 1: It is well known that both assertions may be refined with some care to show that every separable Banach space is a quotient of l^1_N and a subspace of l^∞_N. This shows that l^1_N and l^∞_N are "large" spaces, whereas l^2_N is a "small" space, since all quotients and subspaces are again separable Hilbert spaces.

Remark 2: The first assertion has a very simple interpretation in terms of category theory:

Consider the "forgetful functor" O: $Ban_1 \to Set$ which associates to each Banach space X its unit ball OX (considered as a set) and to each morphism f: $X \to Y$ its restriction $Of = f|_{OX}$.

Since $\|f\| \leq 1$ this is well-defined.
This functor has a left adjoint l^1: Set $\to$ Ban_1, which associates with each set S the Banach space l^1_S and with each map φ: S $\to$ T the morphism

$$l^1_\varphi : l^1_S \to l^1_T$$

defined by

$$l^1_\varphi\,((\xi_s))_t = \sum_{\varphi(s)=1} \xi_s .$$

The adjunction

(*) $$Ban_1(l^1_S, X) = Set\ (S, OX)$$

is given by $\hat{\varphi} \leftrightarrow \varphi$ with

$$\varphi(s) = \hat{\varphi}\ ((\delta^t_s)_{t\in S}) \text{ and } \hat{\varphi}\ ((\xi_s)) = \sum_{s\in S} \xi_s\ \varphi(s).$$

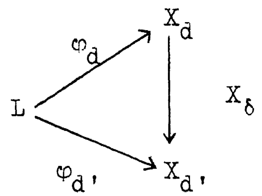
It is easy to verify that (*) is natural in S and X. If we "identify" $s\in S$ and $(\delta^t_s)_{t\in S} \in l^1_S$, then φ is simply the restriction of $\hat{\varphi}$ to S and $\hat{\varphi}$ is the (uniquely determined) continuous linear extension of φ to l^1_S.

The unit of the adjunction (*) (which corresponds to the identity $1_{l^1_S}$ on the left side) is simply this "identification". The counit (corresponding to the identity 1_{OX} on the right) is the quotient map

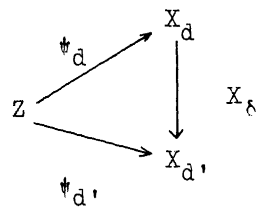
$$\pi_X : l^1_{OX} \to X \text{ given by } \pi_X\ ((\xi_x)_{x\in OX}) = \sum_{x\in OX} \xi_x \cdot x.$$

1.12. Limits: Let D be a (small) category and $X: D \to Ban_1$ a functor. We shall call D an index category and the family of Banach spaces $(X_d)_{d\in D}$ together with the family of morphisms $X_\delta : X_d \to X_{d'}$, for $\delta: d \to d'$ a spectral family.

A Banach space L together with a family $(\varphi_d)_{d\in D}$ of morphisms φ_d: $L \to X_d$ is said to be a (projective) <u>limit</u> of the functor X (or the spectral family (X_d)), if for all $d, d' \in D$ and δ: $d \to d'$ the diagrams



commute and if for all Banach spaces Z and all families $(\psi_d)_{d\in D}$ of morphisms ψ_d: $Z \to X_d$ such that the corresponding diagrams



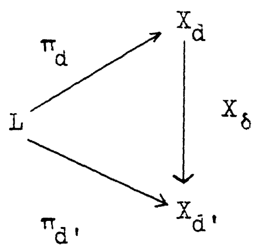
commute, there is a unique morphism $\psi = \underleftarrow{\lim}\, \psi_d$: $Z \to L$, such that $\psi_d = \varphi_d \circ \psi$ for all $d \in D$.
Of course L is uniquely determined (up to an isometric isomorphism) and is denoted by $L = \underleftarrow{\lim}\, X_d$.

<u>Proposition:</u> Let D be a small category. Then for every spectral family $(X_d)_{d\in D}$ the limit $\underleftarrow{\lim}\, X_d$ exists in Ban_1 and coincides with the closed subspace L of $\prod_{d\in D} X_d$ consisting of all $x = (x_d)_{d\in D}$ such that $X_\delta x_d = x_{d'}$, for all $d, d' \in D$ and δ: $d \to d'$.

Proof: L defined as above is a closed subspace because it may be written in the form

$$L = \bigcap_{\substack{d,d' \\ \delta}} \ker (X_\delta \circ \pi_d - \pi_{d'}).$$

Furthermore the diagrams



commute by definition.

For a given family ψ_d: $Z \to X_d$ of morphisms satisfying $X_\delta \circ \psi_d = \psi_{d'}$ for all δ: $d \to d'$ we consider the map $\Pi\psi_d$: $Z \to \Pi X_d$. It is clear that its image belongs to L and so it can be uniquely factored over L with the stated properties.

1.13. Example: Let D be the category $\overset{\to}{\to}$ consisting of two objects and two different morphisms between them. A functor from D to Ban_1 is a family $\{f,g\}$ consisting of two morphisms f,g: $X \to Y$. The limit L of this spectral family is a Banach space L together with two maps $L \to X$ and $L \to Y$ satisfying the following properties:

1) $L \to Y = L \to X \xrightarrow{f} Y = L \to X \xrightarrow{g} Y$

2) For every Banach space Z and morphisms $Z \to X$, $Z \to Y$ with $Z \to Y = Z \to X \xrightarrow{f} Y = Z \to X \xrightarrow{g} Y$ there exists a unique morphism $Z \to L$ such that $Z \to L \to X = Z \to X$ and $Z \to L \to Y = Z \to Y$.

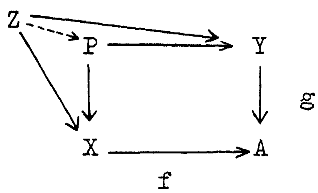
Proposition 1.12. says that L is the subspace of the product $X \times Y$ consisting of all pairs (x,y) such that $y = f(x) = g(x)$, i.e. $(f - g)(x) = 0$.

We call L the difference-kernel of f and g, $L = \ker(f - g)$. It may be identified with the subspace of X consisting of all x satisfying $(f - g)(x) = 0$.

In the special case $g = 0$ we get the kernel ker f of a morphism f. $L = \ker f \to X$ is the isometric injection. On the other hand every isometric embedding of a subspace M of X into X may be interpreted as the kernel of some map, e.g. of the map $X \to X/M$.

Thus the strict monos in Ban_1 are special examples of limits.

1.14. <u>Pullbacks:</u> A pullback is a limit of a spectral family indexed by the category $D = (\bullet \to \circ \leftarrow \circ)$. The universal property of $P = \varprojlim (X \xrightarrow{f} A \xleftarrow{g} Y)$ is visualized by the diagram



By 1.12. P is the subspace of the product $X \times Y \times A$ consisting of all $(x,y,\ a = f(x) = g(y))$.

It can be identified with the subspace of $X \times Y$ consisting of all (x,y) satisfying $f(x) = g(y)$ via the map

$$(x,y) \to (x,y,\ f(x) = g(y)).$$

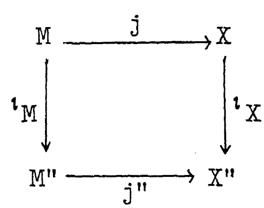
If $g = 0$, then $P = (\ker f) \times Y$.

If g is an isometry then $P = f^{-1}(g(Y))$.

If f and g are isometries then $P = X \cap Y$.

1.15. The following example of a pullback has been given by F.E.J. LINTON [47].

Let ι_X: $X \to X''$ be the canonical embedding of X into its bidual X''. Let M be a closed subspace of X and j: $M \to X$ the corresponding embedding. Then the diagram



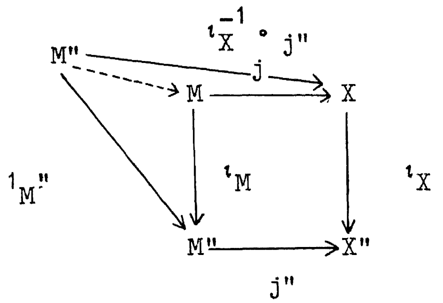
commutes and is a pullback.

Proof: By Hahn-Banach j'' is an isometry. Thus M'' and X may be interpreted as subspace of X''. Our assertion therefore says that $M = X \cap M''$.

Or more explicitly: Let $m'' \in M''$ and $x \in X$ such that $j''m'' = \iota_X x$. Then there is (a uniquely determined) $m \in M$ such that $x = j\,m$ (and then of course $\iota_M m = m''$).

For assume that $x \notin j(M)$ then there would exist some $x' \in X'$ such that $x' | j(M) = 0$ and $\langle x, x' \rangle \neq 0$. But then we would have $0 = \langle 0, m'' \rangle = \langle x' \circ j, m'' \rangle = \langle j'x', m'' \rangle = \langle x', j''m'' \rangle =$ $= \langle x', \iota_X x \rangle = \langle x, x' \rangle \neq 0$, a contradiction.
Thus there is $m \in M$ such that $j\,m = x$ and $j''m'' = \iota_X x = \iota_X jm =$ $= j'' \iota_M m$, i.e. $m'' = \iota_M m$.

This example gives an interpretation in terms of category theory of the proof that a closed subspace of a reflexive Banach space is again reflexive.
Consider the diagram



Since X is reflexive $\iota_X : X \to X''$ is an isomorphism and therefore invertible. Since the inner square is a pullback there is a unique morphism $u : M'' \to M$ such that $\iota_M \circ u = 1_{M''}$. Thus ι_M is surjective and so M is reflexive.

1.16. Let X be a Banach space and let D be the category whose objects are the subspaces $M \subset X$ with finite codimension and whose morphisms are the canonical injections between these spaces.

Every injection $\iota_{MN} : M \to N$ defines a canonical quotient map

$$\pi_{MN}: X/M \to X/N.$$

Then we have

$$\lim_{\overleftarrow{D}} X/M = X''.$$

<u>Proof:</u> It is well known that for a quotient map

$$\pi: X \to X/M$$

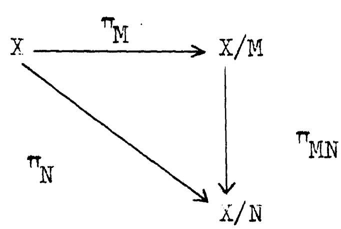
the adjoint map

$$\pi': (X/M)' \to X'$$

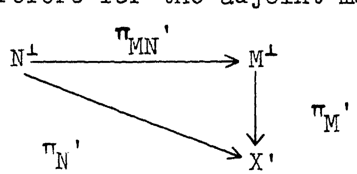
is an isometric imbedding whose image coincides with the w*-closed subspace $M^{\perp} = \{x' \in X' : \langle m, x' \rangle = 0$ for all $m \in M\}$ of X'. Let now M and N be of finite codimension, $M \subseteq N$, and let

$$\pi_M: X \to X/M$$

be the canonical projection. Then the diagram



commutes. Therefore for the adjoint maps the diagram

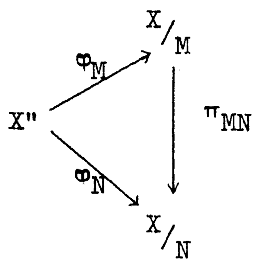


commutes. Since $X/_M$ is finite-dimensional we have

$$(X/_M)'' = (M^\perp)' = X/_M .$$

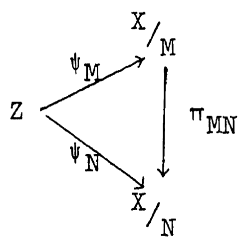
Now define φ_M: $X'' \to X/_M$ by $\varphi_M(x'') = x'' \circ \pi_M'$, i.e. by $\varphi_M = \pi_M''$.

Then the diagram



commutes for all M,N with $M \subseteq N$ and finite codimension.

Now suppose that there is given a Banach space Z and a family of morphisms ψ_M: $Z \to X/_M$ such that the diagrams



commute.

If there exists ψ: $Z \to X''$ such that $\psi_M = \varphi_M \circ \psi$, then we must have $\psi_M = \pi_M'' \circ \psi$, i.e. $\langle \psi_M(z), x' \rangle = \langle \pi_M'(x'), \psi(z) \rangle$ for $x' \in M^\perp = (X/_M)'$.

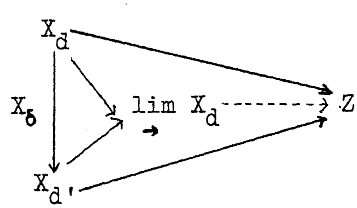
It now suffices to show that ψ is well-defined and $\|\psi\| \leq 1$.

That ψ is well-defined follows from the fact that for $M \subseteq N$ and $x' \in N^{\perp}$ we have

$$<\psi_N(z), x'> = <\pi_{MN}\ \psi_M(z), x'> = <\psi_M(z), \pi'_{MN}\ x'> = <\psi_M(z), x'>.$$

The norm inequality is trivial since $\|\psi_M\| \leq 1$ for all M.

1.17. Colimits or inductive limits: They are defined dually to 1.12. If $(X_d)_{d\in D}$ is a spectral family in Ban_1 then the universal property of $\varinjlim X_d$ is illustrated by the following diagram



Proposition: Let D be a small category. Then $\varinjlim X_d$ exists in Ban_1 and coincides with the quotient space $R = \sum_{d\in D} X_d/_N$, where N is the closed subspace of ΣX_d generated by all elements of the form $\iota_d(x_d) - \iota_{d'}(X_\delta\ x_d)$ for $x_d \in X_d$, $\delta : d \to d'$.

Proof: Given $(\psi_d) : (X_d) \to Z$ there exists $\Sigma\ \psi_d : \Sigma\ X_d \to Z$. Since

$$(\Sigma\ \psi_d)\ (\iota_d(x_d) - \iota_{d'}\ (X_\delta\ x_d)) = \psi_d(x_d) - \psi_{d'}\ (X_\delta\ x_d) = 0$$

the kernel of $\Sigma\ \psi_d$ contains N and we can therefore factorize $\Sigma\psi_d$ over R. All other assertions are clear.

1.18. <u>Difference-cokernels</u>:

A difference-cokernel is the colimit of a spectral family indexed by the category $D = (\circ \rightrightarrows \circ)$.

By 1.17. $\underset{\rightarrow}{\lim}\,(X \overset{f}{\underset{g}{\rightrightarrows}} Y)$ coincides with the space $R = {}^{(X \oplus Y)}/_{N}$ where $X \oplus Y$ is the sum of X and Y in Ban_1 and N is the subspace generated by all elements of the form $(x, -f(x))$ and $(x, -g(x))$, i.e.

$$R = {}^{Y}/_{\overline{(f-g)(X)}}.$$

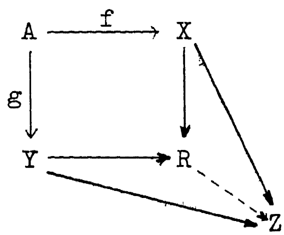
If $g = 0$, then we have $R = {}^{Y}/_{\overline{f(X)}}$ and call this space the cokernel of f.

1.19. <u>Pushouts</u>:

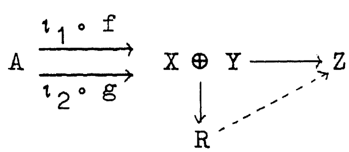
A pushout is a colimit of a spectral family indexed by

$$D = (\circ \leftarrow \circ \rightarrow \circ).$$

It may be visualized by the diagram



Since this diagram is equivalent to the diagram



where $\iota_1(x) = (x,0)$ and $\iota_2(y) = (0,y)$,

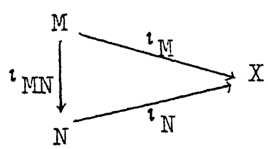
we get by 1.18.

$$R = {}^{(X \oplus Y)}/_{\overline{\{(f(a),\ -g(a)) :\ a \in A\}}}$$

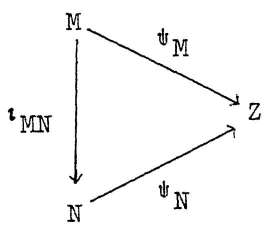
If $g: A \to {}^{A}/_{N}$ is a quotient map we have $R = {}^{X}/_{\overline{f(N)}}$.

1.20. Proposition: Every Banach space X is the colimit of its finite-dimensional subspaces: $X = \varinjlim M$.

Proof: Let D be the set of all finite-dimensional subspaces M of X and $\iota_{MN}: M \to N$ for $M \subseteq N$ the canonical inclusion. Then the diagrams

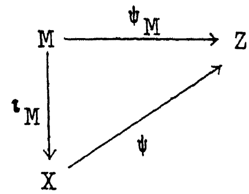


commute, where $\iota_M: M \to X$ is the canonical inclusion. Suppose now that $\psi_M: M \to Z$ is a family of morphisms such that the diagrams



commute.

If there exists $\psi: X \to Z$ with



we must have $\psi(\iota_M m) = \psi_M(m)$, i.e. $\psi(x) = \psi_M(x)$ for $x \in M$. On the other hand this formula defines a mapping ψ. For let $x \in M \subseteq N$, then $\psi_M(x) = \psi_N(\iota_{MN} x)$.

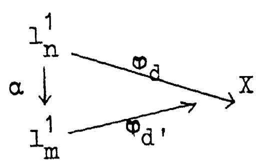
1.21. Proposition: Every Banach space X may be represented as an inductive limit of spaces l_n^1.

Proof: Let X be given. We define a category D in the following way: The objects of D are all pairs $d = (l_n^1, \varphi_d)$ such that $\varphi_d: l_n^1 \to X$ is a morphism. In this case

$$\varphi_d((\xi_1, \ldots, \xi_n)) = \sum_{i=1}^{n} \xi_i x_i$$

for some $x_i \in X$, $\|x_i\| \leq 1$.

The morphisms of D are those morphisms $\alpha: l_n^1 \to l_m^1$ such that



commutes if $d = (l_n^1, \varphi_d)$, $d' = (l_m^1, \varphi_{d'})$.

Let now the spectral family (X_d) be the projection onto the first coordinate of d, i.e. the family

$$X_d = X_{(l_n^1, \varphi_d)} = l_n^1.$$

We assert that $\varinjlim X_d = \varinjlim l_n^1 = X$.

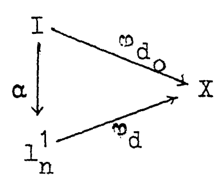
Let now $\varphi_d : l_n^1 \to X$ be given and let $x_o \in X$ and $\xi_o \in l_n^1$ be such that $\varphi_d(\xi_o) = x_o$ and $\xi_o \neq 0$.

Define $d_o = (I, \varphi_{d_o})$ by $\varphi_{d_o}(1) = \dfrac{x_o}{\|\xi_o\|_1}$.

Then $d_o \in D$ since $\|x_o\| \leq \|\xi_o\|_1$.

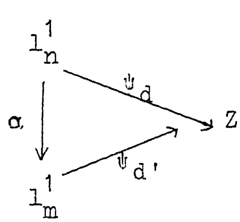
Let $\alpha : I \to l_n^1$ be defined by $\alpha(1) = \dfrac{\xi_o}{\|\xi_o\|_1}$.

Then $\|\alpha\| \leq 1$ and the diagram



commutes.

Let now $\psi_d : X_d \to Z$ be a family of morphisms such that all diagrams



commute.

We have to show that there exists a morphism $\psi : X \to Z$ such that $\psi_d = \psi \circ \varphi_d$. If such a ψ exists it is uniquely determined and satisfies

$$\psi(\varphi_d(\xi)) = \psi_d(\xi).$$

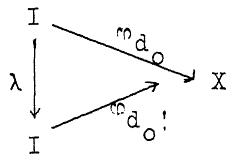
We now want to show that this indeed is a correct definition, in other words: Let $\varphi_d(\xi) = \varphi_{d'}(\xi')$. Then it is to show that

$$\psi_d(\xi) = \psi_{d'}(\xi').$$

Given d, ξ and $\varphi_d(\xi)$ we construct d_o, φ_{d_o} and α as above. In the same way we construct d_o', $\varphi_{d_o'}$, α' for d', ξ' and $\varphi_{d'}(\xi')$. Suppose that $\|\xi'\|_1 \leq \|\xi\|_1$ and let

$$\lambda = \frac{\|\xi'\|_1}{\|\xi\|_1}, \quad |\lambda| \leq 1.$$

Then the diagram

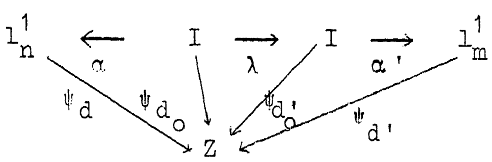


commutes where $\lambda : \xi \to \lambda\xi$.

Therefore

$$1_n^1 \underset{\alpha}{\leftarrow} I \underset{\lambda}{\rightarrow} I \underset{\alpha'}{\longrightarrow} 1_m^1$$

is a diagram in D. Therefore the following diagram commutes



This means $\psi_d\ (\alpha(1)) = \psi_{d_o}(1) = \psi_{d_o'}\ (\lambda) = \psi_{d'}\ (\lambda\ \alpha'(1))$

or equivalently

$$\psi_d\ (\frac{\xi}{\|\xi\|}) = \psi_{d_o}\ (1) = \lambda\ \psi_{d_o'}\ (1) = \lambda\ \psi_{d'}\ (\frac{\xi'}{\|\xi'\|})$$

$$\Rightarrow \psi_d\ (\xi) = \|\xi\|\ \lambda\ \frac{1}{\|\xi'\|}\ \ \psi_{d'}\ (\xi') = \psi_{d'}\ (\xi').$$

This shows that ψ is well defined.

The linearity follows by noting that given $x, y \in X$ the pairs

$$(I,\ \varphi_o),\ (I,\ \varphi_1),\ (I \oplus I,\ \varphi_2)$$

with

$$\varphi_o\ (1) = \frac{x}{\|x\|},\ \varphi_1\ (1) = \frac{y}{\|y\|},\ \varphi_2\ (\lambda, \mu) = \lambda\ \frac{x}{\|x\|} + \mu\ \frac{y}{\|y\|}$$

are elements of D and α_o, α_1 with

$$I \underset{\alpha_o}{\longrightarrow} I \oplus I \underset{\alpha_1}{\longleftarrow} I \quad \text{with} \quad \alpha_o\ (1) = (1,0) \text{ and } \alpha_1(1) = (0,1)$$

are morphisms.

Therefore $\psi : X \to Z$ is a linear mapping and

$$\|\psi\| = \sup_{\|x\| \leq 1}\ \|\psi(x)\| \leq 1$$

because for every $x \in X$ there is

$$d = (I,\ \varphi_d) \quad \text{with} \quad \varphi_d\ (1) = \frac{x}{\|x\|}$$

and therefore

$$\|\psi\ (x)\| = \|\psi\ (\varphi_d(\|x\|))\| = \|\psi_d\ (\|x\|)\| \leq \|x\|.$$

§ 2. The category $\underline{W} = Ban_1^{op}$

2.1. Definition: A Waelbroeck space is a triple $(\mathfrak{X}, K, \tau)$ with the following properties:

a) $\mathfrak{X}$ is a vector space, K a circled convex absorbing subset of $\mathfrak{X}$, and τ a compact (Hausdorff) topology on K.

b) The mapping $x \to \frac{a+x}{2}$ from K to K is continuous for each $a \in K$.

c) The origin 0 of $\mathfrak{X}$ has a base of τ-neighbourhoods in K consisting of circled convex sets ("discs").

Example: Let X be a Banach space, $\mathfrak{X} = X'$ its dual space, $K = O X'$ the unit ball of X' and τ the restriction of the weak star topology $\sigma(X', X)$ to $O X'$.

Then $(\mathfrak{X}, K, \tau)$ is a Waelbroeck space.

The purpose of this chapter is to show that every Waelbroeck space is isomorphic to a dual space of a Banach space and to give a concrete representation of the opposite category of Ban_1 in terms of Waelbroeck spaces.
In order to prove this we need some lemmas. Since K is τ-compact it may be equipped with a (uniquely determined) uniform structure. Our first task is to describe this uniform structure.

2.2. Lemma: A subset $A \subseteq K \times K$ is an entourage of the uniform structure on K if and only if there exists a neighbourhood U of the origin such that A contains all elements $(x,y) \in K \times K$ such that $x - y \in 2U$.

Proof: It is clear that the family of all such sets A is a filter with a symmetric basis and that its intersection is the diagonal. We will show that it is the filter of the entourages of some separated uniform structure on K. In order to do this we have to show that for every neighbourhood U of the origin there exists a neighbourhood V such that $x-z \in 2U$ if $x,y,z \in K$ and $x-y$, $y-z \in 2V$. The mapping $t \to \frac{t}{2}$ is continuous on K by b).

Therefore it is a homeomorphism from K onto $\frac{K}{2}$. Thus for each neighbourhood U of 0 there is a neighbourhood V such that $\frac{U}{2} \supseteq V \cap (\frac{K}{2})$ or $U \supseteq 2V \cap K$. The neighbourhood V may be chosen to be a disc (convex and circled).

Let now $x,y,z \in K$, $x-y \in 2V$, $y-z \in 2V$. Then $\frac{x-z}{2} \in K$ since K is a disc. On the other hand

$$\frac{x-z}{2} = \frac{x-y}{2} + \frac{y-z}{2} \in V + V = 2V.$$

Therefore

$$\frac{x-z}{2} \in 2V \cap K \subseteq U, \text{ i.e. } x-z \in 2U.$$

Hence the sets $(x+2U) \cap K$, U a τ-neighbourhood of 0, form a basis of neighbourhoods of x in the topology induced by the above uniform structure. Each such set is also a τ-neighbourhood of x since it is the inverse image of U by the continuous mapping $y \to \frac{y-x}{2}$. Therefore the topology induced by the uniform structure, which is Hausdorff, is weaker than the compact topology τ and therefore these two topologies coincide.

2.3. <u>Lemma:</u> The mapping $(x,y) \to \frac{x+y}{2}$ from $K \times K$ into K is continuous. More generally the mapping $(x,y) \to \lambda x + \mu y$ for $|\lambda| + |\mu| \leq 1$ is continuous. The proof is obvious.

2.4. Now let $T_{\bar{c}}$ be the strongest locally convex topology on the Waelbroeck space $\mathfrak{X}$ such that the embedding $K \to \mathfrak{X}$ is continuous.

<u>Lemma:</u> A neighbourhood basis of the origin for the topology $T_{\bar{c}}$ is formed by all discs $W \subseteq \mathfrak{X}$ such that for all $r > 0$ there is a τ-neighbourhood V of the origin in K satisfying $V \subseteq rW$.

<u>Proof:</u> Since $K \to \mathfrak{X}$ is continuous, such a V must exist for every convex circled neighbourhood $W \subseteq \mathfrak{X}$ and each $r > 0$. Choosing all such W's we clearly get the strongest locally convex topology such that $K \to \mathfrak{X}$ is continuous.

2.5. Now we define the strongest topology T_c on $\mathfrak{X}$ such that all translations and homotheties (i.e. maps of the form $x \to rx$, $r \in I$) of $\mathfrak{X}$ and the imbedding $K \to \mathfrak{X}$ are continuous. This topology is a priori neither locally convex nor compatible with the linear structure on $\mathfrak{X}$. By definition it is stronger than $T_{\bar{c}}$. The important fact is that these topologies coincide. In order to prove this we need a

<u>Lemma:</u> Let F be a τ-closed subset of K, G open in K and $x \in K$ such that $x \notin F + V \subseteq G$. Then there exists a compact disc V which is a τ-neighbourhood of the origin such that $x \notin (F + V) \cap K \subseteq G$.

<u>Proof:</u> There exists a neighbourhood V_1 of x such that $V_1 \cap F = \emptyset$. We know already that $V_1 \supseteq (x + 2V) \cap K$ for some closed convex circled neighbourhood V of 0.

Then $(x+V) \cap F = \emptyset$ since $V \subseteq 2V$ and

$$(x+V) \cap F \subseteq ((x+2V) \cap K) \cap F \subseteq V_1 \cap F = \emptyset.$$

Once again, this implies that $x \notin F+V$.

Since F is compact V may be chosen such that $(y+V) \cap K \subseteq G$ all $y \in F$.

2.6. Proposition: The topologies T_c and $T_{\bar{c}}$ on $\mathfrak{X}$ coincide and are Hausdorff.

Proof: It suffices to show that T_c is weaker than $T_{\bar{c}}$ and that $T_{\bar{c}}$ is Hausdorff.

Let U be open in the topology T_c. We shall show that U is a $T_{\bar{c}}$-neighbourhood of each element $x \in U$. We may suppose $x = 0$ since translations are continuous. It now suffices to construct a convex circled $T_{\bar{c}}$-neighbourhood W of 0 such that $W \subseteq U$ and such that W does not contain a previously given point $a \neq 0$. Since homotheties are continuous we may suppose $a \in K$.

We know that for every $r > 0$ the set $(rU) \cap K$ is τ-open in K since it is the inverse image of the open set rU by the injection $K \to \mathfrak{X}$.

There exists a τ-closed circled convex neighbourhood V_o of 0 in K such that $a \notin V_o \subseteq U$. Let $K_o = V_o$. The compact disc $\frac{K_o}{3}$ is contained in the τ-open set $\frac{U}{3} \cap K$ and does not contain $\frac{a}{3}$. By the preceding lemma there exists a compact convex circled neighbourhood V_1 of 0 such that

$$\frac{a}{3} \notin \frac{K_o}{3} + \frac{2}{3} V_1 \subset \frac{U}{3} \cap K.$$

The disc $K_1 = \frac{K_o}{3} + \frac{2}{3} V_1$ is compact, contained in K and such that $a \notin 3K_1 \subseteq U$. In this way we get a sequence (K_n) of compact discs in K and a sequence (V_n) of 0-neighbourhoods, which are also compact discs, such that for all $n \geq 0$:

$$a \notin 3^n K_n \subseteq U$$

and

$$3^{n+1} K_{n+1} = 3^n K_n + 2 \cdot 3^n V_{n+1}.$$

The sequence $(3^n K_n)$ is strictly increasing and therefore

$$W = \bigcup_n 3^n K_n$$

is a disc; $a \notin W \subseteq U$. For $r > 0$ choose n such that $\frac{1}{2 \cdot 3^n} < r$. Then we have

$$V_{n+1} \subseteq \frac{3}{2} K_{n+1} = \left(\frac{1}{2 \cdot 3^n}\right) 3^{n+1} K_{n+1} \subseteq r W, \text{ qed.}$$

<u>Corollary</u>: The topology $T_{\bar{c}}$ induces on K the original topology τ.

<u>Proof</u>: This follows from the fact that the embedding $K \to (\mathfrak{X}, T_{\bar{c}})$ is continuous and that $T_{\bar{c}}$ is separated.

2.7. <u>Proposition: (Banach-Dieudonné)</u>: A subset $F \subseteq \mathfrak{X}$ is $T_{\bar{c}}$ - closed if and only if for each $r > 0$ the set $(rF) \cap K$ is closed in K.

Proof: Let $\mathfrak{F}$ be the family of all sets F such that for all $r > 0$, $(rF \cap K)$ is closed in K.

Then $\emptyset$ and $X \in \mathfrak{F}$ and finite unions and arbitrary intersections are in $\mathfrak{F}$. Therefore $\mathfrak{F}$ defines a topology on $\mathfrak{X}$. This topology has the property that the inclusion $K \to \mathfrak{X}$ and all translations and homotheties of $\mathfrak{X}$ are continuous. Since $T_{\bar{c}}$ is the strongest such topology, it is clear that $\mathfrak{F}$ defines a weaker topology than $T_{\bar{c}}$. On the other hand every $T_{\bar{c}}$-closed set belongs to $\mathfrak{F}$ and therefore both topologies coincide.

Corollary (Banach): A linear subspace M of the dual space X' of a Banach space X is w*-closed if and only if $M \cap OX'$ is w*-closed in OX'.

Proof: The dual spaces of $(X', \sigma(X',X))$ and of $(X', T_{\bar{c}})$ both coincide with X. (The second assertion follows from the fact that a linear functional on X' is w*-continuous if and only if its restriction to OX' is w*-continuous. This will also follow from Theorem 2.8.).

Therefore a subspace M is $\sigma(X',X)$-closed if and only if it is $T_{\bar{c}}$-closed. But this means that $M \cap OX'$ is w*closed.

2.8. We consider now the category $\underline{W}$ of all Waelbroeck spaces $(\mathfrak{X}, K, \tau)$. A morphism $\varphi: (\mathfrak{X}, K, \tau) \to (\overline{\mathfrak{X}}, \overline{K}, \overline{\tau})$ will be a linear map $\varphi: \mathfrak{X} \to \overline{\mathfrak{X}}$ such that $\varphi(K) \subseteq \overline{K}$ and $\varphi|K$ is $\tau - \overline{\tau}$ - continuous.

Theorem (Waelbroeck): The category $\underline{W}$ is equivalent to the opposite category Ban_1^{op} to Ban_1.

Proof: We define two contravariant functors $\mathfrak{D}$: $Ban_1 \to W$ and

$D: \underline{W} \to Ban_1$ such that $D\mathfrak{D} = 1_{Ban_1}$ and $\mathfrak{D}D = 1_{\underline{W}}$.

For every $X \in Ban$ let $\mathfrak{D}X = (X', OX', \sigma(X',X)|OX')$ and for every morphism $u: X \to Y$ in Ban_1 let $\mathfrak{D}u = u'$ be the dual map, $u': Y' \to X'$. It is clear that u' is a morphism in $\underline{W}$.

On the other hand define for $\mathfrak{X} \in \underline{W}$ the Banachspace $D\mathfrak{X} = \mathfrak{X}^*$ as the space of all linear functionals f on $\mathfrak{X}$ such that $f|K$ is continuous with the norm $\|f\| = \sup_{x \in K} |f(x)|$. For every morphism $\varphi: \mathfrak{X} \to \mathfrak{Y}$ define

$$\varphi^*: \mathfrak{Y}^* \to \mathfrak{X}^* \text{ by } \varphi^*(g) = g \circ \varphi \text{ for } g \in \mathfrak{Y}^*.$$

We show first that $\mathfrak{D}D = 1_{\underline{W}}$.

Let $\mathfrak{X} \in \underline{W}$. Then $\mathfrak{X}^* = (\overline{c}\ \mathfrak{X})'$ where $\overline{c}\ \mathfrak{X}$ is the linear space $\mathfrak{X}$ with topology $T_{\overline{c}}$ by 2.7. Since $\overline{c}\ \mathfrak{X}$ is Hausdorff, the functionals in $\mathfrak{X}^*$ separate points of $\mathfrak{X}$ and therefore we have a canonical embedding $\mathfrak{X} \to \mathfrak{D}D\ \mathfrak{X} = (\mathfrak{X}^*)'$, given by $x \to \hat{x}$, where $\langle x^*, \hat{x} \rangle = \langle x, x^* \rangle$ for all $x^* \in \mathfrak{X}^*$.

The polar K^o coincides with $O\ \mathfrak{X}^*$ by the definition of the norm in $\mathfrak{X}^*$. Therefore the bipolar K^{oo} coincides with $O(\mathfrak{X}^*)'$.
Since K is τ-compact and the topology $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ is Hausdorff, $\sigma(\mathfrak{X}, \mathfrak{X}^*)|O\mathfrak{X} = \tau$ and therefore $K^{oo} = K$ by the bipolar theorem since

$$\sigma(\mathfrak{X}, \mathfrak{X}^*)|K = \sigma((\mathfrak{X}^*)', \mathfrak{X}^*)|K.$$

We thus get $O(\mathfrak{X}^*)' = K$ and therefore

$$(\mathfrak{X}, K, \tau) = ((\mathfrak{X}^*)', O(\mathfrak{X}^*)', \tau), \text{ i.e. } \mathfrak{D}D\ \mathfrak{X} = \mathfrak{X}.$$

For a morphism $\varphi: \mathfrak{X} \to \mathfrak{Y}$ we have $\langle x, \varphi^*(y^*)\rangle = \langle \varphi(x), y^*\rangle$, or, identifying x with $\hat{x}$, $\langle \varphi^*(y^*), x\rangle = \langle y^*, \varphi(x)\rangle$, i.e. $\langle y^*, \varphi(x)\rangle = \langle y^*, (\varphi^*)'x\rangle$.

This means $(\varphi^*)' = \varphi$.

It remains to show that $D\mathfrak{T} = 1_{Ban_1}$.

Let $X \in$ Ban. Then $D\mathfrak{T}\, X = DX' = (X')^*$.

Since X and $(X')^*$ have the same dual space X' (because $DX' = ((X')^*)' = X'$), they are, by the above reasoning, isometrically isomorphic, qed.

Corollary: Every Waelbroeck space can be considered as a dual space of a Banach space and vice versa.

2.9. Proposition: $T_{\bar{c}}$ is the topology of uniform convergence on the compact subsets of $\mathfrak{X}^*$.

Proof: K being the unit ball of $\mathfrak{X} = (\mathfrak{X}^*)'$, it is easily seen that the restriction to K of the topology of compact convergence coincides with the topology of pointwise convergence, i.e. with $\sigma(\mathfrak{X},\mathfrak{X}^*)$. But $\sigma(\mathfrak{X},\mathfrak{X}^*)|K = \tau$ and $T_{\bar{c}}$ is therefore stronger than the topology of compact convergence. Conversely, let U be some convex, circled and closed $T_{\bar{c}}$-neighbourhood of the origin and $U^o = \{x^* \in \mathfrak{X}^* |\ |\langle x^*, x\rangle| \leq 1\ \forall x \in U\}$ be the polar of U. We have $\mathfrak{X}^* = (\bar{c}\ \mathfrak{X})' \subseteq C(K)$ and $(x + \epsilon\, U) \cap K$ is a τ-neighbourhood of x for each $x \in K$ and $\epsilon > 0$. Therefore U^o is an equicontinuous subset of $C(K)$. Since K is τ-compact, there exists a $\lambda > 0$ such that $K \subset \lambda U$ and so U^o is also bounded.

By the theorem of Ascoli-Arzela U^o is a relatively compact subset of $\mathfrak{X}^*$. By bipolarity we get $U^{oo} = U$ and so U is a neighbourhood for the topology of compact convergence.

Now we want to describe some categorical notions in W explicitly. Let $(\mathfrak{X}, K, \tau)$ be a Waelbroeck space and $\mathfrak{B}$ a linear subspace of $\mathfrak{X}$. We call $\mathfrak{B}$ a <u>(Waelbroeck) subspace</u> of $\mathfrak{X}$ if $\mathfrak{B} \cap K$ is compact in K.

In this case $(\mathfrak{B}, \mathfrak{B} \cap K, \tau|\mathfrak{B} \cap K)$ is a Waelbroeck space and the canonical inclusion $\mathfrak{B} \to \mathfrak{X}$ a morphism in W.

<u>Remark</u>: If $(\mathfrak{X}, K, \tau)$ is given concretely as the dual X' of a Banach space a linear subspace $N \subseteq X'$ is a Waelbroeck subspace if and only if $N \cap OX'$ is w*-compact, i.e. if and only if N is w*-closed (2.7).

In this case $N = M^{\perp}$ for some closed subspace M of X and $\iota : M^{\perp} \to X'$ is the dual $\iota = \pi'$ of the quotient map $\pi : X \to X/M$. Therefore the Waelbroeck subspaces correspond via duality precisely to the quotient spaces of Banach spaces.

Let now $\varphi: \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\underline{W}$.
Let $\mathfrak{B} = \varphi^{-1}(0)$. Since $\mathfrak{B} \cap K$ is compact, $(\mathfrak{B}, \mathfrak{B} \cap K, \tau(\mathfrak{B} \cap K))$ is a Waelbroeck subspace of $\mathfrak{X}$. We call it the kernel of φ, $\mathfrak{B} = \ker \varphi$. The same reasoning as in 1.3. shows now that $\varphi: \mathfrak{X} \to \mathfrak{Y}$ is mono in $\underline{W}$ if and only if φ is injective, i.e. $\ker \varphi = (0)$.

Remark: We could have obtained the same result by considering dual Banach spaces: $u': X' \to Y'$ is mono in $\underline{W}$ if and only if $u: Y \to X$ is epi in Ban_1, if and only if u' is injective.

2.10. Let $\mathfrak{B}$ be a Waelbroeck subspace of the Waelbroeck space $\mathfrak{X}$. Consider the canonical projection $p: \mathfrak{X} \to \mathfrak{X}/\mathfrak{B}$. This will be continuous on K if we define on $p(K)$ the quotient topology, which is compact. Furthermore $\mathfrak{X}/\mathfrak{B}$ becomes a Waelbroeck space. We call it a quotient space.

Remark: We could also give a more concrete definition. The pair $(\mathfrak{B}, \mathfrak{X})$ corresponds to the pair $(M^{\perp}, X')$ for some closed subspace $M \subseteq X$. The canonical projection $p: X' \to X'/M^{\perp}$ may then be identified with $p = \iota'$ where $\iota: M \to X$ is the canonical inclusion of M into X. We therefore see that quotient spaces in $\underline{W}$ correspond via duality to subspaces in Ban_1.

Let now $\varphi: \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\underline{W}$. We define the cokernel $\operatorname{cok} \varphi$ as the quotient space $\mathfrak{Y}/\overline{\varphi(\mathfrak{X})}$ where $\overline{\varphi(\mathfrak{X})}$ is the closure of $\varphi(\mathfrak{X})$ in $\bar{c}\ \mathfrak{Y}$.

The same reasoning as in 1.4. now gives: A morphism $\varphi: \mathfrak{X} \to \mathfrak{Y}$ in $\underline{W}$ is epi if and only if $\varphi(\mathfrak{X})$ is $T_{\bar{c}}$-dense in $\mathfrak{Y}$.

<u>Remark:</u> For a dual map φ': $X' \to Y'$ this may also be formulated as: φ' is epi if and only if $\varphi'(X')$ is w*-dense in Y'. For the w*-closure and the $T_{\bar{c}}$-closure of linear subspaces coincide.

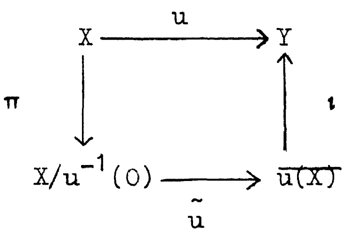
2.11. In the same way as in 1.7. we can now define extreme monos and extreme epis in <u>W</u>.

The same reasoning shows that the extreme monos are those morphisms which are isometric embeddings (in the norm of $\mathfrak{X} = (\mathfrak{X}*)'$) and the extreme monos are the quotient maps in <u>W</u>. We only need a simple

<u>Lemma:</u> Let φ': $X' \to Y'$ be a dual map which is isometric and has w*-dense image. Then φ' is surjective.

<u>Proof:</u> Since $\varphi'(OX') = \varphi'(X') \cap OY'$ is compact, $\varphi'(X')$ is w*-closed in Y' and being w*-dense it coincides with Y'.

2.12. Let u: $X \to Y$ be a morphism in Ban_1 and



its canonical decomposition. The dual diagram looks like

$$\begin{array}{ccc} Y' & \xrightarrow{\ u'\ } & X' \\ {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \overline{u(X)}' & \xrightarrow[\ \tilde{u}'\ ]{} & (X/u^{-1}(0))' \end{array}$$

We know already that ι' is a quotient map, π' an isometric imbedding and $(\tilde{u})'$ injective with w*-dense image. Therefore $\overline{u(X)}' = Y'/\ker u'$ and $(X/u^{-1}(0))' = \overline{u'(Y')}^{w}$ where w denotes the w*-closure in X'.

We call this the canonical Waelbroeck space decomposition of u'. We call u' strict if and only if $\tilde{u}'$ is an isomorphism. Then we have the

Proposition: $u': Y' \to X'$ is strict if and only if $u'(OY') =$
$= u'(Y') \cap OX$.

Proof: If u' is strict then it is the composition of a quotient map in $\underline{W}$ and an isometric embedding. Therefore the equation $u'(OY') = u'(Y') \cap OX'$ holds.
If on the other hand $u'(OY') = u'(Y') \cap OX'$, then $u'(Y')$ is w*-closed in X' since $u'(OY')$ is compact. The mapping $\tilde{u}'$ is bijective on the unit balls and therefore an isomorphism.

2.13. Since every dual space of a Banach space may also be considered as a Banach space we have a forgetful functor $b: \underline{W} \to \mathrm{Ban}$, which assigns to each Waelbroeck space (X', OX', τ) the underlying Banach space X'. One "forgets" that OX' has a compact topology. We write sometimes X_b' instead of X' if we only consider the Banach space structure on X'. It is clear that b is a functor since every dual map u' is a fortiori a morphism in Ban_1.

The composite functor $b \circ \mathfrak{D}$: $\mathrm{Ban}^{op} \to \mathrm{Ban}$, which associates to X the Banach space X'_b is contravariant and self-adjoint on the right, i.e. it satisfies the equation

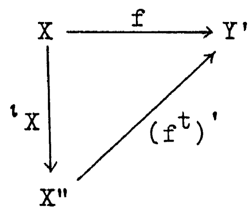
$$\mathrm{Hom}\ (X, Y'_b) = \mathrm{Hom}\ (Y, X'_b).$$

The isometry is given by $f \to f^t$, where the transposed map f^t is defined by

$$\langle y, f\, x\rangle = \langle x, f^t\, y\rangle, \text{ i.e.}$$

$f^t = f' | Y$, the restriction of the dual map f' to the subspace Y, or $f^t = f' \circ \iota_Y$.

Then $\| f^t \| = \| f \|$ and the diagram



commutes.

This implies furthermore the equality

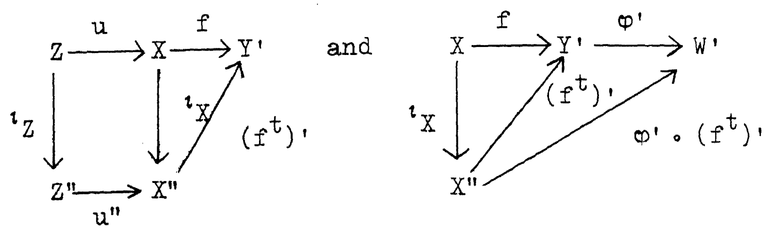
$$(*) \qquad \mathrm{Ban}_1\ (X, Y'_b) = \underline{W}\ (X'', Y')$$

where we write $\mathrm{Ban}_1\ (X, Y'_b)$ instead of $\mathrm{Hom}\ (X, Y'_b)$ in order to make clear the category in which morphism are taken. This equality is given by $f \leftrightarrow (f^t)'$.

It is "natural" in X and Y'. This means the following: let $u: Z \to X$ be a morphism in Ban_1. Then to the element $f \circ u$ on the left hand side corresponds the element $(f^t)' \circ u''$ on the right hand side.

Similarly, if $\varphi': Y' \to W'$ is a morphism in $\underline{W}$, then to the element $\varphi' \circ (f^t)'$ on the right side corresponds the element $\varphi'_b \circ f$ on the left side.

The proof follows from the commutativity of the diagrams



We can now interpret (*) in the following way: The functor $'': Ban_1 \to \underline{W}$ is left adjoint to the forgetful functor $b: \underline{W} \to Ban_1$.

The unit of the adjointness relation (*) (which corresponds to the identity $1_{X''}$) is the canonical embedding $\iota_X: X \to X''$ and the counit (which corresponds to the identity $1_{Y'_b}$) is the map $(\iota_Y)': Y''' \to Y'$.

For later uses we note the following

<u>Proposition:</u> Let $f: M \to X'$. Then f is an isometric embedding if and only if $f^t(OX)$ is w*-dense in OM'.

<u>Proof:</u> Note first that for every $X \in Ban$, OX is w*-dense in OX". This is a well known consequence of the bipolar theorem.

Let now $f: M \to X'$ be extremely mono in Ban_1 then $f': X'' \to M'$ is extremely epi in $\underline{W}$ and therefore a quotient map. Thus $f'(OX'') = OM'$. Since OX is w*-dense in OX" and f' is w*-continuous we get that $f^t(OX)$ is w*-dense in OM'.

On the other hand, let $f^t(OX)$ be w*-dense in OM'. Then f'(OX") is w*-dense in OM' and compact and therefore f'(OX") = OM'. This means that f' is a quotient map in $\underline{W}$ and therefore f is an isometric embedding in Ban_1, qed.

2.14. Let us now consider the forgetful functor O: W → Comp, where Comp denotes the category of all compact Hausdorff spaces and continuous maps. The functor O assigns to a space $X' \in \underline{W}$ its compact unit ball OX' and to each morphism u': X' → Y' its restriction to OX'.

In order to show that this functor has a left adjoint we consider the contravariant functor C: Comp → Ban_1, which assigns to each $T \in$ Comp the Banach space of all continuous functions on T with the sup norm and to each continuous map f: T → S the mapping C(f): C(S) → C(T) given by $C(f)(\varphi) = \varphi \circ f$, $\varphi \in C(S)$. If we combine the contravariant functor C with the contravariant functor $\mathfrak{D}$: $Ban_1 \to \underline{W}$ we get a covariant functor $\mathfrak{W} = \mathfrak{D}C$: Comp → W. It assigns to each $T \in$ Comp the Waelbroeck space $\mathfrak{W}(T) = C(T)'$ of all Radon measures on T and to every f: T → S in Comp the morphism $\mathfrak{W}(f) = C(f)'$.

There is a natural embedding $T \to \mathfrak{W}(T)$ which associates with each $t \in T$ the Dirac measure δ_t, defined by $\delta_t(f) = f(t)$ for all $f \in C(T)$. It is clear that this is a homeomorphism of T onto its image in O $\mathfrak{W}(T)$. The absolutely convex hull of the δ_t's is w*-dense in O $\mathfrak{W}(T)$. This follows immediately from proposition 2.13. applied to the isometric embedding $\psi: C(T) \to (l_T^1)' = l_T^\infty$.

Now we will show that the functor $\mathfrak{M}$ is left adjoint to the forgetful functor O, i.e. that

$$(*) \qquad \underline{W}\,(\mathfrak{M}(T),\, X') = \mathrm{Comp}\,(T, OX')$$

holds naturally in $T \in$ Comp and $X' \in \underline{W}$.

The identification $(*)$ is given by $\varphi \to \varphi|T$ for φ: $\mathfrak{M}(T) \to X'$ and by $f \to \varphi_f$, where φ_f is the unique morphism such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\ f\ } & OX' \\ {\scriptstyle\delta}\downarrow & & \downarrow \\ \mathfrak{M}(T) & \xrightarrow[\ \varphi_f\ ]{} & X' \end{array}$$

is commutative. (The uniqueness of φ_f follows from the fact that the linear combinations of the δ_t's are w^*-dense in $O\mathfrak{M}(T)$. It follows that

$$<x, \varphi_f(\mu)> \; = \int_T <x, f(t)> \, d\mu(t)$$

must hold for $\mu = \Sigma\, \lambda_k\, \delta_{t_k}$ and by w^*-continuity for all μ.

If on the other hand we define φ_f by $\varphi_f(\mu) = \int_T f\, d_\mu$, it is clear that all conditions are fulfilled).

The unit of this adjunction is the identification $t \to \delta_t$ and the counit the quotient map ϵ: $\mathfrak{M}(OX') \to X'$ defined by

$$\epsilon(\mu) = \int_{OX'} x'\, d\mu\,(x'), \text{ i.e. } <x, \epsilon(\mu)> = \int_{OX'} <x, x'>\, d\mu\,(x')$$

for all $x \in X$.

<u>Corollary:</u> Each Waelbroeck space X' is a quotient of a "free Waelbroeck space" $\mathfrak{W}(T)$ for a suitable compact Hausdorff space T.

2.15. It is now easy to describe limits and colimits in $\underline{W}$. From the adjunction $\underline{W}(X'',Y') = \mathrm{Ban}_1(X,Y'_b)$ we get the fact that b commutes with limits and " with colimits, i.e.

$$b\,(\underset{\underline{W}}{\underleftarrow{\lim}}\, X_d') = \underset{\mathrm{Ban}_1}{\underleftarrow{\lim}}\, b\,X_d' \quad\text{and}\quad (\underset{\mathrm{Ban}}{\underrightarrow{\lim}}\, X_d)'' = \underset{\underline{W}}{\underrightarrow{\lim}}\, X_d''$$

From the adjunction

$$\underline{W}\,(\mathfrak{W}(T),X') = \mathrm{Comp}\,(T,OX')$$

we derive on the other hand the fact that

$$O\,(\underset{\underline{W}}{\underleftarrow{\lim}}\, X_d') = \underset{\mathrm{Comp}}{\underleftarrow{\lim}}\, OX_d' \quad\text{and}\quad \mathfrak{W}\,(\underset{\mathrm{Comp}}{\underrightarrow{\lim}}\, T_d) = \underset{\underline{W}}{\underrightarrow{\lim}}\, \mathfrak{W}(T_d).$$

These equations imply e.g. that the product in $\underline{W}$, $\prod_W X_d'$, coincides as Banach space with the product in Ban_1 and that the topology on its unit ball $O(\prod X_d')$ is the product topology in Comp, i.e. the Tychonoff-topology on $\prod OX_d'$.

2.16. <u>The mixed bifunctor L:</u>

Let $L: \underline{W}^{op} \times \mathrm{Ban}_1 \to \mathrm{Ban}_1$ be the bifunctor which associates with a Waelbroeck space X' and a Banach space Y the Banach space L(X',Y) of all bounded linear maps f: X' → Y such that $f|OX'$ is $\sigma(X',X) - \|\,.\,\|$ - continuous, with the norm induced from H(X',Y).

For a fixed Waelbroeck space X' let $L_{X'} = L(X',\cdot)$. This is a covariant functor on Ban_1.

Note that $L_{I'} = 1_{Ban_1}$ and $L(X',I) = X$.

There is an equality

$$(*) \qquad L(X',Y) = L(Y',X)$$

which is natural in X and Y and given by $\varphi \leftrightarrow \varphi^t$, defined by

$$\langle \varphi x', y' \rangle = \langle \varphi^t y', x' \rangle.$$

The mapping φ^t will also be called a transpose of φ.

First of all, φ^t is well-defined: let $\varphi \in L(X',Y)$ and $y' \in Y'$.

Then $\varphi' y' = y' \circ \varphi$ is w^*-continuous on OX', i.e. $\varphi' y' \in \iota_X(X)$.

The mapping φ' with codomain restriced to $X = \iota_X(X) \subseteq X''$ is called φ^t and satisfies the above equation.

All that remains to be shown is the fact, that φ^t belongs to L(Y',X).

Let y_ι' be a net in OY' which converges w^* to $y' \in OY'$.

Then

$$\| \varphi^t y_\iota' - \varphi^t y' \| = \sup_{\|x'\| \le 1} |\langle \varphi x', y_\iota' - y' \rangle| .$$

We know that $\varphi(OX')$ is compact in OY since $\varphi \in L(X',Y)$. For every $\varepsilon > 0$ exists therefore a finite ε-net $y_1, \ldots, y_n \in OY$ for $\varphi(OX')$.

Thus for each $x' \in OX'$ we can find y_k satisfying

$$|<\varphi x', y_\iota' - y'>| \le |<\varphi x' - y_k, y_\iota' - y'>| +$$

$$+ |< y_k , y_\iota' - y'>| \le 2\epsilon + |< y_k , y_\iota' - y'>| .$$

Therefore

$$|<\varphi x', y_\iota' - y'>| \le 2\epsilon + \sup_k |<y_k, y_\iota' - y'>| < 3\epsilon$$

for all $\iota \ge \iota_o$ and all $x' \in OX'$, qed.

2.17. The mixed bifunctor Ω.

We define Ω: $Ban^{op} \times \underline{W} \to \underline{W}$ to be the bifunctor which associates with $X \in$ Ban and $Y' \in \underline{W}$ the space $\Omega(X,Y')$ of all bounded linear maps from X to Y'.

We assert that $\Omega(X,Y')$ can be given the structure of a Waelbroeck space.

Consider the unit ball $O\Omega(X,Y')$. It is clearly an absorbing disc in $\Omega(X,Y')$. Consider on $O\Omega(X,Y')$ the topology of pointwise convergence on OX. In other words consider the mapping which associates with every $f \in O\Omega(X,Y')$ the element

$$(f(x))_{x \in OX} \in \prod_{x \in OX} (OY')_x .$$

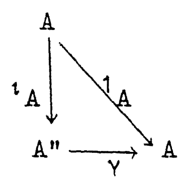
The topology on $O\Omega(X,Y')$ induced in this way from the Tychonoff-topology on $\prod_{x \in OX} (OY')_x$ is compact and satisfies b) and c) of Definition 2.7.

This implies that $\Omega(X,Y')$ belongs to W.

The equality $\Omega(X,Y') = \Omega(Y,X')$ given by $\varphi \leftrightarrow \varphi^t$ has already been considered earlier.

2.18. We conclude this chapter with an interesting result due to Dixmier (1948). [19]

Note first that for a dual space $A = B'$ there exists a projection $\gamma: A'' \to A$ in W such that the diagram



commutes. This morphism γ is uniquely determined (beause OB' is w^*-dense in OB''') and coincides with $(\iota_B)'$. The kernel $K = \ker \gamma$ is then w^*-closed in A''.

The interesting fact is that the reverse assertion also holds.

Proposition (Dixmier): Let A be a Banach space such that there exists a linear contraction $\gamma: A'' \to A$ satisfying $\gamma \iota_A = 1_A$. Then A and γ belong to W if (and only if) $\ker \gamma$ is w^*-closed in A''.

Proof: Let $K = \ker \gamma \subseteq A''$. Since K is w^*-closed there exists a norm closed subspace $B \subseteq A'$ such that $K = B^\perp$.

Let $j: B \to A'$ be the canonical embedding and

$j': A'' \to B'$ the corresponding quotient map in W.

Then we have $j'(a'') = 0$ if and only $\gamma(a'') = 0$, since
$j'a'' = 0 \Leftrightarrow \langle b, j'a'' \rangle = 0$ for $\forall\, b \in B \Leftrightarrow \langle jb, a'' \rangle = 0 \Leftrightarrow$
$\Leftrightarrow a'' \in B^{\perp} = \ker\, \gamma$.

Therefore we have $j'\iota_A\, \gamma = j'$ since $\gamma\, \iota_A \gamma = \gamma$.
Now let $\varphi \colon A \to B'$ be given by $\varphi = j'\, \iota_A$. Then φ is injective, because

$$\varphi(a) = 0 \Rightarrow j'\iota_A(a) = 0 \Rightarrow \gamma(\iota_A(a)) = 0 \Rightarrow a = 0.$$

Furthermore we have $\varphi(OA) = OB'$. For let $b' \in OB'$. Since j' is a quotient map in W there exists $a'' \in OA''$ satisfying $j'a'' = b'$. But then, in addition

$b' = j'a'' = (j'\iota_A\gamma)a'' = (j'\iota_A)(\gamma a'') = \varphi(\gamma(a''))$ and $\gamma(a'') \in OA$.

We have therefore a bijective map $\varphi \colon A \to B'$ such that $\varphi(OA) = OB'$. If we define on OA the topology $\varphi^{-1}(\tau)$ induced by φ from the topology τ on OB', then $\varphi^{-1}(\tau)$ is compact and $(A, OA, \varphi^{-1}(\tau))$ is a Waelbroeck space.

All that remains to be shown is that γ belongs to W.
This follows from $\varphi\, \gamma = j'\iota_A\gamma = j'$, which implies $\gamma = \varphi^{-1} \circ j' \in$ W.

Remark: This result is intimately connected with the following fact: every "-algebra (in the sense of Eilenberg-Moore) is a dual space. Cf. Z. Semadeni [77].

Exercises and complements

1) a) Let $(X_d)_{d \in D}$ be a spectral family (cf 1.12).
Assume that all X_d are monomorphisms and that D is directed downwards, i.e. for all $d, d' \in D$ there exist $d'' \in D$ and morphisms $\delta: d'' \to d$ and $\delta': d'' \to d'$. Assume furthermore that all X_d are linear subspaces of a fixed space Y and that X_δ are the corresponding embeddings. Then the projective limit of (X_d) may be identified with the linear space of $\bigcap_{d \in D} OX_d$ and the norm is given by the Minkowski functional of that set.

b) Dually assume that D is directed upwards.
Then the inductive limit of (X_d) may be identified with the completion of $\bigcup_{d \in D} X_d$, the norm being given by the Minkowski functional of the convex hull of all sets OX_d.

2) For $X \in$ Ban there exist two natural embeddings of X" into $X^{(4)}$, namely the maps ι_X'' and $\iota_{X''}$. Show that the intersection of their images coincides with X.

3) Let $f: X \to Y$ be a compact linear map. Then there exists a dual space Z' and maps $g \in H(X, Z')$, $h \in L(Z', Y)$ such that $f = h \circ g$.

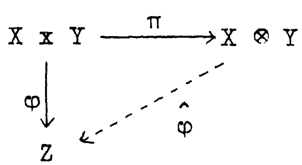
C H A P T E R II

Tensor products of Banach spaces

§ 1. The algebraic and the projective tensor product

1.1. The algebraic tensor product

We assume that the reader is familiar with the algebraic tensor product of linear spaces. Nevertheless we shall give here a brief account that can be generalized later on in the theory of Banach spaces and functors.
Let X, Y be linear spaces over the scalar field I.
The tensor product of X and Y is a linear space $X \otimes Y$ together with a bilinear map $\pi : X \times Y \to X \otimes Y$ which has the following universal property: Any bilinear map $\varphi : X \times Y \to Z$ into an arbitrary linear space Z factors uniquely through π to a linear map $\hat{\varphi} : X \otimes Y \to Z$, i.e. $\hat{\varphi}$ is such that $\varphi = \hat{\varphi} \circ \pi$.



The universal property of the tensor product implies that it is uniquely determined up to a linear isomorphism (for example, by an argument analogous to that used for Banach spaces in 1.5 below). The following "concrete" representation is well known: consider the free linear space M over $X \times Y$ (considered as a set) i.e. the space of all scalar-valued functions on $X \times Y$

which are non-zero at only finitely many points of $X \times Y$. Identify (x,y) with the function that vanishes everywhere except on (x,y), where it has the value 1, then $X \times Y$ is a Hamel-basis for M. Consider the subspace N generated by all elements of the following types:

$$(x + x_1, y) - (x,y) - (x_1,y)$$
$$(x, y + y_1) - (x,y) - (x,y_1)$$
$$\alpha(x,y) - (\alpha x,y)$$
$$\alpha(x,y) - (x,\alpha y) \qquad \text{where } x, x_1 \in X,\ y, y_1 \in Y,\ \alpha \in I.$$

Then $X \otimes Y = M/N$ is easily seen to be a tensor product of X and Y, $\pi : X \times Y \to M/N$ being just the restriction of the canonical projection. We denote the image of (x,y) by $x \otimes y$.

Since we are dealing here with linear spaces only and not with modules over a ring, we can give another representation, which is based on duality arguments.

Taking $Z = I$, then the universal property of $X \otimes Y$ implies that the algebraic dual $(X \otimes Y)^*$ coincides with $\mathfrak{B}(X,Y)$ the space of all bilinear maps from $X \times Y$ to I. (The linearity of the isomorphism is a result of the uniqueness condition). Thus $X \otimes Y$, if it exists, is a subspace of $\mathfrak{B}(X,Y)^*$. Let us denote by $x \otimes y$ the linear functional on $\mathfrak{B}(X,Y)$ which corresponds to evaluation at (x,y), i.e. $\langle f, x \otimes y \rangle = f(x,y)$ for all $f \in \mathfrak{B}(X,Y)$. The construction of our embedding implies that $\pi(x,y)$ must be equal to $x \otimes y$. We now take $X \otimes Y$ as the linear hull of the elements $x \otimes y$ ($x \in X$, $y \in Y$). (It is also a consequence of the universal property that the image of π must generate $X \otimes Y$. This gives us another uniqueness proof). It remains to show that the space we have obtained, together with the map $\pi(x,y) = x \otimes y$, has the desired properties: The bilinearity of π follows immediately from the definition. Given a bilinear map $\varphi: X \times Y \to Z$, we must have $\varphi(x,y) = \hat{\varphi} \circ \pi(x,y) = \hat{\varphi}(x \otimes y)$. $\hat{\varphi}$ is therefore uniquely determined, provided that it exists. Now assume that $\sum_{i=1}^{n} \lambda_i x_i \otimes y_i = 0$ ($\lambda_i \in I$, $x_i \in X$, $y_i \in Y$). If we had $\Sigma \lambda_i \varphi(x_i,y_i) \neq 0$, there would exist some $z^* \in Z^*$ such that $\Sigma \lambda_i z^* \circ \varphi(x_i,y_i) = \langle \Sigma \lambda_i \varphi(x_i,y_i), z^* \rangle \neq 0$. Now $z^* \circ \varphi \in B(X,Y)$ and from our assumption we get $\Sigma \lambda_i z^* \circ \varphi(x_i,y_i) = \langle z^* \circ \varphi, \Sigma \lambda_i x_i \otimes y_i \rangle = 0$, a contradiction. If we set $\hat{\varphi}\left(\sum_{i=1}^{n} \lambda_i x_i \otimes y_i\right) = \sum_{i=1}^{n} \lambda_i \varphi(x_i,y_i)$ ($\lambda_i \in I$, $x_i \in X$, $y_i \in Y$), it follows that $\hat{\varphi}$ is well defined and linear.

1.2. The algebraic tensor product satisfies the following "exponential law": if we write $\mathfrak{L}(X,Y)$ for the space of linear maps from X to Y, then

$$\mathfrak{L}(X \otimes Y, Z) = \mathfrak{B}(X,Y;Z) = \mathfrak{L}(X, \mathfrak{L}(Y,Z)).$$

The first equality holds by the universal property, the second one via the correspondence $\varphi \leftrightarrow \hat{\varphi}$, given by $\varphi(x,y) = \hat{\varphi}(x)(y)$. In an analogous way we have $\mathfrak{B}(X,Y;Z) = \mathfrak{L}(Y, \mathfrak{L}(X,Z))$ (by the correspondence $\varphi(x,y) = \hat{\varphi}(y)(x)$)

and therefore $\mathfrak{L}(X, \mathfrak{L}(Y,Z)) = \mathfrak{L}(Y, \mathfrak{L}(X,Z))$ also holds.
The name exponential law comes from the following notation:
if we write Z^X for $\mathfrak{L}(X,Z)$, the above formula has the form
$Z^{X \otimes Y} = (Z^X)^Y = (Z^Y)^X$.

1.3. We now state two easy results about the elements of $X \otimes Y$:

a) If $x_1,\dots,x_n$ are linearly independent in X and Y and at least one of the elements $y_1,\dots y_n \in Y$ is different from zero, then $\sum_{i=1}^{n} x_i \otimes y_i \neq 0$.

<u>Proof:</u> Assume that $y_1 \neq 0$. Then there exist $x^* \in X^*$ and $y^* \in Y^*$ such that $< x_i, x^* > = \delta_{1i}$ and $< y_1, y^* > \neq 0$. The map $(x,y) \to < x,x^* > < y,y^* >$ clearly defines an element φ of $\mathfrak{B}(X,Y)$ for which we have $\Sigma\, \varphi(x_i,y_i) = \Sigma < x_i x^* > < y_i, y^* > \neq 0$. The result now follows from the universal property.

b) Every non-zero element u of $X \otimes Y$ has a representation $u = \sum_{i=1}^{n} x_i \otimes y_i$ with (x_i) and (y_i) linear independent in X and Y respectively.

<u>Proof:</u> Take any representation $u = \sum_{i=1}^{n} x_i \otimes y_i$ with minimal n (which exists since $u \neq 0$). Suppose e.g. $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$. Then we have $u = \sum_{i=1}^{n-1} x_i \otimes y_i + \sum_{i=1}^{n-1} \alpha_i x_i \otimes y_n$ $= \sum_{i=1}^{n-1} x_i \otimes (y_i + \alpha_i y_n)$, which is a contradiction.

1.4. We suppose now that X and Y are Banach spaces.

B(X,Y) shall denote the space of all bounded bilinear maps from X x Y to I. For $\varphi \in B(X,Y)$ we take $\|\varphi\| = \sup_{\substack{x \in OX \\ y \in OY}} |\varphi(x,y)|$.

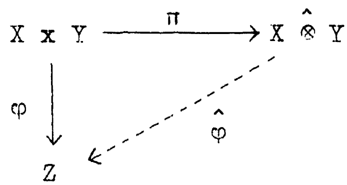
In this way B(X,Y) becomes a Banach space.

As before we consider the map $\varphi : X \times Y \to B(X,Y)'$ given by $< \alpha, \varphi(x,y) > = \alpha(x,y)$, where $\alpha \in B(X,Y)$, $x \in X$, $y \in Y$. It is evidently bilinear and defines therefore a linear map $\hat{\varphi} : X \otimes Y \to B(X,Y)'$. We assert that $\hat{\varphi}$ is injective.

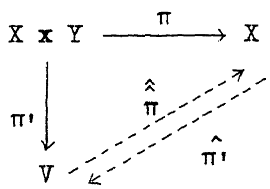
In fact, let (x_i) and (y_i) be linearly independent in X and Y respectively and $u = \sum_{i=1}^{n} x_i \otimes y_i$. By the Hahn - Banach theorem we get functionals $x' \in X'$ and $y' \in Y'$ such that $< x_i, x' > = \delta_{1i}$ and $< y_i, y' > = \delta_{1i}$. $(x,y) \to < x,x' > < y,y' >$ defines an element of B(X,Y) and we have $< < \cdot, x' > < \cdot, y' >, \hat{\varphi}(u) > = 1$, which means $\hat{\varphi}(u) \neq 0$.

1.5. <u>The projective tensor product of Banach spaces</u>.

Let X, Y be Banach spaces. A projective tensor product of X and Y is a Banach space $X \hat{\otimes} Y$ together with a bounded bilinear map $\pi : X \times Y \to X \hat{\otimes} Y$ such that for any bounded bilinear map φ from X x Y into an arbitrary Banach space Z there exists a unique bounded linear map $\hat{\varphi} : X \hat{\otimes} Y \to Z$ such that $\varphi = \hat{\varphi} \circ \pi$. We also require that $\|\varphi\| = \|\hat{\varphi}\|$.



If a projective tensor product exists it is uniquely determined up to isometrical isomorphisms:
Taking $Z = X \hat{\otimes} Y$ and $\varphi = \pi$ one has evidently $\hat{\pi} = 1_{X \hat{\otimes} Y}$ and therefore $\|\pi\| = \|\hat{\pi}\| \leq 1$. Suppose now (V, π') is a second tensor product of X and Y. Then π' factors over π to $\hat{\pi}': X \hat{\otimes} Y \to V$ and π factors over π' to $\hat{\hat{\pi}}$. ($\hat{\hat{}}$ stands for the factorization with respect to the second tensor product).



Since $\pi = \hat{\hat{\pi}} \circ \pi' = \hat{\hat{\pi}} \circ \hat{\pi}' \circ \pi$ and the factorization is unique we have $\hat{\hat{\pi}} \circ \hat{\pi}' = 1_{X \hat{\otimes} Y}$ and similarly $\hat{\pi}' \circ \hat{\hat{\pi}} = 1_V$. From $\|\hat{\hat{\pi}}\| = \|\pi\| \leq 1$ and $\|\hat{\pi}'\| = \|\pi'\| \leq 1$ we see that V and $X \hat{\otimes} Y$ are isometrically isomorphic.
Both constructions sketched in 1.1. may be generalized to give a concrete representation of a projective tensor product. In the first case one can take $l^1(OX \times OY)$ and the quotient with respect to the closure of the subspace considered there. The second way will be described explicitly below. The equivalence of both constructions follows from our uniqueness assertion.
Now for $Z = I$ the universal property says that $(X \hat{\otimes} Y)' = B(X,Y)$ isometrically. Thus $X \hat{\otimes} Y$ can be regarded as a closed subspace of $B(X,Y)'$. As in 1.1. it follows easily that $\pi(x,y) = x \otimes y$ corresponds to the evaluation functional given by

$< f, x \otimes y > = f(x,y)$ (for $f \in B(X,Y)$, $x \in X$, $y \in Y$).
We have already seen that the linear span M generated by these functionals forms an algebraic tensor product of X and Y, and we assert that its norm-closure $\overline{M}$ is a projective tensor product.
Given a bounded bilinear map $\varphi : X \times Y \to Z$, we get a linear map $\hat{\varphi}$ from $M = X \otimes Y$ to Z with $\varphi = \hat{\varphi} \circ \pi$. It remains to show that $\hat{\varphi}$ is bounded:

$$\|\hat{\varphi}(\sum_{i=1}^{n} x_i \otimes y_i)\| = \| \Sigma \varphi(x_i,y_i)\| =$$

$$= \|\varphi\| \sup_{\|z'\| \leq 1} | \Sigma \frac{1}{\|\varphi\|} z' \circ \varphi(x_i,y_i)|$$

$$\leq \|\varphi\| \sup_{\substack{\psi \in B(X,Y) \\ \|\psi\| \leq 1}} | < \psi, \Sigma x_i \otimes y_i > |$$

$$= \|\varphi\| \|\sum_{i=1}^{n} x_i \otimes y_i\|_{B(X,Y)'}$$

Thus $\|\hat{\varphi}\| \leq \|\varphi\|$ and $\hat{\varphi}$ has a unique extension to $\overline{M}$. Since $\|\pi\| \leq 1$ holds by definition and $\varphi = \hat{\varphi} \circ \pi$, we have $\|\varphi\| \leq \|\hat{\varphi}\| \|\pi\| \leq \|\hat{\varphi}\|$.

1.6. We have seen that $X \otimes Y$ is a dense subspace of $X \hat{\otimes} Y$ and shall now give a description of the norm induced on $X \otimes Y$. We claim that $\|x \otimes y\| = \|x\| \|y\|$ for $x \in X$, $y \in Y$.
By the Hahn Banach theorem we get functionals $x' \in OX'$ and $y' \in OY'$ such that $< x,x' > = \|x\|$ and $< y,y' > = \|y\|$.

Now $\langle \cdot, x' \rangle \langle \cdot\cdot, y' \rangle \in OB(X,Y)$ and $\langle \langle \cdot, x' \rangle \langle \cdot\cdot, y' \rangle, x \otimes y \rangle =$

$= \langle x, x' \rangle \langle y, y' \rangle = \|x\| \, \|y\|$, which means that

$\|x \otimes y\| \geq \|x\| \, \|y\|$.

Since $\|\pi\| \leq 1$, equality must hold.

For $u \in X \otimes Y$ we define $\gamma(u) = \inf \{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \}$

It is easy to check that γ is a semi-norm on $X \otimes Y$.

We have $\| \sum_{i=1}^{n} x_i \otimes y_i \|_{X \hat{\otimes} Y} \leq \Sigma \|x_i \otimes y_i\|_{X \hat{\otimes} Y} \leq \Sigma \|x_i\| \, \|y_i\|$

and therefore $\|u\|_{X \hat{\otimes} Y} \leq \gamma(u)$, which means that γ is a norm.

As is easily seen, the completion of $X \otimes Y$ in the norm γ fulfills the universal property of the projective tensor product and the isomorphism constructed in 1.5 acts as the identity on $X \otimes Y$. Thus $\|\cdot\|_{X \hat{\otimes} Y} = \gamma$ and we will denote it by $\|\cdot\|^{\wedge}$ from now on.

As a résumé we have the following

<u>Theorem:</u> The projective tensor product $X \hat{\otimes} Y$ of two Banach spaces X and Y is the completion of the algebraic tensor product in the norm $\|\cdot\|^{\wedge}$, given by

$$\|u\|^{\wedge} = \inf \{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \}.$$

The spaces $H(X, H(Y,Z))$ and $H(X \hat{\otimes} Y, Z)$ are isometrically isomorphic via $\varphi \leftrightarrow \tilde{\varphi}$, where $\varphi(x)(y) = \tilde{\varphi}(x \otimes y)$.

1.7. As a special case we have $(X \hat{\otimes} Y)' = H(X \hat{\otimes} Y, I) = H(X, Y')$. Thus $H(X,Y')$ appears as the dual of a Banach space and has therefore the structure of a Waelbroeck space. Since the unit ball of $H(X,Y')$ is equicontinuous with respect to $X \hat{\otimes} Y$, the restriction of the topology $\sigma(H(X,Y'), X \hat{\otimes} Y)$

coincides with $\sigma(H(X,Y'), X \otimes Y)$, i.e. the topology of pointwise convergence (see Schaefer [73], III, 4.5). So the Waelbroeck structure coincides with that introduced in I, 2.17, i.e. $\Omega(X,Y') = (X \hat{\otimes} Y)'$

Clearly the duality between $X \hat{\otimes} Y$ and $\Omega(X,Y')$ is given by $< x \otimes y, f > = < y, f(x) >$.

1.8. We now collect some elementary properties of the projective tensor product.

a) $X \hat{\otimes} Y = Y \hat{\otimes} X$ via $x \otimes y \rightarrow (x \otimes y)^t = y \otimes x$

By duality we get an isomorphism of $\Omega(Y,X')$ and $\Omega(X,Y')$:
$< {}^t(x \otimes y), f > = < y \otimes x, f > = < x, f(y) > = < y, f^t(x) > =$
$= < x \otimes y, f^t >$ i.e. this isomorphism transforms $f \in \Omega(Y,X')$ into $f^t \in \Omega(X,Y')$ i.e. the map considered in I.2.13.

b) $I \hat{\otimes} X = X \hat{\otimes} I = X$ the isomorphism being given by
$\alpha \otimes x \rightarrow \alpha x$ and $x \rightarrow 1 \otimes x$ (where $\alpha \in I$, $x \in X$)

c) $(X \hat{\otimes} Y) \hat{\otimes} Z = X \hat{\otimes} (Y \hat{\otimes} Z)$ via the identification $(x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$. This map is defined by using the universal property twice (the first time with z fixed). Another proof is obtained from the fact that the duals of both spaces coincide with the Banach space of all bounded trilinear maps from $X \times Y \times Z$ to I and the w*- topology on the unit ball is in both cases the topology of compact convergence (or pointwise convergence).

d) Given $f \in H(X,X_1)$, $g \in H(Y,Y_1)$, then the linear map $f \otimes g : X \otimes Y \rightarrow X_1 \otimes Y_1$ defined by $(f \otimes g)(\Sigma x_i \otimes y_i) =$

$= \Sigma\, f(x_i) \otimes g(y_i)$ extends to a bounded linear map $f \hat{\otimes} g : X \hat{\otimes} Y \rightarrow X_1 \hat{\otimes} Y_1$ which satisfies $\|f \hat{\otimes} g\| = \|f\|\,\|g\|$.
If furthermore $f_1 \in H(X_1, X_2)$, $g_1 \in H(Y_1, Y_2)$ then

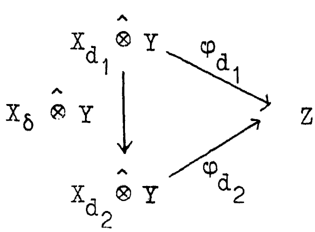
$$(f_1 \hat{\otimes} g_1) \cdot (f \hat{\otimes} g) = (f_1 \cdot f) \hat{\otimes} (g_1 \cdot g).$$

The algebraic part can be checked by using the description of $\|\cdot\|^{\wedge}$ given in 1.6 or by the observation that $(x,y) \rightarrow f(x) \otimes g(y)$ is a bilinear map from $X \times Y$ to $X_1 \hat{\otimes} Y_1$ with norm $\|f\|\,\|g\|$.
The projective tensor product thus becomes a bifunctor and it is easily seen that all isometric equalities of 1.7. and 1.8 a), b), c) are natural in all variables.

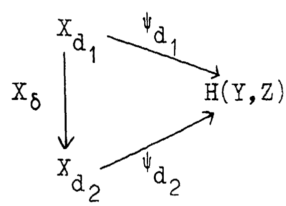
e) If $(X_d)_{d \in D}$ is a spectral family in Ban_1, then

$$(\underrightarrow{\lim}\, X_d) \hat{\otimes} Y = \underrightarrow{\lim}\, (X_d \hat{\otimes} Y) \qquad \text{(see I. 1.17)}$$

The proof is the usual categorical one for the fact that left adjoint functors commute with colimits and uses heavily the naturality of the equation $H(X \hat{\otimes} Y, Z) = H(X, H(Y,Z))$.
We sketch it here:
Let $(\varphi_d)_{d \in D}$ be a family of morphisms in Ban_1 $(\varphi_d : X_d \hat{\otimes} Y \rightarrow Z)$ and ψ_d the corresponding elements of $H(X_d, H(Y,Z))$. Each commutative diagram:



is equivalent to



Thus there is a unique morphism $\psi : \varinjlim X_d \to H(Y,Z)$

such that

$$\begin{array}{ccc} X_d & \xrightarrow{\psi_d} & H(Y,Z) \\ & \searrow & \nearrow \psi \\ & \varinjlim X_d & \end{array}$$

commutes.

If $\varphi \in H((\varinjlim X_d) \hat{\otimes} Y, Z)$ corresponds to ψ, this diagram is in turn equivalent to

$$\begin{array}{ccc} X_d \hat{\otimes} Y & \xrightarrow{\varphi_d} & Z \\ \downarrow & \nearrow \varphi & \\ (\varinjlim X_d) \hat{\otimes} Y & & \end{array}$$

f) As consequences we obtain:

If $f : X \to X/M$ is a quotient map, then $f \hat{\otimes} 1_Y : X \hat{\otimes} Y \to (X/M) \hat{\otimes} Y$ is also a quotient map, whose kernel coincides with the closure of $M \otimes Y$ in $X \hat{\otimes} Y$.

(f is the cokernel of the embedding of M into X, see I. 1.18)

$l^1_S \hat{\otimes} Y = (\sum_{s \in S} I_s) \hat{\otimes} Y = \sum_{s \in S} (I_s \hat{\otimes} Y) = l^1_S(Y)$, where the equivalence is given by $(\xi_s)_{s \in S} \otimes y \to (\xi_s y)_{s \in S}$, and

$l^1_S \hat{\otimes} l^1_T = l^1_{S \times T}$ via $(\xi_s)_{s \in S} \otimes (\eta_t)_{t \in T} \to (\xi_s \eta_t)$.

g) For each $u \in X \hat{\otimes} Y$ and $\epsilon > 0$ there are $(x_n)_{n=1}^{\infty} \subseteq OX$, $(y_n)_{n=1}^{\infty} \subseteq OY$ and $(\lambda_n)_{n=1}^{\infty} \in l^1$ such that

$$u = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n \quad \text{and} \quad \|u\|^{\wedge} \geq \|(\lambda_n)\|_{l^1} - \epsilon$$

Proof: Let $\pi_X : l^1_{0X} \to X$, $\pi_Y : l^1_{0Y} \to Y$ be the quotient maps of (I.1.11). By f) $\pi_X \hat{\otimes} \pi_Y : l^1_{0X \times 0Y} = l^1_{0X} \hat{\otimes} l^1_{0Y} \to$ $\to X \hat{\otimes} Y$ is a quotient map too. The assertion is now immediate.

h) If $f : X \to X_1$ is epi (i.e. it has dense image, see I.1.4) then $f \hat{\otimes} 1_Y : X \hat{\otimes} Y \to X_1 \hat{\otimes} Y$ is also epi, since $f' : X_1' \to X'$ mono implies $(f \hat{\otimes} 1_Y)' = \mathfrak{L}(Y,f')$: $\mathfrak{L}(Y,X_1') \to \mathfrak{L}(Y,X')$ mono.

i) If (Ω,Σ,μ) is a measure space and $L^1_\mu(\Omega,X)$ the space of all X-valued (Bochner-) integrable functions on Ω, then $L^1_\mu(\Omega,X) = L^1_\mu(\Omega) \hat{\otimes} X$.

Proof: To $(f,x) \in L^1_\mu(\Omega) \times X$ we assign the function $\omega \to f(\omega)x$. In this way, we get a linear, contractive map from $L^1_\mu(\Omega) \hat{\otimes} X$ into $L^1_\mu(\Omega,X)$ with dense image. Let $f_i = \sum_{j=1}^m \alpha_{ij} c_{A_j} \in L^1_\mu(\Omega)$ be step-functions and $x_i \in X$ $(i = 1,\dots,n)$. Then $\sum_{i=1}^n f_i \otimes x_i = \sum_{j=1}^m c_{A_j} \otimes (\sum_{i=1}^n \alpha_{ij} x_i)$ and its image has the norm:

$$\|\sum_{j=1}^m c_{A_j}(\sum_{i=1}^n \alpha_{ij}x_i)\|_{L^1_\mu(\Omega,X)}$$

$$= \sum_{j=1}^m \mu(A_j)\|\sum_{i=1}^n \alpha_{ij}x_i\| \le \sum_{i,j} |\alpha_{ij}|\mu(A_j)\|x_i\| =$$

$$= \sum_{i=1}^n \|f_i\|_{L^1_\mu(\Omega)}\|x_i\|.$$

This means that our map is an isometry on a dense subspace of $L^1_\mu(\Omega) \hat{\otimes} X$ and therefore an isometrical isomorphism.

In this case, for an isometry $f : X \to Y$ the map $L^1_\mu(\Omega) \hat{\otimes} f : L^1_\mu(\Omega) \hat{\otimes} X \to L^1_\mu(\Omega) \hat{\otimes} Y$ is also isometric This is wrong in general.

1.9. We want to determine those isometries f such that $f \hat{\otimes} Z$ is an isometry for any Banach space Z.

<u>Definition:</u> $f : X \to Y$, $\|f\| \leq 1$ is said to be a <u>weak retract</u> if there exists a map $h : X' \to Y'$, with $\|h\| \leq 1$ such that $f' \circ h = 1_{X'}$.

This means geometrically that there exists a contractive projection p from Y" onto X", i.e. such that $p \circ f'' = 1_{X''}$: If $f' \circ h = 1_{X'}$ then clearly $h' \circ f'' = 1_{X''}$ and if conversely $p \circ f'' = 1_{X''}$ then $p \circ \iota_Y \circ f = p \circ f'' \circ \iota_X = \iota_X$ and $f' \circ (p \circ \iota_Y)^t = 1_{X'}$.

The canonical embedding $\iota_X : X \to X''$ always is a weak retract, since $(\iota_X)' \circ \iota_{X'} = 1_{X'}$ holds.

<u>Proposition</u>: An isometry $f : X \to Y$ is a weak retract if and only if $f \hat{\otimes} Z : X \hat{\otimes} Z \to Y \hat{\otimes} Z$ is isometric for all Banach spaces Z.

<u>Proof</u>: Choose $Z = X'$: If $X \hat{\otimes} X'$ is a subspace of $Y \hat{\otimes} X'$ via $f \hat{\otimes} 1_{X'}$ we may extend the functional $1_{X'} \in \mathfrak{L}(X',X') = (X' \hat{\otimes} X)'$ to $h \in \mathfrak{L}(X',Y') = (X' \hat{\otimes} Y)'$ with $\|h\| = 1$ (Hahn Banach theorem). This just means that $f' \circ h = 1_{X'}$. If conversely f is a weak retract, $f' \circ h = 1_{X'}$, $\|h\| \leq 1$ and $Z \in$ Ban , we consider the map $(f \hat{\otimes} Z)' = \mathfrak{L}(Z,f') : \mathfrak{L}(Z,Y') \to \mathfrak{L}(Z,X')$.

For $g \in \mathfrak{L}(Z,X')$ we have $\mathfrak{L}(Z,f')(h \circ g) = f' \circ h \circ g = g$ and $\|h \circ g\| \leq \|g\|$.

$\mathfrak{L}(Z,f')$ thus is a quotient map and $f \hat{\otimes} Z$ isometric.

<u>Remark</u>: In § 3 we will see that for an isometry f, $f \hat{\otimes} Z$ need not be injective. The above geometric description of weak retracts shows that even for finite dimensional spaces X and Y, $f \hat{\otimes} Z$ is in general not an isometry. If one drops the condition $\|h\| \leq 1$ in the above definition of a weak retract and assumes mere continuity, one gets a necessary and sufficient condition for an isometry f to satisfy the condition that $f \hat{\otimes} Z$ maps $X \hat{\otimes} Z$ onto a closed subspace of $Y \hat{\otimes} Z$, for any Banach space Z.

1.10. The map $(x,x') \rightarrow \langle x,x' \rangle$ is evidently bilinear and contractive on $X \times X'$. It extends therefore to a linear and contractive map $tr : X \hat{\otimes} X' \rightarrow I$, called the trace functional.

For Banach spaces X and Y we associate to each pair $(x',y) \in X' \times Y$ the function $x \rightarrow \langle x,x' \rangle y$. It is an easy consequence of 1.3 b) that the corresponding map of $X' \otimes Y$ into $H(X,Y)$ is injective. Its image consists of all bounded, finite dimensional, linear maps from X into Y. If X = Y is finite dimensional then clearly $H(X,X) = X' \otimes X$ via the above identification. In this way the trace of a tensor coincides with the trace of the matrix, which represents the corresponding map. On the other hand, for a Banach space X without the approximation property the map from $X' \hat{\otimes} X$ into

H(X,X) which extends the above embedding, is not injective and the trace can now be defined only for the tensor and not for the induced map (see 3.4.).

A short computation shows that the duality action in $(X \hat{\otimes} Y)' = \mathfrak{L}(X,Y')$ is given by $< u,v > = \mathrm{tr}\,((v \hat{\otimes} 1_Y)(u)) = \mathrm{tr}\,((1_X \hat{\otimes} v^t)(u))$. ($u \in X \hat{\otimes} Y$, $v \in H(X,Y')$; it suffices to verify the equation for finite tensors u and then to use continuity on both sides)

§2. General tensor products of Banach spaces.

2.1. Let X and Y be Banach spaces.
For $(x',y') \in X' \times Y'$ we define a linear map on $X \otimes Y$ by $< x \otimes y, x' \otimes y' > = < x,x' > < y,y' >$ (use the universal property).
The assignment is clearly bilinear and we get a map of $X' \otimes Y'$ into $(X \otimes Y)^*$, which is injective by 1.3 b).

Definition: A norm α on $X \otimes Y$ is said to be reasonable, if each element $(x',y') \in X' \times Y'$ defines a continuous functional on $X \otimes_\alpha Y$ and if both bilinear maps:

$X \times Y \to X \otimes_\alpha Y$ defined by $(x,y) \to x \otimes y$
$X' \times Y' \to (X \otimes_\alpha Y)'$ defined by $(x',y') \to x' \otimes y'$
have norm ≤ 1.

Here $X \otimes_\alpha Y$ means the completion of $X \otimes Y$ in the norm α.

If α is reasonable we have:

$$\|x \otimes y\|_{X \underset{\alpha}{\otimes} Y} = \|x\| \, \|y\| \text{ and } \|x' \otimes y'\|_{(X \underset{\alpha}{\otimes} Y)'} = \|x'\| \, \|y'\|.$$

<u>Proof</u>: ' $\leq$ ' follows from the definition.

$$\|x \otimes y\|_{X \underset{\alpha}{\otimes} Y} = \sup_{\varphi \in O(X \underset{\alpha}{\otimes} Y)'} |< x \otimes y, \varphi >| =$$

$$\geq \sup_{\|x' \otimes y'\|_{(X \underset{\alpha}{\otimes} Y)'} \leq 1} |< x \otimes y, x' \otimes y' >|$$

$$\geq \sup_{\|x'\|, \|y'\| \leq 1} |< x, x' > < y, y' >| = \|x\| \, \|y\|$$

$$\|x' \otimes y'\|_{(X \underset{\alpha}{\otimes} Y)'} \geq \sup_{\|x \otimes y\| \leq 1} |< x \otimes y, x' \otimes y' >| = \|x'\| \, \|y'\|$$

2.2. To $(x,y) \in X \times Y$ corresponds a bounded bilinear map on $X' \times Y'$ by $(x',y') \to < x,x' > < y,y' >$. This defines a map of $X \otimes Y$ into $B(X',Y')$, which is injective by 1.3. b)

<u>Definition</u>: The closure of $X \otimes Y$ in $B(X',Y')$ is called the <u>inductive tensor product</u> and is denoted by $X \overset{\wedge}{\otimes} Y$.
It is the completion of $X \otimes Y$ in the norm

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|^{\wedge} = \sup_{\substack{\|x'\| \leq 1 \\ \|y'\| \leq 1}} \left| \sum_{i=1}^{n} < x_i, x' > < y_i, y' > \right| \quad \text{where}$$

the right hand expression is independent of the representation in $X \otimes Y$.

For $f \in H(X,X_1)$, $g \in H(Y,Y_1)$, the map $f \otimes g$ has clearly a continuous extension to $X \overset{\wedge}{\otimes} Y$, which is denoted by

$f \hat{\hat{\otimes}} g$ and satisfies $\|f \hat{\hat{\otimes}} g\| \le \|f\| \, \|g\|$, i.e. the inductive tensor product defines a bifunctor.

Since $B(X',Y') = H(X',Y'')$ isometrically and since the image of $X \otimes Y$ in $H(X',Y'')$ is contained in the closed subspace $L(X',Y)$ of all linear maps whose restriction to OX' is w^*- norm - continuous, we may consider $X \hat{\hat{\otimes}} Y$ as a subspace of $L(X',Y)$. (see I. 2.16.). The embedding of $X \otimes Y$ into $L(X',Y)$ is given by $x \otimes y \to (x' \to \langle x,x' \rangle y)$.

By means of this embedding the transposition $t : X \hat{\hat{\otimes}} Y \to Y \hat{\hat{\otimes}} X$ corresponds to the transposition $t : L(X',Y) \to L(Y',X)$ defined in I.2.12. Consequently $X \hat{\hat{\otimes}} Y = Y \hat{\hat{\otimes}} X$.

Since the embedding of $X \hat{\hat{\otimes}} Y$ into $L(X',Y)$ is easily seen to be natural, we conclude that $X \hat{\hat{\otimes}} f$ is an isometry whenever f is.

2.3. Proposition: The reasonable crossnorms are exactly those norms α that satisfy $\|\cdot\|^{\hat{\hat{}}} \le \alpha \le \|\ \|^{\hat{}}$

Proof: Let α be a reasonable norm on $X \otimes Y$.

For any representation $u = \sum_{i=1}^{n} x_i \otimes y_i$ we have

$$\alpha(u) = \alpha\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \le \sum_{i=1}^{n} \alpha(x_i \otimes y_i) \le \sum_{i=1}^{n} \|x_i\| \, \|y_i\|,$$

and thus $\alpha(u) \le \|u\|^{\hat{}}$.

On the other hand:

$$\alpha(u) = \alpha(\Sigma\, x_i \otimes y_i)$$
$$= \sup_{\varphi \in O(X \otimes_\alpha Y)'} |\langle \Sigma\, x_i \otimes y_i, \varphi \rangle|$$

$$\geq \sup_{\|x' \otimes y'\|_{(X \otimes_\alpha Y)'} \leq 1} |< \Sigma x_i \otimes y_i, x' \otimes y' >|$$

$$\geq \sup_{\|x'\|, \|y'\| \leq 1} | \Sigma < x_i, x' > < y_i, y' >| = \|u\|^{\hat{}}.$$

If conversely a norm α on $X \otimes Y$ satisfies $\|\cdot\|^{\hat{}} \leq \alpha \leq \| \|^{\hat{}}$, then $\alpha(x \otimes y) \leq \|x \otimes y\|^{\hat{}} = \|x\| \|y\|$.

We have $|< \sum_{i=1}^{n} x_i \otimes y_i, x'_0 \otimes y'_0 >| \leq$

$$\leq \|x'_0\| \|y'_0\| \sup_{\|x'\|, \|y'\| \leq 1} |< \Sigma x_i \otimes y_i, x' \otimes y' >|$$

$$= \|x'_0\| \|y'_0\| \|\Sigma x_i \otimes y_i\|_{X \hat{\otimes} Y}$$ and this means

$$\|x'_0 \otimes y'_0\|_{(X \hat{\otimes} Y)'} \leq \|x'_0\| \|y'_0\|.$$ Finally

$$\|x' \otimes y'\|_{(X \otimes_\alpha Y)'} \leq \|x' \otimes y'\|_{(X \hat{\otimes} Y)'} \leq \|x'\| \|y'\|.$$

2.4. Let T, S be locally compact topological spaces and X be a Banach space. We denote by $C_0(T,X)$ the Banach space of all X- valued, continuous functions on T that vanish at infinity. We write simply $C_0(T)$ for $C_0(T,I)$.

<u>Theorem:</u> For locally compact topological spaces T, S and a Banach space X, we have the equations:

$C_0(T) \hat{\otimes} X = C_0(T,X)$ and $C_0(T) \hat{\otimes} C_0(S) = C_0(T \times S)$.

<u>Proof</u>: Consider the linear map which assigns to $\sum_{i=1}^{n} f_i \otimes x_i \in C_0(T) \otimes X$ the function $(t \to \sum_{i=1}^{n} f_i(t)x_i) \in C_0(T,X)$.

This map is an isometry for the inductive norm since

$$\|\Sigma f_i \otimes x_i\|^{\hat{}} = \sup_{\|x'\| \leq 1} \|\Sigma < x_i, x' > f_i\|_{C_o(T)}$$

$$= \sup_{\|x'\| \leq 1} \sup_{t \in T} |\Sigma < x_i, x' > f_i(t)|$$

$$= \sup_{t \in T} \|\Sigma f_i(t) x_i\|_X = \|\sum_i f_i(\cdot) x_i\|_{C_o(T,X)}$$

For the first assertion it remains to show that $C_o(T) \otimes X$ is dense in $C_o(T,X)$. Now fix some element $g \in C_o(T,X)$ and $\epsilon > 0$, and consider the sets $A = \{ t \in T : \|g(t)\| \geq \epsilon\}$ and $V_t = \{s \in T : \|g(s) - g(t)\| < \epsilon/2\}$, where $t \in T$ is arbitrary. $\{V_t\}_{t \in A}$ is an open cover of the compact set A in T, hence there are $t_1, \ldots, t_n \in A$ such that $\{V_{t_i}\}_{i=1}^n$ covers A . There exists a partition of unity subordinate to the cover $V_{t_1}, \ldots, V_{t_1}$, $(T \backslash A) \cup \{\infty\}$ of the one point compactification $T \cup \{\infty\}$ of T, i.e. there are continuous functions $p_1, \ldots, p_n, p_{n+1}$ on T such that $0 \leq p_i \leq 1$ $(1 \leq i \leq n+1)$, $p_i(t) = 0$ in $T \backslash V_{t_i}$ $(1 \leq i \leq n)$, $p_{n+1}(t) = 0$ in A and $\sum_{i=1}^{n+1} p_i(t) = 1$ on T. Since p_i vanishes outside the compact set $\overline{V}_{t_i}$ $(1 \leq i \leq n)$, we have $p_1, \ldots, p_n \in C_o(T)$. Denote $g(t_i)$ by x_i $(1 \leq i \leq n)$, and consider $\sum_{i=1}^n p_i \otimes x_i \in C_o(T) \otimes X$. We conclude that $p_i(t) \cdot \|g(t) - g(t_i)\| \leq \epsilon\, p_i(t)$ $(1 \leq i \leq n)$, since either $p_i(t) = 0$ or the second factor is smaller than ϵ.
Consequently

$$\|g(t) - \sum_{i=1}^n p_i(t) x_i\| = \|\sum_{i=1}^{n+1} p_i(t) g(t) - \sum_{i=1}^n p_i(t) x_i\| \leq$$

$$\leq \sum_{i=1}^{n} p_i(t)\|g(t) - g(t_i)\| + p_{n+1}(t)\|g(t)\| < \varepsilon.$$

For the second assertion we use the map

$f \otimes g \in C_0(T) \otimes C_0(S) \rightarrow ((t,s) \rightarrow f(t)g(s)) \in C_0(T \times S)$.

As above it is shown that the map is an isometry, and the Stone-Weierstrass theorem tells us that $C_0(T) \otimes C_0(S)$ is dense in $C_0(T \times S)$.

Corollary:

a) Let S, T be compact topological spaces and X a Banach space. Then $C(T) \hat{\hat{\otimes}} X = C(T,X)$, $C(S) \hat{\hat{\otimes}} C(T) = C(S \times T)$.

b) Let S be a set (considered as a discrete topological space) and let $S \cup \{\infty\}$ be its one point compactification. Then $C(S \cup \{\infty\})$ is the set of all nets $(x_s)_{s \in S}$ which converge unconditionally, i.e. for each $\varepsilon > 0$ there exists a finite subset E of S, such that $|x_s - x| < \varepsilon$ for all $s \in S \setminus E$, where x is the limit of (x_s).
$C(S \cup \{\infty\}, X)$ is the space of all unconditionally convergent nets in X and we conclude:

$$C(S \cup \{\infty\}, X) = C(S \cup \{\infty\}) \hat{\hat{\otimes}} X.$$

In particular, we get $c(X) = c \hat{\hat{\otimes}} X$, where c is the space of all convergent sequences.

c) If S is as above, then $C_0(S)$ is the space of all nets which converge unconditionally to 0.

Therefore $C_o(S) \hat{\hat{\otimes}} X = C_o(S,X)$.

A special case is $c_o = C_o(\mathbb{N})$: $c_o \hat{\hat{\otimes}} X = c_o(X)$.

2.5. Theorem: $l^1_S \hat{\otimes} X$ is the space of all (unconditionally) summable families $(x_s)_{s \in S}$ in X, i.e. those $(x_s)_{s\in S}$ for which $\sum_{s\in E} x_s$ converges , where E runs through the system of all finite subsets of S, directed by inclusion.

First we need a simple lemma.

Lemma: If $(\xi_s)_{s \in S}$ is a family of complex numbers such that for each finite subset $E \subseteq S$ we have $|\sum_{s\in E} \xi_s| \leq \varepsilon$, then for each finite subset $\sum_{s\in E} |\xi_s| \leq 4\varepsilon$ holds.

Proof: $|\sum_{s\in E} \text{Re}\ \xi_s| = |\text{Re} \sum_{s\in E} \xi_s| \leq |\sum_{s\in E} \xi_s| \leq \varepsilon$,

similarly $|\sum_{s\in E} \text{Im}\ \xi_s| \leq \varepsilon$.

Now $\text{Re}\ \xi_s$ and $\text{Im}\ \xi_s$ are ≥ 0 or < 0 and for the subset F of E where $\text{Re}\ \xi_s \geq 0$

$\sum_{s\in F} |\text{Re}\ \xi_s| = |\sum_{s\in F} \text{Re}\ \xi_s| \leq \varepsilon$. The other cases are similar.

Thus $\sum_{s\in E} |\text{Re}\ \xi_s| \leq 2\varepsilon$, $\sum_{s\in E} |\text{Im}\ \xi_s| \leq 2\varepsilon$ and consequently $\sum_{s\in E} |\xi_s| \leq 4\varepsilon$.

Proof of the theorem:

$X \hat{\otimes} l^1_S$ is isometrically contained in $L(l^{1\,\prime}_S, X) = L(l^\infty_S, X)$.

For $s_o \in S$ we put $\varepsilon_{s_o} = (\delta_{s_o s})_{s\in S} \in l^\infty_S$

To $\varphi \in L(l^\infty_S, X)$ we assign the family $(x_s)_{s\in S}$ defined by $x_s = \varphi(\varepsilon_s)$. By I.2.16 we know that φ' maps X' into the subspace l^1_S of $(l^\infty_S)'$ and therefore

$$\|\varphi\| = \|\varphi'\| = \sup_{x' \in OX'} \|\varphi'(x')\| = \sup_{x' \in OX'} \sum_{s \in S} |<\epsilon_s, \varphi'(x')>|$$
$$= \sup_{x' \in OX'} \sum_{s \in S} |<x_s, x'>|.$$

If we order the finite subsets of S by inclusion, it is easily seen that $\sum_{s \in E} \epsilon_s$ converges in the w*- topology of l^∞_S to the constant function. Consequently $\sum_{s \in E} x_s = \varphi(\sum_{s \in E} \epsilon_s)$ converges in the norm topology.

From the above norm equation it follows that φ is uniquely determined by the (x_s).

Now let conversely (x_s) be a summable family in X and $\epsilon > 0$. Then there is a finite subset E of S such that for each finite subset $F \subseteq S \setminus E$ we have

$\|\sum_{s \in F} x_s\| < \epsilon$, i.e. $\sup_{x' \in OX'} |\sum_{s \in F} < x_s, x' >| < \epsilon$.

Using the lemma above we conclude that

$$\sup_{x' \in OX'} \sum_{s \in F} |<x_s, x'>| \leq 4\epsilon.$$

For $s_0 \in S$ take $e_{s_0} = (\delta_{ss_0})_{s \in S} \in l^1_S$. To each finite subset E of S we assign the element $\varphi_E = \sum_{s \in E} x_s \otimes e_s \in X \otimes l^1_S \subseteq L(l^\infty, X)$. Obviously $\varphi_E(\epsilon_s) = x_s$ for $s \in E$ and $\varphi_E(\epsilon_s) = 0$ elsewhere. By the above argument φ_E is a Cauchy-net in $L(l^\infty_S, X)$ and converges therefore to $\varphi \in X \hat{\hat{\otimes}} l^1_S \subseteq L(l^\infty_S, X)$. $\varphi(\epsilon_s) = x_s$, which yields the desired conclusion.

2.6. <u>Remarks</u>: a) The proof of theorem 2.5. also shows that $l^1_S \hat{\hat{\otimes}} X = L(l^\infty_S, X)$, a fact which also is an easy consequence of 3.5.

b) It has been shown by Dvoretzky and Rogers [22] that each infinite-dimensional Banach space X contains a sequence which is summable but not absolutely summable, i.e. $l^1 \hat{\otimes} X \neq l^1 \hat{\hat{\otimes}} X$. Thus if $\| \ \|^\wedge$ and $\| \ \|^{\hat{\wedge}}$ are equivalent on $l^1 \otimes X$ then X is finite dimensional. The construction of such a sequence is based on the observation that $x_n = (0,0,\ldots,0, \frac{1}{n}, 0,\ldots)$ constitutes an example of a summable sequence in l^2 for which $\Sigma \|x_n\| = \Sigma \frac{1}{n} = \infty$.

2.7. Let α be a reasonable norm on $X \otimes Y$. Since by 2.3. $\| \cdot \|^\wedge \geq \alpha \geq \| \cdot \|^{\hat{\wedge}}$, the identity on $X \otimes Y$ has continuous extensions: $X \hat{\otimes} Y \to X \underset{\alpha}{\otimes} Y \to X \hat{\hat{\otimes}} Y$. Both maps are contractive and have dense images. Thus the adjoint maps

$$(X \hat{\hat{\otimes}} Y)' \to (X \underset{\alpha}{\otimes} Y)' \to (X \hat{\otimes} Y)' = H(X,Y')$$

are monomorphic in $\underline{W}$, i.e. they are injective.

<u>Proposition</u>: If α is a reasonable norm on $X \otimes Y$, then $(X \underset{\alpha}{\otimes} Y)'$ coincides with the space of all $f \in H(X,Y')$ such that the linear functional $\Sigma x_i \otimes y_i \to \Sigma < y_i, f(x_i) >$ on $X \otimes Y$ is continuous in the α-norm.

We have $\|f\|_{(X \underset{\alpha}{\otimes} Y)'} = \sup \{ |\Sigma < y_i, f(x_i) >| :$

$: \alpha(\Sigma x_i \otimes y_i) \leq 1 \}$

Moreover $(X \hat{\hat{\otimes}} Y)'$ is always contractively contained in $(X \underset{\alpha}{\otimes} Y)'$.

Proof: Clear by the preceding argument.

2.8. For a reasonable norm α on $X \otimes Y$ it is easily seen that $(X \underset{\alpha}{\otimes} Y)'$ induces a reasonable norm α' on $X' \otimes Y'$. Starting with $\alpha = \| \ \|^\wedge$, the "greatest" crossnorm, one gets $\alpha' = \| \ \|^\star$ the "least" crossnorm. On the other hand, we need not have $\alpha' = \| \ \|^\wedge$ for $\alpha = \| \ \|^{\hat{\wedge}}$. (see 3.9).
We will now treat a special case:

Lemma: For a finite dimensional space M and an arbitrary Banach space X one has $(M \overset{\star}{\otimes} X)' = M' \hat{\otimes} X'$.

Proof: We use the map of 2.1. Since M is finite dimensional every bounded linear functional on $M \overset{\hat{\wedge}}{\otimes} X$ has a representation in $M' \otimes X'$ and by 2.3 the norm induced on $M' \otimes X'$ is smaller than $\| \ \|^\wedge$.
For $M = l_n^\infty$ one has $l_n^\infty \overset{\hat{\wedge}}{\otimes} X = l_n^\infty(X)$ and

$l_n^\infty(X)' = l_n^1(X')$.

Now consider an arbitrary finite dimensional M. Take $u \in M' \otimes X'$ and $\varepsilon > 0$. Since OM' is compact in the norm topology, one can find a finite ε-mesh $\{y_1', \ldots, y_n'\}$ therein. Define $\varphi : M \to l_n^\infty$ by $y \to (< y, y_1' >, \ldots, < y, y_n' >)$. If $y \in M$ and $\|y\| = 1$, there exists an element $y' \in OM'$ such that $< y, y' > \geq 1 - \varepsilon$. Then $\|y' - y_i'\| \leq \varepsilon$ for a certain index i and therefore $|< y, y_i' >| \geq |< y, y' >| - |< y, y_i' - y' >| > 1 - \varepsilon - \varepsilon =$

$= 1 - 2\epsilon$. It follows that $\|y\| \geq \|\varphi(y)\| \geq (1 - 2\epsilon)\ \|y\|$ for all $y \in M$. If we assume in addition that $\epsilon < \frac{1}{2}$ and take $\varphi(M) = M_1$, then $\varphi\colon M \to M_1$ is an isomorphism with $\|\varphi^{-1}\| \leq \frac{1}{1-2\epsilon}$ and $\|\varphi\| \leq 1$.

$u_1 = (\varphi^{-1} \hat{\hat{\otimes}} 1_X)'(u)$ is a functional on $M_1 \hat{\hat{\otimes}} X$ and $\|u_1\| \leq \frac{\|u\|}{1-2\epsilon}$. Since $M_1 \hat{\hat{\otimes}} X$ is isometrically contained in $l_n^\infty \hat{\hat{\otimes}} X$, there exists an extension u_2 of u_1 with $\|u_2\| \leq \frac{\|u\|}{1-2\epsilon}$. Now $u_2 \in (l_n^\infty \hat{\hat{\otimes}} X)' = l_n^1 (X')$ is represented by $(x_1', \ldots, x_n')$ and the action has the following form:

$$\langle y \otimes x,\ u\rangle = \langle \varphi(y) \otimes x,\ u_2\rangle = \langle(\langle y,y_1'\rangle, \ldots, \langle y,y_n'\rangle) \otimes x, u_2\rangle =$$
$$= \sum_{i=1}^{n} \langle x,x_i'\rangle\ \langle y,y_i'\rangle \text{ for all } y \in M,\ x \in X.$$

This means that $\sum_{i=1}^{n} y_i' \otimes x_i'$ is a representation of u for which the equation $\sum_{i=1}^{n} \|y_i'\|\ \|x_i'\| \leq \Sigma\ \|x_i'\| \leq \frac{\|u\|}{1-2\epsilon}$ holds. Consequently $\|u\|^\wedge \leq \|u\|$.

<u>Corollary 1:</u> Let X be a Banach space, N a finite dimensional subspace of X", $j\colon N \to X''$ the inclusion. Then there exists a net $(u_\iota) \subset OH(N,X)$ such that $\lim u_\iota = j$ in the w*-topology of $\Omega(N,X'')$.

<u>Proof:</u> $H(N,X) = N' \hat{\hat{\otimes}} X$ because N is finite dimensional. Consequently $H(N,X)' = (N' \hat{\hat{\otimes}} X)' = N \hat{\otimes} X'$ and

$H(N,X)'' = H(N,X'')$. It is easily seen that the canonical inclusion of $H(N,X)$ into $H(N,X)''$ corresponds to the embedding $u \to \iota_X \circ u$. Now the result follows immediately from the fact that $OH(N,X)$ is w^*-dense in $OH(N,X)''$.

Corollary 2: Let X, Y be Banach spaces. Then $X \hat{\hat{\otimes}} Y \cap OH(X',Y)$ is dense in $X'' \hat{\hat{\otimes}} Y \cap OH(X', Y)$ for the topology of compact convergence.

Proof: The topology of compact convergence on $OH(X',Y)$ coincides with the topology of pointwise convergence (compare the proof of 3.8). Take $u = \sum_{i=1}^{n} x_i'' \otimes y_i \in X'' \otimes Y \cap OH(X',Y)$. We use Cor. 1 for $N = < x_1'',\ldots,x_n'' >$. u may be viewed as an element of $N \otimes Y$. Then $(u_\iota \otimes 1_Y)(u) \in X \otimes Y \cap OH(X',Y)$ and $\lim(u_\iota \otimes 1_Y)(u) = u$ for the topology of pointwise convergence, since $(u_\iota \otimes 1_Y)(u) = \Sigma\, u_\iota(x_i'') \otimes y_i$ and $(u_\iota \otimes 1_Y)(u)(x') =$ $= \Sigma < u_\iota(x_i''), x' > y_i \to \Sigma < x', x_i'' > y_i$ for all $x' \in X'$.

2.9. For Banach spaces X, Y let $I_1(X,Y)$ be the space of all maps $u \in H(X,Y)$ such that $\iota_Y \circ u \in (X \hat{\hat{\otimes}} Y')'$ (compare 2.7), equipped with the norm $\|u\|_{I_1} = \|\iota \circ u\|_{(X \hat{\otimes} Y')'}$.

$I_1(X,Y)$ thus becomes a Banach space. It follows easily from the definitions that for $v \in H(W,X)$ and $w \in H(Y,Z)$ we have $w \circ u \circ v \in I_1(W,Z)$ and $\|w \circ u \circ v\|_{I_1} \leq$ $\leq \|w\|\,\|v\|\,\|u\|_{I_1}$, i.e. I_1 is a contra-covariant bifunctor.

Lemma: Let X and Y be Banach spaces, then the following assertions hold:

a) $u \in I_1(X,Y)$ implies $\iota_Y \circ u \in I_1(X,Y'')$ and $\|u\|_{I_1} = \|\iota_Y \circ u\|_{I_1}$

b) $u \in I_1(X,Y)$ implies $u' \in I_1(Y',X')$ and $\|u\|_{I_1} = \|u'\|_{I_1}$

c) $I_1(Y,X') = (Y \hat{\hat{\otimes}} X)'$

Proof: a) $\iota_Y \circ u \in I_1(X,Y'')$ follows from the above argument. We have furthermore: $\|u\|_{I_1} = \|\iota_Y \circ u\|_{(X \hat{\hat{\otimes}} Y')'} =$

$= \|(1_X \hat{\hat{\otimes}} \iota_{Y'})'(\iota_{Y''} \circ \iota_Y \circ u)\|_{(X \hat{\hat{\otimes}} Y')'} \leq \|\iota_{Y''} \circ \iota_Y \circ u\|_{(X \hat{\hat{\otimes}} Y''')'} =$

$= \|\iota_Y \circ u\|_{I_1} \leq \|u\|_{I_1}$

c) For $u \in I_1(Y,X')$ it follows that $\|u\|_{(Y \hat{\hat{\otimes}} X)'} \leq \|u\|_{I_1}$ since $Y \hat{\hat{\otimes}} X$ is isometrically contained in $Y \hat{\hat{\otimes}} X''$.

If $v = \sum_{i=1}^{n} y_i \otimes x_i''$ we construct as in the proof of 2.8. Cor. 2 a net $(v_\iota) \subseteq OH(N,X)$ such that

$\lim v_\iota(x_i'') = x_i''$ in the w^*-topology.

For $u \in (Y \hat{\hat{\otimes}} X)'$ we conclude then:

$< v, \iota_Y \circ u > = \Sigma < u(y_i), x_i'' > = \lim \Sigma < v_\iota(x_i''), u(y_i) > =$

$\lim < (1_Y \hat{\hat{\otimes}} v_\iota)(v), u >$. Consequently $|< v, \iota_Y \circ u >| \leq$ $\leq \|v\|_{Y \hat{\hat{\otimes}} X''} \cdot \|u\|_{(Y \hat{\hat{\otimes}} X)'}$, which means that $u \in I_1(Y,X')$ and $\|u\|_{I_1} \leq \|u\|_{(Y \hat{\hat{\otimes}} X)'}$.

b) follows immediately from the equation $I_1(Y',X') =$ $= (Y' \hat{\hat{\otimes}} X)'$.

§ 3. The approximation property for Banach spaces.

3.1. Some notations: If X and Y are Banach spaces, let us denote by $K(X,Y)$ the Banach space of all compact linear maps from X into Y and by $K_o(X,Y)$ the closure of all finite dimensional maps therein. Both $K(X,X)$ and $K_o(X,X)$ are closed two-sided ideals in the Banach algebra $H(X,X)$. It is easily seen that $K_o(X,Y)$ is isometrically isomorphic to $X' \hat{\otimes} Y$ for each Y.

Let A be a Banach algebra. A left [right] approximate identity for A is a net $(u_s)_{s \in S}$, where S is some directed system, such that for all $a \in A$ $\lim u_s a = a$ [$\lim a u_s = a$] in the norm topology. The approximate identity is said to be bounded if $\|u_s\| \leq 1$ for all $s \in S$. an approximate identity is two-sided if it is both a right and left one. A left Banach-A-module is a Banach space together with a contractive Banach algebra homomorphism $A \to H(V,V)$ (denoted by $a \to (v \to av)$). It is called essential if the linear span of $A \cdot V$ is dense in V. (see Chapter III for further information and a detailed account of properties of Banach modules).
For each Y the space $K(Y,X)$ [$K_o(Y,X)$] is a left Banach module with respect to $K(X,X)$ [$K_o(X,X)$] .

<u>Definition</u>: A Banach space X is said to have the [c-bounded] <u>approximation property</u>, if for each compact subset K of X there exists a continuous finite dimensional linear map $u : X \to X$ [with $\|u\| \leq c$ for a fixed constant $c > 0$], such that for all $x \in K$

$$\| u(x) - x\| \leq 1.$$

If X has the bounded approximation property with $c = 1$, then we say that X has the <u>metric approximation property</u>.

3.2. To prove some properties that are equivalent to the approximation property we need three lemmas. The first two of them are essentially well known, the third one is due to Grothendieck [32].

<u>Lemma a)</u>: Let X be a Banach space and C a compact subset. Then there exists a null sequence (x_n) such that C is contained in the closed, absolutely convex hull of the (x_n).

<u>Proof</u>: C is precompact. Let F_1 be a finite $\frac{1}{2^2}$ - mesh in C. Take $C_1 = (C - F_1) \cap \frac{1}{2^2}$ OX. Then C_1 is compact (since F_1 is finite) and $C \subset C_1 + F_1$. By induction, we construct a sequence of finite sets F_k and compact sets C_k, such that F_k is a finite $\frac{1}{2^{2k}}$ - mesh in C_{k-1} and $C_k = (C_{k-1} - F_k) \cap \frac{1}{2^{2k}}$ OX. Then clearly $C_{k-1} \subset C_k + F_k$,

from which it follows that $C \subset C_k + F_1 + \ldots + F_k$.
We multiply the elements of F_k by 2^k and arrange them into a sequence, which by construction converges to zero. Each $x \in C$ has a representation as a convergent series $x = \sum_{k=1}^{\infty} y_k$, where $y_k \in F_k$. Then $x = \sum_{k=1}^{\infty} \frac{1}{2^k} 2^k y_k$ is contained in the closed absolutely convex hull of $\bigcup_{k=1}^{\infty} 2^k F_k$.

Lemma b): Let X and Y be Banach spaces. The dual of H(X,Y) for the topology of pointwise convergence is (algebraically) isomorphic to $X \otimes Y'$.

Proof: Each element $(x,y') \in X \times Y'$ defines a linear functional on H(X,Y) by $u \in H(X,Y) \to \langle u(x), y' \rangle$. The topology of pointwise convergence on H(X,Y) is induced by the product topology on Y^X (the space of all mappings of X into Y). According to the Hahn Banach theorem each continuous functional on H(X,Y) may be extended to Y^X. Let φ be such a functional on Y^X. Since the subspace of finite sequences is dense in Y^X, there exist elements $(y'_x)_{x \in X} \subset Y'$ such that $\varphi((y_x)) = \sum_{x \in X} \langle y_x, y'_x \rangle$. If countably many y'_x were different from zero, we could find elements $(y_x) \subset Y$ for which the above series would be divergent. Hence $\varphi((y_x)) = \sum_{i=1}^{n} \langle y_{x_i}, y'_i \rangle$. It follows now from 1.3.b) that the correspondence is bijective.

Lemma c): Let X and Y be Banach spaces. The dual of $H(X,Y)$ for the topology of compact convergence is (algebraically) isomorphic to a quotient of $X \hat{\otimes} Y'$.

Proof: The map of $X \times Y'$ into $H(X,Y)'$ defined in b) is continuous and therefore extends to $X \hat{\otimes} Y'$. As a consequence of 1.8 g) each element $v \in X \hat{\otimes} Y'$ has a representation $v = \sum_{n=1}^{\infty} x_n \otimes y_n'$, where $(x_n)_{n=1}^{\infty} \in c_o(X)$ and $(y_n') \in l^1(Y')$. By continuity, the functional which corresponds to v has the form $u \in H(X,Y) \to \sum_{n=1}^{\infty} < u(x_n), y_n' >$. Since every null-sequence is compact, this functional is continuous for the topology of compact convergence. It remains to show that each functional can be represented in this way. To a given map $u \in H(X,Y)$ we assign $c_o(u) \in H(c_o(X), c_o(Y))$ by $(x_n) \to (u(x_n))$. This defines an isometric embedding of $H(X,Y)$ into $H(c_o(X), c_o(Y))$. By Lemma a) the topology of pointwise convergence on $H(c_o(X), c_o(Y))$ induces the topology of compact convergence on $H(X,Y)$. If we start with a functional v on $H(X,Y)$, which is continuous for the topology of compact convergence, we may extend it to $H(c_o(X), c_o(Y))$ and get by Lemma b) a representation in

$c_o(X) \otimes c_o(Y)'$. Now $c_o(Y)' = l^1(Y')$ and for $(x_n) \otimes (y_n') \in c_o(X) \otimes l^1(Y')$ and $u \in H(X,Y)$ we have $< c_o(u), (x_n) \otimes (y_n') > = < c_o(u)((x_n)), (y_n') > = = \Sigma < u(x_n), y_n' >$, i.e. $(x_n) \otimes (y_n')$ corresponds to $\Sigma\, x_n \otimes y_n' \in X \hat{\otimes} Y'$. Since this property is inherited by finite sums, we get the desired representation.

3.3. Theorem: Let X be a Banach space. Then the following statements are equivalent:

a) X has the approximation property

b) 1_X is a clusterpoint of $X' \otimes X$ in $H(X,X)$ for the topology of compact convergence.

c) $Y' \otimes X$ is dense in $H(Y,X)$ for the topology of compact convergence, for all Banach spaces Y.

d) $X' \otimes Y$ is dense in $H(X,Y)$ for the topology of compact convergence, for all Banach spaces Y.

Proof: The implications b) ⇒ a), c) ⇒ b) and d) ⇒ b) are obvious.

a) ⇒ b) Since for any compact subset K of X and $\lambda > 0$, λK is also compact, we get for every $\varepsilon > 0$ a finite dimensional linear map $u: X \to X$, i.e. $u \in X' \otimes X$, such that $\|u(x) - x\| < \varepsilon$ for all $x \in K$.

b) $\Rightarrow$ c) Let v be an element of $H(Y,X)$, K compact in Y and $\varepsilon > 0$. $v(K)$ also is compact and consequently there exists an element u of $X' \otimes X$ such that $\|u \circ v(y) - v(y)\| < \varepsilon$ for all $y \in K$. Since $u \circ v = (v' \otimes 1_X)(u) \in Y' \otimes X$ the conclusion follows.

The proof of b) $\Rightarrow$ d) is similar to that of b) $\Rightarrow$ c).

3.4. <u>Theorem</u>: Let X be a Banach space. Then the following statements are equivalent:

a) X has the approximation property.

b) for every $u \in X' \hat{\otimes} X$ which induces the null-map in $H(X,X)$ we have $\operatorname{tr} u = 0$.

c) the canonical map of $X' \hat{\otimes} X$ into $H(X,X)$ is injective.

d) for every Banach space Y the canonical map of $Y \hat{\otimes} X$ into $L(Y',X)$ is injective.

e) $X' \otimes X$ is w^*-dense in $\mathfrak{L}(X',X') = \mathfrak{L}(X,X'')$.

f) for every Banach space Y it follows that $Y \otimes X'$ is w^*-dense in $\mathfrak{L}(Y',X')$.

<u>Proof</u>: a) $\Rightarrow$ d): Assume that $u \in Y \hat{\otimes} X$ induces the zero-map. It is an easy consequence of 1.8 g) that u

has a representation $u = \sum_{n=1}^{\infty} y_n \otimes x_n$ where $(y_n) \in l^1(Y)$ and $(x_n) \in c_o(X)$. Let $(u_\iota) \subset X' \otimes X$ be a net converging to 1_X in the topology of compact convergence. Then $u_\iota(x_n) \to x_n$ uniformly for all n and therefore $(1_Y \otimes u_\iota)(u) \to u$ in $Y \hat{\otimes} X$. Since the map of $Y \hat{\otimes} X$ into $L(Y',X)$ is easily seen to be natural and its restriction to $Y \otimes X$ is injective, we have $(1_Y \otimes u_\iota)(u) = 0$ and therefore $u = 0$.

The implications d) $\Rightarrow$ c) $\Rightarrow$ b) are obvious.

b) $\Rightarrow$ a): Assume that 1_X is not contained in the closure of $X' \otimes X$ for the topology of compact convergence. Then there exists a functional w on $H(X,X)$, which is continuous for the topology of compact convergence, such that $< 1_X, w > \neq 0$ and $w|_{X' \otimes X} = 0$. By 3.2 Lemma c) w can be represented by an element u of $X \hat{\otimes} X'$. The conditions on w imply that tr $u \neq 0$ and that the canonical image of u in $H(X',X')$ is zero, from which the conclusion follows immediately.

d) $\Rightarrow$ f) The canonical map of $Y' \otimes X$ into $L(Y'',X)$ and hence into $\mathfrak{L}(Y,X'')$ is injective. By transposition we get the canonical map of $Y \hat{\otimes} X'$ into $\mathfrak{L}(Y',X')$, which consequently has w*-dense image (by I.2.13).

f) ⇒ e) is obvious

e) ⇒ c) If the canonical map of $X \hat{\otimes} X'$ into $\mathfrak{L}(X',X')$ has w*-dense image, then, by transposition, the canonical map of $X' \hat{\otimes} X$ into $\mathfrak{L}(X,X'')$ is injective.

3.5. Theorem: Let X be a Banach space. Then the following statements are equivalent:

a) X has the approximation property

b) $X \hat{\hat{\otimes}} Y = L(X',Y) = L(Y',X)$ for all Banach spaces Y.

c) $Y' \hat{\hat{\otimes}} X = K(Y,X)$ for all Banach spaces Y.

d) $K_o(X,X)$ has a left approximate identity and $L(Y',X)$ is an essential left $K_o(X,X)$-module for all Banach spaces Y.

Proof: a) ⇒ b) Assume that $v \in L(Y',X)$ and $\varepsilon > 0$. Since v is compact, there exists an element $u \in X' \otimes X$ such that $\|u(v(y')) - v(y')\| \leq \varepsilon$ for all $y' \in OY'$ and this means $\|u \circ v - v\| \leq \varepsilon$. $v'(X')$ is contained in Y by I.2.16 and consequently $u \circ v = (v' \otimes 1_X)(u)$ has a representation in $Y \otimes X$. We conclude that $Y \otimes X$ is dense in $L(Y',X)$ and by 2.2 the latter space induces the inductive norm on $Y \otimes X$.

b) ⇒ c) We show that $K(Y,X) = L(Y'',X)$:

If $u \in L(Y'',X)$ then clearly its restriction to Y belongs to $K(Y,X)$.

Conversely if $v \in K(Y,X)$, then $C = \overline{v(OY)}$ is a norm-compact subset of X, and therefore also $\sigma(X,X')$-compact. Since v" is $\sigma(Y'',Y') - \sigma(X'',X')$-continuous and OY is $\sigma(Y'',Y')$-dense in OY", it follows that $v''(OY'') \subset C$. This means that v" can be written as $v'' = \iota_Y \circ w$, where $w \in K(Y'',X)$ is $\sigma(Y'',Y') - \sigma(X,X')$ continuous. Since the restrictions of the norm topology and the weak topology to a norm-compact subset coincide, $w \in L(Y'',X)$. It is easily seen that both transformations are linear, contractive and inverse to each other.

c) $\Rightarrow$ a) Assume that C is a compact subset of X. By lemma a) in 3.2, C is contained in the closed, absolutely convex hull of some null-sequence (x_n). Take $y_n = x_n/\|x_n\|^{\frac{1}{2}}$ (where $\frac{0}{0} = 0$) and D the closed, absolutely convex hull of the y_n. Since y_n is also a null sequence, it follows easily that D is precompact and consequently compact. Let Y be the linear span of D in X, i.e. $Y = \bigcup_{n=1}^{\infty} nD$, and take as norm the Minkovski-functional of D, i.e. $\|y\|_Y = \inf \{ \lambda > 0 : y \in \lambda D \}$. Since D is closed in X, D coincides with the unit ball of Y. Assume that $(z_n) \subset D$ is a Cauchy-sequence in Y. Since D is compact in X, it has a cluster-point z in X. For $\epsilon > 0$ there exists an index $N_0(\epsilon)$ such that

$z_n - z_m \in \varepsilon D$ for all $n,m \geq N_o(\varepsilon)$. Consequently $z_n - z \in \varepsilon D$ for all $n \geq N_o(\varepsilon)$ and that means $\|z_n - z\|_Y \leq \varepsilon$. We conclude that Y is complete.

The embedding $j : Y \to X$ is by definition compact. For $\varepsilon > 0$ there exists by assumption an element $j_1 \in Y' \otimes X$ such that $\|j - j_1\| < \varepsilon$. j is injective and so j' has w^*-dense image. Since by 3.2 Lemma c) or by I.2.8 Prop., I.2.7 and I.2.6 the dual of Y' for the topology of compact convergence coincides with Y, $j'(X')$ is also dense for the topology of compact convergence (by the Hahn-Banach theorem). Now according to our construction $\|x_n\|_Y \leq \|x_n\|^{\frac{1}{2}}$, from which it follows that C is contained in Y and compact therein. Consequently there exists $u \in X' \otimes X$ such that $\|j_1(y) - u \circ j(y)\| \leq \varepsilon$ for all $y \in C$ (since $u \circ j = (j' \otimes 1_X)(u)$). Take $K = \max(1, \|x_n\|)$, then $\|j(y) - j_1(y)\| < K\varepsilon$ for all $y \in C$. Combining these facts, we obtain the inequality: $\|x - u(x)\| < (K+1)\varepsilon$ for all $x \in C$. For $\varepsilon < 1/(K+1)$ we get the desired result.

a),b) $\Rightarrow$ d): If X has the approximation property, then $K_o(X,X)$ has a left approximate identity since every member of $K_o(X,X)$ is compact. If $u \in Y \otimes X$, then im u is a finite dimensional subspace of X. Let $x_1,\dots,x_n$ be a basis of im u and extend the coefficient functionals $x'_1, \ldots, x'_n$ to the whole of X. Then $v = \sum_{i=n}^{n} x'_i \otimes x_i$ acts as the identity on im u, i.e. $u = v \circ u$.

The implication d) ⇒ b) is obvious.

Corollary: A dual space X' has the approximation property if and only if $K(X,Y) = X' \hat{\hat{\otimes}} Y$ holds for all Banach spaces Y.

Proof: It follows from the argument in b) ⇒ c) that $K(X,Y) = L(X'',Y) = L(Y',X')$ (see also I.2.16). A short computation shows that the identification is compatible with the embedding of $X' \hat{\hat{\otimes}} Y$. Now the result follows from b).

Remark: In 3.5 d) $L(Y',X)$ may be replaced by $K(Y,X)$ (since $K(Y,X) = L(Y'',X)$). It seems to be unknown whether it is sufficient that the assertions of b), c), d), hold for some particular choice of Y, e.g. for Y = X .

3.6. Theorem: Let X be a Banach space such that X' has the approximation property. Then X also has the approximation property and $X \otimes Y$ is dense in $H(X',Y)$ for the topology of compact convergence for all Y.

Proof: The following diagram is easily seen to be commutative:

$$\begin{array}{ccc} X' \hat{\otimes} X & \longrightarrow & H(X,X) \\ \downarrow & & \downarrow \\ X \hat{\otimes} X' & \longrightarrow & H(X',X') \end{array}$$

where all maps are the canonical ones.

By 3.4 d) the map of $X \hat{\otimes} X'$ into $H(X',X')$ is injective. Consequently X has the approximation property by 3.4 c).

OX is w^*-dense in OX'' and the restriction of the w^*-topology to OX'' coincides with the topology of compact convergence. It follows that $X \otimes Y$ is dense in $X'' \otimes Y$ for the topology of compact convergence induced by $H(X',Y)$. Now the second statement is a consequence of 3.3.

Corollary: If X' has the approximation property, then $K_o(X,X)$ has a right approximate identity and $K(X,Y)$ is an essential right $K_o(X,X)$-module.

Proof: Let $(u_\iota) \subset X' \otimes X$ be a net such that (u'_ι) converges to $1_{X'}$ in the topology of compact convergence. If $v \in K_o(X,X)$, then $v' \in K_o(X',X')$ is compact. Consequently $\|u'_\iota \circ v' - v'\| \to 0$ and this implies that $\|v \circ u_\iota - v\| \to 0$. $K(X,Y) = X' \hat{\hat{\otimes}} Y$ by 3.5, Cor. If $v \in X' \otimes Y$ then $v' \in H(Y',X')$ has finite dimensional range. Since on a finite dimensional space the w^*-topology coincides with the norm topology, there exists an element $u \in X \otimes X'$ such that $u \circ v' = v'$ (compare the proof of 3.5). Transposition now gives the result.

3.7. Remarks: Most of the 'classical' Banach spaces even have the metric approximation property (compare the examples in 3.12). It had been conjectured that every Banach space has this property. In 1972 Enflo [23] gave the first example of a Banachspace without the approximation property. A simplified exposition of Enflo's main ideas may be found in [27]. His example may be modified to construct subspaces of c_o and l^p $(2 < p < \infty)$ without the approximation property. Other examples, based on similar constructions, were given by Davie [15] and Kwapien [42].

We would like to point out that it seems to be unknown whether the following spaces have the approximation property:

a) $H^\infty(D)$, the algebra of bounded analytic functions on the open unit disc with supremum-norm.

b) $CB^{(n)}(\mathbb{R})$ $(1 \leq n < \infty)$, the space of all functions on the real line that have bounded continuous derivatives up to order n equipped with the norm $\|f\|_{(n)} = \sup_{\nu=0,\dots,n} \sup_{x \in \mathbb{R}} |f^{(\nu)}(x)|$

c) $H(l^2, l^2)$

While the metric approximation property and the approximation property are easily seen to be preserved by sums, it can be shown that a countable product of finite dimensional spaces need not have the approximation property [37]. This gives also an example of a Banach space with the metric approximation property, whose dual does not have any approximation property.

3.8. We will now state conditions for the metric approximation property corresponding to 3.3 - 3.6.

<u>Theorem</u>: Let X be a Banach space. Then the following statements are equivalent:

a) X has the metric approximation property.

b) 1_X is a clusterpoint of $X' \otimes X \cap OH(X,X)$ for the topology of pointwise convergence.

c) $Y' \otimes X \cap OH(Y,X)$ is dense in $OH(Y,X)$ for the topology of compact convergence, for all Banach spaces Y.

d) $X' \otimes Y \cap OH(X,Y)$ is dense in $OH(X,Y)$ for the topology of compact convergence, for all Banach spaces Y.

<u>Proof</u>: We show that the restriction to $OH(X,Y)$ of the topology of compact convergence coincides with the topology of pointwise convergence. The rest of the

proof is analogous to that of 3.3. If $u \in OH(X,Y)$, C is compact in OX and $\epsilon > 0$, take a finite ϵ-mesh $\{x_1,\ldots,x_n\}$ in C. Assume that $v \in OH(X,Y)$ and $\|u(x_i) - v(x_i)\| < \epsilon$ for $i = 1,\ldots,n$. If $x \in C$ there exists an index i such that $\|x - x_i\| < \epsilon$. Consequently $\|u(x) - v(x)\| \leq \|u(x - x_i)\| + \|u(x_i) - v(x_i)\| + \|v(x_i - x)\| < 3\epsilon$.

3.9. __Theorem__: Let X be a Banach space. Then the following statements are equivalent:

a) X has the metric approximation property.

b) $X' \hat{\otimes} X$ is isometrically contained in $I_1(X,X)$

c) $X \hat{\otimes} Y$ is isometrically contained in $I_1(X',Y)$ for all Banach spaces Y.

d) $X' \otimes X \cap O\Omega(X,X'')$ is w^*-dense in $O\Omega(X,X'')$

e) $X' \otimes Y \cap O\Omega(X,Y'')$ is w^*-dense in $O\Omega(X,Y'')$ for all Banach spaces Y.

__Proof__: a) $\Rightarrow$ e) By 3.8. $X' \otimes Y'' \cap OH(X,Y'')$ is dense in $OH(X,Y'')$ for the topology of compact convergence. If $u \in X \hat{\otimes} Y'$, then by 1.8 g) u can be written in the form $u = \sum_{n=1}^{\infty} x_n \otimes y_n'$ with $(x_n) \in c_o(X)$ and $(y_n') \in l^1(Y')$. In particular, (x_n) is a compact subset of X. It follows that the topology of compact convergence is stronger than the w^*-topology with respect to $X \hat{\otimes} Y'$, Consequently $X' \otimes Y'' \cap O\Omega(X,Y'')$ is w^*-dense too. By 2.8 Cor. 2

the unit-ball of $X' \hat{\hat{\otimes}} Y$ is w*-dense in the unit-ball of $X' \hat{\hat{\otimes}} Y''$. Therefore $X' \otimes Y \cap O\mathcal{L}(X,Y'')$ is w*-dense in $X' \otimes Y'' \cap O\mathcal{L}(X,Y'')$.

e) $\Rightarrow$ c) : Transposing the map of $X' \hat{\hat{\otimes}} Y'$ into $\mathcal{L}(X,Y''')$ one gets the canonical map of $X \hat{\otimes} Y''$ into $(X' \hat{\hat{\otimes}} Y')' = I_1(X',Y'')$ (2.9 c)), which is therefore an isometry by (I.2.13 Prop.).

The following diagram is commutative:

$$\begin{array}{ccc} X \hat{\otimes} Y & \longrightarrow & I_1(X',Y) \\ {\scriptstyle 1_X \hat{\otimes} \iota_Y}\downarrow & & \downarrow {\scriptstyle I_1(X',\iota_Y)} \\ X \hat{\otimes} Y'' & \longrightarrow & I_1(X',Y'') \end{array}$$

(the horizontal maps are the canonical ones).

$1_X \hat{\otimes} \iota_Y$ is an isometry by 1.9. Since $I_1(X',\iota_Y)$ is contractive, the map of $X \hat{\otimes} Y$ into $I_1(X',Y)$ must also be an isometry.

c) $\Rightarrow$ b): The map of $I_1(X,X)$ into $I_1(X',X')$ given by $u \to u'$ is an isometry by 2.9 b).

Now the assertion follows from the commutativity of the following diagram (the maps are the obvious ones):

$$\begin{array}{ccc} X' \hat{\otimes} X & \longrightarrow & I_1(X,X) \\ \downarrow & & \downarrow \\ X \hat{\otimes} X' & \longrightarrow & I_1(X',X') \end{array}$$

b) ⇒ d) By the same argument as before $X \hat{\otimes} X'$ is isometrically contained in $I_1(X',X') = (X' \hat{\hat{\otimes}} X)'$ (2.9 b) By transposition and (I.2.13 Prop.) it follows that the unit-ball of $X' \hat{\hat{\otimes}} X$ is w^*-dense in $O\mathfrak{L}(X,X'')$.

d) ⇒ a) Let $(u_\iota) \subset X' \otimes X$ be a net converging to $\iota_X \in \mathfrak{L}(X,X'')$ in the w^*-topology. Fixing $x_1 \in X$ this means in particular that $u_\iota(x_1) \to x_1$ with respect to $\sigma(X'',X')$. Since $u_\iota(X) \subset X$ it follows that $u_\iota(x_1) \to x_1$ weakly. Now the weak closure of a convex set coincides with the norm closure (see [73] p. 65). Therefore we can find a net (v_ι) consisting of convex combinations of the (u_ι) such that $v_\iota \to \iota_X$ in the w^*-topology and $v_\iota(x_1) \to x_1$ in the norm topology. Iterating this process we get for given $x_1,\dots,x_n \in X$ a net (w_ι) such that $w_\iota(x_i) \to x_i$ for $i = 1,\dots,n$ in the norm topology. The (w_ι) are still convex combinations of the (u_ι) and therefore $(w_\iota) \subset OH(X,X)$ provided the (u_ι) also satisfy this condition. By 3.8 b) X has the metric approximation property.

3.10. <u>Theorem</u>: Let X be a Banach space. Then the following statements are equivalent:

a) X has the metric approximation property.

b) $K_o(X,X)$ has a bounded left approximate identity

c) $K_o(X,X)$ has a bounded left approximate identity and $L(Y',X)$ and $K(Y,X)$ are essential left $K_o(X,X)$-modules for all Banach spaces Y.

Proof: a) $\Rightarrow$ c) is proved as in theorem 3.5 ,

c) $\Rightarrow$ b) is obvious

b) $\Rightarrow$ a) Take $x_1,\ldots,x_n \in X$. There exists a finite dimensional map $u \in X' \otimes X$ such that $u(x_i) = x_i$ for $i = 1,\ldots,n$. (Hahn-Banach theorem). To $\varepsilon > 0$ there exists by assumption an element $v \in K_o(X,X)$ such that $\|v \circ u - u\| < \varepsilon$ and $\|v\| \leq 1$. Since $X' \otimes X$ is dense in $K_o(X,X)$ we may assume $v \in X' \otimes X$. Consequently $\|v(x_i) - x_i\| < \varepsilon \max_{i=1,\ldots,n} \{\|x_i\|\}$ for $i = 1,\ldots,n$. Since ε is arbitrary, theorem 3.8 b) yields the result.

3.11. Theorem: Let X be a Banach space such that X' has the metric approximation property. Then X also has the metric approximation property and $X \otimes Y \cap OH(X',Y)$ is dense in $OH(X',Y)$ for the topology of compact convergence, for all Y.

Proof: The map $u \to u'$ from $I_1(X,X)$ into $I_1(X',X')$ is an isometry by 2.9 b).

The following diagram is commutative:

$$\begin{array}{ccc} X' \hat{\otimes} X & \longrightarrow & I_1(X,X) \\ \downarrow & & \downarrow \\ X \hat{\otimes} X' & \longrightarrow & I_1(X',X') \end{array}$$

By 3.9 c) $X' \hat{\otimes} X$ is isometrically contained in $I_1(X,X)$ and consequently X has the metric approximation property by 3.9 b). The second part of the proof is similar to that of 3.6 , using 3.8 d) and 2.8 Cor. 2.

Corollary: If X' has the metric approximation property, then $K_o(X,X)$ has bounded right approximate identities and K(X,Y) is an essential right $K_o(X,X)$-module.

Proof: As in 3.6.

3.12. Remarks: There exist Banach spaces with the approximation property which fail to have the metric approximation property (see [26]). On the other hand for the class of reflexive spaces and of separable dual spaces both notions coincide (see [33] p. 181). It seems to be unknown if they coincide for dual spaces in general. Another open question is whether every space with the bounded approximation property can be equivalently renormed to have the metric approximation property.

A stronger conjecture states that every space with the bounded approximation property has a basis (compare 3.13). It also seems to be unknown whether the existence of bounded right approximate identities for $K_o(X,X)$ implies the metric approximation property for X'.

3.13. Examples:

a) Let (Ω, Σ, μ) be a measure space and $1 \leq p < \infty$. For each finite family $\pi = \{B_1,\dots,B_n\}$ of disjoint measurable sets with $0 < \mu(B_i) < \infty$ define the conditional expectation operator E_π on $L^p(\Omega, \Sigma, \mu)$ by $E_\pi(f) = \sum_{i=1}^{n} (\int f\, d\mu)/\mu(B_i) \cdot c_{B_i}$. It is easily seen that E_π is linear and $\|E_\pi\| \leq 1$. One defines a preorder by: $\pi = \{B_1,\dots,B_n\} \leq \pi' = \{C_1,\dots,C_m\}$ if for each i with $1 \leq i \leq m$ there exists either an index j with $1 \leq j \leq n$ such that $\mu(C_i \setminus B_j) = 0$ or $\mu(C_i \cap B_j) = 0$ for all j. In this way $\{E_\pi\}$ becomes a net of contractions. It follows immediately that $\lim E_\pi f = f$ for all integrable step functions, i.e. finite combinations of characteristic functions, and by a density argument $\lim E_\pi f = f$ holds for all $f \in L^p$. Consequently each space $L^p(\Omega, \Sigma, \mu)$ has the metric approximation property.

b) Let T be a locally compact space. We consider families $\{V_1,\ldots,V_n\}$ of open subsets with compact closure and an open subset V_{n+1} such that $\bigcup_1^{n+1} V_i = T$

There exists a partition of unity $\{p_1,\ldots,p_{n+1}\}$ subordinate to $V_1,\ldots,V_n$, $V_{n+1}\cup\{\infty\}$. We choose $t_i \in V_i$ and consider the map $f \in C_o(T) \to \sum_{i=1}^{n} f(t_i)p_i$, which is evidently contractive and linear. If the families $\{V_1,\ldots,V_{n+1}\}$ are ordered by inclusion, the same argument as in 2.4 shows that the corresponding maps form an approximate unit, bounded by 1, i.e. $C_o(T)$ has the metric approximation property.

Another proof could be based on the fact that by Kakutani's representation theorem $C_o(T)' = M(T)$ may be represented as an L^1-space. Theorem 3.11 can then be used.

c) A basis for a Banach space X consists of a family $\{x_n\}_{n=1}^{\infty} \subset X$ such that every element $x \in X$ has a unique representation as a convergent sum $x = \sum_{n=1}^{\infty} \alpha_n x_n$. An application of the closed graph theorem (see [73]p. 115) shows that the coefficient functionals $x \to \alpha_n$ are continuous. By the Banach-Steinhaus theorem the projections $P_k(x) = \sum_{n=1}^{k} \alpha_n x_n$

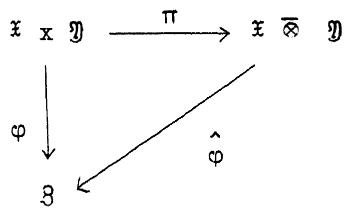
are uniformly bounded. Take $c = \sup_k \|P_k\|$, then the P_k's form an approximate unit bounded by c, i.e. X has the bounded approximation property and if c = 1 even the metric approximation property.

Exercises

1) If X, Y are Banach spaces show that $\|\cdot\|^{\wedge}$ is the Minkowsky functional of the convex hull of the set $\{x \otimes y \,,\ \|x\| \leq 1,\ \|y\| \leq 1\}$ in $X \otimes Y$. (cf. Schaefer [73], III, 6.3)

2) Projective tensor product of Waelbroeck spaces:

Let $\mathfrak{X}, \mathfrak{Y} \in \underline{W}$. A projective tensor product of $\mathfrak{X}, \mathfrak{Y}$ is a Waelbroeck space $\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y}$ together with a bilinear bounded map $\pi : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y}$ which is continuous from $O\mathfrak{X} \times O\mathfrak{Y}$ into $\|\pi\| \cdot O(\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y})$ in the compact topologies (a so-called bilinear $\underline{W}$ - map) such that each bilinear $\underline{W}$ - map $\varphi : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ into any $\underline{W}$ - space $\mathfrak{Z}$ factors uniquely over π to $\hat{\varphi} : \mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y} \to \mathfrak{Z}$, i.e. $\varphi = \hat{\varphi} \circ \pi$.



Show that $\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y}$ is given by $\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y} = L(\mathfrak{X}, \mathfrak{Y}^*)'$,
$\pi : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y}$ is given by
$< f, \pi(x,y) > = < f(x), y >$.
Show that $\mathfrak{X} \otimes \mathfrak{Y}$ is dense in $\overline{c}(\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y})$ (I, §2) and that $(\mathfrak{X} \otimes \mathfrak{Y}) \cap O(\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y})$ is dense in $O(\mathfrak{X} \,\overline{\otimes}\, \mathfrak{Y})$.

Show that $\overline{c}(\mathfrak{X} \overline{\otimes} \mathfrak{Y}) = \overline{c}\,\mathfrak{X} \hat{\otimes} \overline{c}\mathfrak{Y}$, where $\hat{\otimes}$ denotes the completion of the projective tensorproduct of locally convex spaces (cf. Grothendieck).

3) Tensor products of Hilbert spaces:

Let X,Y,Z be Hilbert spaces.

We denote by X^* the conjugate space of X, i.e. scalar multiplication is replaced by $(\lambda, x) \to \overline{\lambda}x$. X^* is also a Hilbert space and canonically isomorphic to the dual of X (Theorem of Riesz-Fischer). Show that $(X \hat{\otimes} Y^*)' = B^*(X,Y) = H(X,Y)$, where $B^*(X,Y)$ stands for the space of sesquilinear maps $\varphi : X \times Y \to Z$. Each element $v \in K(X,Y)$ has a representation as a pointwise convergent sum $v = \sum_{i=0}^{\infty} s_i \psi_i^* \otimes \varphi_i$, where (ψ_i) and (φ_i) are orthonormal systems in X and Y respectively and (s_i) is a uniquely determined, decreasing sequence of nonnegative real numbers. Show that for $v \in K(X,Y)$ $\|v\|^{\wedge} = s_0$. $X^* \hat{\otimes} Y$ consists exactly of those operators $v \in K(X,Y)$ for which $\sum_{i=0}^{\infty} s_i < \infty$ holds and furthermore $\|v\|^{\wedge} = \Sigma s_i$. Show that $I_1(X,Y) = X^* \hat{\otimes} Y$. (Hint: Show first that each $v \in I_1(X,Y)$ is a Hilbert-Schmidt operator i.e. $\sum_{i=0}^{\infty} \|v(e_i)\|^2 < \infty$ for any orthonormal system in X).

C H A P T E R III

Banach modules

§ 1 Banach algebras and Banach modules

In this chapter we shall study Banach modules over a Banach algebra A. Our aim is to generalize some aspects of Banach space theory to this more general setting.

In order to obtain a reasonable theory which nevertheless contains the most important examples we shall impose some restrictions on the Banach algebra A and the Banach modules V.

1.1. Many Banach algebras arising in applications do not have a unit element e. But most of them have a useful substitute, which is called approximate unit or approximate identity.

Definition: Let A be a Banach algebra. An approximate left (right) unit (u_ι) in A is a net (u_ι) of elements of A satisfying $\|u_\iota\| \leq 1$ and $\lim_\iota u_\iota a = a$ (resp. $\lim au_\iota = a$) for all $a \in A$.
If (u_ι) is both a left and a right approximate unit, it is called a two-sided approximate unit.

Remark: In the literature there appear various useful modifications of the concept "approximate unit" with less stringent assumptions (cf. e.g. H.Reiter [69]). We do not consider these concepts here, because we are only interested in typical results.

1.2. Let $A = c_o$ be the set of all sequences $a = (a_1, a_2, a_3, \dots)$ of complex numbers a_n which satisfy $\lim_{n\to\infty} a_n = 0$.
Then A is a Banach algebra with coordinatewise algebraic operations and the norm $\|a\|_\infty = \sup |a_n|$. It is in fact a closed subalgebra of the Banach algebra l^∞ of all bounded sequences. An approximate (two-sided) unit is given by the sequence $u_n = (1,1,\dots,1,0,0,\dots)$.

1.3. Let H be a separable Hilbert space. Let K(H) be the algebra of all compact operators on H with the operator norm.
Then K(H) is a Banach algebra with a two-sided approximate unit: let $(e_k)_{k=1}^{\infty}$ be an orthonormal base of H and let $P_n = \sum_{k=1}^{n} e_k \otimes e_k^*$ be the orthoprojection onto the subspace spanned by the first n elements of the base (e_k).
(Here $e \otimes f^*$ denotes the operator defined by $(e \otimes f^*)(x) = (x|f)e$).
Then P_n is an approximate unit: it is clear that $\lim_{n\to\infty} P_n x = x$ for all $x \in H$. Now let $A \in K(H)$ be finite-dimensional, $A = \sum_{i=o}^{k} s_i \varphi_i \otimes \psi_i^*$.
Then

$$\|P_n A - A\| = \|\sum_{i=o}^{k} s_i (P_n \varphi_i - \varphi_i) \otimes \psi_i^*\| \leq \sum_{i=o}^{k} s_i \|P_n \varphi_i - \varphi_i\| \|\psi_i\|$$

and therefore $\lim_{n\to\infty} \|P_n A - A\| = 0$.
For an arbitrary $B \in K(H)$ and any $\epsilon > 0$ there is a finite-dimensional A such that $\|A - B\| < \epsilon$.
This implies

$$\|P_n B - B\| \leq \|P_n (B - A)\| + \|P_n A - A\| + \|A - B\| < 2\epsilon + \|P_n A - A\| < 3\epsilon$$

for all sufficiently large n. The fact that P_n is also a right approximate identity follows from the equation

$$\|BP_n - B\| = \|P_n^* B^* - B^*\| = \|P_n B^* - B^*\|$$

because $B^* \in K(H)$.

1.4. Now let X be an arbitrary Banachspace and let

$F(X) = X' \hat{\otimes} X \subsetneqq H(X,X)$ (see II.2.2).

Then F(X) is a closed subalgebra of H(X,X).

If X satisfies the metric approximation condition then F(X) coincides with the algebra of all compact operators on X. In this case there exists a left approximate unit consisting of finite-dimensional operators. This has already been shown in II. 3.10.

1.5. Let G be a locally compact topological group. Then $L^1(G)$ is a Banach algebra with respect to convolution which has a two-sided approximate unit (cf.[69]).

1.6. Let A be a Banach algebra. A Banach space V is called a left (Banach) A-module, if there is a morphism $\nu : A \hat{\otimes} V \to V$, which is of course determined by $\nu(a \otimes v) = av$, such that $b(av) = (ba)v$ holds for all $a,b \in A$, $v \in V$. Since ν is a morphism in Ban_1 the map $(a,v) \to av$ is bilinear and satisfies $\|av\|_V \leq \|a\|_A \|v\|_V$

<u>Remark</u>: In the same way we define a right A-module W. In this case there is a morphism $\nu : W \hat{\otimes} A \to W$ with

analogous properties. In this case we write

$\nu(w,a) = wa$.

We shall also have to consider bimodules;

let A and B be Banach algebras. A Banach space Z is called an A-B-bimodule, if Z is at the same time an A-module and a B-module and if furthermore these module operations commute.

Unless otherwise stated, in the following pages the statement "Z is an A-B-bimodule" shall mean that Z is a left A-module and a right B-module. In this case $(az)b = a(zb)$ holds for all $a \in A$, $b \in B$, $z \in Z$.

1.7. Let $A = c_o$. A linear space n of sequences $x = (x_k)$ of complex numbers x_k which is a Banach space under a norm $\|\cdot\|_n$ is called a <u>sequence-space</u> (in order to avoid a cumbersome terminology) if

na) n is a l^∞-module (in the sense of 1.2.)

nb) For every i the sequence $e_i = (\delta_i^k)_{k=1}^\infty \in n$ and $\|e_i\|_n = 1$

nc) For each $v \in n$ the relation $\lim_{k\to\infty} \|u_k v\|_n = \|v\|_n$ holds, where $u_k = (1,1,\ldots,1,0,0,\ldots)$.

The sequence space is called symmetric if, moreover, the following condition holds:

nd) $v \in n \Rightarrow (v_{\pi(k)}) \in n$ and

$\|(v_{\pi(k)})\|_n = \|(v_k)\|_n$ for each permutation π of the natural numbers.

Concrete examples are the spaces l^p, $1 \leq p \leq \infty$, and c_o itself. Further examples may be found in [29].

<u>Proposition</u>: Each sequence space n satisfies $l^1 \subseteq n \subseteq l^\infty$.

Furthermore $\|x\|_\infty \leq \|x\|_n \leq \|x\|_1$ for each $x \in n$,

where $\|x\|_1 = \infty$ if $x \notin l^1$.

<u>Proof</u>: Let $x = (x_k)$. Then

$$|x_k| = \|xe_k\|_n \leq \|e_k\|_\infty \|x\|_n = \|x\|_n$$

and therefore $\|x\|_\infty = \sup_k |x_k| \leq \|x\|_n$.

On the other hand

$$\|u_k x\|_n = \|\sum_{i \leq k} x_i e_i\|_n \leq \sum_{i \leq k} |x_i| \|e_i\| \leq \|x\|_1 .$$

Now nc) implies $\|x\|_n = \sup_k \|u_k x\|_n \leq \|x\|_1$.

As an easy consequence we get

$$\lim_{j \to \infty} \|(x_j^1 - x^1, \ldots, x_j^k - x^k, 0, 0, \ldots)\|_n = 0$$

if $\lim_{j \to \infty} x_j^l = x^l$, $l = 1,2,\ldots,k$.

1.8. Let $A = L^\infty[0,1]$ be the Banach algebra of all (equivalence classes of) bounded, real-valued measurable (with respect to Lebesgue measure on $[0,1]$) functions f on $[0,1]$ with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |f(t)| .$$

A linear space N of (equivalence classes of) measurable functions on $[0,1]$ which is a Banach space under a norm $\|.\|_N$ will be called a <u>function space</u> if the following properties hold:

Na) N is an L^∞-module with the pointwise operations

Nb) $L^\infty \subseteq N \subseteq L^1$ and $\|f\|_1 \leq \|f\|_N \leq \|f\|_\infty$ for all $f \in N$, where L^1 denotes the space of all integrable functions on $[0,1]$ and $\|f\|_\infty = \infty$ if $f \notin L^\infty$.

Nc) N has the Fatou property: for each increasing sequence $f_n \in N$ which converges pointwise almost everywhere to a function $f \in N$ we have $\lim_{n\to\infty} \|f_n\|_N = \|f\|_N$.

1.9. Let A = K(H) for a separable Hilbert space H.

A subspace $N \subseteq K(H)$ is called a <u>norm ideal</u> in K(H) if

a) N is a K(H)-bimodule which contains all finite-dimensional operators

b) For each one-dimensional operator $x \otimes y^*$ the equation $\|x \otimes y^*\|_N = \|x \otimes y^*\| = \|x\|\|y\|$ holds

c) For each $v \in N$ the equation
$$\lim_{k\to\infty} \|P_k v\|_N = \|v\|_N$$
holds, where P_k is the sequence of projections defined in 1.3.

Numerous examples can be found in [29].

1.10. Let $A = L^1(G)$. An important class of examples of $L^1(G)$-modules are the Segal-algebras studied in [69] .

1.11. Let A be a Banach algebra, X a Banach space, V a left A-module. Then the Banach space H(V,X) becomes a right A-module if we define fa by $(fa)(v) = f(av)$ for $f \in H(V,X)$.

As a special case we get that the dual space V' is a right A-module if V is a left A-module.

If W is a right A-module then H(W,X) becomes a left A-module by defining $(af)(w) = f(wa)$.

1.12. For a sequence space n and a Banach space X let n(X) be the set of all sequences $x = (x_1, x_2, x_3, \dots)$ such that $x_i \in X$ and $(\|x_1\|, \|x_2\|, \|x_3\|, \dots) \in n$.

Then n(X) is a Banach space with norm $\|x\| = \|(\|x_1\|, \|x_2\|, \|x_3\|, \dots)\|_n$ and a c_o-module if we define $ax = (a_k x_k)$ for $a = (a_k) \in c_o$.

<u>Proof</u>: The only nontrivial assertion is that n(X) is a Banach space: let $(x^{(n)})$ be a Cauchy sequence in n(X). For each $\epsilon > 0$ there exists $N(\epsilon)$ such that $\|x^{(m)} - x^{(n)}\| < \epsilon$ for all $m,n \geq N(\epsilon)$.

This implies $\|x_k^{(m)} - x_k^{(n)}\| = \|e_k(x^{(m)} - x^{(n)})\| < \epsilon$ for each k. Therefore $\lim_{n \to \infty} x_k^{(n)} = x_k$ exists for each k.

The inequality $\left| \|x_k^{(m)}\| - \|x_k^{(n)}\| \right| \leq \|x_k^{(m)} - x_k^{(n)}\|$

implies that $((\|x_k^{(n)}\|_k))$ is a Cauchy sequence in n,

which converges evidently to $(\|x_k\|)$.

Consequently $x = (x_k) \in n(X)$.

Then $\|u_k(x^{(n)} - x)\| = \lim\limits_{m \to \infty} \|u_k(x^{(n)} - x^{(m)}\| \leq \epsilon$

for all k and therefore

$\|x^{(n)} - x\| = \sup\limits_k \|u_k(x^{(n)} - x)\| \leq \epsilon$.

1.13. Let N be a function space and X a Banach space. We denote by N(X) the set of all (equivalence classes of) Bochner-integrable X-valued functions G on $[0,1]$ such that $\|G(\cdot)\|_X \in N$. Then N(X) is a Banach space with respect to the norm $\|G\|_{N(X)} = \| \, \|G(\cdot)\|_X \|_N$. Moreover N(X) is an L^∞-module with pointwise algebraic operations.

Proof: Again the only nontrivial assertion is the completeness of N(X).

It suffices to show that for each sequence $(f_n) \in N(X)$ such that $\Sigma \|f_n\|_{N(X)} < \infty$ the series $\Sigma f_n(t)$ converges almost everywhere to some element $f \in N(X)$ and that furthermore

$$\lim_{n \to \infty} \|f - \sum_{k=1}^{n} f_n\|_{N(X)} = 0.$$

Now let $g(t) = \sum_1^\infty \|f_n(t)\|_X$.

Then $g \in N$, because the partial sums are Cauchy in N:

$$\|\sum_m^p \|f_k(t)\|_N \leq \sum_m^p \|f_k\|_{N(X)} < \epsilon \quad \text{for all} \quad m,n \geq N(\epsilon).$$

Since $N \subseteq L^1$ we have $g \in L^1$ and therefore

$$\Sigma\|f_k(t)\| < \infty \text{ a.e.}$$

$$\text{Let} \quad f(t) = \sum_1^\infty f_k(t) \quad \text{if} \quad \Sigma\|f_k(t)\| < \infty$$

$$= 0 \qquad \text{else.}$$

Then

$$\| \|f\|_X - \|\sum_1^n f_k\|_X\|_N \leq \|f - \sum_1^n f_k\|_{N(X)} \leq$$

$$\leq \|\sum_{k=n+1}^\infty \|f_k(t)\|_X\|_N \leq \sum_{k=n+1}^\infty \|f_k\|_{N(X)} < \epsilon$$

for all sufficiently large n.

The last inequality follows from Nc) since

$$\|\sum_{k=n+1}^\infty \|f_k(\cdot)\|_N = \sup_p \|\sum_{k=n+1}^p \|f_k(\cdot)\|_X\|_N \leq$$

$$\leq \sup_p \sum_{k=n+1}^p \|f_k\|_{N(X)} = \sum_{k=n+1}^\infty \|f_k\|_{N(X)}.$$

1.14. Under the same assumptions as in 1.13 let $N_o(X)$ be the subspace of N(X) spanned by all elements of the form $\sum_{i=1}^n x_i f_i$, $x_i \in X$, $f_i \in N$.

Then $N_o(X)$ is an L^∞-module.

Of special imortance is the space $L_o^\infty(X)$, which can also be identified with $L^\infty \hat{\otimes} X$.

1.15. <u>The factorization theorem of Cohen-Hewitt</u>.

<u>Theorem</u>: (Cohen-Hewitt): Let A be a Banach algebra with approximate left unit (u_ι) and let V be a left A-module. Then the following statements about an element $x \in V$ are equivalent:

1) $\lim_\iota \|u_\iota x - x\| = 0$

2) There exist elements $a \in A$ and $y \in V$ such that $x = ay$. (These elements may be chosen such that $\|a\| \leq 1$ and $\|x - y\| \leq r$ for any preassigned $r > 0$).

<u>Proof</u>: 1) $\Rightarrow$ 2): Denote by A_1 the Banach algebra of all formal sums of the form $a + \lambda e$ with $ae = ea = a$ for all $a \in A$ and $\|a + \lambda e\| = \|a\| + |\lambda|$.

Then V becomes an A_1-module if one defines

$(a + \lambda e)v = av + \lambda v$.

Now let $(u_n)_1^\infty$ be a sequence of elements $u_n \in A$ such that $\|u_n\| \leq 1$. Given such a sequence (u_n) we define a sequence (a_n) with $a_n \in A_1$ by setting $a_o = e$ and

$$a_{n+1} = \frac{u_{n+1} + 2e}{3} a_n = \frac{2}{3} a_n + \frac{u_{n+1} a_n}{3} = (\tfrac{2}{3})^{n+1} e + a'_{n+1}$$

with $a'_{n+1} \in A$ for $n \geq 0$.

By induction we see that $\|a_n\| \leq 1$ for all n.

We claim that all the a_n are invertible in A_1 and that $\|a_n^{-1}\| \leq 3^n$. This is also shown by induction. For $n = 0$ this is trivial. If the result is assumed for n we get

$$a_{n+1}^{-1} = [\tfrac{2}{3}(e+\tfrac{u_{n+1}}{2})a_n]^{-1} = \tfrac{3}{2}\, a_n^{-1}(e+\tfrac{u_{n+1}}{2})^{-1} =$$

$$= \tfrac{3}{2}\, a_n^{-1}(e - \frac{u_{n+1}}{2} + \frac{u_{n+1}^2}{2^2} - + \ldots).$$

Therefore

$$\|a_{n+1}^{-1}\| \le \tfrac{3}{2}\|a_n^{-1}\| \sum_{k=o}^{\infty} \frac{1}{2^k} = 3\cdot 3^n = 3^{n+1}.$$

Furthermore we have the formulas

$$a_{n+1} - a_n = \frac{u_{n+1} - e}{3}\, a_n \quad \text{and}$$

$$a_{n+1}^{-1} - a_n^{-1} = a_{n+1}^{-1} - a_{n+1}^{-1}\,\frac{2e+u_{n+1}}{3} =$$

$$= \frac{a_{n+1}^{-1}}{3}\,(e-u_{n+1}).$$

Now define $y_n = a_n^{-1}x$.

Then

$$\|y_{n+1} - y_n\| = \|(a_{n+1}^{-1} - a_n^{-1})x\| \le$$

$$\le \|\frac{a_{n+1}^{-1}}{3}(e - u_{n+1})x\| \le 3^n\|x - u_{n+1}x\|.$$

Let $\epsilon_n = \frac{1}{2^{n+1}}\,\frac{r}{3^n}$, $n = 0,1,2,\ldots.$

Now we specify the sequence u_n : we choose u_{n+1} from the approximate unit such that

$$\|u_{n+1}a_n' - a_n'\| < \epsilon_n$$

and

$\|u_{n+1}x - x\| < \epsilon_n$, which is possible by assumption.

$$\Rightarrow \|y_{n+1} - y_n\| \le 3^n\|x - u_{n+1}x\| <$$

$$< 3^n\,\frac{1}{2^{n+1}}\,\frac{r}{3^n} = \frac{r}{2^{n+1}}$$

and

$$\|a_{n+1} - a_n\| = \frac{1}{3} \|u_{n+1}a_n - a_n\| =$$

$$= \frac{1}{3} \|(\tfrac{2}{3})^n u_{n+1} - (\tfrac{2}{3})^n e + u_{n+1}a_n' - a_n'\| \le (\tfrac{2}{3})^{n+1} + \frac{\varepsilon_n}{3} .$$

This shows that $y = \lim y_n$ and $a = \lim a_n = \lim a_n'$ exist and belong to A.

Since $y_n = a_n^{-1}x$, we have $x = a_n y_n$ for all n and therefore $x = ay$.

Furthermore $\|a\| = \lim \|a_n\| \le 1$ and

$$\|x - y\| = \lim_{n\to\infty} \|y_o - y_n\| \le \|y_o - y_1\| + \|y_1 - y_2\| + \ldots \le$$

$$\le r \sum_o^{\infty} \frac{1}{2^{n+1}} = r.$$

2) ⇒ 1) : If $x = ay$, then

$$\|u_\iota x - x\| = \|u_\iota ay - ay\| = \|(u_\iota a - a)y\| \le$$

$$\le \|u_\iota a - a\| \, \|y\| \to 0.$$

<u>Remark</u>: Of course an analogous theorem holds also for right modules W.

The factorization theorem has a partial converse (M. Altman [1]), but there are also Banach algebras without approximate units which are factorable (see e.g. W.L. Paschke [60], M. Leinert [43]).

The factorization theorem has some interesting applications:

1.16. Theorem: Let A be a Banach algebra with approximate left unit (u_ι) and let V be a left A-module. Then the following statements are equivalent:

1) The mapping $A \hat{\otimes} V \to V$ defined by $a \otimes v \to av$ has dense image.

2) $\lim_\iota u_\iota v = v$ for each $v \in V$

3) For each $v \in V$ there exists $a \in A$ and $v' \in V$ such that $v = av'$.

Proof: 1) $\Rightarrow$ 2): Let $v \in V$ and $\epsilon > 0$ be given. Then there exist $a_1,\ldots,a_n \in A$ and $v_1,\ldots,v_n \in V$ such that $\|v - \sum_{k=1}^{n} a_k v_k\| < \epsilon/3$. Choose ι_o such that for $\iota \geq \iota_o$

$$\|a_k - u_\iota a_k\| < \frac{\epsilon}{3n\|v_k\|}\quad, \quad k = 1,2,\ldots,n.$$

$$\Rightarrow \|v - u_\iota v\| \leq \|v - \sum_{k=1}^{n} a_k v_k\| + \|\Sigma\, a_k v_k - \Sigma\, u_\iota a_k v_k\| +$$

$$+ \|u_\iota(\Sigma\, a_k v_k) - u_\iota v\| < \epsilon.$$

2) $\Rightarrow$ 3) by theorem 1.

3) $\Rightarrow$ 1) trivial.

1.17. Definition: An A-module V satisfying one of the conditions of theorem 2 is called an essential A-module.

For every A-module V the closure of all elements of the form $\sum_{k=1}^{n} a_k v_k$ is a submodule V_e of V which is called the essential part of V. The above reasoning implies that V_e consists of all $v \in V$ which are factorable, i.e. which can be written in the form $v = av'$ for some elements $a \in A$, $v' \in V$. It is clear that every submodule of an essential A-module again is essential.

Remark: If A does not have approximate units the situation may be quite different. We can define the essential part of an A-module V as the closure of all elements of the form $\sum_{k=1}^{n} a_k v_k$ in V. But it may happen that a submodule of an essential module ceases to be essential. For let A be a factorable Banach algebra ($AA = A$) such that there exists $a \in A$ satisfying $a \notin \overline{Aa}$. Then A is an essential A-module. Now let $V \subseteq A$ be the closure of all elements of the form $ba + \lambda a$ with $b \in A$. Then V is an A-module satisfying $V_e \subseteq \overline{Aa} \subsetneq V$.

1.18. Examples:

1) Let $A = c_0$ and $V = l^\infty$. Then $(l^\infty)_e = c_0$. For $x \in l^\infty$ and $a \in c_0$ we have $ax \in c_0$ and on the other hand each $y \in c_0$ has the factorization $y = y \cdot e$ with $y \in c_0$ and $e = (1,1,1,\ldots) \in l^\infty$.

2) Let A be a Banach algebra and let V be an essential A-module. Then $c_0(V) = \{x = (v_1, v_2, \ldots),\ v_i \in V,\ \lim_{i\to\infty} \|v_i\| = 0\}$ is an essential A-module with norm $\|x\| = \sup \|v_i\|$.

Proof: The set of all elements of the form $(v_1, v_2, v_3, \dots v_n, 0, 0, \dots)$ with $v_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}$, $v_{ij} \in V$, is dense in $c_o(V)$ since V is essential. Let $\overline{v_{ij}} = (0, \dots, 0, v_{ij}, 0, \dots)$ with v_{ij} in the i-th position. Then $\overline{v_{ij}} \in c_o(V)$ and $(v_1, v_2, \dots, v_n, 0, 0, 0, \dots) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} a_{ij} \overline{v_{ij}}$.

§ 2. Banach module homomorphisms.

2.1. For left A-modules V_1 and V_2 we define an A-module homomorphism $\varphi: V_1 \to V_2$ to be a bounded linear map such that $\varphi(av_1) = a\varphi(v_1)$ for all $a \in A$ and $v_1 \in V_1$ holds. The set of all A-module homomorphisms is denoted by $H_A(V_1, V_2)$.

It is clear that $H_A(V_1, V_2)$ is a closed subspace of $H(V_1, V_2)$, because $\varphi = \lim \varphi_n$ implies $\varphi(av_1) = \lim \varphi_n(av_1) = \lim a\varphi_n(v_1) = a\varphi(v_1)$.

For right modules W_1, W_2 the Banach space $H^A(W_1, W_2)$ consists of all $\varphi \in H(W_1, W_2)$ such that $\varphi(w_1 a) = \varphi(w_1)a$ for all $w_1 \in W_1$ and $a \in A$.

2.2. Let n_1 and n_2 be sequence spaces. Then every $\varphi \in H_{c_o}(n_1, n_2)$ is a multiplication operator associated with some element

$\lambda \in l^{\infty}$, i.e. $\varphi(x) = \lambda x$.

<u>Proof</u> : Let $\varphi \in H_{c_o}(n_1,n_2)$. Then $\varphi(e_k) = \varphi(e_k e_k) =$

$= e_k \varphi(e_k) = \lambda_k e_k$ with $\lambda_k \in I$.

$|\lambda_k| = \|\lambda_k e_k\|_{n_2} = \|\varphi(e_k)\|_{n_2} \leq \|\varphi\| < \infty$ for all k.

Therefore $\lambda \in l^{\infty}$ and $\|\lambda\|_{\infty} \leq \|\varphi\| < \infty$.

Now let $x = (x_k) \in n_1$ be arbitrary. Then

$(\varphi(x))_k e_k = \varphi(x) e_k = e_k \varphi(x) = \varphi(e_k x) =$

$= \varphi(e_k x_k e_k) = e_k x_k \varphi(e_k) = \lambda_k x_k e_k$

$\Rightarrow (\varphi(x))_k = \lambda_k x_k \quad \Rightarrow \quad \varphi(x) = \lambda x$.

It follows that $H_{c_o}(n_1,n_2)$ consists of all those multiplication operators $\varphi(x) = \lambda x$ with $\lambda \in l^{\infty}$ such that $\lambda x \in n_2$ for all $x \in n_1$.

The only thing that remains to be proved is that every such operator is bounded. But this follows from the fact that the graph of φ is closed.

<u>Remarks</u>: $H_{c_o}(n_1,n_2)$ is again a sequence space.

Consider the map $\varphi \to \lambda$ for $\varphi(x) = \lambda x$. This identifies φ with a sequence $\lambda \in l^{\infty}$. We define $a\varphi$ for $a \in l^{\infty}$ and $\varphi \in H_{c_o}(n_1,n_2)$ by $(a\varphi)(v) = \varphi(av) = a\varphi(v)$.

Then to $a\varphi$ corresponds the sequence $a\lambda$ and $\|a\varphi\| = \sup_{\|x\|_{n_1} \leq 1} \|a\varphi(x)\| \leq \|a\|_{\infty} \|\varphi\|$. Therefore na) is fulfilled.

The element $\lambda = e_i \in l^\infty$ defines the element $\varphi_i \in H_{c_o}(n_1,n_2)$ given by $\varphi_i(x) = x_i e_i \in n_2$. We have

$$\|\varphi_i\| = \sup_{\|x\|_{n_1}\le 1} \|\varphi_i(x)\|_{n_2} = \sup_{\|x\|_{n_1}\le 1} \|x_i e_i\|_{n_2} = \sup_{\|x\|_{n_1}\le 1} |x_i| = 1.$$

This proves nb).

Condition nc) follows from the equation

$$\|\varphi\| = \sup_{\|x\|_{n_1}\le 1} \|\varphi(x)\|_{n_2} = \sup_{\|x\|_{n_1}\le 1} \|\lambda x\|_{n_2} =$$

$$= \sup_{\|x\|_{n_1}\le 1} \sup_k \|u_k \lambda x\|_{n_2} = \sup_k \|u_k \varphi\|.$$

2.3. Let $1 \le q \le p \le \infty$. Then $H_{c_o}(l^p,l^q) = l^r$ with $\frac{1}{q}-\frac{1}{p} = \frac{1}{r}$. If $p < q$ then $H_{c_o}(l^p,l^q) = l^\infty$.

<u>Proof</u>: Consider first the case $p \ge q$.

For $x \in l^p$ and $\lambda \in l^r$ we have $\lambda x \in l^q$ because $|\lambda|^q|x|^q \in l^1$. (Note that $|x|^q \in l^{\frac{p}{q}}$, $|\lambda|^q \in l^{\frac{r}{q}}$ and $\frac{q}{p}+\frac{q}{r} = 1$).

Now let $\lambda \in l^\infty$ be such that $\lambda x \in l^q$ for all $x \in l^p$.

Then $|\lambda x|^q \in l^1$ for all $|x|^q \in l^{\frac{p}{q}}$.

$\Rightarrow |\lambda|^q$ defines a continuous linear functional

$\Rightarrow |\lambda|^q \in (l^{\frac{p}{q}})' = l^{\frac{r}{q}} \Rightarrow \lambda \in l^r$.

Furthermore we have

$$\sup_{\|x\|_p \leq 1} \|\lambda x\|_q = \sup_{\|z\|_{\frac{p}{q}} \leq 1} \sqrt[q]{\|\lambda^q z\|_1} = \sqrt[q]{\|\lambda^q\|_{\frac{r}{q}}} = \|\lambda\|_r,$$

i.e. the mapping $\varphi \to \lambda$ is an isometry of $H_{c_o}(l^p, l^q)$ into l^r.

Now the case $q > p$: we know already that each $\varphi \in H_{c_o}(l^p, l^q)$ has the form $\varphi(x) = \lambda x$ with $\|\lambda\|_\infty \leq \|\varphi\|$.

Now let $\lambda \in l^\infty$. Then

$\|\lambda x\|_q \leq \|\lambda\|_\infty \|x\|_q \leq \|\lambda\|_\infty \|x\|_p$

$\Rightarrow \|\varphi\| \leq \|\lambda\|_\infty$.

We have used the fact that $\|x\|_q \leq \|x\|_p$ for $q > p$, i.e. the fact that for each x with $\|x\|_p = 1$ we have $\|x\|_q \leq 1$. But this is trivial since $\Sigma|x_k|^p = 1$ and $q > p$ implies $\Sigma|x_k|^q \leq 1$.

<u>Remark</u>: It would be more appropriate to write $(l^1)^{\frac{1}{p}}$ instead of l^p because l^p is the set of all x satisfying $|x|^p \in l^1$.

Since $H_{c_o}(l^p, l^q) = \{\lambda: \lambda x \in l^q \text{ for all } x \in l^p\}$ we could write formally

$H_{c_o}(l^p, l^q) = (l^1)^{\frac{1}{q}} : (l^1)^{\frac{1}{p}}$. This formalism would immediately give the right result for $p \geq q$.

2.4. Let n and m be sequence spaces (1.7) and $X,Y \in$ Ban. Then $H_{c_o}(n(X),m(Y))$ consists of all operators φ of the form $\varphi((x_n)) = (\lambda_n(x_n))$ with $\lambda_n \in H(X,Y)$ satisfying $(\|\lambda_n\|) \in H_{c_o}(n,m)$; i.e. $H_{c_o}(n(X),m(Y)) = H_{c_o}(n,m)(H(X,Y))$.

<u>Proof</u>: Let φ be given by $\varphi((x_n)) = (\lambda_k(x_k))$. Then

$$\|\varphi((x_n))\|_{m(Y)} = \lim_{j\to\infty} \|u_j(\|\lambda_k(x_k)\|_Y)\|_m$$

$$\leq \lim_{j\to\infty} \|u_j(\|\lambda_k\|\|x_k\|)\|_m = \|(\|\lambda_k\|)(\|x_k\|)\|_m$$

$$\leq \|(\|\lambda_k\|)\|_{H_{c_o}(n,m)} \|(\|x_k\|)\|_n .$$

If on the other hand $\varphi \in H_{c_o}(n(X),m(Y))$ there exist $\lambda_k \in H(X,Y)$ such that $\varphi((x_k)) = (\lambda_k(x_k))$ by the same argument as in 2.2. Then

$$\|(\|\lambda_k\|)\|_{H_{c_o}(n,m)} = \sup_{\|(\mu_k)\|_n\leq 1} \|(\mu_k\|\lambda_k\|)\|_m =$$

$$= \sup_{j\in\mathbb{N}} \sup_{\|(\mu_k)\|_n\leq 1} \|u_j(\mu_k\|\lambda_k\|)\|_m =$$

$$= \sup_{x_k\in OX} \sup_{\|(\mu_k)\|_n\leq 1} \sup_{j\in\mathbb{N}} \|u_j(\|\lambda_k(\mu_k x_k)\|_Y)\|_m$$

$$\leq \sup_{\|(\|x_k\|)\|_n\leq 1} \|(\lambda_k(x_k))\|_{m(Y)} = \|\varphi\|_{H_{c_o}(n(X),m(Y))}. \qquad \text{qed.}$$

2.5. Let N_1 and N_2 be norm ideals in $K(H)$ and let $\varphi \in H_{K(H)}(N_1,N_2)$. Then φ is a multiplication operator with an element

$b \in B(H) = H(H,H)$, i.e. $\varphi(v) = vb$ for all $v \in N_1$.

Proof: Let $h \otimes e^*$ be a one-dimensional operator.

Then $\varphi(h \otimes e^*) = \varphi((h \otimes e^*)(\frac{e}{\|e\|^2} \otimes e^*)) =$

$= (h \otimes e^*)\varphi(\frac{e}{\|e\|^2} \otimes e^*) = h \otimes f^*$ for some $f \in H$ which depends on e. We set $f = a(e)$. Then a is linear and $\|ae\|\|h\| = \|h \otimes f^*\| = \|\varphi(h \otimes e^*)\| \leq \|\varphi\|\|h\|\|e\| \Rightarrow$ $\|ae\| \leq \|\varphi\|\|e\| \Rightarrow a \in B(H)$ and $\|a\| \leq \|\varphi\|$. Now let be $b = a^*$.

Then $\varphi(h \otimes e^*) = (h \otimes e^*)b$ for all one-dimensional operators $h \otimes e^*$.

Now let $v \in N_1$ be arbitrary. Then for all $e,f \in H$ we have

$(e \otimes f^*)\varphi(v) = \varphi((e \otimes f^*)v) = [(e \otimes f^*)v]b = (e \otimes f^*)vb$

$\Rightarrow$ For all $x \in H$ we have $(\varphi(v)x|f)e = (vbx|f)e$

$\Rightarrow \varphi(v) = vb$ for all $v \in N_1$.

2.6. Let $A = X' \hat{\hat{\otimes}} X \subseteq H(X,X)$ for some Banach space X. Then $H_A(A,A) = H(X',X')$ and $H^A(A,A) = H(X,X)$.

Proofs: Consider first $H^A(A,A)$.

Let $x \in X$, $x' \in X'$, $\psi \in H^A(A,A)$. Choose some $y' \in X'$ with $\langle x,y' \rangle = 1$. Then we have

$\psi(x' \otimes x) = \psi((y' \otimes x)(x' \otimes x)) = \psi((y' \otimes x))(x' \otimes x) =$

$= x' \otimes \psi(y' \otimes x)(x) = x' \otimes b(x)$.

It is easily shown that b is linear and bounded.

$\Rightarrow$ $\psi = 1_{X'} \otimes b$ with $b \in H(X,X)$.

In terms of operators: $\psi(h) = b \circ h$, $h \in A \subseteq H(X,X)$.

It is clear that, conversely, each such ψ belongs to $H^A(A,A)$.

Now consider the space $H_A(A,A)$:

Let $x \in X$, $x' \in X'$, $\varphi \in H_A(A,A)$. Choose some $y \in X$ with $\langle y,x' \rangle = 1$. Then we have

$\varphi(x' \otimes x) = \varphi((x' \otimes x)\cdot(x' \otimes y)) = (x' \otimes x)\varphi(x' \otimes y) = z' \otimes x$. It is easily shown that $z' = a(x')$ for some $a \in H(X',X')$. Therefore $\varphi = a \otimes 1_X$ with $a \in H(X',X')$.

In terms of operators $\varphi(h) = h'' \circ a^t$.

2.7. Let N be a function space and $X \in$ Ban.

A linear map ψ: $N \to X$ is called <u>summable</u> (in the sense of V.L. Levin [44]) if there is a constant $K > 0$, such that for all finite subsets $\{g_1,\dots,g_n\} \subseteq N$ the inequality

$$\sum_{i=1}^{n} \|\psi(g_i)\|_X \leq K\|\sum_{i=1}^{n} |g_i|\|_N$$

holds. The infimum over all such K's is denoted by $\|\psi\|_{S(N,X)}$.

The set of all summable maps from N into X is a Banach space $S(N,X)$ with $\|\cdot\|_{S(N,X)}$ as norm and is an L^∞-module with the module operation

$(f\varphi)(g) = \varphi(fg)$ for $\varphi \in S(N,X)$, $g \in N$, $f \in L^\infty$.

The only nontrivial assertion is the completeness of $S(N,X)$. But this follows easily by observing that $\|\psi\|_{H(N,X)} \leq \|\psi\|_{S(N,X)}$ and that $\Sigma\|\psi_n\|_{S(N,X)} < \infty$ implies $\psi = \Sigma\psi_n \in H(N,X)$ with $\|\psi\|_{S(N,X)} < \infty$.

<u>Proposition</u>: $H_{L^\infty}(N, S(L^\infty, X)) = S(N,X)$

for each sequence space N and $X \in$ Ban.

<u>Proof</u>: Let $\psi \in S(N,X)$. Then T, defined by $T(g) = \psi(g\cdot)$, belongs to $H_{L^\infty}(N, S(L^\infty, X))$:

a) $T(g) \in S(L^\infty, X)$ for $g \in N$:

This follows from

$$\Sigma\|\psi(gf_k)\|_X \leq$$

$$\leq \|\psi\|_{S(N,X)} \; \|\Sigma|gf_k|\|_N \leq$$

$$\leq \|\psi\|_{S(N,X)} \; \|g\|_N \; \|\Sigma|f_k|\|_\infty$$

Moreover we get $\|T\| \leq \|\psi\|_{S(N,X)}$

b) T is an L^∞-module homomorphism:

$T(fg) = \psi(fg\cdot) = f\psi(g\cdot) = fT(g)$ for $f \in L^\infty$, $g \in N$.

This follows from the definition of the module operation on $S(N,X)$.

Now let $T \in H_{L^\infty}(N, S(L^\infty, X))$.

Then $T(g_1)(f_1) = T(g_2)(f_2)$ if $g_1, g_2 \in N$,
$f_1, f_2 \in L^\infty$ satisfying $g_1 f_1 = g_2 f_2$.
For $T(g_1)(f_1) = T(g_1)(f_1 \cdot 1) = f_1 T(g_1)(1) =$
$= T(f_1 g_1)(1) = T(f_2 g_2)(1) = T(g_2)(f_2)$.

Now we define $\psi: N \to X$ by $\psi(g) = T(g)(1)$ and assert that $\psi \in S(N,X)$.
Let $g_1, \ldots, g_n \in N$. Then

$$\Sigma\|\psi(g_k)\|_X = \Sigma\|T(g_k)(1)\|_X =$$

$$= \Sigma\|T(\Sigma|g_i|)\left(\frac{g_k}{\Sigma|g_i|}\right)\|_X \le$$

$$\le \|T(\Sigma|g_i|)\|_{S(L^\infty,X)} \left\|\sum_k \frac{|g_k|}{\Sigma|g_i|}\right\|_\infty \le$$

$$\le \|T\| \, \|\Sigma|g_i|\|_N$$

This means that $\psi \in S(N,X)$ and $\|\psi\|_{S(N,X)} \le \|T\|$.

2.8. In order to understand the role of summable maps we prove

<u>Proposition</u>: For every $X \in$ Ban $L_0^\infty(X)' = S(L^\infty, X')$ holds.

<u>Proof</u>: We define a bijection between the two spaces by
$\xi \in L_0^\infty(X)' \leftrightarrow \varphi \in S(L^\infty, X')$ if
$\langle xf, \xi \rangle = \langle x, \varphi(f) \rangle$ for $x \in X$ and $f \in L^\infty$.

For given $\xi \in L_0^\infty(X)'$ we have

$$\sum_{k=1}^{n} \|\varphi(f_k)\|_{X'} = \sup_{\|x_k\| \leq 1} \left| \sum_{k=1}^{n} < x_k, \varphi(f_k) > \right| =$$

$$= \sup_{\|x_k\| \leq 1} \left| \sum_{k=1}^{n} < x_k f_k, \xi > \right| =$$

$$= \sup_{\|x_k\| \leq 1} \left| < \Sigma\, x_k f_k, \xi > \right| \leq$$

$$\leq \|\xi\| \sup_{\|x_k\| \leq 1} \|\Sigma\, x_k f_k\|_{L_0^\infty(X)} \leq \|\xi\| \, \|\Sigma\, |f_k|\|_\infty .$$

Therefore $\varphi \in S(L^\infty, X')$ and $\|\varphi\|_{S(L^\infty,X')} \leq \|\xi\|$.

Now let $\varphi \in S(L^\infty, X')$ be given. Let A_k be disjoint measurable subsets of $[0,1]$. Then

$$|< \Sigma\, x_k c_{A_k}, \xi >| = |\Sigma < x_k, \varphi(c_{A_k}) >| \leq$$

$$\leq \Sigma \|x_k\| \|\varphi(c_{A_k})\| = \Sigma \|\varphi(\|x_k\| c_{A_k})\| \leq$$

$$\leq \|\varphi\|_{S(L^\infty,X')} \|\sum_k \|x_k\| c_{A_k}\|_\infty =$$

$$= \|\varphi\|_{S(L^\infty,X')} \|\sum_k x_k c_{A_k}\|_{L_0^\infty(X)} .$$

Since the elements of the form $\Sigma\, x_k c_{A_k}$ are dense in $L_0^\infty(X)$ we get

$$\xi \in (L_0^\infty(X))' \quad \text{and} \quad \|\xi\| \leq \|\varphi\|_{S(L^\infty,X')} .$$

It is clear that the module structure which we have defined on $S(L^\infty, X')$ is the same as that induced by $L_0^\infty(X)'$.

2.9. Let V be a left A-module. Then the Banach space $H_A(A,V)$ is also a left A-module with $(a\varphi)(b) = \varphi(ba)$ for $\varphi \in H_A(A,V)$ and $a,b \in A$. If Z is an A-B-bimodule (1.6) for Banach algebras A and B, then $H_A(A,Z)$ is again an A-B-bimodule via

$$(a\varphi)(a_1) = \varphi(a_1 a) \quad \text{and} \quad (\varphi b)(a) = \varphi(a)b.$$

<u>Definition</u>: We will call $H_A(A,V)$ the (left) A-completion of V and denote it by $\overline{V}$. The reason will become clear in 2.13 and 2.14.

2.10. We are now interested in the connections between V and $H_A(A,V)$.

If A has a unit element e with $ev = v$ for all $v \in V$, then these A-modules coincide. For let $\varphi \in H_A(A,V)$ and $\varphi(e) = v$. Then for each $a \in A$ we have

$\varphi(a) = \varphi(ae) = a\varphi(e) = av = \varphi_v(a)$.

The map $v \to \varphi_v$ is an isometry from V into $H_A(A,V)$ and an A-module homomorphism.

If A does not have a unit element which acts as an identity on V then the situation becomes more complicated. Since we are not interested in situations which, from our point of view, are pathological we introduce a class of nice A-modules.

<u>Definition</u>: An A-module V is called <u>strong</u> if $\|v\| = \sup_{\|a\| \leq 1} \|av\|$ for all $v \in V$.

2.11. Proposition: Let A be a Banach algebra. Then the following assertions hold:

1) V is strong if and only if the canonical mapping $v \to \varphi_v$ from V into $\overline{V} = H_A(A,V)$, defined by $\varphi_v(a) = av$, is an isometry from V into $\overline{V}$.

2) If V is strong, then $\overline{V}$ is strong too.

3) If A has an approximate left or right unit, then $\overline{V}$ is strong.

Proof:

1) We have

$$\sup_{\|a\|\leq 1} \|av\| = \sup_{\|a\|\leq 1} \|\varphi_v(a)\| = \|\varphi_v\| .$$

Therefore $v \to \varphi_v$ is an isometry, if and only if V is strong.

2) Let $\overline{v} \in H_A(A,V)$. Then

$$\sup_{\|a\|\leq 1} \|a\overline{v}\| = \sup_{\|a\|\leq 1} \|\varphi_{\overline{v}(a)}\| = \sup_{\|a\|\leq 1} \|\overline{v}(a)\| = \|\overline{v}\| ,$$

because $(a\overline{v})(b) = \overline{v}(ba) = b\overline{v}(a) = \varphi_{\overline{v}(a)}(b)$.

3) $$\sup_{\|a\|\leq 1} \|a\overline{v}\| = \sup_{\|a\|\leq 1} \sup_{\|b\|\leq 1} \|(a\overline{v})(b)\| =$$

$$= \sup_{\|a\|\leq 1} \sup_{\|b\|\leq 1} \|\overline{v}(ba)\| = \sup_{\|c\|\leq 1} \|\overline{v}(c)\| = \|\overline{v}\|.$$

2.12. Let us suppose in the following that A has a two-sided approximate unit (u_ι) and that all A- modules V are strong.

Lemma: Each algebraic A-module homomorphism φ from A into V is continuous.

Proof: We have to show that a linear mapping $\varphi: A \to V$ satisfying $\varphi(ab) = a\varphi(b)$ for all $a,b \in A$ is continuous. It suffices to show that $\lim a_n = 0$ implies $\lim \varphi(a_n) = 0$. Now $(a_n) \in c_o(A)$ and $c_o(A)$ is an essential A-right module, because A has right approximate units. Therefore $(a_n) = (b_n)c$ with $(b_n) \in c_o(A)$ and $c \in A$. This implies $\lim \varphi(a_n) = \lim \varphi(b_n c) = \lim b_n\varphi(c) = 0$.

2.13. Definition: The strict topology on V is the topology on V induced by the strong operator topology on $\overline{V}$ via the embedding $v \to \varphi_v$.

Lemma: $\overline{V}$ is complete in the strict topology and V is dense in $\overline{V}$.

Proof: Let (φ_ι) be a strict Cauchy-net. Then for each $a \in A$ the net $(\varphi_\iota(a))$ is Cauchy in V and therefore converges to some element $\varphi(a) \in V$. Since $\varphi_\iota(ab) = a\varphi_\iota(b)$ it follows that $\varphi(ab) = a\varphi(b)$ and therefore $\varphi \in H_A(A,V)$ by lemma 2.12.
Let now $\varphi \in H_A(A,V)$ and $v_\iota = \varphi(u_\iota)$. Then $\varphi(a) = \lim \varphi(au_\iota) = \lim a\varphi(u_\iota) = \lim av_\iota =$ $= \lim \varphi_{v_\iota}(a)$ for all $a \in A$, i.e. $\varphi = \lim \varphi_{v_\iota}$ in the strict topology on $\overline{V}$.

For each strong A-module V we have the isometric inclusions $V_e \subseteq V \subseteq \overline{V}$.
From the definition of the essential part it follows that $(V_e)_e = V_e$.

Observe that for an essential A-module V_1 and an arbitrary A-module V_2 we have

$H_A(V_1,V_2) = H_A(V_1,(V_2)_e)$.

For let $v \in V_1$. Then $v = av_1$ and

$\varphi(v) = \varphi(av_1) = a\varphi(v_1) \in (V_2)_e$.

Therefore $H_A(A,\overline{V}) = H_A(A,(\overline{V})_e)$.

In order to compute this space we need.

2.14. Lemma: For each A-module V we have $(\overline{V})_e = V_e$.

Proof: We can assume that V is essential since $\overline{(V_e)} = \overline{V}$ by the foregoing remark.

By the factorization theorem of Cohen-Hewitt $\varphi \in (\overline{V})_e$ if and only if $\varphi = a\varphi_1$ for some $a \in A$ and $\varphi_1 \in \overline{V}$. But then $\varphi(b) = (a\varphi_1)(b) = \varphi_1(ba) = b\varphi_1(a) = \varphi_v(b)$ with $v = \varphi_1(a)$.

Corollary: $\overline{(\overline{V})} = \overline{V}$.

Proof: $\overline{\overline{V}} = H_A(A,\overline{V}) = H_A(A,(\overline{V})_e) = H_A(A,V_e) = H_A(A,V) = \overline{V}$.

2.15. Remark: We know that for each $\varphi \in \overline{V}$ $a\varphi$ belongs to V_e.

Now $(a\varphi)(b) = \varphi(ba) = b\varphi(a) = \varphi_{\varphi(a)}(b)$.

Therefore $a\varphi = \varphi(a)$ with the obvious identification.

For right modules W we have of course a similar theory. This follows from the fact that a right A-module is just a left A^{op}-module, where A^{op} is the opposite Banach algebra (with reversed multiplication).

2.16. Let us now consider the case of bimodules. Let Z be an A-B-bimodule, where A and B are Banach algebras with approximate units. We define the essential part of Z to be the set of all elements of the form azb for $z \in Z$, $a \in A$, $b \in B$. Then $Z_e = AZ \cap ZB$.

We have only to show that each $u \in AZ \cap ZB$ has a representation of the form $u = azb$.

Now $u \in ZB$, i.e. $u = wb$ for some $w \in Z$ and $b \in B$. The proof of the Cohen - Hewitt factorization theorem shows that $u = ay$ for some $a \in A$ and $y \in Z$ of the form

$y = \lim y_n = \lim a_n^{-1} u = \lim a_n^{-1} wb$.

Therefore $y_n \in ZB$ and therefore also $y \in ZB$.

This means $y = zb \Rightarrow u = ay = azb \in Z_e$.

An immediate consequence is: denote by $H_A^B(Z,Z')$ the set of all A-B-bimodule homomorphisms from Z to Z'. If Z is essential then $H_A^B(Z,Z') = H_A^B(Z,(Z')_e)$.

2.17. For each A-B-bimodule Z we define the A-B-completion $\overline{Z}$ to be the space

$\overline{Z} = \mathbb{B}_A^B(A \times B, Z)$

of all bilinear maps $\lambda: A \times B \to Z$

such that $a \to \lambda(a,b) \in H_A(A,Z)$ and

$b \to \lambda(a,b) \in H^B(B,Z)$.

It is clear that $\mathbb{B}_A^B(A \times B,\ Z)$ is an A-B-bimodule via $(a\,\lambda\,b)(a_1,b_1) = \lambda(a_1 a,\ bb_1)$ and that the canonical mapping $z \to \lambda_z$ from Z into $\overline{Z}$, defined by $\lambda_z(a,b) = azb$ is an A-B-bimodule homomorphism: $\lambda_{a_1 z b_1}(a,b) = aa_1\ zb_1\ b =$ $= \lambda_z(aa_1,\ b_1 b) = (a_1 \lambda_z b_1)(a,b)$.

The bimodule Z is called strong if $\|z\| = \sup\limits_{\substack{\|a\| \leq 1 \\ \|b\| \leq 1}} \|azb\|$.

This is equivalent to the fact that $z \to \lambda_z$ is an isometry.

Proposition: $(\overline{Z})_e = Z_e$.

Proof: Let $\lambda \in (\overline{Z})_e \Rightarrow \lambda = a\lambda_1 b$ with $\lambda_1 \in \overline{Z}$.

$\Rightarrow \lambda(a_1,b_1) = (a\lambda_1 b)(a_1,b_1) = \lambda_1(a_1 a, bb_1) = a_1\lambda_1(a,b)b_1 =$
$= \lambda_{\lambda_1(a,b)}(a_1,b_1)$.

$\Rightarrow \lambda = \lambda_{\lambda_1(a,b)}$. Now $a = \overline{a}\,\overline{\overline{a}}$, $b = \overline{\overline{b}}\,\overline{b}$ since A and B have approximate units.

$\Rightarrow \lambda_1(a,b) = \lambda_1(\overline{a}\,\overline{\overline{a}},\ \overline{\overline{b}}\,\overline{b}) = \overline{a}\ \lambda_1(\overline{\overline{a}},\ \overline{\overline{b}})\overline{b} \in Z_e$.

Corollary: $\overline{\overline{Z}} = \overline{Z}$.

Proof: $\overline{\overline{Z}} = \mathbb{B}_A^B(A \times B,\ \overline{Z}) = \mathbb{B}_A^B(A \times B,\ (\overline{Z})_e) =$
$= \mathbb{B}_A^B(A \times B,\ Z_e) = \mathbb{B}_A^B(A \times B,\ Z) = \overline{Z}$.

2.18. <u>Definition</u>: The strict topology on Z is the topology on Z induced by the topology of pointwise convergence on $\mathbb{B}_A^B(A \times B, Z)$ via the embedding $z \to \lambda_z$.

<u>Proposition</u>: $\overline{Z}$ is complete in the strict topology and Z is dense in $\overline{Z}$.

<u>Proof</u>: Let (λ_ι) be a strict Cauchy net. Then for each pair $(a,b) \in A \times B$ the net $\lambda_\iota(a,b)$ is Cauchy in Z and therefore converges to some element $\lambda(a,b) \in Z$. From $\lambda_\iota(a_1a_2,b_2b_1) = a_1\lambda_\iota(a_2,b_2)b_1$ it follows that $\lambda(a_1a_2, b_2b_1) = a_1\lambda(a_2,b_2)b_1$. This means that $a \to \lambda(a,b) \in H_A(A,Z)$ and $b \to \lambda(a,b) \in H^B(B,Z)$. Therefore λ is separately continuous on A x B. By the Banach-Steinhaus theorem λ is continuous and therefore belongs to $\mathbb{B}_A^B(A \times B, Z)$.

Let now $\lambda \in \mathbb{B}_A^B(A \times B, Z)$ and $z_{\iota\varkappa} = \lambda(u_\iota, w_\varkappa)$ where (u_ι) is an approximate unit in A and and $(w_\varkappa)$ one in B. Then $\lambda(a,b) = \lim\limits_{\iota,\varkappa} \lambda(au_\iota, w_\varkappa b) =$

$= \lim a\lambda(u_\iota,w_\varkappa)b = \lim az_{\iota\varkappa}b = \lim\limits_{\iota,\varkappa} \lambda_{z_{\iota\varkappa}}(a,b)$.

<u>Remark</u>: The case of left and right modules is included in the case of bimodules. Consider e.g. a left A-module V as an A-I-bimodule.

2.19. We have now the following situation for an A-B-bimodule Z:

$$\begin{array}{ccccc} & & Z & & \\ & \overset{\varphi}{\swarrow} & \Big\downarrow \lambda & \overset{\psi}{\searrow} & \\ H_A(A,Z) & & & & H^B(B,Z) \\ & \searrow & & \swarrow & \\ & & \overline{Z} & & \end{array}$$

Here all arrows are isometric A-B-module homomorphisms.

$z \to \varphi_z$ is defined by $\varphi_z(a) = az$.

We have $\varphi_{\bar{a}z\bar{b}}(a) = a\bar{a}z\bar{b} = (\bar{a}\varphi_z\bar{b})(a)$.

$z \to \psi_z$ is defined by $\psi_z(b) = zb$. Here we also have

$\psi_{\bar{a}z\bar{b}}(b) = \bar{a}z\bar{b}b = (\bar{a}\psi_z\bar{b})(b)$.

The inclusion $H_A(A,Z) \to \overline{Z}$ is given by

$\varphi \to \{(a,b) \underset{\lambda_\varphi}{\to} \varphi(a)b\}$.

This is an A-B-module homomorphism because

$\lambda_{\bar{a}\varphi\bar{b}}(a,b) = (\bar{a}\varphi\bar{b})(a)b = \varphi(a\bar{a})\bar{b}b = (\bar{a}\lambda_\varphi\bar{b})(a,b)$.

$$\|\lambda_\varphi\| = \sup_{\substack{\|a\|\le 1\\ \|b\|\le 1}} \|\lambda_\varphi(a,b)\| = \sup_{\substack{\|a\|\le 1\\ \|b\|\le 1}} \|\varphi(a)b\| = \|\varphi\|$$

since Z is strong.

In view of the above inclusions we can consider $H_A(A,Z)$ and $H^B(B,Z)$ as closed submodules of $\overline{Z}$. Therefore

$$\Delta(Z) = H_A(A,Z) \cap H^B(B,Z)$$

is also a closed A-B-submodule of $\overline{Z}$.

This module is of special interest. It consists of all $\lambda \in \overline{Z}$ which are of the form

$\lambda(a,b) = \varphi(a)b = a\psi(b)$ for some (uniquely determined) $\varphi \in H_A(A,Z)$ and $\psi \in H^B(B,Z)$.

It is clear that $\|\lambda\| = \sup\limits_{\substack{\|a\|\leq 1\\ \|b\|\leq 1}} \|\lambda(a,b)\| =$

$$= \sup_{\substack{\|a\|\leq 1\\ \|b\|\leq 1}} \|\varphi(a)b\| = \|\varphi\| = \sup_{\substack{\|a\|\leq 1\\ \|b\|\leq 1}} \|a\psi(b)\| = \|\psi\|.$$

On $\Delta(Z)$ we also have a strict topology, i.e. the restriction of the product topology of $H_A(A,Z) \times H^B(B,Z)$ where in each factor the strong operator topology is considered.

A net $\lambda_\iota \in \Delta(Z)$ converges to λ in this strict topology if and only if for all pairs $(a,b) \in A \times B$ $(\varphi_\iota(a), \psi_\iota(b))$ converges to $(\varphi(a), \psi(b))$.

Here $\lambda_\iota(a,b) = \varphi_\iota(a)b = a\psi_\iota(b)$ and $\lambda(a,b) = \varphi(a)b = a\psi(b)$.

2.20. Of special importance is the case $Z = A$, a Banach algebra. In this case $\Delta(A)$ can be given the structure of a Banach algebra too.

By the foregoing $\Delta(A)$ consists of all pairs (f,g), $f \in H_A(A,A)$, $g \in H^A(A,A)$, such that $f(a)b = ag(b)$ for all $(a,b) \in A \times A$. We define a multiplication in $\Delta(A)$ by

$(f_1,g_1)(f_2,g_2) = (f_2 f_1, g_1 g_2)$.

Then $\Delta(A)$ is a Banach algebra which isometrically contains A via the embedding $a \to (f_a, g_a)$ given by $f_a(b) = ba$, $g_a(b) = ab$.

$\Delta(A)$ has a unit element $(1_A, 1_A)$.

For each pair (f_1, g_1) and $(f_2, g_2) \in \Delta(A)$ we have $f_2 \circ g_1 = g_1 \circ f_2$. For let (u_ι) be an approximate identity in A. Then

$f_2(g_1(a))u_\iota = g_1(a)g_2(u_\iota) = g_1(ag_2(u_\iota))$ and

$g_1(f_2(a))u_\iota = g_1(f_2(a)u_\iota) = g_1(ag_2(u_\iota))$.

In the limit we get the equality $f_2 \circ g_1 = g_1 \circ f_2$.

<u>Proposition</u>: A becomes a $\Delta(A)$-bimodule with the operation

$$(f_1, g_1)a(f_2, g_2) = g_1(f_2(a)) = f_2(g_1(a))$$

<u>Proof</u>: Consider e.g. $(f_1, g_1)[(f_2, g_2)a] = (f_1, g_1)g_2(a) = g_1(g_2(a))$

This is the same element as

$[(f_1, g_1)(f_2, g_2)]a = (f_2 f_1,\ g_1 g_2)a = (g_1 g_2)(a)$.

The special case $(f_b, g_b)a(f_c, g_c) = g_b(f_c(a)) = bac$ shows that this module operation is a natural extension of the bimodule operation of A on itself.

$\Delta(A)$ is called the double centralizer algebra of A.

§ 3. A-module tensor products and related constructions

3.1. For an A-bimodule Z we define the end of Z as the Banach space of all $z \in Z$ such that $az = za$ for all $a \in A$.

In symbols

$$\text{end}(Z) = \int_A Z = \{z \in Z : az = za \ \forall a \in A\}.$$

Dually we define the coend of Z as the quotient space

$$\text{coend}(Z) = \int^A Z = Z/_N$$

where N is the closed subspace of Z spanned by all elements of the form $az - za$.

Examples and properties:

3.2. Example: Let V_1 and V_2 be left A-modules. Then $H(V_1,V_2)$ is an A-bimodule with the module operations $(a\varphi b)(v) = a \cdot \varphi(bv)$. In this case the end of the bimodule $H(V_1,V_2)$ coincides with the Banach space $H_A(V_1,V_2)$ of all A-module homomorphisms, since

$$\int_A H(V_1,V_2) = \{\varphi \in H(V_1,V_2) : a\varphi = \varphi a\} =$$
$$= \{\varphi \in H(V_1,V_2) : a \cdot \varphi(x) = \varphi(ax)\}.$$

In the same way we see that

$$\int_A H(W_1,W_2) = H^A(W_1,W_2)$$

for two right A-modules W_1, W_2.

3.3. Let Z be an A-bimodule. Then

$$\int_A H(Z,X) = H(\int^A Z, X).$$

This equation is to be understood in the following way: For each $\varphi \in H(Z,X)$ satisfying $a\varphi = \varphi a$, i.e. $\varphi(za) = (a\varphi)(z) = (\varphi a)(z) = \varphi(az)$, there exists a uniquely determined $\tilde{\varphi} \in H(\int^A Z, X)$ such that $\|\tilde{\varphi}\| = \|\varphi\|$ and the diagram

$$(*) \qquad \begin{array}{ccc} Z & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\varphi} & \\ \int^A Z & \xrightarrow[\bar{\varphi}]{} & X \end{array}$$

commutes, where $\pi : Z \to \int^A Z = Z/N$ denotes the canonical quotient map, and conversely.

The proof is immediate, since $a\varphi = \varphi a$ is equivalent to $\varphi(za - az) = 0$, i.e. $\varphi | N = 0$.

The coend $\int^A Z$ could be defined by the property $(*)$.

3.4. Let Z be an A-bimodule. Then

$$\int_A H(X,Z) = H(X, \int_A Z).$$

For $a\varphi = \varphi a \Leftrightarrow (a\varphi)(x) = (\varphi a)(x) \Leftrightarrow a\cdot\varphi(x) = \varphi(x)\cdot a$ $\Leftrightarrow \varphi \in H(X, \int_A Z)$.

3.5. For each $Z \in \mathrm{Ban}_A^A$ (the category of all A-bimodules and contractive A-bimodule homomorphisms) let

$\pi_Z : Z \longrightarrow \int^A Z$ be the canonical quotient map.
Then (π_Z) defines a natural transformation from the forgetful functor $\iota : Z \to Z$ from Ban_A^A into Ban (which forgets the A-module structures) into the functor $\int^A Z$.

<u>Proof</u>: Consider the diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{\ \varphi\ } & Z_2 \\ \pi_{Z_1}\downarrow & & \downarrow \pi_{Z_2} \\ \int^A Z_1 & \dashrightarrow & \int^A Z_2 \end{array}$$

for an A-bimodule homomorphism φ. Then
$\pi_{Z_2} \circ \varphi \in H(Z_1, \int^A Z_2)$.
Since $(\pi_{Z_2} \circ \varphi)(az) = \pi_{Z_2}(a\varphi(z)) = \pi_{Z_2}(\varphi(z)a) =$
$= (\pi_{Z_2} \circ \varphi)(za)$ there exists a unique factorization over $\int^A Z_1$ given by $\pi_{Z_2} \circ \varphi = \int^A \varphi \circ \pi_{Z_1}$.
This defines $\int^A \varphi$ and gives $\|\int^A \varphi\| \leq \|\varphi\|$.
The uniqueness of $\int^A \varphi$ gives immediately
$\int^A \varphi_2 \circ \varphi_1 = \int^A \varphi_2 \circ \int^A \varphi_1$ and $\int^A 1_Z = 1_{\int^A Z}$.

3.6. In the same way the canonical injections
$\iota_Z : \int_A Z \to Z$ define a natural transformation from the functor $\int_A Z : \mathrm{Ban}_A^A \to \mathrm{Ban}$ into the forgetful functor $\iota : \mathrm{Ban}_A^A \to \mathrm{Ban}$.

This follows immediately from the diagram

$$\begin{array}{ccc} \int_A Z_1 & \xrightarrow[\int_A \varphi]{} & \int_A Z_2 \\ \downarrow \iota_{Z_1} & & \downarrow \iota_{Z_2} \\ Z_1 & \xrightarrow[\varphi]{} & Z_2 \end{array}$$

We have only to show that $az_1 = z_1 a$ implies $a\varphi(z_1) = \varphi(z_1)a$. But this is clear since φ is an A-bimodule homomorphism. It is also clear that

$$\int_A \varphi_2 \circ \varphi_1 = \int_A \varphi_2 \circ \int_A \varphi_1 \quad \text{and} \quad \left\|\int_A \varphi\right\| \leq \|\varphi\|$$

hold.

3.7. In the following we need a "Fubini - like" theorem: let Z be an A- and B bi-module, such that the different bimodule operations commute.

Then

$$\int_A \int_B Z = \int_B \int_A Z.$$

<u>Proof</u>: We show that $\int_B Z$ is an A-bimodule.

This follows from $b(az) = (ba)z = (ab)z = a(bz) = $ $= a(zb) = (az)b$, since this equation means that $z \in \int_B Z$ implies $az \in \int_B Z$.

Now we can prove the assertion: both sides consist of all $z \in Z$ such that both $az = za$ holds for all $a \in A$

and $bz = zb$ holds for all $b \in B$.

3.8. The projective A-module tensor product.

Let W be a right A-module and V a left A-module.
Then $W \hat{\otimes} V$ is an A-bimodule, where the module operations are given by $a(\Sigma w_i \otimes v_i)b = \Sigma w_i b \otimes av_i$.

Definition: By the projective A-module tensor product $W \hat{\otimes}_A V$ we understand the coend

$$W \hat{\otimes}_A V = \int^A W \hat{\otimes} V$$

of the bimodule $W \hat{\otimes} V$.

We want to give a description of $W \hat{\otimes}_A V$ in terms of a universal property analogous to that used in the definition of $W \hat{\otimes} V$.
To this end we define the Banach space of all A-bilinear mappings from W x V into a space X by $B_A(W,V;X) = \int_A B(W,V;X)$.

By definition of the end of a bimodule an element $\varphi \in B(W,V;X)$, i.e. a bilinear mapping $\varphi : W \times V \to X$, belongs to $B_A(W,V;X)$, i.e. is A-bilinear if and only if $a\varphi = \varphi a$. This means
$\varphi(wa,v) = \varphi(w,av)$ for all $v \in V$, $w \in W$.

From the defining equation

$$H(W \hat{\otimes} V, X) = B(W,V;X)$$

of the projective tensor product we get immediately

$$B_A(W,V;X) = \int_A B(W,V;X) = \int_A H(W \hat{\otimes} V, X) =$$

$$= H(\int^A W \hat{\otimes} V,\ X) = H(W \underset{A}{\hat{\otimes}} V,\ X).$$

This equation means that

for each A-bilinear mapping φ: W x V $\to$ X

there exists a uniquely determined bounded linear map $T_\varphi : W \underset{A}{\hat{\otimes}} V \to X$, such that $\|T_\varphi\| = \|\varphi\|$ and such that the diagram

$$\begin{array}{ccc} W \times V & \xrightarrow{\omega_A} & W \underset{A}{\hat{\otimes}} V \\ \varphi \downarrow & \swarrow T_\varphi & \\ Z & & \end{array}$$

commutes. Here $\omega_A = \pi_{W \hat{\otimes} V} \circ \omega$ is A-bilinear.

(ω is the linear map from the definition of $W \underset{A}{\hat{\otimes}} V$ and $\pi_{W \hat{\otimes} V} : W \hat{\otimes} V \to \int^A W \hat{\otimes} V$ in the map of 3.3).

By the above universal property the pair $(W \underset{A}{\hat{\otimes}} V,\ \omega_A)$ is uniquely determined up to an isomorphism.

3.9. Properties of $W \underset{A}{\hat{\otimes}} V$:

1) $W \underset{A}{\hat{\otimes}} V = (W \hat{\otimes} V)/_N$, where N is the closed subspace

of $W \hat{\otimes} V$ spanned by all elements of the form $(wa \otimes v - w \otimes av)$.

2) $(W \underset{A}{\hat{\otimes}} V)' = H_A(V,W') = H^A(W,V')$

This follows immediately from the equation

$$H(W \underset{A}{\hat{\otimes}} V, I) = H(\int^A W \hat{\otimes} V, I) = \int_A H(W \hat{\otimes} V, I) =$$

$$= \int_A H(W,V') = \int_A H(V,W').$$

3) Let V, V_1 be left A-modules, W, W_1 be right A-modules, and $\varphi \in H_A(V,V_1)$, $\psi \in H^A(W,W_1)$.

Then there exists a uniquely determined linear map $\psi \underset{A}{\otimes} \varphi \in H(W \underset{A}{\hat{\otimes}} V, W_1 \underset{A}{\hat{\otimes}} V_1)$ such that

$(\psi \underset{A}{\otimes} \varphi)(w \otimes v) = \psi(w) \otimes \varphi(v)$ for $w \otimes v \in W \underset{A}{\otimes} V$ and $\|\psi \underset{A}{\otimes} \varphi\| \leq \|\psi\| \|\varphi\|$.

<u>Proof</u>: Define $\psi \otimes \varphi : W \hat{\otimes} V \to W_1 \hat{\otimes} V_1$ by $(\psi \otimes \varphi)(w \otimes v) = \psi(w) \otimes \varphi(v)$.

Then $\psi \otimes \varphi$ is an A-bimodule homomorphism.

Therefore $\psi \underset{A}{\otimes} \varphi = \int^A \psi \otimes \varphi \in H(W \underset{A}{\hat{\otimes}} V, W_1 \underset{A}{\hat{\otimes}} V_1)$ and $\|\int^A \psi \otimes \varphi\| \leq \|\psi \otimes \varphi\| \leq \|\psi\| \|\varphi\|$.

4) Let A and B be Banach algebras, Z an A-B-bimodule, X a left B-module and V a left A-module.

Then the so-called 'exponential law' holds:

$$H_A(Z \mathop{\hat{\otimes}}_B X,\ V) = H_B(X,\ H_A(Z,V)).$$

This isometry is natural in all variables.

Proof:

$$H_A(Z \mathop{\hat{\otimes}}_B X,\ V) = \int_A H(\int^B Z \hat{\otimes} X,\ V) =$$

$$= \int_A \int_B H(Z \hat{\otimes} X,\ V) = \int_B \int_A H(Z \hat{\otimes} X,\ V) =$$

$$= \int_B \int_A H(X,\ H(Z,V)) = \int_B H(X,\ \int_A H(Z,V)) =$$

$$= H_B(X,\ H_A(Z,V)).$$

5) Let X be a Banach space and Z be an A-bimodule.
Then $X \hat{\otimes} \int^A Z = \int^A (X \hat{\otimes} Z)$

Proof: $H(X \hat{\otimes} \int^A Z,\ Y) = H(\int^A Z,\ H(X,Y)) =$

$$= \int_A H(Z,\ H(X,Y)) = \int_A H(X \hat{\otimes} Z,\ Y) = H(\int^A (X \hat{\otimes} Z),\ Y)$$

6) For an A-bimodule Z we have $\int^A \int^B Z = \int^B \int^A Z$.

Proof: $H(\int\int^{AB} Z,\ X) = \int_A H(\int^B Z,\ X) = \int\int_{AB} H(Z,X) =$

$$\int\int_{BA} H(Z,X) = H(\int\int^{BA} Z,\ X).$$

7) Let W be a right A-module, Z an A-B-bimodule, and V a

left B-module. Then we have

$$W \hat{\otimes}_A (Z \hat{\otimes}_B V) = (W \hat{\otimes}_A Z) \hat{\otimes}_B V .$$

This isomorphism is given by $w \otimes (z \otimes v) \to (w \otimes z) \otimes v$ and is natural in all variables.

Proof:

$$W \hat{\otimes}_A (Z \hat{\otimes}_B V) =$$

$$= \int^A W \hat{\otimes} (\int^B Z \hat{\otimes} V) = \int\int^{AB} W \hat{\otimes} (Z \hat{\otimes} V) =$$

$$= \int\int^{AB} (W \hat{\otimes} Z) \hat{\otimes} V = \int\int^{BA} (W \hat{\otimes} Z) \hat{\otimes} V =$$

$$= \int^B (\int^A (W \hat{\otimes} Z)) \hat{\otimes} V = (W \hat{\otimes}_A Z) \hat{\otimes}_B V$$

3.10. Theorem: If A has an approximate left identity (u_ι), then $A \hat{\otimes}_A V = V_e$.

Proof: It suffices to show that (V_e, ω) with $\omega(a,v) = av$ is a projective A-tensor product of A and V.
Let therefore $X \in$ Ban and $\varphi: A \times V \to X$ be A-bilinear.
We have to show that there exists a uniquely determined linear map $T_\varphi : V_e \to X$ such that $\|T_\varphi\| = \|\varphi\|$ satisfying $\varphi(a,v) = T_\varphi(av)$ for all $a \in A$, $v \in V$.
To show this observe first that for each $v \in V_e$ the limit $\lim_\iota \varphi(u_\iota, v)$ exists. For let $v = av'$ then $\varphi(u_\iota, v) = \varphi(u_\iota, av') = \varphi(u_\iota a, v')$ and this converges to $\varphi(a,v')$.

Now we can define T_φ by $T_\varphi(v) = \lim_\iota \varphi(u_\iota, v)$.

Then it is clear that T_φ is linear and satisfies

$T_\varphi(av) = \varphi(a,v)$.

The norm equality follows from

$$\|\varphi\| = \sup_{\substack{\|a\|\le 1 \\ \|v\|\le 1}} \|\varphi(a,v)\| = \sup_{\substack{\|a\|\le 1 \\ \|v\|\le 1}} \|T_\varphi(av)\| \le \|T_\varphi\|$$

and

$$\|T_\varphi\| = \sup_{\|v\|\le 1} \|\lim_\iota \varphi(u_\iota, v)\| \le \|\varphi\|.$$

Remark: For right modules we have of course a similar theorem: Let A have a right approximate identity then $W \hat{\otimes}_A A = W_e$ for each right A-module W.

From now on we shall assume A has a two-sided approximate identity.

3.11. A special case of the above theorem is that $A \hat{\otimes}_A A = A$ and that $\Delta(A) \hat{\otimes}_{\Delta(A)} A = A = A \hat{\otimes}_{\Delta(A)} \Delta(A)$, where $\Delta(A)$ has a unit element and A is a $\Delta(A)$-bimodule.

A $\Delta(A)$-module is called strict $\Delta(A)$-module if the module operation is continuous for the strict topology on $\Delta(A)$. E.g. we have that A is a strict $\Delta(A)$-bimodule. For let $(f_\iota, g_\iota) \to (f,g)$ in the strict topology of $\Delta(A)$. Then $\lim_\iota (f_\iota, g_\iota)a = \lim_\iota g_\iota(a) = g(a)$ and $\lim_\iota a(f_\iota, g_\iota) = \lim_\iota f_\iota(a) = f(a)$.

Proposition: Any essential left A-module V is a strict $\Delta(A)$-module and vice versa.

The same holds for right A-modules and A-bimodules.

Proof: Let V be an essential left A-module.

Then $V = A \hat{\otimes}_A V = (\Delta(A) \hat{\otimes}_{\Delta(A)} A) \hat{\otimes}_A V =$

$= \Delta(A) \hat{\otimes}_{\Delta(A)} (A \hat{\otimes}_A V) = \Delta(A) \hat{\otimes}_{\Delta(A)} V$

Therefore V is a $\Delta(A)$-module. It is strict because A is strict.

For the converse observe that $(f_{u_\iota}, g_{u_\iota}) \to (1_A, 1_A)$ strict.

Therefore $(f_{u_\iota}, g_{u_\iota})v = u_\iota v \to v$, which says that V is an essential A-module.

3.12. Let Z be an essential A-bimodule.

Then $\int_A Z = \int_{\Delta(A)} Z$.

Proof: $\{z \in Z: az = za\} \supseteq \{z: (f,g)z = z(f,g)\}$ is trivial.

Let now $z \in Z$ be such that $az = za$ for all $a \in A$.

Then $(f,g)z = \lim_\iota u_\iota((f,g)z) = \lim_\iota (u_\iota(f,g))z =$

$= \lim_\iota f(u_\iota)z = \lim_\iota zf(u_\iota) = \lim_\iota zu_\iota(f,g) = z(f,g).$

This gives the other inclusion.

This result may hold even for non-essential A-bimodules, as is shown by:

$H_A(V_1,V_2) = H_{\Delta(A)}(V_1,V_2)$ if V_1,V_2 are essential.

Proof: Again $H_{\Delta(A)}(V_1,V_2) \subseteq H_A(V_1,V_2)$ is trivial. If

$\varphi \in H_A(V_1,V_2)$, then for $(f,g) \in \Delta(A)$ we have:

$$\varphi((f,g)v_1) = \varphi(\lim u_i((f,g)v_1)) = \varphi((\lim u_i(f,g))v_1)$$
$$= \varphi(\lim f(u_i)v_1) = \lim f(u_i)\varphi(v_1) = (\lim u_i(f,g))\varphi(v_1)$$
$$= (f,g)\varphi(v_1).$$

3.13. Let Z be an essential A-bimodule.

Then $\int^{A} Z = \int^{\Delta(A)} Z$.

Proof: We have to show that the closed subspace N_o of Z spanned by all elements of the form $(az - za)$ coincides with the closed subspace N spanned by all elements of the form $((f,g)z - z(f,g))$.

The inclusion $N_o \subseteq N$ being trivial, we show that $N \subseteq N_o$: It suffices to show show that each element of the form $(f,g)z - z(f,g)$ can be approximated by elements from N_o. This follows from the inequality

$$\|((f,g)z - z(f,g)) - ((f,g)u_\iota z - z(f,g)u_\iota)\| \leq$$
$$\leq \|(f,g)(z - u_\iota z)\| + \|z(((f,g) - (f,g)u_\iota))\| \leq$$
$$\leq \|(f,g)\| \, \|z - u_\iota z\| + \|z(f,g) - z(f,g)u_\iota\|$$

since Z is essential.

Example: $W \hat{\otimes}_A V = W \hat{\otimes}_{\Delta(A)} V$ if both V and W are essential.

3.14. Example:

Let m and n be sequence spaces. If $m \hat{\otimes}_{c_o} n$ is a strong c_o-module, then it is again a sequence space.

Proof: We use the map $j: x \otimes y \to xy$ which is clearly a c_o-module homomorphism of $m \otimes_{c_o} n$ into l^∞.

Assume that $v \in m \otimes_{c_o} n$, $v \neq 0$. By II.1.8g) $v = \sum_{i=1}^{\infty} x_i \otimes y_i$, where $\Sigma \|x_i\|_m \|y_i\|_n < \infty$. Since $m \hat{\otimes}_{c_o} n$ is strong, there exists some $k > 0$ such that $u_k v \neq 0$.

$u_k v = \sum_{i=1}^{\infty} u_k x_i \otimes y_i = \sum_{i=1}^{\infty} u_k \otimes u_k x_i y_i = u_k \otimes \sum_{i=1}^{\infty} u_k x_i y_i$.

$\sum_{i=1}^{\infty} u_k x_i y_i$ converges in n because of 1.7 Prop. and coincides with $u_k(\sum_{i=1}^{\infty} x_i y_i) = u_k\ j(v)$. Consequently $j(v) \neq 0$.

If $\Sigma\ x_i y_i = e_i$ then $\sum_{i=1}^{\infty} \|x_i\|_m \|y_i\|_n \geq \sum_{i=1}^{\infty} \|e_i x_i\|_m \|e_i y_i\|_n \geq 1$.

Consequently $\|e_i \otimes e_i\|_{m \hat{\otimes}_{c_o} n} = 1$.

Some special cases are worth mentioning:

a) $l^1 \hat{\otimes}_{c_o} n = l^1$ for each sequence space n.

b) $l^p \hat{\otimes}_{c_o} l^t = l^q$ if $\frac{1}{p} + \frac{1}{t} = \frac{1}{q}$ and $\frac{1}{1-\frac{1}{t}} \leq p \leq \infty$ $1 \leq t \leq \infty$.

Proof: The same methods as in 2.3.

3.15. Example:

For each sequence space n we have

$$n(X) = l^\infty(X) \hat{\otimes}_{l^\infty} n$$

<u>Proof</u>: Let $\omega: l^\infty(X) \times n \to n(X)$ be the l^∞-bilinear map $(f,g) \to fg$. We have to show that for each Banach space Y and each l^∞-bilinear map $\varphi: l^\infty(X) \times n \to Y$ there exists a unique continuous linear map $T_\varphi: n(X) \to Y$ such that $\varphi(f,g) = T_\varphi(\omega(f,g))$ and $\|T_\varphi\| = \|\varphi\|$.

Now $\omega(f_1,g_1) = \omega(f_2,g_2)$ implies $\varphi(f_1,g_1) = \varphi(f_2,g_2)$ because this means $f_1g_1 = f_2g_2$ and therefore

$$\varphi(f_1,g_1) = \varphi(f_1 \cdot \frac{g_1}{|g_1|+|g_2|}, |g_1| + |g_2|) =$$

$$= \varphi(f_2 \frac{g_2}{|g_1|+|g_2|}, |g_1| + |g_2|) = \varphi(f_2,g_2).$$

Each element $w \in n(X)$ is the image of an element of $l^\infty(X) \times n$, e.g. of $(\frac{w}{\|w\|}, \|w\|)$. Now define T_φ by $T_\varphi(fg) = \varphi(f,g)$. Then T_φ is well defined and linear because

$$T_\varphi(\Sigma f_i g_i) = T_\varphi(\frac{\Sigma f_i g_i}{\Sigma |g_k|} \cdot \Sigma |g_k|) =$$

$$= \varphi(\frac{\Sigma f_i g_i}{\Sigma |g_k|}, \Sigma |g_k|) = \sum_i \varphi(\frac{f_i g_i}{\Sigma |g_k|}, \Sigma |g_k|) =$$

$$= \sum_i T_\varphi(f_i g_i).$$

$T_\varphi(\omega(f,g)) = \varphi(f,g) \Rightarrow \|\varphi\| \leq \|T_\varphi\|$.

on the other hand

$$\|T_\varphi(w)\| = \|T_\varphi(\frac{w}{\|w\|} \cdot \|w\|)\| = \|\varphi(\frac{w}{\|w\|}, \|w\|)\| \leq$$

$$\leq \|\varphi\| \, \| \|w\| \|_n$$

$$\Rightarrow \|T_\varphi\| \leq \|\varphi\|.$$

Corollary: For n essential we have

$$n(X) = c_o(X) \hat{\otimes}_{c_o} n.$$

Proof: $n(X) = l^\infty(X) \hat{\otimes}_{l^\infty} n = l^\infty(X) \hat{\otimes}_{l^\infty} (c_o \hat{\otimes}_{c_o} n) =$

$$= (l^\infty(X) \hat{\otimes}_{l^\infty} c_o) \hat{\otimes}_{c_o} u = c_o(X) \hat{\otimes}_{c_o} n .$$

3.16. Example: For each function space N we have

$$N_o(X) = L_o^\infty(X) \hat{\otimes}_{L^\infty} N .$$

Proof: Let $\omega : L_o^\infty(X) \times N \to N_o(X)$ be defined by

$\omega(F,g) = Fg$ for $F \in L_o^\infty(X)$ and $g \in N$.

Then the same reasoning as above is applicable.

3.17. Example: $l^\infty \hat{\otimes}_{c_o} l^\infty$ is not a sequence space.

Proof: Let $f_o = (0,1,0,1,0,1,\ldots)$ and $f_1 = (1,0,1,0,\ldots)$.

Choose Banach limits F_o, F_1 such that $F_o(f_o) = 1$, $F_1(f_1) = 1$, then $F_o, F_1 \in H_{c_o}(l^\infty, I)$, where $c_o \cdot I = (0)$.

Then $F_o \otimes_{c_o} F_1 \in (l^\infty \hat{\otimes}_{c_o} l^\infty)'$

Since $(F_o \otimes_{c_o} F_1)(f_o \otimes f_1) = 1 \neq 0$ we have that $f_o \otimes f_1 \neq 0$ in $l^\infty \hat{\otimes}_{c_o} l^\infty$.

If $l^\infty \hat{\otimes}_{c_o} l^\infty$ were a sequence space, it would be a strong c_o module so we would have $\|f_o \otimes f_1\| = \lim_n \|u_n f_o \otimes f_1\| = 0$, a contradiction.

3.18. Now we want to introduce a notion which in some ways resembles the notion of dual space.

Definition: Let V be a left A-module. Then we call the right A-module $V^o = H_A(V,A')$ the associate module of V.
For a right module W the associate is the left module $W_o = H^A(W,A')$.

The associate modules are easy to compute in terms of dual spaces: $V^o = (V_e)'$ and $W_o = (W_e)'$.
We prove the first equation:

$$V^o = \int_A H(V,H(A,I)) = \int_A H(A \hat{\otimes} V,I) = H(A \hat{\underset{A}{\otimes}} V,I) = (V_e)'.$$

The correspondence $v^o \longleftrightarrow v_e'$ is given by

$$< a,v^o(v) > = < av,v' >.$$

In the same way for a right module we have

$$< a,w_o(w) > = < wa,w' >.$$

It is easy to see that the mappings $v^o \to v'$ and $w^o \to w'$ are A-module homomorphisms.

Proposition: Each associate module is A-complete, i.e. satisfies $\overline{V^o} = V^o$.
This follows from the equation

$$V^o = \int_A H(V,A') = \int_A H(A,V') = H^A(A,V') = \overline{V'}$$

3.19. Proposition: Let V be a left and W a right A-module. Then $H_A(V,W_o) = H^A(W,V^o)$ where the correspondence is given by $\varphi \longleftrightarrow \overline{\Phi}$ with $\overline{\Phi}(w)(v) = \varphi(v)(w)$.

Proof: This follows from the equations

$$\begin{aligned} H_A(V,W_o) &= \int_A H(V, \int_A H(W,A')) \\ &= \int_A \int_A H(V, H(W,A')) \\ &= \int_A \int_A H(W \hat{\otimes} V, A') \\ &= \int_A \int_A H(W, H(V,A')) \\ &= \int_A H(W, \int_A H(V,A')) \\ &= H^A(W,V^o) \end{aligned}$$

3.20. Proposition: Every strong A-module V is isometrically contained in ${V^o}_o$ via the canonical embedding $i : V \to {V^o}_o$ given by $i(v)(v^o) = v^o(v)$. This embedding is an A-module homomorphism.

Proof: In the equation $H_A(V,{V^o}_o) = H^A(V^o,V^o)$ the map i corresponds to the identity on the right side.

We only need to verify that i is an isometry:

$$\|i(v)\| = \sup_{\|v^o\|\leq 1} \|v^o(v)\| = \sup_{\|v^o\|\leq 1} \sup_{\|a\|\leq 1} |\langle a,v^o(v)\rangle| =$$

$$= \sup_{\|a\|\leq 1} \sup_{\|v'\|\leq 1} |\langle av,v'\rangle| = \sup_{\|a\|\leq 1} \|av\| = \|v\|.$$

Since $(\overline{V})^o = (\overline{V}_e)' = (V_e)' = V^o$ we have $V \subseteq \overline{V} \subseteq V^o_o$.

3.21. <u>Definition</u>: A left A-module V is called <u>A-reflexive</u> if the natural embedding $i: V \to V^{o}{}_{o}$ is an isomorphism.

<u>Proposition</u>: An A-module V is A-reflexive if and only if it is A-complete and satisfies $(V^{o}{}_{o})_e = V_e$.

<u>Proof</u>: Let $V = V^{o}{}_{o}$. Then $V = \overline{V}$ since $V \subseteq \overline{V} \subseteq V^{o}{}_{o}$.
Therefore V is A-complete. Furthermore $V_e = (V^{o}{}_{o})_e$.
Let now V be A-complete and $V_e = (V^{o}{}_{o})_e$.
Then
$V = \overline{V} = H_A(A,V) = H_A(A,V_e) = H_A(A,V^{o}_{o}) = \overline{V^{o}{}_{o}} = V^{o}{}_{o}$.

3.22. <u>Examples</u>:

1) Let $A = c_o$ and $V = n$ a sequence space.
In this case $n^o = H_{c_o}(n,c_o') = H_{c_o}(n,l^1)$.
This implies that n^o is again a sequence space.
It should be noted that 2.4 implies that
$n^o(X) = H_{c_o}(n,l^1(X))$ for each $X \in$ Ban holds,
whereas $n(X)^o = H_{c_o}(n(X),l^1) = H_{c_o}(n,l^1)(H(X,I))$
$= H_{c_o}(n,l^1)(H(I,X')) = H_{c_o}(n,l^1(X'))$.
A sequence space n is c_o-reflexive if and only if it is c_o-complete: it suffices to show that $(n^o_o)_e = n_e$.
This holds if we can show that $\|u_k x\|_{n_e} = \|u_k x\|_{n^o_o}$ for each k. But this is obvious since $n^o = (n_e)'$
and therefore

$$\|u_k x\|_{n_e} = \sup_{\|y\|_{n_e'} \leq 1} \|u_k xy\|_1 = \sup_{\|y\|_{n^o} \leq 1} \|u_k xy\| = \|u_k x\|_{n^o_o} .$$

2) Let $A = L^{\infty}$ and $V = N$ a function space.

In this case $N^{o} = H_{L^{\infty}}(N, (L^{\infty})') =$

$= H_{L^{\infty}}(N, S(L^{\infty}, I)) = S(N,I)$

We conclude this chapter with some observations about the dual space V' of a left A-module V. Similar results hold for right A-modules.

3.23. V is essential if and only if V' is strong. Then V' is even A-complete.

Proof: If V is essential, then $\overline{V'} = H^{A}(A,V') = (A \hat{\otimes}_{A} V)' = V'$.

If V is not essential, then there exists some $v' \in V'$ such that $v' \neq 0$ and $v' | V_e = 0$. For each $a \in A$ and $v \in V$ we have then $\langle v, v'a \rangle = \langle av, v' \rangle = 0$. This implies $v'a = 0$ for all $a \in A$, so $\sup_{\|a\| \leq 1} \|v'a\| = 0 \neq \|v'\|$ and V' is not strong.

3.24. We assume that V_e is reflexive as a Banach space. Then $V^{o}{}_{o} = (V_e{}')_{o} = \overline{V_e{}''}$ (using the equation from 3.18 Proposition) $= \overline{V_e} = \overline{V}$. This means that $\overline{V}$ is A-reflexive and $V_e'' = V_e$ is strong, so V_e' is essential by 3.23 and V_e is A-complete. Consequently $V_e = \overline{V}$ is A-reflexive. If V is strong, then $V \subseteq \overline{V} = V_e$ and so $V = V_e$. Since $V' = V_e{}'$ is again reflexive and essential, we may apply the same procedure and arrive at the following

<u>Proposition</u>: Let V be a strong A-module such that V_e is reflexive as a Banach space. Then V and V' are both essential, A-complete and A-reflexive.

3.25. <u>Lemma</u>: Let $l : V'_e \to V_e'$ be defined by $l(v') = v'|_{V_e}$. Then l is an isometric A-module homomorphism, whose image coincides with $V_e{}'_e$, i.e. $V'_e = V_e{}'_e$. Thus $l' : V_e'' \longrightarrow V'_e{}' = V^o{}_o$ is a quotient map, which coincides with the natural homomorphism $V_e'' \to \overline{V_e''} = V^o{}_o$.

<u>Proof</u>: For $v' \in V'_e$ we have

$$\|v'\|_{V'_e} = \sup_{\|a\|\le 1} \|v'a\|_{V'} =$$

$$= \sup_{\|a\|\le 1} \sup_{\|v\|\le 1, v\in V} |\langle v, v'a \rangle|$$

$$= \sup_{\|a\|\le 1} \sup_{\|v\|\le 1} |\langle av, v' \rangle|$$

$$= \sup_{\|v\|\le 1, v\in V_e} |\langle v, v' \rangle| = \|l(v')\|_{V_e'} .$$

If $v' \in V_e{}'_e$ then by 1.16 $v' = av_1'$ with $a \in A$, $v_1' \in V_e'$. Let $w_1' \in V'$ be any extension of v_1'. Then $w' = aw_1'$ satisfies $w' \in V'_e$ and $l(w') = v'$.
Next note that $V^o{}_o = (V^o)_e' = (\overline{V'})_e' = V'_e{}'$.
So only the last assertion remains to be shown. Let be $\alpha \in V^o$, $\varphi \in V^o{}_o$. We write again α and φ for the corresponding elements of isomorphic left A-modules.

For $a \in A$, $v \in V$ we have:

$\alpha \in V_e{}' = V^o = \overline{V'}$

$< av, \alpha > = < a, a(v) > = < v, \alpha(a) >$

The equations for φ look like this:

$\varphi \in \overline{V''_e} = (V_e{}')_o = V^o_o = (V^o)_e{}' = (\overline{V'})'_e$

$$\underset{\alpha \in V'_e}{< \alpha, \varphi(a) >} = \underset{\alpha \in V'_e}{< a, \varphi(\alpha) >} = \underset{\alpha \in V^o}{< a, \varphi(\alpha) >} = \underset{\alpha \in V^o}{< \alpha a, \varphi >} = \underset{\alpha \in \overline{V'}}{< \alpha a, \varphi >}$$

By 2.14 $(\overline{V'})_e = V'_e$, where $v' \in V'_e$ is mapped onto $\varphi_{v'}(a) = v'a$. The corresponding element of V'_e satisfies the equation $< av, \varphi_{v'} > = < v, \varphi_{v'}(a) > = < v, v'a > =$ $= < av, v' >$, i.e. it coincides with $1(v')$.

The natural homomorphism $V''_e \to \overline{V''_e}$ is again defined by $v'' \to \varphi_{v''}$, $\varphi_{v''}(a) = av''$.

This means that the corresponding element $\varphi \in V'_e{}'$ satisfies: $< \alpha a, \varphi > = < \alpha, \varphi_{v''}(a) > = < \alpha, av'' > =$ $= < \alpha a, v'' > = < f(\alpha, a), v'' >$ i.e. $\varphi = 1'(v'')$.

3.26. Proposition: If V is A-reflexive and V" is strong, then V is a reflexive Banach space.

Proof: V" is strong, so by 3.23 V' is essential, thus strong, so V is essential. Then 3.25 implies that $V'' = V_e{}'' = \overline{V_e{}''} = V^o_o = V$ via the canonical embedding $V \to V''$.

Corollary 1: Any two of the three properties of 3.26 implies the third one.

Proof: Combine 3.24 and 3.26.

Corollary 2: If $\bar{V}$ is A-reflexive, then $V_e{}''_e = V_e$.

Proof: $V_e = (V^o{}_o)_e = (\overline{V_e{}''})_e = V_e{}''_e$.

Exercises

The following exercises show some connections between Chapter III and the theory of operator ideals on a separable Hilbert space X.

1) Let N be a norm ideal in K(X) as defined in 1.9., which is also an H(X)-bimodule. For an orthonormal base $(e_k)_{k=1}^{\infty}$ in X put $n = \{(a_k)_1^{\infty} \mid \sum_{k=1}^{\infty} a_k \, e_k \otimes e_k^* \in N\}$.
   Show that n is a symmetric sequence space which is contained in c_o and does not depend on the special choice of the orthonormal base.

2) Show conversely that to each symmetric sequence space $n \subseteq c_o$ there exists a norm ideal N in K(X), which is also an L(X)-bimodule (use the construction of [29] Ch. III, § 4). Show that the correspondences defined by 1) and 2) are inverse to each other and bijective.

3) Show that n is an essential c_o-module if and only if N is an essential K(X)-module.

CHAPTER IV

Functors on Categories of Banach Spaces

§ 1. Functors on Ban

1.1. A covariant functor $F : \mathrm{Ban} \to \mathrm{Ban}$ assigns Banach spaces $F(X)$ to Banach spaces X and morphisms $F(f) : F(X) \to F(Y)$ to morphisms $f : X \to Y$ in such a way that the following statements hold:

(a) $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined in Ban.

(b) $F(1_X) = 1_{F(X)}$ for all Banach spaces X.

(c) Each map $f \mapsto F(f)$ from $H(X,Y)$ into $H(F(X),F(Y))$ is linear and contractive.

A contravariant functor $G : \mathrm{Ban}^{op} \to \mathrm{Ban}$ assigns Banach spaces $G(X)$ to Banach spaces X and morphisms $G(f) : G(Y) \to G(X)$ to morphisms $f : X \to Y$ in Ban in such a way that the following statements hold:

(a') $G(f \circ g) = G(g) \circ G(f)$ whenever $f \circ g$ is defined in Ban.

(b') $G(1_X) = 1_{G(X)}$ for all $X \in \mathrm{Ban}$.

(c') Each map $f \to G(f)$ from $H(X,Y)$ into $H(G(Y),G(X))$ is linear and contractive.

Remark: Properties (a), (b) and (a'), (b') are those generally required to hold for functors. Properties (c) and (c') state that we consider only strong functors or Ban-functors in terms of relative category theory (DUBUC [20], FISCHER - PALMQUIST and NEWELL [28].

1.2. Nearly all of the constructions of chapters I and II can be interpreted as functors. We recall some of them. Of fundamental importance is, of course, the lifted Hom-functor H. It is a contra-covariant bifunctor (cf. § 2) into Ban and we will often consider its covariant partial functors $H_A = H(A,.)$, the action on morphisms being given by

$$H(A,f)g = f \circ g, \ f : X \to Y, \ g \in H(A,X),$$

and the contravariant partial functors $H^A = H(. \ , A)$, the action on morphisms being given by

$$H(f,A)g = g \circ f \ , \ f : X \to Y, \ g \in H(Y,A).$$

In the literature H_A is often denoted by Ω_A.
As important as H is the co-covariant bifunctor $.\hat{\otimes}.$ defined by the projective tensor product $X \hat{\otimes} Y$ in Ban. Its partial functor $X \hat{\otimes}.$ is frequently denoted by Σ_X and its action on morphisms is given by

$$(X \hat{\otimes} f)(\Sigma x_i \otimes y_i) = \Sigma x_i \otimes f(y_i), \ f : Y \to Z (\text{cf. II.1.8d})$$

The equation $H(X \hat{\otimes} Y,Z) = H(X,H(Y,Z))$ holds naturally (cf. II.1.6.) and shows that the functor $Y \hat{\otimes}.$ is left adjoint to the functor H_Y. Thus $Y \hat{\otimes}.$ commutes with colimits and H_Y commutes with limits in Ban_1 (cf. II, 1.8 e)).
The following equation shows that the covariant functor H^Z is adjoint on the right to itself:

$$H(X, H(Y,Z)) = H(X \hat{\otimes} Y,Z) = H(Y,H(X,Z)).$$

Thus H^Z transforms colimits into limits in Ban_1,
a special case being $(\varinjlim X_d)' = \varprojlim X_d'$ (cf. I,2.15)

Proposition: The projective tensor product is completely determined by the property that all its partial functors commute with colimits. Thus there is essentially only one pair of adjoint functors on Ban_1.
On the other hand H is uniquely determined by the property that all its contravariant partial functors transform colimits into limits.

The first assertion is the essential content of SEMADENI-WIWEGER [78].

Proof: Suppose $F : Ban \to Ban$ commutes with colimits in Ban_1 and $G : Ban^{op} \to Ban$ transforms colimits in limits.
Each $X \in Ban$ is colimit of finite dimensional spaces l_n^1 (cf.I,1.21), $X = \varinjlim l_n^1$, naturally in X. Then we have

$$F(X) = F(\varinjlim l_n^1) = \varinjlim F(l_n^1) = \varinjlim l_n^1(F(I))$$
$$= \varinjlim (l_n^1 \hat{\otimes} F(I)) = (\varinjlim l_n^1) \hat{\otimes} F(I) = X \hat{\otimes} F(I).$$
$$G(X) = G(\varinjlim l_n^1) = \varprojlim G(l_n^1) = \varprojlim l_n^\infty (G(I))$$
$$= \varprojlim H(l_n^1, G(I)) = H(\varinjlim l_n^1, G(I)) = H(X, G(I)).$$

qed.

1.3. Given two functors F and F_1 from Ban into Ban a natural transformation $\alpha : F \to F_1$ is a family $(\alpha_X)_{X \in Ban}$ of morphisms $\alpha_X : F(X) \to F_1(X)$ such that for each $f : X \to Y$

the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & F_1(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F_1(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & F_1(Y) \end{array}$$

commutes,

and furthermore $\|\alpha\| := \sup_X \|\alpha_X\| < \infty$ holds.
By Nat (F,F_1) we denote the Banach space of all natural transformations $F \to F_1$ with coordinate-wise operàtions. The unit ball of $\mathrm{Nat}(F,F_1)$ is the set of all natural transformations $F \to F_1$, where F and F_1 are regarded as functors $\mathrm{Ban}_1 \to \mathrm{Ban}_1$.

1.4. <u>Theorem</u>: (Yoneda lemma): For all functors
$F : \mathrm{Ban} \to \mathrm{Ban}$ (resp. $G : \mathrm{Ban}^{op} \to \mathrm{Ban}$) and $A \in \mathrm{Ban}$
we have $\mathrm{Nat}(H_A, F) = F(A)$
(resp. $\mathrm{Nat}\,(H^A, G) = G(A)$) naturally in A
and F.(resp. G).

This theorem is a variant of the ordinary Yoneda lemma for strong functors - thus the conclusion is stronger: = means isometric isomorphism. In order to be complete and intelligible we sketch the proof.

<u>Proof</u>: One uses the assignments

$\varphi \to \varphi_A(1_A) \in F(A)$, $\varphi \in \mathrm{Nat}\,(H_A, F)$ and

$f_A \in F(A) \to (f \in H(A,X) \to F(f)\, f_A)$, which are linear, contractive, natural and inverse to each other.

Remark: We can interpret this result in the following manner: Nat $(H,F) = F$, and so H behaves like an identity - compare this with the equation $H(I,X) = X$ in Ban. As a special case we have Nat $(H_A, H_B) = H(B,A) = \underline{W}(A',B')$ and so we can define a functorial embedding $\underline{W} \to \mathrm{Ban}^{\mathrm{Ban}}$ if we assign H_A to A'. Its inverse (defined on the image) is given by $H_A \to H(A,I) = A'$, since we can extract A (i.e. the weak * - topology on A') out of H_A in a functorial manner by Nat $(H_A, H_I) = H(I, A) = A$. Similar considerations show that $A \to H^A$ is a functorial embedding of Ban into $\mathrm{Ban}^{\mathrm{Ban}^{op}}$.

1.5. Theorem: (i) For all $F : \mathrm{Ban} \to \mathrm{Ban}$ and $A \in \mathrm{Ban}$ Nat $(A \hat{\otimes} ., F) = H(A,F(I))$ holds naturally in A and F.

(ii) Nat $(A \hat{\otimes}(.'), G) = H(A,G(I))$ holds naturally in $A \in \mathrm{Ban}$ and $G : \mathrm{Ban}^{op} \to \mathrm{Ban}$.

Proof: (i) $\varphi \in$ Nat $(A \hat{\otimes} ., F) \mapsto \varphi_I \in H(A,F(I))$ and $f \in H(A,F(I)) \mapsto (a \otimes x \mapsto F(\hat{x})f(a))$, where for $x \in X$ we have $\hat{x} \in H(I,X)$ by $\hat{x}(\lambda) = \lambda x$, are easily seen to be linear, contractive, natural in A and F and inverse to each other.

(ii) $\varphi \in \mathrm{Nat}\,(A \hat{\otimes} (.'), G) \mapsto \varphi_I \in H(A,G(I))$ and $f \in H(A,G(I)) \mapsto (a \otimes x' \mapsto G(x')f(a))$ have the same properties.

1.6. We can interpret 1.5. (i) in the following way: $A \to A \hat{\otimes} .$ is a functorial embedding from Ban into $\mathrm{Ban}^{\mathrm{Ban}}$ as is seen by $\mathrm{Nat}\,(A \hat{\otimes} ., B \hat{\otimes} .) = H(A,B)$ and the functor $F \to F(I)$ from $\mathrm{Ban}^{\mathrm{Ban}}$ into Ban is the right adjoint to this embedding by 1.5 (i). We call $F \to F(I)$ a forgetful functor and its left adjoint $A \to A \hat{\otimes} .$ the associated free functor. The unit of this adjunction is the trivial isomorphism $A \to A \hat{\otimes} I$, but it is worthwile to look at the counit $\epsilon^F_X : F(I) \hat{\otimes} X \to F(X)$. It is given by $\epsilon^F_X\,(a \otimes x) = F(\hat{x})a$ and we have $\|\epsilon^F\| = 1$ whenever $F(I) \neq 0$. Also ϵ^F_X is natural in F and X, since the adjunction is natural.

In a similar manner we can interprete 1.5 (ii): the forgetful functor $G \to G(I)$ has a left adjoint $A \to A \hat{\otimes}(.')$ from Ban into $\mathrm{Ban}^{\mathrm{Ban}^{op}}$. The counit of this adjunction is the map $\epsilon^G_X : G(I) \hat{\otimes} X' \to G(X)$, given by $\epsilon^G_X(a \otimes x') = G(x')a$. It is contractive, $\|\epsilon^G\| = 1$ whenever $F(I) \neq 0$ (since the adjunction is isometric) and it is natural in G and X.

1.7. <u>Lemma</u>: $\epsilon^F_X \mid F(I) \otimes X$ and $\epsilon^G_X \mid G(I) \otimes X'$ are always injective.

Proof: Let $\sum_{i=1}^{n} a_i \otimes x_i \in F(I) \otimes X$ with (a_i) linearly independent and $\epsilon_X^F (\Sigma a_i \otimes x_i) = \Sigma F(\hat{x}_i)a_i = 0$.
Then for all $x' \in X'$ we have

$$0 = F(x') \Sigma F(\hat{x}_i)a_i = \Sigma F(< x_i, x' > 1_I)a_i$$
$$= \Sigma < x_i, x' > a_i ;$$

since (a_i) are linearly independent we conclude that $< x_i, x' > = 0$, $i = 1,\dots,n$ for all $x' \in X'$. Thus $x_i = 0$, $i = 1,\dots,n$ and $\Sigma a_i \otimes x_i = 0$.
In a similar manner the second assertion can be proved.

1.8. Proposition: $\|.\|_{F(X)}$ and $\|.\|_{G(X)}$ induce reasonable tensor norms α and β on the subspaces $F(I) \otimes X$ and $G(I) \otimes X'$ of $F(X)$ and $G(X)$ respectively.

Proof: Since $\|\epsilon_X^F\|, \|\epsilon_X^G\| \leq 1$ we have $\alpha, \beta \leq \|.\|^{\wedge}$.
Let $\sum_{i=1}^{n} a_i \otimes x_i \in F(I) \otimes X$, then

$$\|\Sigma a_i \otimes x_i\|^{\wedge} = \sup_{\|x'\| \leq 1} \|\Sigma < x_i, x' > a_i\|_{F(I)}$$
$$= \sup_{\|x'\| \leq 1} \|F(x')\Sigma F(\hat{x}_i)\, a_i\|_{F(I)}$$
$$\leq \|\epsilon_X^F(\Sigma a_i \otimes x_i)\|_{F(X)}$$
$$= \alpha(\Sigma a_i \otimes x_i).$$

If $\sum_{i=1}^{n} b_i \otimes x_i' \in G(I) \otimes X'$, then

$$\begin{aligned}\|\Sigma\, b_i \otimes x_i'\|^{\wedge} = \|\Sigma\, \hat{b}_i \circ x_i'\| &= \sup_{\|x\|\leq 1} \|\Sigma < x, x_i' > b_i\|_{G(I)} \\ &= \sup_{\|x\|\leq 1} \|G(\hat{x})\; \Sigma\, G(x_i')\, b_i\|_{G(I)} \\ &\leq \|\epsilon_X^G\, (\Sigma\, b_i \otimes x_i')\|_{G(X)} \\ &= \beta\, (\Sigma\, b_i \otimes x_i').\end{aligned}$$

1.9. Given a functor F : Ban $\rightarrow$ Ban we consider the canonical decomposition of ϵ_X^F (cf I,1.6):

$$\begin{array}{ccc} F(I)\hat{\otimes} X & \xrightarrow{\ \epsilon_X^F\ } & F(X) \\ {\scriptstyle \text{coim}\ \epsilon_X^F}\downarrow & & \uparrow{\scriptstyle \text{im}\ \epsilon_X^F} \\ F^{\wedge}(X) & \xrightarrow{\ \tilde{\epsilon}_X^F\ } & F_e(X) \end{array} ,$$

where $F^{\wedge}(X) = F(I)\hat{\otimes} X/(\epsilon_X^F)^{-1}(0)$ and $F_e(X)$ is the closure of the image of ϵ_X^F in $F(X)$. Since ϵ_X^F is natural in F and X we conclude that $F_e(X)$ and $F^{\wedge}(X)$ define functors $F_e, F^{\wedge}$: Ban $\rightarrow$ Ban and that the assignations $F \rightarrow F_e$ and $F \rightarrow F^{\wedge}$ are functorial too.

By 1.7 $F(I) \otimes X$ is a dense subspace in $F_e(X)$ and also in $F^{\wedge}(X)$ and, by 1.8, $F_e(X)$ is the completion of $F(I) \otimes X$ in a reasonable crossnorm α which is "functorial" in X since F_e is a functor. The norm induced on $F(I) \otimes X$ by $\|.\|_{F^\wedge(X)}$ lies between $\|.\|^\wedge$ and α and is thus a reasonable crossnorm and functorial in X.

Given a functor $G : Ban^{op} \to Ban$ we consider the canonical decomposition of ϵ_X^G:

$$\begin{array}{ccc} G(I)\hat{\otimes}X' & \xrightarrow{\epsilon_X^G} & G(X) \\ \Big\downarrow{\scriptstyle \text{coim}\, \epsilon_X^G} & & \Big\uparrow{\scriptstyle \text{im}\, \epsilon_X^G} \\ G^\wedge(X) & \xrightarrow{\tilde{\epsilon}_X^G} & G_e(X) \end{array} ,$$

where again $G^\wedge(X) = G(I)\hat{\otimes} X'/(\epsilon_X^G)^{-1}(0)$ and $G_e(X)$ is the closure of the image of ϵ_X^G in $G(X)$. Again by the naturality of ϵ_X^G we conclude that $G_e(X)$ and $G^\wedge(X)$ define functors $G_e, G^\wedge : Ban^{op} \to Ban$ and that the assignations $G \to G_e, G^\wedge$ are functorial, $G_e(X)$ and $G^\wedge(X)$ are the completion of $G(I)\otimes X'$ in reasonable crossnorms respectively which are both functorial in X.

1.10. <u>Definition</u>: Given F and G as above, then F_e and G_e are called the subfunctors of type Σ of F and G respectively or the essential parts. F, G are said to be of type Σ or essential functors if $F = F_e$, $G = G_e$ respectively via the isometric natural transformations im ϵ_X^F, im ϵ_X^G respectively.

Examples: $F^\wedge$, $G^\wedge$ always are of type Σ.

$(H_A)_e = A' \hat{\hat{\otimes}} .$, $A \hat{\otimes} .$ are of type Σ,

$(H^A)_e = (.') \hat{\hat{\otimes}} A$, $A \hat{\otimes} (.')$ are of type Σ.

If X is finite dimensional then $F(X) = F_e(X)$ holds for all F, since we can write $1_X = \sum_{i=1}^{n} \hat{x}_i \circ x_i'$ and then

$\xi = F(1_X)\xi = \Sigma F(\hat{x}_i)\ (F(x_i')\ \xi) =$

$= \epsilon_X^F(\Sigma\ (F(x_i')\xi) \otimes x_i)$ for all $\xi \in F(X)$.

1.11. Corollary: For an arbitrary functor F (resp. G) we have $\|F(f)\| = \|f\|$ (resp. $\|G(f)\| = \|f\|$) for all $f \in H(X,Y)$ if and only if $F(I) \neq (0)$ (resp. $G(I) \neq (0)$).

Proof: If $F(I) \neq 0$ then there are $a \in F(I)$ and $x \in X$ such that $\|a\| = \|x\| = 1$ and $\|f(x)\| \geq \|f\| - \epsilon$. Then

$\|F(f)\ \epsilon_X^F\ (a \otimes x)\| = \|\epsilon_Y^F\ (a \otimes f(x))\|$

$= \|a\|\|f(x)\| \geq \|f\| - \epsilon$ and $\|\epsilon_X^F\ (a \otimes x)\| = 1$.

If $F(I) = (0)$, then $F(x') = 0$ for all $x' \in X' = H(X,I)$.

For G choose $y' \in Y'$, $\|y'\| = 1$ and $\|f'(y')\| \geq \|f\| - \epsilon$.

1.12. Lemma: If F: Ban $\to$ Ban is of type Σ and F_1 is arbitrary, then $\mathrm{Nat}(F,F_1) = \mathrm{Nat}(F,F_{1e})$ and the map $\mathrm{Nat}(F,F_1) \to H(F(I),F_1(I))$, $\eta \to \eta_I$, is injective.

Proof: Since ϵ_X^F from 1.6 is natural in F and X we conclude that $\epsilon_X^{F_1} \circ \eta_X = \eta_X \circ \epsilon_X^F$ for any natural transformation $\eta: F \to F_1$; hence any natural transformations maps essential parts into essential parts and the first assertion follows. For all $x \in X$ we have $\eta_X \circ F(\hat{x}) = F_1(\hat{x}) \circ \eta_I$, so if $\eta_I = 0$ then η_X vanishes on all elements of the form $F(\hat{x})a$, $x \in F(I)$, i.e. on the whole of $F(X)$ since F is essential. qed.

1.13. Proposition: Let $F : Ban \to Ban$ and $G : Ban^{op} \to Ban$ be functors. Then we have:

(i) For all $f \in X' \hat{\hat{\otimes}} Y = K_o(X,Y) \subseteq H(X,Y)$ we have $F(f)\, F(X) \subseteq F_e(Y)$ and $G(f)G(Y) \subseteq G_e(X)$.

(ii) If X has the metric approximation property and (u_ι) is a left approximate unit in $K(X,X)$ (cf. II. 3.10), then for $f_X \in F(X)$ we have:
$f_X \in F_e(X)$ iff $\|F(u_\iota)f_X - f_X\| \to 0$.
If $f_X \in F_e(X)$ then for any $\varepsilon > 0$ there are $f \in K(X,X)$ with $\|f\| \le 1$ and $f_X' \in F_e(X)$ with $\|f_X - f_X'\| < \varepsilon$ and $F(f)f_X' = f_X$.

(iii) If X' has the metric approximation property and (v_ι) is a right approximate unit in $K(X,X)$ (cf II.3.11 Cor), then for $g_X \in G(X)$ we have:
$g_X \in G_e(X)$ iff $\|G(v_\iota)g_X - g_X\| \to 0$.
If $g_X \in G_e(X)$ then for any $\varepsilon > 0$ there are $f \in K(X,X)$ and $g_X' \in G_e(X)$ with $\|f\| \le 1$, $\|g_X - g_X'\| < \varepsilon$ and $g_X = G(f)g_X'$.

Proof: (i) $F(\sum_{i=1}^{n} \hat{y}_i \circ x_i')f_X = \sum_i F(\hat{y}_i)\, F(x_i')f_X =$

$= \epsilon_Y^F\, (\sum_i F(x_i')f_X \otimes y_i) \in F_e(X)$, and maps of the form $\Sigma\, \hat{y}_i \circ x_i'$ are dense in $K_o(X,Y)$. The argument for G is similar.

(ii) F(X) is a left Banach-K(X,X)-module and its essential submodule is easily seen to coincide with $F_e(X)$. Now use III.1.15.

(iii) By an analogous argument.

1.14. Proposition: If X has the metric approximation property then for all functors F of type Σ the (natural) map $\varphi_X^F : F(X) \to L(X', F(I))$, defined by $\varphi_X^F(f_X)(x') =$ $= F(x')f_X \in F(I)$ is injective, thus $F(I) \hat{\otimes} X \subseteq F(X) \subseteq$ $\subseteq F(I) \hat{\hat{\otimes}} X$.

If X' has the metric approximation property then for all functors G of type Σ the (natural) map $\varphi_X^G : G(X) \to K(X, G(I))$, defined by $\varphi_X^G(g_X)(x) = G(\hat{x})g_X \in G(I)$ is injective thus $G(I) \hat{\otimes} X' \subseteq G(X) \subseteq G(I) \hat{\hat{\otimes}} X'$.

Proof: If $\varphi_X^F(f_X) = 0$ then $\varphi_X^F(f_X)(x') = 0$ for all $x' \in X'$, so $F(\sum_{i=1}^{n} \hat{x}_i \circ x_i')f_X = 0$ and $F(f)f_X = 0$ for all $f \in K(X,X)$, so $f_X = 0$ by 1.13 (ii). The argument for contravariant functors is similar.

1.15. If F : Ban → Ban is a functor of type Σ, then we have maps

$$F(I) \hat{\otimes} X \xrightarrow{\epsilon_X^F} F(X) \xrightarrow{\varphi_X^F} F(I) \hat{\hat{\otimes}} X.$$

which are natural in X and contractive and whose restriction to $F(I) \otimes X$ is the identity: both maps have dense image, i.e. are epi in Ban. Thus the maps

$$(F(I) \hat{\otimes} X)' \xrightarrow{\varphi'} F(X)' \xrightarrow{\varepsilon'} H(F(I),X')$$

are mono in $\underline{W}$, i.e. they are injective, and the duality action looks like:

$$< \varepsilon (\Sigma a_i \otimes x_i), u > = < \Sigma a_i \otimes x_i, \varepsilon'(u) >$$

$$= \Sigma < x_i, \varepsilon'(u) (a_i) >.$$

Thus $F(X)'$ consists of all $f \in H(F(I),X')$ which define a bounded linear functional on $F(X)$ by

$\Sigma a_i \otimes x_i \to \Sigma < x_i, f(a_i) >$ and

$\|f\|_{F(X)'} = \sup \{|\Sigma < x_i, f(a_i) >| : \|\Sigma F(\hat{x}_i)a_i\|_{F(X)} \leq 1\}.$

Since the unit ball $OF(X)'$ is equicontinuous on $F(X)$ and $F(I) \otimes X$ is dense in $F(X)$ we conclude by SCHAEFER [73], III, 4.5, that the compact topology on $OF(X)'$ is given by $\sigma(F(X)', F(I) \otimes X) | OF(X)'$; i.e. it is the topology of pointwise weak *- convergence in OX'. This follows more directly from the fact that $\varepsilon' | OF(X)'$ is injective and $\underline{W}$-continuous, thus a homeomorphism of $OF(X)'$ into $O\Omega(F(I),X')$. (cf. I, 2.12).

We also conclude that $F(X)'$contains (not isometrically) the space $(F(I) \hat{\otimes} X)' = I_1(F(I),X')$ of all integral operators $F(I) \to X'$.(cf. II, 2,9).

Similar considerations show that for a contravariant functor G

of type Σ the space $G(X)'$ consists of all $f \in H(G(I),X'')$ which define a bounded linear functional on $G(X)$ by
$\Sigma\, b_i \otimes x_i' \mapsto \Sigma < x_i', f(b_i) >$ and
$\|f\|_{G(X)'} = \sup \{ |\Sigma < x_i', f(b_i) >| : \| \Sigma\, G(x_i')b_i \|_{G(X)} \leq 1 \}$.
The Waelbroeck-structure on $OG(X)'$ is again given by the topology of pointwise weak * - convergence in OX''.

1.16. Theorem: (i) Nat $(F, H(.',A)) = H(F(I),A)$ holds naturally in $F \in Ban^{Ban}$ and $A \in Ban$.
(ii) Nat $(G,H(.,A)) = H(G(I),A)$ holds naturally in $G \in Ban^{Ban^{op}}$ and $A \in Ban$.

Proof: (i) The maps $\alpha \mapsto \alpha_I \in H(F(I),A)$, $\alpha \in$ Nat $(F,H(.',A))$ and $f \mapsto (\xi_X \mapsto (x' \mapsto f \circ F(x')\xi_X))$, $f \in H(F(I),A)$, $\xi_X \in F(X)$, $x' \in X'$ are easily seen to be linear, contractive, natural and inverse to each other.

(ii) The same proof works where the second map looks like
$f \mapsto (\xi_X \mapsto (x \mapsto f \circ G(\hat{x})\ \xi_X))$, $f \in H(G(I),A)$,
$\xi_X \in G(X)$, $x \in X$, $\hat{x} \in H(I,X)$, $\hat{x}\ (r) = r\, x$.

1.17. The last result gives an adjunction: the functor $A \to H(.',A)$ is right adjoint to the forgetful functor $F \to F(I)$.
The unit of this adjunction is the map
$\varphi_X^F : F(X) \to H(X',F(I))$, given by
$\varphi_X^F\ (\xi_X)\ (x') = F(x')\ \xi_X$, $\xi_X \in F(X)$, $x' \in X'$.

The same interpretation is true for the contravariant case: the unit here has the form:

$\varphi_X^G : G(X) \to H(X,G(I))$, given by

$\varphi_X^G (\xi_X)(x) = G(\hat{x})\, \xi_X,\ \xi_X \in G(X),\ x \in X.$

Both maps are clearly contractive and natural in X and F, G.

Definition: F, G is said to be a total functor, if for all X the map φ_X^F, respectively φ_X^G is injective.
F (resp. G) is total if and only if maps of the form $F(x')$, $x' \in X$ (resp. $G(\hat{x})$, $x \in X$) separate points on F(X) (resp. G(X)).

1.18. Remark: The same theory is of course valid for functors $F : \underline{K} \to$ Ban, $G : \underline{K}^{op} \to$ Ban, where $\underline{K}$ is a full subcategory of Ban which contains I. We will use all results in this (formally) greater generality without hesitation.

1.19. Example: Let n be a sequence space (III, 1.7).
Then the construction of III, 1.12, which assigns the space n(X) to each Banach space X, defines a functor n(.): Ban → Ban.
n(X) is the space of all sequences $x = (x_1, x_2, \ldots .)$ in X such that $(\|x_1\|, \|x_2\|, \ldots) \in n$. If $f \in H(X,Y)$, then $n(f) : n(X) \to n(Y)$ is given by
$n(f)((x_i)) = ((fx_i))$. It is easily checked that n(.) becomes a functor in this way. Clearly n(I) = n.

We had $l^\infty(X) \hat{\otimes}_{l^\infty} n = n(X)$ for each sequence space (III, 3.15), and it is easily seen that $n(f)$ coincides with the canonically given mapping $l^\infty(f) \hat{\otimes}_{l^\infty} n : l^\infty(X) \hat{\otimes}_{l^\infty} n \to l^\infty(Y) \hat{\otimes}_{l^\infty} n$ which is induced by $l^\infty(f)$ via this identification. If n is an essential c_o-module, then the equation $n(X) = c_o(X) \hat{\otimes}_{c_o} n$ of III, 3.15, Corollary is natural in X also.

We now determine the essential part $n(.)_e$ of the functor $n(.)$, which is in general different from the functor $n_e(.)$, derived from the (c_o-)essential submodule of n.

The map $\epsilon^{n(.)}_X : X \hat{\otimes} n \to n(X)$ from 1.6 is easily seen to be $\epsilon^{n(.)}_X(x \otimes (a_i)) = (a_i x)$, so an inspection of the subspace $X \otimes n$ in $n(X)$ shows that $n(.)_e(X)$ is the closure of the space of all elements of $n(X)$ which lie as sequences in a finite dimensional subspace of X.

Since this property is invariant under the action of l^∞ on $n(X)$ we conclude that $n(.)_e(X)$ is again a l^∞-module.

If n is an essential c_o-module, then $n(.)$ is an essential functor, since $X \otimes n$ clearly contains elements $\sum_{i=1}^{k} x_k \otimes e_k = (x_1, x_2, \ldots x_k, 0, \ldots) \in n(X)$, and these elements are dense in $n(X)$ since for any $(x_i) \in n(X)$ we have $\|(0,\ldots,0,x_{k+1},x_{k+2},\ldots)\|_{n(X)} =$
$= \|(0,\ldots,0, \|x_{k+1}\|, \|x_{k+2}\|,\ldots.)\|_n \to 0.$

If n is not essential as c_o-module, then $n(.)$ is not of type Σ. To see that choose $X = l^1$ and let $u_k : l^1 \to l^1$ be the projection onto the first k coordinates. Then

(u_k) is an approximate unit bounded by 1 in $K(l^1, l^1)$ (II,3.8) and we would have $\lim_{k\to\infty} n(u_k)(x_i) = (x_i)$ for all $(x_i) \in n(l^1)$ if $n(.)$ were of type Σ by 1.13. Now choose $(a_i) \in n \setminus n_e$ and set $x_i = a_i e_i \in l^1$, then clearly $(x_i) \in n(l^1)$, but $\|(u_k(a_i e_i) - a_i e_i)\|_{n(l^1)} = \|u_k(|a_i|) - (|a_i|)\|_n$ and this does not converge to 0.

1.20. Lemma: Let n be a sequence space and $(x_i) \in n(X)$. Then $(x_i) \in n(.)_e(X)$ if and only if for each $\delta > 0$ there is a finite dimensional subspace $M \subseteq X$ with $\|(\pi_M(x_i))\|_{n(X/M)} < \delta$, where $\pi_M : X \to X/M$ is the canonical projection.

Proof: If $(x_i) \in n(.)_e(X)$ and $\delta > 0$ then there is $(y_i) \in X \otimes n \subseteq n(X)$ with $\|(x_i) - (y_i)\|_{n(X)} < \delta$, i.e. the sequence (y_i) lies in a finite dimensional subspace M of X. Then $\|\pi_M(x_i)\|_{X/M} \leq \|x_i - y_i\|$, so $\|(\pi_M(x_i))\|_{n(X/M)} = \|(\|\pi_M(x_i)\|)\|_n \leq$ $\leq \|(\|x_i - y_i\|)\|_n = \|(x_i) - (y_i)\|_{n(X)} < \delta$.

Now suppose that the condition of the lemma is fulfilled. Let $\delta > 0$ and let M be the corresponding finite dimensional subspace of X. We have to construct $(y_i) \in X \otimes n \subseteq n(X)$ with $\|(x_i) - (y_i)\|_{n(X)} < \delta$. Since M is finite dimensional for each i there is $y_i \in M$ with $\|x_i - y_i\| = \|\pi_M(x_i)\| = \inf \{\|x_i - y\| : y \in M\}$ (use compactness of $3\|x_i\| OM$ and continuity of $y \to \|x_i - y\|$, $y \in M$). Then

$\|y_i\| \leq \|x_i\| + \|x_i - y_i\| = \|x_i\| + \|\pi_M(x_i)\|$, so

$\|(y_i)\|_{n(X)} = \|(\|y_i\|)\|_n \leq \|(\|x_i\| + \|\pi_M(x_i)\|)\|_n$

$\leq \|(x_i)\|_{n(X)} + \|(\pi_M(x_i))\|_{n(X/M)} \leq \|(x_i)\|_{n(X)} + \delta$,

so $(y_i) \in n(X)$ and clearly $(y_i) \in X \otimes n \subseteq n(X)$.

Furthermore $\|(x_i) - (y_i)\|_{n(X)} = \|(\|x_i - y_i\|)\|_n =$

$= \|(\|\pi_M(x_i)\|)\|_n = \|(\pi_M(x_i))\|_{n(X/M)} < \delta$. qed.

1.21. <u>Example</u>: Let N be a function space (III, 1.8). Then the construction of III, 1.13, which assigns the space $N(X)$ to each Banach space X, defines a functor $N(.) : \mathrm{Ban} \to \mathrm{Ban}$.

$N(X)$ is the space of all (equivalence classes of) Bochner integrable X-valued functions G on the interval $[0,1]$ such that the function $t \mapsto \|G(t)\|_X$ is an element of N. If $f \in H(X,Y)$ then $N(f) : N(X) \to N(Y)$ is defined by $N(f)(G) = f \circ G$. Since $\|f \circ G(.)\|_X \leq \|f\| \, \|G(.)\|_X$ almost everywhere we conclude that $\|N(f)\| \leq \|f\|$. All the other properties of a functor are trivially verified.

Clearly $N(I) = N$.

We now determine the essential part of the functor $N(.)$. It is clear that the map $\varepsilon_X^{N(.)} : X \hat{\otimes} N \to N(X)$ looks like

$\varepsilon_X^{N(.)}(\Sigma\, x_i \otimes f_i) = \Sigma\, f_i(.)x_i \in N(X)$.

So $N(.)_e(X)$ is the closure in $N(X)$ of the subspace consisting of all $G \in N(X)$ such that $G(t)$ lies in a finite dimensional subspace of X for almost all $t \in [0,1]$, or the closure of all elements of the form $\sum_{i=1}^{n} f_i(.)x_i$, $f_i \in N$, $x_i \in X$. Thus $N(.)_e(X)$ is exactly the space $N_o(X)$, introduced in III 1.14. By III, 3.11 we have: $N_o(X) = L_o^\infty(X) \hat{\otimes}_{L^\infty} N$, and $N_o(f) = N(.)_e(f)$ is easily seen to coincide with the canonically given map $L_o^\infty(f) \hat{\otimes}_{L^\infty} N$.

1.22. Example: Let N be a function space as in 1.21. For each Banach space X we consider the space $S(N,X)$ of all summable maps $N \to X$ of III, 2.7. If $f \in H(X,Y)$ and $\psi \in S(N,X)$, then it is easy to check that $f \circ \psi \in S(N,Y)$, $\|f \circ \psi\|_S \leq \|f\| \, \|\psi\|_S$, so $S(N,.) : \mathrm{Ban} \to \mathrm{Ban}$ is clearly a functor.

By III, 2.7 and 2.8 we see that

$S(N,I) = H_{L^\infty}(N,S(L^\infty,I)) = H_{L^\infty}(N,(L^\infty)') =$

$= H_{L^\infty}(L^\infty, N') = N'$, which can also easily be derived directly. So we see that, as expected, $S(N,.)_e(X)$ is the closure of finite dimensional maps $N \to X$ in $S(N,X)$.

§ 2. Bifunctors on Ban

2.1. Remark: It is possible to derive the theory of bifunctors from the theory of functors by introducing bilinear categories or tensor products of categories: a bifunctor M would look like an ordinary functor $M: \mathrm{Ban} \hat{\otimes} \mathrm{Ban} \to \mathrm{Ban}$, where objects in $\mathrm{Ban} \hat{\otimes} \mathrm{Ban}$ would be pairs (X,Y) of objects and the spaces of morphisms would look like $H(X,X_1) \hat{\otimes} H(Y,Y_1)$. Then for example the Yoneda lemma (2.2 below) for bifunctors would be a special case of the Yoneda lemma for Ban_1-based relative category theory. But we prefer the ordinary approach and we will sketch proofs of the well known category results in our special setting too in order to provide a feeling for the theory. Throughout this section G stands for a contra-covariant bifunctor: $\mathrm{Ban}^{op} \times \mathrm{Ban} \to \mathrm{Ban}$ and M stands for a covariant one: $\mathrm{Ban} \times \mathrm{Ban} \to \mathrm{Ban}$.

2.2. Theorem: (i) For all $X,Y \in \mathrm{Ban}$ and G we have

$$\mathop{\mathrm{Nat}}_{\mathrm{Ban}^{op} \times \mathrm{Ban}} (H(.,X) \hat{\otimes} H(Y,..),G) = G(X,Y)$$

naturally in X,Y and G.

(ii) For all $X,Y \in \mathrm{Ban}$ and M the relation

$$\mathop{\mathrm{Nat}}_{\mathrm{Ban} \times \mathrm{Ban}} (H(X,.) \hat{\otimes} H(Y,..),M) = M(X,Y)$$

holds naturally in all respects.

Proof: We use the maps

$\varphi \in \mathrm{Nat}\ (H(.,X) \hat{\otimes} H(Y,.),G) \mapsto \varphi_{XY}(1_X \otimes 1_Y) \in G(X,Y)$,

$g_{XY} \in G(X,Y) \mapsto (f \otimes h \mapsto G(f,h)g_{XY})$,

which are easily seen to be linear, contractive, natural in X,Y and G and inverse to each other. The second part can be proved similarly.

2.3. Theorem: (i) $\underset{Ban^{op} \times Ban}{\mathrm{Nat}} ((.') \hat{\otimes} A \hat{\otimes} ..,G) = H(A,G(I,I))$

holds naturally in A and G.

(ii) $\underset{Ban \times Ban}{\mathrm{Nat}} (. \hat{\otimes} A \hat{\otimes} ..,M) = H(A,M(I,I))$

holds naturally in A and M.

Proof: We use the maps

$\varphi \mapsto \varphi_{II} \in H(A,G(I,I))$ and

$f \in H(A,G(I,I)) \mapsto (x' \otimes a \otimes y \mapsto G(x',\hat{y})f(a))$, which again are obviously linear, contractive, natural in A and G and inverse to each other. The proof of (ii) uses similar maps.

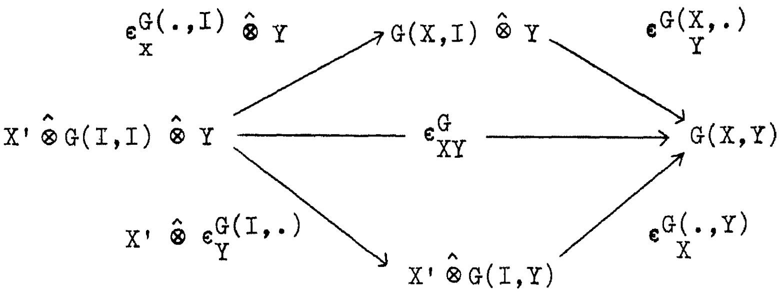
2.4. We can interpret 2.3 (i) as an adjunction between the free functor $A \to (.') \hat{\otimes} A \hat{\otimes} ..$ from Ban into $Ban^{Ban^{op} \times Ban}$ and the forgetful functor $G \to G(I,I)$. The unit of this adjunction turns out to be the trivial isomorphism $A = I' \hat{\otimes} A \hat{\otimes} I$ and the counit

ϵ^G_{XY}: $X' \hat{\otimes} G(I,I) \hat{\otimes} Y \to G(X,Y)$ is given by

$\epsilon^G_{XY}(\Sigma x_i' \otimes a_i \otimes y_i) = \Sigma\, G(x_i', y_i)\, a_i$;

Clearly it is contractive and natural in X,Y and G.

For all X, Y and G the following diagram commutes, where the notation is partly from 1.5 - 1.7:



If we bear in mind that a reasonable tensor norm on $X \otimes Y$ induces a reasonable tensor norm on $X_1 \otimes Y$ for all $X_1 \subset X$, then this implies via 1.8, 1.9:

<u>Proposition</u>: (i) For all X, Y $\in$ Ban and all G the map $\epsilon^G_{XY}|(X' \otimes G(I,I) \otimes Y)$ is injective.

$\|\ .\ \|_{G(X,Y)}$ induces a reasonable tensor norm α (i.e. $\|\ .\ \|_{X' \check{\otimes} G(I,I) \check{\otimes} Y} \leq \alpha \leq \|.\|_{X' \hat{\otimes} G(I,I) \hat{\otimes} Y}$ on the "subspace" $X' \otimes G(I,I) \otimes Y$ of $G(X,Y)$)

(ii) For all X,Y $\in$ Ban and all M the map $\epsilon^M_{XY}|X \otimes M(I,I) \otimes Y$ is injective. $\|.\|_{M(X,Y)}$ induces a reasonable tensor norm α on the subspace $X \otimes M(I,I) \otimes Y$ of

$M(X,Y)$.

The proof of the second part is similar to that of the first part.

2.5. For all X, Y we consider the canonical factorization of ϵ^G_{XY}:

$$\begin{array}{ccc} X' \hat{\otimes} G(I,I) \hat{\otimes} Y & \xrightarrow{\epsilon^G_{XY}} & G(X,Y) \\ {\scriptstyle \mathrm{coim}\, \epsilon^G_{XY}} \downarrow & & \uparrow {\scriptstyle \mathrm{im}\, \epsilon^G_{XY}} \\ G^{\wedge}(X,Y) & \xrightarrow{\epsilon^{\tilde{G}}_{XY}} & G_e(X,Y), \end{array}$$

where $G^{\wedge}(X,Y) = X' \hat{\otimes} G(I,I) \hat{\otimes} Y/(\epsilon^G_{XY})^{-1}(0)$ and $G_e(X,Y)$ is the closure of the image of ϵ^G_{XY} in $G(X,Y)$. By the naturality of ϵ^G_{XY} we obtain contra-covariant bifunctors $G^{\wedge}$, G_e and the maps $G \mapsto G_e$, $G \mapsto G^{\wedge}$ are functorial too. By 2.4 $G_e(X,Y)$ is the completion of $X' \otimes G(I,I) \otimes Y$ in a reasonable cross norm, which is functorial in X and Y and since the norm of $G^{\wedge}(X,Y)$ lies between that of $G_e(X,Y)$ and the projective one it is reasonable too.

We have the corresponding factorization of ϵ^M_{XY}:

$$\begin{array}{ccc} X \hat{\otimes} M(I,I) \hat{\otimes} Y & \xrightarrow{\epsilon^M_{XY}} & M(X,Y) \\ {\scriptstyle \mathrm{coim}\, \epsilon^M_{XY}} \downarrow & & \uparrow {\scriptstyle \mathrm{im}\, \epsilon^M_{XY}} \\ M^{\wedge}(X,Y) & \xrightarrow{\epsilon^{\tilde{M}}_{XY}} & M_e(X,Y), \end{array}$$

where again $M^{\wedge}(X,Y) = X \hat{\otimes} M(I,I) \hat{\otimes} Y/(\epsilon_{XY}^{M})^{-1}(0)$ and $M_e(X,Y)$ is the closure of the image of ϵ_{XY}^{M} in $M(X,Y)$. Both $M^{\wedge}$ and M_e are again co-covariant bifunctors and $M \mapsto M^{\wedge}, M_e$ are functorial actions.

$M_e(X,Y)$ and $M^{\wedge}(X,Y)$ are the completions of $X \otimes M(I,I) \otimes Y$ in reasonable cross norms which are functorial in X and Y.

2.6. <u>Definition</u>: G_e, M_e are called the partial functors of type Σ or the essential parts of G,M respectively. If $G_e = G$, $M_e = M$ via im ϵ_{XY}^{G}, im ϵ_{XY}^{M}, then G, M is called a bifunctor of type Σ or an essential bifunctor.

Clearly $G_e(X,Y) = G(X,Y)$ and $M_e(X,Y) = M(X,Y)$ holds whenever X and Y are finite-dimensional by an argument like that in 1.10 or by the following result.

2.7. <u>Proposition</u>: (i) If $G(I,.)$ or $G(.,I)$ is a functor of type Σ, then $G_e(X,Y) = G(.,Y)_e(X)$ or $G_e(X,Y) = G(X,.)_e(Y)$ respectively.

(ii) If we introduce the notation $G(.,Y)_e(X) = G_{le}(X,Y)$ and $G(X,.)_e(Y) = G_{re}(X,Y)$, then $G_e = (G_{le})_{re} = (G_{re})_{le}$.

(iii) If X' or Y has the metric approximation property, then $G_e(X,Y) = G(X,.)_e(Y) \cap G(.,Y)_e(X)$.

<u>Remark</u>: Similar assertions hold for a co-covariant functor M: in (iii) the hypothesis is: If X or Y has the metric

approximation property.

It is not clear whether (iii) holds without assumptions on the metric approximation property.

Proof: (i) and (ii) are immediately clear by looking at the diagram in 2.4. To prove (iii) we proceed as follows:

If we set

$G_1(X,Y) = G(X,.)_e(Y) \cap G(.,Y)_e(X)$, then G_1 is an isometric subfunctor of G with $G_1(I,I) = G(I,I)$, so $(G_1)_e = G_e$ and $G_e(X,Y) \subseteq G_1(X,Y)$. To show the converse inclusion take $u \in G_1(X,Y)$ and $\varepsilon > 0$.

By the construction of G_1 there are $v = \sum_{i=1}^{n} G(X,\hat{y_i})b_i$ and $v_1 = \sum_{j=1}^{m} G(x_j',Y)c_j$ in $G_1(X,Y)$ with $\|u - \sum_i G(X,\hat{y_i})b_i\| < \frac{\varepsilon}{2}$ and $\|u - \sum_j G(x_j',Y)c_j\| < \frac{\varepsilon}{2}$.

If Y has the metric approximation property, there exists by 1.13 a finite dimensional map $w: Y \to Y$ with $\|w\| = 1$ and $\|u - G(X,w)u\| < \frac{\varepsilon}{2}$. Then $\|u - G(X,w)v_1\| \leq$
$\leq \|u - G(X,w)u\| + \|G(X,w)u - G(X,w)v_1\| < \varepsilon$ and $G(X,w)v_1 \in G_e(X,Y)$. If X' has the metric approximation property, take a finite dimensional $w_1: X \to X$ such that $\|u - G(w_1 Y)u\| < \frac{\varepsilon}{2}$ and consider $\|u - G(w_1,Y)v'\|$.

2.8. Theorem: (i) $\underset{Ban^{op} \times Ban}{\text{Nat}} (G, H(.\hat{\otimes}..',A)) = H(G(I,I),A)$

holds naturally in A and G.

$$\text{(ii)} \quad \underset{\text{Ban} \times \text{Ban}}{\text{Nat}} (M, H(.'\hat{\otimes}..',A)) = H(M(I,I),A)$$

holds naturally in A and M.

Proof: (i) Define $\text{Nat}\ (\ldots) \underset{\theta}{\overset{\psi}{\rightleftarrows}} H(G(I,I),A)$

by $\psi(\alpha) = \alpha_{I,I'}$, $\alpha \in \text{Nat}\ (\ldots)$, and $\theta(f)_{XY}(g)\ (x \otimes y') = f \circ G(\hat{x},y')g$, $f \in H(G(I,I),A)$, $g \in G(X,Y)$, $x \otimes y' \in X \otimes Y'$.

It is a routine matter to verify that ψ, θ are linear, contractive, inverse to each other and natural in X, Y and G.

(ii) is proved in a similar manner

2.9. This result is an adjunction too: the functor $A \to H(.\ \hat{\otimes}\ ..',A)$ is right adjoint to the forgetful functor $G \to G(I,I)$. The unit of this adjunction is the map

$\varphi^G_{XY}: G(X,Y) \to H(X\ \hat{\otimes}\ Y',\ G(I,I))$, given by

$\varphi^G_{XY}(g)\ (x \otimes y') = G(\hat{x},y')g$, $g \in G(X,Y)$. Clearly φ^G_{XY} is contractive and natural in X, Y and G.

The unit of the adjunction for M is the map

$\varphi^M_{XY}: M(X,Y) \to H(X'\ \hat{\otimes}\ Y', M(I,I))$, given by

$\varphi^M_{XY}(m)(x' \otimes y') = M(x',y')m$, $m \in M(X,Y)$.

Definition: G (resp. M) is said to be a total bifunctor if the maps φ^G_{XY} (resp. φ^M_{XY}) are injective for all X,Y.

G (resp. M) is total if and only if maps of the form $G(\hat{x},y')$, $x \in X$, $y' \in Y'$ (resp. $M(x',y)$, $x' \in X'$, $y' \in Y'$) separate points on $G(X,Y)$ (resp. $M(X,Y)$) for all X,Y.

2.10. For the remainder of this section we suppose that G and M satisfy the condition $G(I,I) = I$ and $M(I,I) = I$.

<u>Definition</u>: A tensor product on Ban is a co-covariant bifunctor M: Ban x Ban $\to$ Ban of type Σ with $M(I,I)= I$. This definition is justified by 4.4, since $X \otimes Y$ is dense in $M(X,Y)$, $\|.\|_{M(X,Y)}$ is a reasonable crossnorm on $X \otimes Y$ by 2.4 - 2.7, and the tensor product is bifunctorial, i.e the map $f \otimes g: X \otimes Y \to X_1 \otimes Y_1$ extends to $M(f,g): M(X,Y) \to M(X_1,Y_1)$ with $\|f \otimes g\| = \|M(f,g)\| \leq \|f\| \, \|g\|$ for all $f \in H(X,X_1)$, $g \in H(Y,Y_1)$. We will sometimes write $X \,\overline{\otimes}_M Y$ for $M(X,Y)$.

2.11. Given a contra-covariant bifunctor G with $G(I,I) = I$, then the canonical map $\varphi^G_{XY}: G(X,Y) \to H(X \hat{\otimes} Y',I)$ actually takes its values in $H(X,Y'')$ and is given by $< y', \varphi^G_{XY}(g)(x) > =$ $= G(\hat{x},y')g \in I$, $g \in G(X,Y)$.
Now suppose that G is a total bifunctor.
Since φ^G is natural and contractive, the action of the bifunctor $H(.,..'')$ coincides with that of G on $G(X,Y)$, if we consider $G(X,Y)$ (G now a total functor) to be a (generally non closed) subspace of $H(X,Y'')$ via φ^G_{XY}; the norm of $G(X,Y)$ is

greater than that of $H(X,Y'')$; and we express this fact by saying that $G(X,Y)$ is contractively contained in $H(X,Y'')$, or that G is a subfunctor of $H(\cdot,\cdot\cdot'')$ (in contrast, a partial functor is always an isometrically contained functor, 1.9, 1.10). Now via canonical maps we have $X' \otimes Y \subset G(X,Y) \subset H(X,Y'')$. To verify that all these inclusions are the obvious ones we must check that

$$\begin{array}{ccc} X' \hat{\otimes} Y & \xrightarrow{e^G_{XY}} & G(X,Y) \\ & \searrow & \downarrow \varphi^G_{XY} \\ & & H(X,Y'') \end{array}$$

commutes, where $X' \hat{\otimes} Y \to H(X,Y'')$ is the canonical map $x' \otimes y \to (x \to i_y(\langle x,x' \rangle y))$. But this is obvious. Since $\|\cdot\|_{G(X,Y)}$ induces a reasonable cross norm on $X' \otimes Y$ we have

$X' \subset G(X,I) \subset H(X,I) = X'$

$Y \subset G(I,Y) \subset H(I,Y'') = Y''$, where the first inclusions are isometric. Thus $G(X,I) = X'$ for all X. However, the covariant part does not behave as well; we must distinguish two cases.

<u>Definition:</u> A total bifunctor $G: \mathrm{Ban}^{op} \times \mathrm{Ban} \to \mathrm{Ban}$ with $G(I,I) = I$ is said to be of type (I), if $G(I,Y) = Y$ holds for any $Y \in \mathrm{Ban}$ via the above inclusions.

If $G(I,Y) = Y''$ for all $Y \in \mathrm{Ban}$, then G is said to be of type (II).

Remark: There exists a total bifunctor G with $G(I,I) = I$ which is neither of type (I) nor of type (II) (cf. exercise 3). Since we can factor φ^G_{XY} as $G(X,Y) \to H(X,G(I,Y)) \to H(X,Y'')$, where the first map is given by $g \mapsto (x \mapsto G(\hat{x},Y)g)$, for $g \in G(X,Y)$, the canonical map φ^G_{XY} actually takes its values in $H(X,Y)$, if G is of type (I), and thus the expression $< \varphi^G_{XY}(g)(x),y' > = G(\hat{x},y')g$, $g \in G(X,Y)$ is weak*-continuous and well defined.

2.12. Definition: A bifunctor A of type (I) is called an operator ideal.

To justify this notation we will show that it coincides with the usual notion of a Banach operator ideal (see PIETSCH [63], [64], or GORDON-LEWIS-RETHERFORD [30] for a quick account and examples): A class A of bounded linear maps between Banach spaces is a Banach operator ideal, if its components $A(X,Y) = A \cap H(X,Y)$ are linear subspace of $H(X,Y)$, which are Banach spaces with a norm $\| \cdot \|_A$ and fulfill the following conditions:

(i) $x' \in X'$, $y \in Y$ implies $x' \otimes y \in A(X,Y)$ and $\|x' \otimes y\|_A = \|x'\| \, \|y\|$.

(ii) $f \in H(X_1,X)$, $g \in A(X,Y)$, $h \in H(Y,Y_1)$ implies $h \circ g \circ f \in A(X_1,Y_1)$ and $\|h \circ g \circ f\|_A \leq \|h\| \, \|g\|_A \, \|f\|$.

Thus each Banach operator ideal in the usual sense clearly becomes a bifunctor of type (I) by putting $A(f,h)g = h \circ g \circ f$. Conversely each bifunctor A of type (I) is a Banach operator ideal, condition

(ii) being subsumed in the functorial property:

$g \in \Lambda(X,Y)$, $f \in H(X_1,X)$, $G \in H(Y,Y_1)$, then

$h \circ g \circ f = H(f,h)\ \varphi_{XY}(g) = \varphi_{X_1Y_1} \circ \Lambda(f,h)g,$

$\|h \circ g \circ f\|_\Lambda = \|\Lambda(f,h)g\|_{\Lambda(X_1,Y_1)} \leq \|f\|\ \|h\|\ \|g\|_{\Lambda(X,Y)}$, where we have identified g and $\varphi^\Lambda_{XY}(g)$ for short.

§ 3. Tensor products of functors

In this section we develop the very useful theory of tensor products of functors. There are several possible methods of introducing it – the most powerful and general one being certainly the use of ends and coends of bifunctors (compare our treatment of the A-module tensor product in III, § 3). Compare MICHOR [56], where this theory is developed for semicategories. Here we want to use a method based on spaces of natural transformations and the **Yoneda**-lemma and depending heavily on duality theory. The disadvantage is that it cannot be used for arbitrary bifunctors. The advantage is that certain special properties of the tensor product can be derived very quickly.

3.1. Let $\underline{K}$ be a full subcategory of Ban, let $G : \underline{K}^{op} \to \text{Ban}$ be a contravariant and $F : \underline{K} \to \text{Ban}$ a covariant functor. For any Banach space Z we have a covariant functor $H(G(.),Z) : \text{Ban} \to \text{Ban}$, and the assignment $Z \to H(G(.),Z)$ is a covariant functor $\text{Ban} \to \text{Ban}^{\text{Ban}}$.
Now we consider the covariant functor
$Z \to \underset{\underline{K}}{\text{Nat}}\ (F(.),H(G(.),Z))$.
We want to investigate whether this functor is representable.

<u>Definition</u>: The Banach space

$$G \hat{\otimes}_{\underline{K}} F := \underset{\underline{\text{Ban}}}{\text{Nat}}\ (\underset{\underline{K}}{\text{Nat}}\ (F,H(G,.)),\ \text{Id}_{\text{Ban}})$$

is called the tensor product of G and F over $\underline{K}$.

Natural transformations $\varphi : F \to F_1$, $\psi : G \to G_1$ over $\underline{K}$ clearly induce natural transformations

$$\operatorname{Nat}_{\underline{K}} (\varphi, H(\psi,.)) : \operatorname{Nat}_{\underline{K}} (F_1, H(G_1,.)) \to \operatorname{Nat}_{\underline{K}} (F, H(G,.))$$

and so give rise to morphisms

$$\varphi \hat{\otimes}_{\underline{K}} \psi : G \hat{\otimes}_{\underline{K}} F \to G_1 \hat{\otimes}_{\underline{K}} F_1,$$

and the assignation $G, F \mapsto G \hat{\otimes}_{\underline{K}} F$ is a co-covariant bifunctor $\mathrm{Ban}^{\underline{K}^{op}} \times \mathrm{Ban}^{\underline{K}} \to \mathrm{Ban}$.

Clearly $\|\varphi \hat{\otimes}_{\underline{K}} \psi\| \leq \|\varphi\|\|\psi\|$.

To motivate this definition: If U is a Banach space which represents $\operatorname{Nat}_{\underline{K}} (F,H(G,.))$ in Ban_1, i.e. there is a natural isometric equivalence $H(U,.) = \operatorname{Nat}_{\underline{K}}(F,H(G,.))$, then by 1.4 we have $\operatorname{Nat}_{\mathrm{Ban}} (\operatorname{Nat}_{\underline{K}}(F,H(G,.)),\mathrm{Id}_{\mathrm{Ban}}) = \operatorname{Nat}_{\mathrm{Ban}} (H(U,.),\mathrm{Id}_{\mathrm{Ban}}) = U$.

3.2. Lemma: The map

$$G \hat{\otimes}_{\underline{K}} F = \operatorname{Nat}_{\mathrm{Ban}} (\operatorname{Nat}_{\underline{K}} (F,H(G,.)),.) \to \operatorname{Nat}_{\underline{K}} (F,G')',$$

given by $\eta \mapsto \eta_{\mathbb{I}}$ is isometric.

Proof: Given $\eta \in G \hat{\otimes}_{\underline{K}} F$ there is an $X \in \mathrm{Ban}$ such that $\|\eta\| = \sup_{Y \in \mathrm{Ban}} \|\eta_Y\| \leq \|\eta_X\| + \epsilon$ and there is a $\varphi \in \operatorname{Nat}_{\underline{K}}(F,H(G,X))$ with $\|\varphi\| = 1$ such that $\|\eta_X(\varphi)\|_X \leq \|\eta_X\| + \epsilon$. There is an $x' \in X'$, $\|x'\| = 1$ such that $|\langle \eta_X(\varphi), x' \rangle| = \|\eta_X(\varphi)\|_X$.

Then $\|\eta_I\| \leq \|\eta\| \leq \|\eta_X\| + \varepsilon \leq |\langle \eta_X(\varphi), x' \rangle| + 2\varepsilon$

$= |\eta_I \circ \operatorname{Nat}_K (F, H(G, x'))(\varphi)| + 2\varepsilon$

$\leq \|\eta_I\| \|x'\| \|\varphi\| + 2\varepsilon = \|\eta_I\| + 2\varepsilon.$ qed.

3.3. Thus $G \hat{\otimes}_{\underline{K}} F$ is isometric to a subspace of $\operatorname{Nat}_{\underline{K}} (F,G')'$. We shall show how $G \hat{\otimes}_{\underline{K}} F$ can be canonically identified as a subspace of $\operatorname{Nat}_{\underline{K}} (F,G')'$.

For any $X \in \underline{K}$ there is a contractive bilinear map $\sigma_X : G(X) \times F(X) \to \operatorname{Nat}_{\underline{K}} (F,G')'$, given by

$\langle \eta, \sigma_X(g_X, f_X) \rangle = \langle g_X, \eta_X(f_X) \rangle$.

This family of maps $(\sigma_X)_{X \in \underline{K}}$ has the following property:

For any $f : X \to Y$ in $\underline{K}$, $g_Y \in G(Y)$, $f_X \in F(X)$ we have

$\sigma_X(G(f)g_Y, f_X) = \sigma_Y(g_Y, F(f)f_X)$, since for all $\eta \in \operatorname{Nat}_{\underline{K}} (F,G')$

$\langle \eta, \sigma_X(G(f)g_Y, f_X) \rangle = \langle G(f)g_Y, \eta_X(f_X) \rangle$

$= \langle g_Y, G(f)' \eta_X(f_X) \rangle = \langle g_Y, \eta_Y F(f) f_X \rangle$

$= \langle \eta, \sigma_Y(g_Y, F(f)f_X) \rangle$.

Now let M denote the closed linear subspace of $\operatorname{Nat}_{\underline{K}}(F,G')'$ generated by the images of all maps σ_X, $X \in \underline{K}$.

We assert that M represents the functor $\operatorname{Nat}_{\underline{K}}(F,H(G,.)) : \text{Ban} \to \text{Ban}$, i.e. there is a natural isometric equivalence of functors

$T : H(M,.) \to \operatorname{Nat}_{\underline{K}}(F,H(G,.))$.

If $Z \in \text{Ban}$ and $\alpha \in H(M,Z)$, then let

$(T_Z\alpha)_X(f_X)(g_X) = \alpha(\sigma_X(g_X, f_X)) \in Z$ for $X \in \underline{K}$.

If $f : X \to Y$ in $\underline{K}$, then

$$[H(G(f),Z)\circ(T_Z\alpha)_X(f_X)](g_Y) = (T_Z\alpha)_X(f_X)(G(f)g_Y)$$

$$= \alpha(\sigma_X(G(f)g_Y,f_X)) = \alpha(\sigma_Y(g_Y,F(f)f_X))$$

$$= (T_Z\alpha)_Y(F(f)f_X)(g_Y), \text{ and}$$

$$\|T_Z\alpha\| = \sup_{X\in\underline{K}}\ \sup_{\|f_X\|\le 1}\ \sup_{\|g_X\|\le 1} \|(T_Z\alpha)_X(f_X)(g_X)\|_Z$$

$$= \sup_{X}\ \sup_{\|f_X\|\le 1}\ \sup_{\|g_X\|\le 1} \|\alpha(\sigma_X(g_X,f_X))\|_Z$$

$$\le \|\alpha\|.$$

So $T_Z\alpha \in \mathrm{Nat}_{\underline{K}}(F,H(G,.))$ and $\|T_Z\alpha\| \le \|\alpha\|$.

If $Z \in \mathrm{Ban}$ and $\varphi \in \mathrm{Nat}_{\underline{K}}(F,H(G,Z))$, let

$$(T_Z^{-1}\varphi)\left(\sum_{i=1}^{n}\sigma_{X_i}(g_{X_i},f_{X_i})\right) = \sum_i \varphi_{X_i}(f_{X_i})(g_{X_i}) \in Z.$$

$$\left\|(T_Z^{-1}\varphi)\left(\sum_{i=1}^{n}\sigma_{X_i}(g_{X_i},f_{X_i})\right)\right\|_Z =$$

$$= \sup_{\|z'\|\le 1} \left|< \sum_i \varphi_{X_i}(f_{X_i})(g_{X_i}),z' >\right|$$

$$= \sup_{\|z'\|\le 1} \left| \sum_i [H(G,z')\circ\varphi]_{X_i}(f_{X_i})(g_{X_i})\right|$$

$$\le \sup_{\|z'\|\le 1} \left|< H(G,z')\circ\varphi, \sum_i \sigma_{X_i}(g_{X_i},f_{X_i}) >\right|$$

$$\le \sup_{\|z'\|\le 1} \|H(G,z')\circ\varphi\|_{\mathrm{Nat}_{\underline{K}}(F,G')} \cdot \left\|\sum_i \sigma_{X_i}(g_{X_i},f_{X_i})\right\|_M$$

$$\le \|\varphi\|_{\mathrm{Nat}_{\underline{K}}(F,H(G,.))} \cdot \left\|\sum_i \sigma_{X_i}(g_{X_i},f_{X_i})\right\|_M .$$

This computation shows that $T_Z^{-1}\varphi$ is well definded, linear, and $\|T_Z^{-1}\varphi\| \le \|\varphi\|$, so that $T_Z^{-1}\varphi$ is defined on the whole of M. T_Z^{-1} is obviously the inverse of T_Z and it is equally simple to see that T_Z is natural in Z. So we have:

Theorem: $G \hat{\otimes}_{\underline{K}} F$ represents the functor $\underset{\underline{K}}{\mathrm{Nat}}(F,H(G,.))$,

i.e. $H(G \hat{\otimes}_{\underline{K}} F,.) = \underset{\underline{K}}{\mathrm{Nat}}(F,H(G,.))$ as functors Ban → Ban.

$G \hat{\otimes}_{\underline{K}} F$ is uniquely determined by this property up to isometric isomorphisms and is isometrically isomorphic to the closed linear hull M of all elements $\sigma_X(g_X,f_X)$, $X \in \underline{K}$ in $\underset{\underline{K}}{\mathrm{Nat}}(F,G')'$.

Proof: It only remains to show that $G \hat{\otimes}_{\underline{K}} F = M$:

$$G \hat{\otimes}_{\underline{K}} F = \underset{\mathrm{Ban}}{\mathrm{Nat}}\ (\underset{\underline{K}}{\mathrm{Nat}}(F,H(G,.)),\ \mathrm{Id}_{\mathrm{Ban}})$$
$$= \underset{\mathrm{Ban}}{\mathrm{Nat}}\ (H(M,.),\ \mathrm{Id}_{\mathrm{Ban}}) = M \text{ by } 1.4.$$

In a similar manner, by 1.4, we have $M = M_1$ if $H(M,.) = H(M_1,.)$.

3.4. Let $\underline{K}$ be small with respect to Ban, so that the coproduct $\sum_{X \in \underline{K}} G(X) \hat{\otimes} F(X)$ exists in Ban. (cf. I, 1.10). Then we may consider the map

$$\sigma = \sum_{X \in \underline{K}} \sigma_X :\ \sum_{X \in \underline{K}} G(X) \hat{\otimes} F(X) \to \underset{\underline{K}}{\mathrm{Nat}}(F,G')'.$$

Proposition: σ is a strict morphism (I, 1.6) and so has the following canonical factorization, where $\tilde{\sigma}$ is an isometric isomorphism and $\sigma^{-1}(0)$ is the closed linear subspace of $\sum_{X \in \underline{K}} G(X) \hat{\otimes} F(X)$ generated by all elements of the form

$g_Y \otimes F(f)f_X - G(f)g_Y \otimes f_X$; $f : X \to Y$ in $\underline{K}$.

$$\begin{array}{ccc}
\sum_{X\in\underline{K}} G(X)\,\hat{\otimes}\,F(X) & \xrightarrow{\quad\sigma\quad} & \underset{\underline{K}}{\mathrm{Nat}}(F,G')' \\
\text{coim } \sigma \Big\downarrow & & \Big\uparrow \text{ im } \sigma \\
\sum_{X\in\underline{K}} G(X)\,\hat{\otimes}\,F(X)/_{\sigma^{-1}(0)} & \xrightarrow{\quad\tilde{\sigma}\quad} & G\,\hat{\otimes}_{\underline{K}}\,F
\end{array} .$$

Remark: Here we have one of the reasons why we call this space a tensor product. Compare III, § 3.

Proof: If M is the subspace of $\underset{\underline{K}}{\mathrm{Nat}}(F,G')'$ considered above, then clearly the image of σ is dense in M, and $M = G\,\hat{\otimes}_{\underline{K}}\,F$. So we are done if we know that σ is strict. Let N be the closed linear subspace generated by elements $g_Y \otimes F(f)f_X - G(f)g_Y \otimes f_X$ in $\sum_{X\in\underline{K}} G(X)\,\hat{\otimes}\,F(X)$. It suffices to show that $(\sum_{X\in\underline{K}} G(X)\,\hat{\otimes}\,F(X)/_N)' = \underset{\underline{K}}{\mathrm{Nat}}(F,G')$ and that $(\mathrm{im}\ \sigma)\circ\tilde{\sigma}$ is just the canonical embedding into the bidual. This is seen as follows:

$(\sum_{X\in\underline{K}} G(X)\,\hat{\otimes}\,F(X)/_N)' = N^{\perp} \subseteq \prod_{X\in\underline{K}} (G(X)\,\hat{\otimes}\,F(X))'$

$= \coprod_{X\in\underline{K}} H(F(X),G(X)')$, and $N^{\perp}$ consists exactly of all families $(\varphi_X)_{X\in\underline{K}}$ of maps $\varphi_X : F(X) \to G(X)'$

with $\sup_X \|\varphi_X\| < \infty$ and $< N,\varphi > = 0$, i.e.

$< g_Y \otimes F(f)f_X,\varphi > = < G(f)g_Y \otimes f_X,\varphi >$ or

$< g_Y,\varphi_Y F(f)f_X > = < g_Y,G(f)'\varphi_X(f_X) >$, i.e.

$\varphi_Y \circ F(f) = G(f)' \circ \varphi_X$ for all $f : X \to Y$ in $\underline{K}$;

thus $N^{\perp} = \underset{\underline{K}}{\mathrm{Nat}}(F,G')$.

$(\mathrm{im}\ \sigma) \circ \tilde{\sigma}$ coincides with the canonical embedding into the bidual by the construction of (σ_X).

3.5. Remark: We have a canonical natural equivalence

$\mathrm{Nat}_{\underline{K}}(F,H(G,.)) = \mathrm{Nat}_{\underline{K}}(G,H(F,.))$ which comes from the componentwise natural equality $H(F(X),H(G(X),Z)) = H(G(X),H(F(X),Z))$ (1.2). We could have used this to introduce $G \hat{\otimes}_{\underline{K}} F$.

We will use the formula

$$H(G \hat{\otimes}_{\underline{K}} F,.) = \mathrm{Nat}_{\underline{K}}(G,H(F,.))$$

which is compatible via the above equalities with the definition of $G \hat{\otimes}_{\underline{K}} F$.

3.6. Theorem: Let $\underline{K}$ and $\underline{L}$ be full subcategories of Ban, $M:\underline{K}^{op} \times \underline{L} \to \mathrm{Ban}$ a contra-covariant bifunctor, $G:\underline{L}^{op} \to \mathrm{Ban}$ contravariant and $F:\underline{K} \to \mathrm{Ban}$ covariant. Then we have $G \hat{\otimes}_{\underline{L}} (M \hat{\otimes}_{\underline{K}} F) = (G \hat{\otimes}_{\underline{L}} M) \hat{\otimes}_{\underline{K}} F$.

Proof: From 3.1 it follows that $M \hat{\otimes}_{\underline{K}} F : \underline{L} \to \mathrm{Ban}$ is a functor, likewise $G \hat{\otimes}_{\underline{L}} M : \underline{K}^{op} \to \mathrm{Ban}$. Now we compute using 3.3 and 3.5.

$$H(G \hat{\otimes}_{\underline{L}} (M \hat{\otimes}_{\underline{K}} F),.) = \mathrm{Nat}_{\underline{L}}(G,H(M \hat{\otimes}_{\underline{K}} F,.)) \quad \text{(by 3.5)}$$

$$= \mathrm{Nat}_{\underline{L}}(G, \mathrm{Nat}_{\underline{K}}(F,H(M,.))) \quad \text{(by 3.3)}$$

$$= \mathrm{Nat}_{\underline{K}}(F, \mathrm{Nat}_{\underline{L}}(G,H(M,.))) \quad \text{(using the pointwise}$$

equality $H(F(X), H(G(Y), H(M(X,Y),Z))) =$

$$= H(G(Y),H(F(X),H(M(X,Y), Z)))$$

$$= \underset{\underline{K}}{\mathrm{Nat}}(F,H(G\ \hat{\otimes}_{\underline{L}}\ M,.)) \qquad \text{(by 3.5)}$$
$$= H((G\ \hat{\otimes}_{\underline{L}}\ M)\ \hat{\otimes}_{\underline{K}}\ F,.) \qquad \text{(by 3.3)}$$

Remark: Using 3.4 the equality is given by:

$\sigma_X(g_X,\sigma_Y(m_{XY},f_Y)) = \sigma_Y(\sigma_X(g_X,m_{XY}),f_Y)$, or, in shorter form, by $\quad g_X \otimes (m_{XY} \otimes f_Y) = (g_X \otimes m_{XY}) \otimes f_Y$.

3.7. Theorem: Let $\underline{K}$ and $\underline{L}$ be full subcategories of Ban, $M : \underline{L} \times \underline{K} \to \mathrm{Ban}$ a contra-covariant bifunctor and $F_1 : \underline{L} \to \mathrm{Ban}$ and $F_2 : \underline{K} \to \mathrm{Ban}$ be covariant functors.

Then the "exponential law " holds:

$$\underset{\underline{K}}{\mathrm{Nat}}(M\ \hat{\otimes}_{\underline{L}}\ F_1,F_2) = \underset{\underline{L}}{\mathrm{Nat}}(F_1,\underset{\underline{K}}{\mathrm{Nat}}(M,F_2)).$$

This equality is natural with respect to natural transformations of all functors. Hence $F_1 \to M\ \hat{\otimes}_{\underline{L}}\ F_1$ is left adjoint to $F_2 \to \underset{\underline{K}}{\mathrm{Nat}}(M,F_2)$.

Proof: By 3.3 we have

$$\prod_{X\in\underline{K}} H(M(.,X)\ \hat{\otimes}_{\underline{L}}\ F_1,F_2(X)) = \prod_{X\in\underline{K}} \underset{\underline{L}}{\mathrm{Nat}}(F_1,H(M(.,X),F_2(X))$$

and the subspace $\underset{\underline{K}}{\mathrm{Nat}}(M\ \hat{\otimes}_{\underline{L}}\ F_1,F_2)$ of the left hand side corresponds exactly to the space of families $(\varphi_X)_{X\in\underline{K}}$ in the left hand side, which are natural in X, i.e. to the space $\mathrm{Nat}(F_1,\underset{\underline{K}}{\mathrm{Nat}}(M,F_2))$.

To prove the naturality of the equation is a routine task.

Remark: 1) Using 3.4 and the shorthand of the remark in 3.6 the correspondence is given by $\varphi \longleftrightarrow \hat{\varphi}$ where

$$\varphi_X(\sigma_Y(m_{XY}, f^1{}_Y)) = (\hat{\varphi}_Y(f^1{}_Y))_X(m_{XY}) \text{ or}$$
$$\sigma_X(m_{XY} \otimes f^1{}_Y) = (\hat{\varphi}_Y(f^1{}_Y))_X(m_{XY}).$$

2) Of course there are corresponding exponential laws:

$\operatorname{Nat}_{\underline{K}} (G \hat{\otimes}_{\underline{L}} M, F) = \operatorname{Nat}_{\underline{L}} (G, \operatorname{Nat}_{\underline{K}} (M,F))$ where G is contravariant and F may be co -or -contravariant according to M.

We will use all those without hesitation.

3.8. <u>Proposition:</u> Let H: $\underline{K}^{op} \times \underline{K} \to \mathrm{Ban}$ be the restriction of the contra-covariant Hom-functor to $\underline{K}$. Then for any F: $\underline{K} \to \mathrm{Ban}$ and G: $\underline{K}^{op} \to \mathrm{Ban}$ we have $H \hat{\otimes}_{\underline{K}} F = F$ and $G \hat{\otimes}_{\underline{K}} H = G$ naturally in F and G.

<u>Proof:</u> $H(H(.,..) \overset{(.)}{\hat{\otimes}}_{\underline{K}} F(.),...) = \operatorname{Nat}_{(.)\in\underline{K}} (H(.,..), H(F(.),...))$
$= H(F(..),...)$ by 1.4.

<u>Remark:</u> These equations reduce to the trivial relations $I \hat{\otimes} X = X = X \hat{\otimes} I$ for $\underline{K} = \{I\}$.

3.9. As a simple application of the theory of tensor products we can compute all Kan-extensions between full subcategories of Ban.

If $\underline{K}$ and $\underline{L}$ are full subcategories of Ban and if S: $\underline{K} \to \underline{L}$ is a functor, then it defines the "restriction via S" functor: $\mathrm{Ban}^{\underline{L}} \to \mathrm{Ban}^{\underline{K}}$ by $F \to FS$. The left Kan extension Lan_S is a left adjoint to this functor, the right Kan extension is a right adjoint (cf. Mac LANE, [49], chapter X for more details). The restriction functor $F \to FS$ can be written either as tensor product or as space of natural transformations:

$$FS(.) = H(..,S(.)) \hat{\otimes}_{(..)\in \underline{L}} F(..) \quad \text{by 3.8.}$$

$$FS(.) = \mathrm{Nat}_{(..)\in \underline{L}} (H(S(.),..),F(..)) \quad \text{by 1.4.}$$

Proposition: $\mathrm{Ran}_S F_1 = \mathrm{Nat}_{(..)\in \underline{K}} (H(.,S(..)),F_1(..))$,

$$\mathrm{Lan}_S F_1 = H(S(..),.) \hat{\otimes}_{(..)\in \underline{K}} F_1(..)$$

for all $F_1 \in \mathrm{Ban}^{\underline{K}}$.

For $S: \{I\} \longrightarrow \mathrm{Ban}$ this reduces to 1.17. and 1.6.

Proof: Let $F \in \mathrm{Ban}^{\underline{L}}$, $F_1 \in \mathrm{Ban}^{\underline{K}}$. Then $\mathrm{Nat}_{(.)\in \underline{K}} (FS(.),F_1(.)) =$

$$= \mathrm{Nat}_{(.)\in \underline{K}} (H(..,S(.)) \hat{\otimes}_{(..)\in \underline{L}} F(..),F_1(.)) =$$

$$= \mathrm{Nat}_{(..)\in \underline{L}} (F(..), \mathrm{Nat}_{(.)\in \underline{K}} (H(..,S(.)),F_1(.))).$$

$$\mathrm{Nat}_{(.)\in \underline{K}} (F_1(.),FS(.)) = \mathrm{Nat}_{(.)\in \underline{K}} (F_1(.), \mathrm{Nat}_{(..)\in \underline{L}} (H(S(.),..),F(..)))$$

$$= \mathrm{Nat}_{(..)\in \underline{L}} (H(S(.),..) \hat{\otimes}_{(.)\in \underline{K}} F_1(.),F(..)).$$

3.10. Proposition: $\mathrm{Nat}_{\mathrm{Ban}} (F_1,F_2(X \hat{\otimes} .)) = \mathrm{Nat}_{\mathrm{Ban}} (F_1 H_X,F_2)$ holds naturally in $F_1,F_2 \in \mathrm{Ban}^{\mathrm{Ban}}$ and $X \in \mathrm{Ban}$.

<u>Proof:</u> $\mathrm{Nat}\,(F_1(.),F_2(X\hat{\otimes}\,.)) =$

$= \mathrm{Nat}\,(F_1(.),\ \mathrm{Nat}\,(H(X\hat{\otimes}\,.,..),F_2(..)))$

$= \mathrm{Nat}\,(H(X\hat{\otimes}\,.,..)\hat{\otimes}F_1(.),F_2(..))$

$= \mathrm{Nat}\,(H(.,H(X,..))\hat{\otimes}F_1(.),F_2(..))$

$= \mathrm{Nat}\,(F_1H_X,F_2)$

<u>Remark:</u> This result shows that

$\mathrm{Lan}_{X\hat{\otimes}\,.}F_1 = F_1H_X$ and $\mathrm{Ran}_{H_X}F_2 = F_2(X\hat{\otimes}\,.)$ hold, which is true for any pair of adjoint functors. But on Ban_1 there is only one such pair (1.2.).

3.11. <u>Proposition:</u> Let $\underline{K}$ and $\underline{L}$ be full subcategories of Ban, $S\colon \underline{K}\to\underline{L}$, $G\colon \underline{L}^{op}\to \mathrm{Ban}$ and $F\colon \underline{K}\to \mathrm{Ban}$ be functors.

Then $G\circ S\,\hat{\otimes}_{\underline{K}}\,F = G\,\hat{\otimes}_{\underline{L}}\,\mathrm{Lan}_S F$.

If $S\colon \underline{L}\to\underline{K}$, then we have $G\,\hat{\otimes}_{\underline{L}}\,F\circ S = \mathrm{Lan}_S G\,\hat{\otimes}_{\underline{K}}\,F$.

<u>Proof:</u> $G\circ S\,\hat{\otimes}_{\underline{K}}\,F = (G(.)\ \hat{\otimes}_{(.)\in\underline{L}}\ H(S(.),..))\ \hat{\otimes}_{(..)\in\underline{K}}\ F(..) =$

$= G(.)\ \hat{\otimes}_{(.)\in\underline{L}}\ (H(S(.),..)\ \hat{\otimes}_{(..)\in\underline{K}}\ F(..)) = G\,\hat{\otimes}_{\underline{L}}\,\mathrm{Lan}_S F.$

Note that for contravariant G we obviously have

$\mathrm{Lan}_S\, G = G(.)\ \hat{\otimes}_{(.)\in\underline{L}}\ H(..,S(.))$, going along the lines of 3.9. for contravariant F and F_1.

$G\,\hat{\otimes}_{\underline{L}}\,F\circ S = G(.)\ \hat{\otimes}_{(.)\in\underline{L}}\ (H(..,S(.))\ \hat{\otimes}_{(..)\in\underline{K}}\ F(..))$

$= (G(.)\ \hat{\otimes}_{(.)\in\underline{L}}\ H(..,S(.)))\ \hat{\otimes}_{(..)\in\underline{K}}\ F(..) = \mathrm{Lan}_S G\,\hat{\otimes}_{\underline{K}}\,F.$

Remark: 1) If $K = \{I\}$ and $S: \{I\} \to \mathrm{Ban}$ is the embedding, then $F = X \in \mathrm{Ban}$ and $\mathrm{Lan}_S X = X \hat{\otimes} .$. So we have

$$G \underset{\mathrm{Ban}}{\hat{\otimes}} (X \hat{\otimes} .) = G \underset{\mathrm{Ban}}{\hat{\otimes}} \mathrm{Lan}_S X = GS \underset{\{I\}}{\hat{\otimes}} X = G(I) \hat{\otimes} X,$$

analogously $(X \hat{\otimes} (.)') \underset{\mathrm{Ban}}{\hat{\otimes}} F = X \hat{\otimes} F(I)$.

2) This result is the dual one to the property by which we defined Kan extensions (3.9.). We could have used this property to define them. Note that $(\mathrm{Lan}_S F)' = \mathrm{Ran}_S F'$ and $(\mathrm{Lan}_S G)' = \mathrm{Ran}_S G'$ hold.

3.12. Proposition: Let G be contravariant, F covariant and one of type Σ. Then $G \underset{\mathrm{Ban}}{\hat{\otimes}} F = G(I) \underset{\alpha}{\otimes} F(I)$, where $\otimes_\alpha$ denotes the completion of the algebraic tensor product in a reasonable crossnorm α (II., 2.1.).

Proof: Suppose that F is essential. Since

$$\sigma_I' : (G \underset{\mathrm{Ban}}{\hat{\otimes}} F)' = \mathrm{Nat}\,(F, G') \to H(F(I), G(I)') = (G(I) \hat{\otimes} F(I))'$$

is injective, the mapping $\sigma_I : G(I) \hat{\otimes} F(I) \to G \underset{\mathrm{Ban}}{\hat{\otimes}} F$ has dense image. We have natural transformations

$$G(I) \hat{\otimes} (.)' \xrightarrow{\varepsilon^G} G \xrightarrow{\varphi^G} H(., G(I)) \quad (1.6., 1.14., 1.15.).$$

Both are contractive and induce therefore contractive maps

$$G(I) \hat{\otimes} F(I) = (G(I) \hat{\otimes} (.)') \underset{\mathrm{Ban}}{\hat{\otimes}} F \longrightarrow G \hat{\otimes}_{\mathrm{Ban}} F \longrightarrow$$

$$\longrightarrow H(., G(I)) \underset{\mathrm{Ban}}{\hat{\otimes}} F = F(G(I)).$$

Since $F(G(I)) \to F(I) \hat{\hat{\otimes}} G(I)$ has dense image (F is of type Σ) the proof is finished.

3.13. Proposition: If X has the metric approximation property, then for any F: Ban $\to$ Ban we have $(.' \hat{\hat{\otimes}} X) \underset{\text{Ban}}{\hat{\otimes}} F = F_e(X)$.

If X' has the metric approximation property, then for any G: $\text{Ban}^{op} \to \text{Ban}$ we have $G \underset{\text{Ban}}{\hat{\otimes}} (X' \hat{\hat{\otimes}} .) = G_e(X)$.

Proof: We prove the first assertion, the second one can be proved similarly.

By 3.12. $(.' \hat{\hat{\otimes}} X) \underset{\text{Ban}}{\hat{\otimes}} F(.)$ is a completion of $X \otimes F(I)$, since $(.' \hat{\hat{\otimes}} X)$ is essential. We have natural inclusions $F_e \to F$ and $.' \hat{\hat{\otimes}} X \to H(.,X)$, which induce linear contractions $(.' \hat{\hat{\otimes}} X) \underset{\text{Ban}}{\hat{\otimes}} F_e \to (.' \hat{\hat{\otimes}} X) \underset{\text{Ban}}{\hat{\otimes}} F \to H(.,X) \underset{\text{Ban}}{\hat{\otimes}} F = F(X)$.

So we have to show that $(.' \hat{\hat{\otimes}} X) \underset{\text{Ban}}{\hat{\otimes}} F_e \to F(X)$ is an isometry, i.e. we have to show that the adjoint map $F(X)' \to \text{Nat}\,(.' \hat{\hat{\otimes}} X, F_e(.)')$ is a quotient map. It is clearly given by

$F(X)' \to \text{Nat}\,(H(.,X),F(.)') \to \text{Nat}\,(K(.,X),F(.)') =$

$= \text{Nat}\,(K(.,X),F_e(.))$, where the first map is the Yoneda-map from 1.4. and the second one is restriction. $F(X)' \to F_e(X)'$ is a quotient map, so it suffices to show that the "Yoneda map" τ: $F_e(X)' \to \text{Nat}(K(.,X),F_e(.)')$ is an isometric isomorphism:

for $z \in F_e(X)'$ let $(\tau z)_Y(f) = F_e(f)'z$, $f \in K(Y,X)$.

Let (u_i) be an approximate unit in $K(X,X)$. Then

$$\|z\|_{F_e(X)'} = \sup_{\substack{v\in F_e(X)\\ \|v\|\leq 1}} |\langle v,z\rangle \leq \sup_{\|v\|\leq 1} \sup_i |\langle F(u_i)v,z\rangle| \quad (1.13.)$$

$$= \sup_{\|v\|\leq 1} \sup_i |\langle v, F_e(u_i)'z\rangle|$$

$$= \sup_i \|(\tau z)_X(u_i)\|_{F_e(X)'} \leq \|(\tau z)_Y\|$$

$$\leq \|\tau z\| \leq \|z\|.$$

So τ is an isometry and it remains only to show that it is onto. Let $\beta \in \mathrm{Nat}\,(K(.,X),F_e(.)')$. Since $\beta_X(u_i)$ is bounded in $F_e(X)'$ it has a weak*-cluster point $z \in F_e(X)'$. But then for any $f \in K(Y,X)$ and $v \in F_e(Y)$ we have

$$\langle v, \beta_Y(f)\rangle = \lim_i \langle v, \beta_Y(u_i \circ f)\rangle = \lim_i \langle v, \beta_Y(K(f,X)u_i)\rangle =$$

$$= \lim_i \langle v, F_e(f)' \beta_X(u_i)\rangle$$

$$= \lim \langle F_e(f)v, \beta_X(u_i)\rangle = \langle F_e(f)v,z\rangle$$

$$= \langle v, F_e(f)'z\rangle = \langle v, (\tau z)_Y(f)\rangle.$$

So $\tau z = \beta$ and τ is onto.

CHAPTER V

Duality of functors

§ 1. Duality of functors

Several notions of duality of functors have been studied in the literature until now. We will give a unified general approach to them in this section, starting with a very general situation and then specializing to get more detailed results. We restrict ourselves to dualities for covariant functors; a similar theory for contravariant functors can be developed along the same lines and we will outline the theory at the end of this section.

$\underline{K}$ will always denote a full subcategory of Ban. By $\mathrm{Ban}^{\underline{K}}$ we mean the category of all admissible functors $\underline{K} \to \mathrm{Ban}$ (IV., 1.1.).

1.1. Definition: A duality for covariant functors is a contravariant functor $D: (\mathrm{Ban}^{\underline{K}})^{op} \to \mathrm{Ban}^{\underline{K}}$ which is admissible (linear and contractive on Hom-spaces) and self-adjoint on the right, i.e. $\mathrm{Nat}_{\underline{K}}(F_1, DF_2) = \mathrm{Nat}_{\underline{K}}(F_2, DF_1)$ holds naturally in F_1 and F_2 via an isometric isomorphism $\eta_{F_1F_2}$ with $\eta_{F_1F_2}^{-1} = \eta_{F_2F_1}$.

1.2. Example: This example will later on be seen to be the only possible one. We consider co-covariant bifunctors $G: \underline{K} \times \underline{K} \to \mathrm{Ban}$ (see IV., 2.1.). Such a bifunctor G is said to

be symmetric, if there is an isometric isomorphism $^t: G(X,Y) \to G(Y,X)$ which is natural in X and Y and is an involution (i.e. $^{tt} = Id_G$). ^{t} is called transposition.
Examples of symmetric bifunctors abound:
$X \hat{\otimes} Y$, $L^1(X',Y)$, $X \hat{\hat{\otimes}} Y$, $L(X',Y)$, $H(X',Y'')$, $K(X',Y'')$, $H(X,Y')'$, $K(X,Y')'$, $K_o(X',Y'')$, $I_1(X,Y')'$ etc. Some of them are contracovariant in their natural form. We have made them co-covariant by duality.
All of them fulfill $G(I,I) = I$. It is easy to construct more general ones: $G(X,Y) \hat{\otimes} A$ or $H(A,G(X,Y))$, where G is as above and A is a fixed Banach space.
Given any symmetric bifunctor $G: \underline{K} \times \underline{K} \to Ban$ we construct a duality, the so-called G-duality $D^G: (Ban^{\underline{K}})^{op} \to Ban^{\underline{K}}$ by

$$D^G F(X) = \underset{\underline{K}}{Nat} (F, G(X,.)).$$

It is clear that D^G is a contravariant functor on $Ban^{\underline{K}}$.
Since Nat is the Hom-functor of $Ban^{\underline{K}}$, D^G is obviously admissible, and we show now that the symmetry of G implies that D^G is adjoint to itself on the right:

$$Nat (F_1, D^G F_2) = \underset{(..)}{Nat} (F_1(..), \underset{(.)}{Nat}(F_2(.), G(..,.)))$$

$$= \underset{(..,.)}{Nat} (F_1(..) \hat{\otimes} F_2(.), G(..,.))$$

$$= \underset{(..,.)}{Nat} (F_2(.) \hat{\otimes} F_1(..), G(.,..)) \text{ via } ^t$$

$$= \underset{(.)}{Nat} (F_2(.), \underset{(..)}{Nat} (F_1(..), G(.,..))) = Nat (F_2, D^G F_1).$$

1.3. Given a duality D on $Ban^{\underline{K}}$ we use its self-adjointness on the right to define the notion of D-reflexive functors: $\underline{K} \to Ban$. We have naturally in F (setting $F_1 = DF$): Nat $(DF, DF) = $ Nat (F, DDF), and the morphism in the right-hand-space that corresponds to the identity in the left-hand-space is called the canonical morphism and denoted by ι^F: $F \to DDF$. A functor F is said to be D-reflexive, if ι^F is an isometry onto.

1.4. <u>Theorem:</u> There is a one-one correspondence between dualities D on $Ban^{\underline{K}}$ and co-covariant symmetric bifunctors G on $\underline{K} \times \underline{K}$; thus each duality has the form D^G of 1.2.

<u>Proof:</u> Given a duality D we define a bifunctor G_D by $G_D(X,Y) = DH_X(Y)$. It is clearly a functor in Y and since $f: X_1 \to X_2$ defines a natural transformation $H_f: H_{X_2} \to H_{X_1}$ we can define $G_D(f,Y) = (DH_f)_Y: DH_{X_1}(Y) \to DH_{X_2}(X)$. $G_D(.,Y)$ is thus covariant and G_D becomes a bifunctor since for $g: Y_1 \to Y_2$ we have $G_D(f,Y_2) \circ G_D(X_1,g) =$
$= (DH_f)_{Y_2} \circ DH_{X_1}(g) = DH_{X_2}(g) \circ (DH_f)_{Y_1} = G_D(X_2,g) \circ G_D(f,Y_1)$
since $DH_f: DH_{X_1} \to DH_{X_2}$ is a natural transformation.
G_D is symmetric since we can define ${}^t: G_D(X,Y) \to G_D(Y,X)$ by the following equation: $G_D(X,Y) = DH_X(Y) = $ Nat (H_Y, DH_X) by (IV., 1.4.),
$=$ Nat (H_X, DH_Y) by self-adjointness of D,
$= DH_Y(X)$ by (IV., 1.4.) again
$= G_D(Y,X)$ naturally in X,Y, and the iterated process furnishes identity.

Now $D^{(G_D)} = D$ since $D^{(G_D)}F(X) = \mathrm{Nat}\ (F, G_D(X,.)) =$
$= \mathrm{Nat}\ (F, DH_X) = \mathrm{Nat}\ (H_X, DF) = DF(X)$.
On the other hand we have $G_{(D^G)} = G$ for each symmetric bifunctor G on $\underline{K} \times \underline{K}$ since $G_{(D^G)}(X,Y) = D^G H_X(Y) =$
$= \mathrm{Nat}\ (H_X, G(Y,.)) = G(Y,X) = G(X,Y)$.

1.5. So it suffices to study the dualities derived from symmetric bifunctors G. We are able to write down the form of the canonical map $\iota^F: F \to D^G D^G F$ of 1.3.:
$\iota_X^F(f_X)_Y(\mu) = {}^t(\mu_X(f_X))$ for $f_X \in F(X)$, $\mu \in D^G F(Y) = \mathrm{Nat}(F, G(Y,.))$
as is easily seen by following up the equation in 1.2. which we used to prove the self-adjointness of D^G.
Furthermore we have the following result:

<u>Proposition:</u> $D^G(\iota^F) \circ \iota^{D^G F} = 1_{D^G F}: D^G F \to D^G F$ for all $F: \underline{K} \to \mathrm{Ban}$. So $D^G(\iota^F)$ is a quotient map and $\iota^{D^G F}$ is an isometry for all F.

<u>Proof:</u> The last assertion follows since both maps are contractive. Let $\mu \in D^G F(Y)$, $f_X \in F(X)$. Then

$$[[D^G(\iota^F) \circ \iota^{D^G F}]_Y(\mu)]_X(f_X) =$$
$$= [D^G(\iota^F)_Y \circ \iota^{D^G F}{}_Y(\mu)]_X(f_X) =$$
$$= [\mathrm{Nat}(\iota^F, G_Y)(\iota^{D^G F}{}_Y(\mu))]_X(f_X) =$$
$$= [\iota^{D^G F}{}_Y(\mu) \circ \iota^F]_X(f_X) =$$

$$= \iota^{D^G F}{}_Y(\mu)_X(\iota^F{}_X(f_X)) =$$
$$= {}^t(\iota^F_X(f_X)_Y(\mu)) = {}^{tt}(\mu_X(f_X)) = \mu_X(f_X).$$

1.6. We recall from IV., 2.9. that G is said to be total, if $G(x',y')$, $x' \in X'$, $y' \in Y'$ separates point on $G(X,Y)$ for all $X,Y \in \underline{K}$.

We have maps (compare IV., 2.9.):
$\varphi^{D^G F}: D^G F \to H(.',D^G F(I))$, and
$j^F: D^G F \to H(F(I),G(.,I))$, given by
$\varphi^{D^G F}{}_X(\mu)(x') = D^G F(x')(\mu) = G(x',.) \circ \mu \in D^G F(I)$
and $j^F{}_X(\mu) = \mu_I$ for $\mu \in D^G F(X)$, x' in X'.
$\varphi^{D^G F}$ and j^F are easily seen to be natural in F and even in G and contractive.

<u>Proposition:</u> If G is total on $\underline{K}$ then both $\varphi^{D^G F}$ and j^F are injective.

<u>Proof:</u> $0 = \varphi^{D^G F}{}_X(\mu)(x') = G(x',.) \circ \mu$ for all $x' \in X'$ implies $G(x',Y') \circ \mu_Y = 0$ for all $y' \in Y'$, $Y \in \underline{K}$, thus $\mu_Y = 0$ for all $Y \in \underline{K}$, so $\mu = 0$.
Similarly the commutative diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\mu_Y} & G(X,Y) \\ {\scriptstyle F(y')}\downarrow & & \downarrow{\scriptstyle G(X,y')} \\ F(I) & \xrightarrow{\mu_I} & G(X,I) \end{array}$$

shows that $\mu_Y = 0$ for all $Y \in \underline{K}$ if $\mu_I = 0$ for $\mu \in D^G F(X)$.

1.7. <u>Theorem</u>: Let F be a functor. If G is total, then $D^G F$ is D^G-reflexive if and only if

$$\iota^{D^G F}{}_I : D^G F(I) \to (D^G)^3 F(I)$$ is an epimorphism in Ban.

<u>Proof</u>: The stated condition is clearly necessary. If conversely $\iota^{D^G F}{}_I$ is epi, then we may suppose that it is an isometric isomorphism since $\iota^{D^G F}$ is always isometric by 1.5. Thus its left inverse $D^G(\iota^F)_I$ (see 1.5) is also an isometric isomorphism and so is a right inverse, i.e. we have

$$\iota^{D^G F}{}_I \circ D^G(\iota^F)_I = 1_{(D^G)^3 F(I)}$$

It suffices to show that $\iota^{D^G F} \circ D^G(\iota^F) = 1_{(D^G)^3 F}$, since the converse is always true by 1.5, and for that it is enough to show that

$\varphi^{(D^G)^3 F} \circ \iota^{D^G F} \circ D^G(\iota^F) = \varphi^{(D^G)^3 F}$, since by 1.6 $\varphi^{(D^G)^3 F}$ is mono. Let $\alpha \in (D^G)^3 F(X)$ and $x' \in X'$:

$$\varphi^{(D^G)^3 F}{}_X(\iota^{D^G F}{}_X(D^G(\iota^F)_X(\alpha)))(x') =$$

$$= (D^G)^3 F(x')(\iota^{D^G F}{}_X(D^G(\iota^F)_X(\alpha))) =$$

$$= [\iota^{D^G F}{}_I \circ D^G F(x') \circ D^G(\iota^F)_X](\alpha) ,$$

since $\iota^{D^G F} : D^G F \to (D^G)^3 F$ is natural,

$$= [\iota^{D^G F}{}_I \circ D^G(\iota^F)_I \circ (D^G)^3 F(x')](\alpha) ,$$

since $D^G(\iota^F) : (D^G)^3 F \to D^G F$ is natural,

$$= [1_{(D^G)^3 F(I)} \circ (D^G)^3 F(x')](\alpha)$$

$$= \varphi^{(D^G)^3 F}{}_X(\alpha)(x').$$

1.8. Definition: Let A be a Banach space. A subfunctor $\Lambda(.',A)$ of $H(.',A)$ is a functor $X \to \Lambda(X',A)$ together with an injective natural transformation $\Lambda(.',A) \to H(.',A)$ such that $\Lambda(I,A) = H(I,A) = A$. We may consider $\Lambda(.',A)$ to be an algebraic partial functor of $H(.',A)$.

We can derive from this definition the following properties of subfunctors:

(i) $\|f \circ g'\|_\Lambda \leq \|f\|_\Lambda . \|g'\|$, $f \in \Lambda(X',A)$, $g \in H(X,Y)$, because $\Lambda(.',A)$ is a functor.

(ii) $\Lambda(X',A) \supseteq X \otimes A$, the space of all finite-dimensional weak*- continuous maps $X' \to A$, because $\sum_{i=1}^{n} \hat{a}_i < x_i, . > = \Sigma\, \hat{a}_i \circ (\hat{x}_i)' = \Sigma\, \Lambda((\hat{x}_i)',A)\hat{a}_i$ and $A = \Lambda(I,A)$.

(iii) $\|.\|_\Lambda$ is a reasonable crossnorm (II,2.1) on $X \otimes A$ by IV, 1.8.

By 1.6 we may consider $D^G F$ to be a subfunctor of $H(.',D^G F(I))$ if G is total.

1.9. If G is a symmetric co-covariant bifunctor on $\underline{K} \times \underline{K}$ we may consider the contractive natural map $\varphi^G_{XY} : G(X,Y) \to H(X' \hat{\otimes} Y', G(I,I))$, defined by $\varphi^G_{XY}(g)(\Sigma\, x_i' \otimes y_i') = \Sigma\, G(x_i',y_i')g$ (compare IV, 2.9 where we considered a similar map). Clearly $G(I,I) = H(I \hat{\otimes} I, G(I,I))$, thus we may consider G as a subfunctor of $H((.') \hat{\otimes} (..'), G(I,I))$ in the obvious generalization of the notion of 1.7, if G is total, since then φ^G is injective.

If $F : \underline{K} \to \text{Ban}$ is any functor, we consider the natural contractive map $\varphi^F : F \to H(.',F(I))$, given by $\varphi^F_X(f_X)(x') = F(x')f_X$ (compare IV, 1.14) and consider its canonical factorization (I, 1.6)

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi^F_X} & H(X',F(I)) \\ \text{coim}\,\varphi^F_X \downarrow & & \uparrow \text{im}\,\varphi^F_X \\ \Lambda^F(X',F(I)) & \xrightarrow{\tilde{\varphi}^F_X} & \overline{\varphi^F_X(F(X))} \end{array},$$

where $\Lambda^F(X',F(I)) = F(X)/(\varphi^F_X)^{-1}(0)$; it is clear that $\Lambda^F(.',F(I))$ becomes a quotient functor of F (compare IV, 1.9, where we used a similar construction), which is a subfunctor of $H(.',F(I))$, called the subfunctor associated to F.

<u>Lemma</u>: For $f_Y \in F(Y)$ and $\mu \in D^G F(X)$ we have $\varphi^G_{XY}(\mu_Y(f_Y)) =$
$\varphi^G_{X,I} \circ \mu_I \circ \varphi^F_Y(f_Y)$ in $H(X' \hat{\otimes} Y', G(I,I)) = H(Y', H(X',G(I,I)))$.

<u>Proof</u>: Let us denote by ^{s} the isomorphism

$$^s : H(X' \hat{\otimes} Y',G(I,I)) \to H(Y',H(X',G(I,I))).$$

Then we have

$$\varphi^G_{XY}(\mu_Y(f_Y))(x' \otimes y') = G(x',y')\mu_Y(f_Y) =$$
$$= G(x',I)G(X,y')\mu_Y(f_Y) = G(x',I)\mu_I(F(y')f_Y))$$
$$= [G(x',I) \circ \mu_I \circ (\varphi^F_Y(f_Y))](y') =$$
$$= \varphi^G_{XI}(\mu_I((\varphi^F_Y(f_Y))y')(x' \otimes 1)$$
$$= [\varphi^G_{XI} \circ \mu_I \circ \varphi^F_Y(f_Y)](y')(x')$$
$$= {}^s[\varphi^G_{XI} \circ \mu_I \circ \varphi^F_Y(f_Y)](x' \otimes y') .$$

1.10. If G is total, then lemma 1.9 implies that any $\mu \in D^G F(X)$ factors through φ^F , i.e. $D^G F(X)$ is determined by the associated quotient functor $\Lambda^F(.',F(I))$ of F. So we have:

<u>Corollary</u>: If G is total then we have for any F:

$$D^G F = D^G(\Lambda^F(.',F(I))), \text{ and } F(I) = (0) \text{ implies } D^G F = (0).$$

1.11 It remains to compute D^G on arbitrary subfunctors $\Lambda(.',A)$ of $H(.',A)$ if G is total.

<u>Theorem</u>: Let G be a total symmetric bifunctor on $\underline{K} \times \underline{K}$ and $\Lambda(.',A)$ be any subfunctor of $H(.',A)$. Then $D^G\Lambda(.',A)(X)$ consists of all $f \in H(A,G(X,I))$ (via $j_X^\wedge$) which fulfill the following conditions:

a) $f \bullet g \in G(X,Y)$ (via φ_{XY}^G) for all $g \in \Lambda(Y',A)$, $Y \in \underline{K}$.

b) $\|f \circ g\|_{G(X,Y)} \leq \rho \cdot \|g\|_\Lambda$ for all $g \in \Lambda(Y',A)$, $Y \in \underline{K}$.

We then have $\|f\|_{D^G\Lambda(.',A)(X)} = \inf \rho$.

<u>Proof</u>: If $\mu \in D^G\Lambda(.',A)(X)$, then by 1.6 $j_X^\wedge(\mu) = \mu_I \in H(A,G(X,I))$ injectively and by lemma 1.9 conditions (a) and (b) are fulfilled.

If conversely $f \in H(A,G(X,I))$ has the stated properties, then $g \mapsto f \bullet g$ (forgetting $(\varphi_{XY}^G)^{-1} \circ \varphi_{XI}^G$) defines a map: $\Lambda(Y',A) \to G(X,Y)$ of norm less than ρ and it acts on $\hat{a} \in \Lambda(I,A)$, $a \in A$ by $\hat{a} \to f \bullet \hat{a} = \widehat{f(a)}$. Thus its image under $j_X^\wedge$ is just f. This map is natural in Y, since for $h : Y \to Z$ we have

$\Lambda(G',A)g = g \circ h' \mapsto f \circ g \circ h' =$
$= H(h',G(X,I))(f \circ g)$ and $(\varphi^G_{XY})^{-1} \circ \varphi^G_{XI}$ is natural. Lemma 1.9 now concludes the proof that the two constructions are inverse to each other.

<u>Remark</u>: (i) In this theorem we identified $D^G\Lambda(.',A)(X)$ with its image under the injective map $j^{\wedge}_X$ in $H(A,G(X,I))$ (1.6) and $G(X,Y)$ with its image under φ^G_{XY} in the space $H(Y',H(X',G(I,I))) = H(X' \hat{\otimes} Y',G(I,I))$ (1.9).

(ii) If $\underline{K}$ = Ban or the category $\underline{A}$ of spaces with metric approximation property or the category Hilb of Hilbert spaces or the category $\underline{R}$ of reflexive spaces or similar categories, then condition (a) implies (b) by the following argument: If (a) holds but (b) does not, then there are spaces $Y_i \in \underline{K}$, $i \in \mathbb{N}$, $g_i \in \Lambda(Y_i',A)$ with $\|g_i\|_\Lambda \le \frac{1}{2^i}$ such that $\|f \circ g_i\|_{G(X,Y_i)} \ge i$.

Let $Y = \sum_i Y_i$ if $\underline{K}$ = Ban or $\underline{A}$,

$Y = \sum_i^{(2)} Y_i = \{(y_i),\ y_i \in Y_i : \|(y_i)\| = (\sum_i \|y_i\|^2)^{1/2} < \infty\}$

if $\underline{K}$ is Hilb or $\underline{R}$, and

$g = \sum_i g_i \circ \pi_i$, where $\pi_i : Y \to Y_i$ is the projection.

Then $\|g\|_\Lambda \le \Sigma \|g_i\|_\Lambda \|\pi_i\| \le 2$ so $g \in \Lambda(Y,A)$ by completeness, but $\|f \circ g\|_{G(X,Y)} \ge \|f \circ g \circ i_i\|_{G(X,Y_i)} = \|f \circ g_i\|_{G(X,Y_i)} \ge i$ for all i, where $i_i : Y_i \to Y$ is the injection, so $f \circ g \notin G(X,Y)$, a contradiction to (a).

1.12. We consider now a bifunctor $G : \underline{K} \times \underline{K} \to$ Ban with $G(I,I) = I$. Then the following assertions hold:

The map $\varphi^G_{XY} : G(X,Y) \rightarrow H(X',Y'')$, defined by $< y', \varphi^G_{XY}(g)(x') > \; = G(x',y')g$, $g \in G(X,Y)$ is natural in X,Y,G and contractive; it is the map φ^G_{XY} of 1.9, followed by the isomorphism $H(X' \hat{\otimes} Y', G(I,I)) = H(X',Y'')$. If G is symmetric, then φ^G is compatible with the symmetries of G and $H(X',Y'')$ respectively, i.e. $\varphi^G_{XY}({}^t g) = {}^t(\varphi^G_{XY}(g))$. If G is total, then φ^G_{XY} is injective and we may consider $G(X,Y)$ as a "subfunctor" of $H(X',Y'')$. The diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\;\epsilon\;} & G(X,Y) \\ & \searrow^{\tau} & \downarrow \varphi \\ & & H(X',Y'') \end{array}$$

commutes, where

ϵ is the embedding of the algebraic tensor product (IV, 1.2.4) and τ is given by $\tau(\Sigma\, x_i \otimes y_i) = \iota_Y \bullet \Sigma\, \hat{y_i} \circ < x_i, . >$.

<u>Proof</u>: $< x', {}^t(\varphi^G_{XY}(g))(y') > \; = \; < y', \varphi^G_{XY}(g)(x') >$

$= G(x',y')g = {}^t(G(x',y')g) = G(y',x')({}^t g) =$

$= \; < x', \varphi^G_{XY}({}^t g)(y') >$, since ${}^t : G(I,I) = I \rightarrow I$ is 1_I without loss of generality: if not then it is -1_I and we may consider $-t$ instead of t . The other assertions are clear.

<u>Definition</u>: In analogy to IV,2,11 a symmetric co-covariant bifunctor G on $\underline{K} \times \underline{K}$ is said to be of type A if $G(I,I) = I$ and $G(X,I) = X$, $G(I,X) = X$ via the maps

$$X = I \hat{\otimes} X \xrightarrow{\epsilon} G(I,X) \xrightarrow{\varphi} H(I,X'') = X''$$

$$X = X \hat{\otimes} I \xrightarrow{\epsilon} G(X,I) \xrightarrow{\varphi} H(X',I) = X'' ,$$

i.e. $\varphi \circ \epsilon = \iota_X$.

We did not suppose G to be total (as in IV 2.11 in the definition of functors of type (I)), since our main example $X \hat{\otimes} Y$ is not total on Ban.

If G is symmetric of type A, then the map $\varphi^G_{XY} : G(X,Y) \to H(X',Y'')$ factors over $G(X,Y) \to H(X',Y) \to H(X',Y'')$ where the first map, again called φ^G_{XY} , is given by $< \varphi^G_{XY}(g)(x'),y' > = G(x',y')g$, and this expression is weak*-continuous in y' on the unit ball OY' of Y' for fixed x' and g. Furthermore the image of φ^G_{XY} consists of weak*- weak - continuous mappings $X' \to Y$.

Proof: That im $\varphi^G_{XY} \subseteq H(X',Y) \subseteq H(X',Y'')$ is seen by factoring φ^G_{XY} in the following way:
$G(X,Y) \to H(X',G(I,Y)) = H(X',Y) \subseteq H(X',Y'')$ (compare 1.9).
Now if $\varphi^G_{XY}(g) \in H(X',Y)$ for $g \in G(X,Y)$, then its adjoint $(\varphi^G_{XY}(g))' : Y' \to X''$ maps Y' into X, since it coincides with the transposed mapping in $H(Y',X'')$: ${}^t(\varphi^G_{XY}(g)) = \varphi^G_{YX}({}^tg)$, and is therefore again an element of $H(Y',X)$.
This property is equivalent to the fact that $\varphi^G_{XY}(g)$ is weak*- weak - continuous as a map $X' \to Y$.

1.13. From now on let G be a symmetric bifunctor of type A.

<u>Lemma:</u> For all $F: \underline{K} \to \mathrm{Ban}$ the map $j_I^F: D^G F(I) \to F(I)'$ is isometric.

<u>Proof:</u> Let $\mu \in D^G F(I) = \mathrm{Nat}(F, \mathrm{Id})$. Then

$$\|\mu_X\| = \sup_{\|f_X\|<1} \|\mu_X(f_X)\|_X$$
$$= \sup_{\|f_X\|\leq 1} \sup_{\|x'\|\leq 1} |\langle \mu_X(f_X), x'\rangle|$$
$$= \sup_{\|f_X\|\leq 1} \sup_{\|x'\|\leq 1} |\mu_I(F(x')f_X)| \leq \|\mu_I\|.$$

Thus $\|\mu\| = \sup_X \|\mu_X\| \leq \|\mu_I\| = \|j_I^F(\mu)\|_{F(I)'}$.

1.14. <u>Lemma:</u> If G is symmetric, of type A and total, then $(D^G)^2 F(X)$ contains the space

$\{f \mid \mathrm{im}\, j_I^F,\ f \in G(F(I),X) \subseteq H(F(I)',X)\}$

via the injective map $j_X^{D^G F}: (D^G)^2 F(X) \to H(D^G F(I), X)$ for all functors F.

<u>Proof:</u> $j_X^{D^G F}$ is injective by 1.6. For $\mu \in D^G F(Y)$ we have

$[j_I^F \circ \varphi_Y^{D^G F}(\mu)](y') = j_I^F(D^G F(y')(\mu)) =$

$j_I^F\ (G(y', .) \circ \mu) = G(y', I) \circ \mu_I = y' \circ \mu_I = (\mu_I)'(y')$.

So $j_I^F \circ \varphi_Y^{D^G F}(\mu) = (\mu_I)': Y' \to F(I)'$.

If $f \in G(F(I),X) \subseteq H(F(I)',X)$, then for all $g \in \Lambda^{D^G F}(Y', D^G F(I))$, i.e. for all $g = \varphi_Y^{D^G F}(\mu)$, $\mu \in D^G F(Y)$ we have

$(f|\text{im } j_I^F) \circ g = f \circ j_I^F \circ \varphi_Y^{D^G F}(\mu) =$

$= f \circ (\mu_I)' = G(\mu_I,X)(f) \in G(Y,X) \subseteq H(Y',X)$, since $G(Y,X)$ is a subfunctor of $H(Y',X)$.

This is condition (a) of 1.11 for $f|\text{im } j_I^F$;

(b) is fulfilled too, since

$\|(f|\text{im } j_I^F) \circ g\|_G = \|G(\mu_I,X)(f)\|_G \leq \|\mu_I\|\|f\|_{G(F(I),X)} \leq$

$\leq \|\mu\|\|f\|_G$, so $\|f\|_{(D^G)^2F(X)} \leq \|f\|_G$.

1.15. __Theorem__: Let G be symmetric, of type A and total. If F is a functor such that $F(I)$ is a reflexive Banach space, then D^GF is D^G-reflexive.

__Proof__: By 1.7 we have only to show that

$\iota_I^{D^GF}: D^GF(I) \to (D^G)^3F(I)$ is epi . If $F(I)$ is a reflexive Banach space, then the isometric subspace

$B: = D^GF(I) \subseteq F(I)'$ (1.13) is reflexive too. Again by 1.13 the space $(D^G)^2F(I)$ is an isometric subspace of B' via $j_I^{D^GF}$ and by 1.14 $(D^G)^2F(I)$ contains the space

$\{f|B \text{ , } f \in G(F(I),I) = F(I) = F(I)''\} = F(I)''/B^{\perp} = B'$,

again via $j_I^{D^GF}$. Thus $(D^G)^2F(I) = B'$. Repeating this argument we obtain $(D^G)^3F(I) = B'' = B$.

For $\mu \in D^GF(I)$, $\nu \in (D^G)^2F(I)$ we have $\iota_I^{D^GF}(\mu)_I(\nu) =$

$= {}^t(\nu_I(\mu)) = \nu_I(\mu) \in I$, i.e.

$j_I^{(D^G)^2F} \circ \iota_I^{D^GF} = \iota_B \circ j_I^F$, or the diagram

$$\begin{array}{ccc} D^G F(I) & \xrightarrow{\;j_I^F\;} & B \subseteq F(I)' \\ {\scriptstyle \iota_I^{D^G F}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_B} \\ (D^G)^3 F(I) & \xrightarrow[\;j_I^{(D^G)^2 F}\;]{} & B'' \end{array}$$

commutes, where $\iota_B : B \to B''$ is the canonical map. Therefore the equality $(D^G)^3 F(I) = B'' = B = D^G F(I)$ is established by $\iota_I^{D^G F}$ and so this map is an isomorphism.

1.16. <u>Example:</u> Let G be symmetric, of type A, and total. Then H_A and $G(A,.)$ are D^G-dual to each other and D^G-reflexive. Furthermore $D^G(A \hat{\otimes} .) = H_A$.

<u>Proof:</u> $D^G(A \hat{\otimes} .)(X) = \mathrm{Nat}(A \hat{\otimes} ., \; G(X,.)) = H(A, G(X,I)) = H(A,X)$ naturally in A,X and G by IV, 1.5.

$D^G(H_A)(X) = \mathrm{Nat}(H_A, G(X,.)) = G(X,A)$ naturally in A,X and G by IV, 1.4.

$j_X^{G(A,.)} : D^G G(A,.)(X) \to H(A,X)$ is contractive and injective (see 1.6.). If $f \in H(A,X)$, then $G(f,.) : G(A,.) \to G(X,.)$ is a natural transformation, $\|G(f,.)\| \leq \|f\|$, and $G(f,I) = f$, so $j_X^{G(A,.)}$ is isometric and onto. To conclude the proof we show that e.g.

$$H(A,I) \xrightarrow{\;\iota_I^{H_A}\;} (D^G)^2 H_A(I) = D^G G(.,A)(I) \xrightarrow{\;D^{G(t)}{}_I\;} D^G G(A,.)(I)$$

$\xrightarrow{\;j_I^{G(A,.)}\;} H(A,I)$ is the identity, then $\iota_I^{H_A}$ is isometric onto since $j^{G_A}_I$ is and we may invoke 1.7.

Let $a' \in A' = H(A,I)$ and $a \in A$.

$< a, [j^{G(A,.)}_I \cdot D^G(^t)_I \cdot \iota^{H_A}_I](a') > =$

$= [D^G(^t)_I \cdot \iota^{H_A}_I (a')]_I(a) = \iota^{H_A}_I(^t a')_I(a) = < a,a' >$, using the last argument of the proof of 1.15.

§ 2. The dual functor of Mitiagin-Shvarts

The duality D, studied by Shvarts [79], Mitiagin & Shvarts [57], Pothoven [67], Wick-Negrepontis [58], Herz & Pelletier [35], Michor [51] is the form of the Eckmann-Hilton duality valid in the category Ban_1. Let $\underline{K}$ again be a full subcategory of Ban which contains I.

2.1. Definition: $D: (Ban^{\underline{K}})^{op} \to Ban^{\underline{K}}$ is given by taking $X \hat{\otimes} Y$ for the symmetric bifunctor G. So $DF(X) = \operatorname{Nat}_{\underline{K}} (F, X \hat{\otimes} .)$.
$X \hat{\otimes} Y$ is a symmetric bifunctor of type A, but is not total in general (cf. II, § 3).
There should be no confusion possible between D as used here and the general duality D in the beginning of § 1.

2.2. We immediately compute examples:
$DH_A = A \hat{\otimes} .$, $D(A \hat{\otimes} .) = H_A$ and both are (D-) reflexive (1.16). Taking $A = l^1$, then $l^1 \hat{\otimes} X = l^1(X)$ (II,1.8) and $D[l^1(.)] = H_{l^1} = l^\infty(.)$. Thus the functors $l^1(.)$ and $l^\infty(.)$ (IV,1.19) are dual to each other and reflexive. We see that D-duality does not resemble the duality of Banach spaces but looks rather like the Köthe dual or the associated module of III, 3.18. For that reason Mitiagin & Shvarts [57] conjectured that any functor of the form DF was reflexive. This turns out to be true if $\underline{K} \subset \underline{A}$, the category of all Banach spaces with the metric approximation property, and F(I) is reflexive (1.15). In general this conjecture is wrong: the next section is devoted to a counterexample.

2.3. Example: Let n be a sequence space (III, 1.7) and let $n(.) : \underline{K} \to Ban$ be the functor of IV, 1.19. Then $D(n(.)) = n^{\circ}(.)$, the functor, defined by the sequence space $n^{\circ} = H_{c_0}(n, l^1)$ (see III, 3.18 and III, 3.22). We have $n^{\circ}(X) = H_{c_0}(n, l^1(X))$.

Proof: By 2.2 we have $Dl^{\infty} = l^1$. Then

$D(n(.))(X) = Nat(n(.), X \hat{\otimes} .) =$

$= Nat\ (l^{\infty}(.) \hat{\otimes}_{l^{\infty}} n, X \hat{\otimes} .)$, using IV, 1.19,

$= H_{l^{\infty}}(n, Nat(l^{\infty}(.), X \hat{\otimes} .)$, using the "pointwise" equality $H(l^{\infty}(Y) \hat{\otimes}_{l^{\infty}} n, X \hat{\otimes} Y) = H_{l^{\infty}}(n, H(l^{\infty}(Y), X \hat{\otimes} Y))$ of III, 3.9, 4); we continue then:

$= H_{l^{\infty}}(n, D(l^{\infty}(.))(X)) =$

$= H_{l^{\infty}}(n, l^1(X)) = n^{\circ}(X)$ by III, 3.22.

2.4. Example: Let N be a function space (III, 1.8) and let $N_o(.) : \underline{K} \to Ban$ be the functor considered in IV, 1.21. Then $D(N_o(.)) = S(N,.)$, the functor considered in IV, 1.22, via $j^{N_o(.)}$.

Proof: We first prove this for the function space L^{∞}; we assert that $D(L_o^{\infty}(.)) = S(L^{\infty},.)$.

$j = j_X^{L_o^{\infty}(.)} : D(L_o^{\infty}(.))(X) \to H(L^{\infty},X)$, $j(\eta) = \eta_I$,

is injective, since $L_o^{\infty}(.)$ is of type Σ (compare the beginning of the proof of 2.5). Let $X = l_n^1$, let

$(f_1,\ldots,f_n) \in L^{\infty}(l_n^1)$. Then $\varphi_{l_n^1}((f_1,\ldots,f_n)) =$

$= (\varphi_I(f_1),\ldots, \varphi_I(f_n)) \in X \hat{\otimes} l_n^1 = l_n^1(X)$.

So $\Sigma\|\varphi_I(f_j)\|_X = \|(\varphi_I(f_1),\ldots,\varphi_I(f_n))\|_{A \hat{\otimes} l^1_n} \leq$

$\leq \|\varphi\|_{DL_o^\infty(.)} \|(f_1,\ldots,f_n)\|_{L^\infty(l^1_n)} = \|\varphi\|_{DL_o^\infty(.)} \|\Sigma|f_k|\|_{L^\infty}$.

So $\varphi_I \in S(L^\infty,X)$ and $\|\varphi_I\|_{S(L^\infty,X)} \leq \|\varphi\|_{DL_o^\infty(.)}$.

Now let conversely $g \in S(L^\infty,X)$. Then we define

$\varphi_Z : L_o^\infty(Z) \to X \hat{\otimes} Z$ by

$\varphi_Z(\Sigma\, z_k f_k) = \Sigma\, z_k \otimes g(f_k)$ (compare IV.1.21). Then

$\Sigma\|x_k\|\|g(f_k)\| = \Sigma\,\|g(\|x_k\|f_k)\|_X \leq$

$\leq \|g\|_{S(L^\infty,X)} \|\Sigma\|x_k\|f_k\|_{L^\infty}$. So φ_Z may be continuously extended to the whole of $L_o^\infty(X)$ with $\|\varphi_Z\| \leq \|g\|_{S(L^\infty,X)}$.

Clearly the family (φ_Z) defines a natural transformation $\varphi: L_o^\infty(.) \to X \hat{\otimes}\ .$ with $\varphi_I = g$.

Now we prove the general case.

$D(N_o(.))(X) = Nat(L_o^\infty(.) \hat{\otimes}_{L^\infty} N,\ X \hat{\otimes}\ .)$, using IV, 1.21, and by the same argument as in 2.3, applying III, 3.9, 4) pointwise, this equals $H_{L^\infty}(N, Nat(L_o^\infty(.), X \hat{\otimes}\ .)) =$ $H_{L^\infty}(N, DL_o^\infty(.)(X)) = H_{L^\infty}(N, S(L^\infty,X)) = S(N,X)$ by III, 2.7.

2.5. <u>Theorem</u>: If $F : \underline{K} \to Ban$ is a functor of type Σ, then for any $X \in \underline{K}$ such that $X' \in \underline{K}$ we have $DF(X) =$ $= \{f \in H(F(I),X) : \iota_X \circ f \in F(X')'\}$ via j_X^F, where $\iota_X: X \to X''$ is the canonical embedding and $\|f\|_{DF(X)} =$ $= \|\iota_X \circ f\|_{F(X')'}$.

<u>Proof</u>: We consider $F(X')'$ as a subspace of $H(F(I),X'')$ in a natural way via $(\epsilon^F_{X'})'$, as in IV, 1.15. So $F(X')'$ consists

of all $h \in H(F(I),X'')$ which define a bounded linear functional on $F(X')$ by $\Sigma a_i \otimes x_i' \to \Sigma < x_i', h(a_i) >$, $\Sigma a_i \otimes x_i' \in F(I) \otimes X' \subseteq F(X')$.

The map $j = j_X^F : DF(X) = Nat(F, X \hat{\otimes} .) \to H(F(I),X)$, defined by $j(\eta) = \eta_I$, is clearly contractive and injective since F is of type Σ : let $\eta \in DF(X)$ and $\eta_I = 0$. For $z \in Z \in \underline{K}$ the diagram

$$\begin{array}{ccc} F(I) & \xrightarrow{\eta_I} & X \hat{\otimes} I = X \\ {\scriptstyle F(\hat{z})}\downarrow & & \downarrow {\scriptstyle 1_X \hat{\otimes} \hat{z}} \\ F(Z) & \xrightarrow{\eta_Z} & X \hat{\otimes} Z \end{array}$$

commutes, so for all $a \in F(I)$ we have $\eta_Z F(\hat{z})a = (1_X \hat{\otimes} \hat{z})\, \eta_I(a) = 0$.

Thus $\eta_Z (\sum_i F(\hat{z}_i)a_i) = 0$ for all $\Sigma a_i \otimes z_i \in F(I) \otimes Z$, the latter space is dense in $F(Z)$, $\eta_Z = 0$ and since Z was arbitrary, $\eta = 0$.

Now take any $\eta \in Nat(F, X \hat{\otimes} .)$. Then we assert that $\iota_X \circ \eta_I \in F(X')'$. The diagram

$$\begin{array}{ccc} F(I) & \xrightarrow{\eta_I} & X \\ {\scriptstyle F(\hat{x}')}\downarrow & & \downarrow {\scriptstyle 1_X \hat{\otimes} \hat{x}'} \\ F(X') & \xrightarrow{\eta_{X'}} & X \hat{\otimes} X' \end{array}$$

commutes for all $x' \in X'$. Let $tr : X \hat{\otimes} X' \to I$ be the trace functional (II,1.10), corresponding to $\iota_X \in H(X,X'') = (X \hat{\otimes} X')'$, $tr\,(x \otimes x') = < x,x' >$. Then for all $a \in F(I)$ and $x' \in X'$ we have: $< \eta_I(a),x' > = tr\,(\eta_I(a) \otimes x') =$

$= tr \circ (1_X \hat{\otimes} \hat{x}') \circ \eta_I(a) = tr \cdot \eta_{X'} \circ F(\hat{x}')(a)$.

For $\sum_{i=1}^{n} a_i \otimes x_i' \in F(I) \otimes X'$ we compute:

$$< \Sigma\, a_i \otimes x_i',\ \iota_X \circ \eta_I > = \Sigma < x_i', \iota_X \circ \eta_I(a_i) >$$

$$= \Sigma < \eta_I(a_i), x_i' > = \Sigma\ tr \circ \eta_{X'} \circ F(\hat{x}_i')(a_i)$$

$$= tr \circ \eta_{X'} \circ \epsilon^F_{X'}\ (\Sigma\, a_i \otimes x_i')\ ,$$

where $\epsilon^F_{X'} : F(I) \otimes X' \to F(X')$ is the map of IV,1.6.

Thus $|< \Sigma\, a_i \otimes x_i',\ \iota_X \circ \eta_I >| = |tr \circ \eta_{X'} \circ \epsilon^F_X(\Sigma\, a_i \otimes x_i')|$

$\leq \|tr\| \|\eta_{X'}\| \|\Sigma\, F(\hat{x}_i')a_i\|_{F(X')}$,

i.e. $\|\iota_X \circ \eta_I\|_{F(X')'} \leq \|\eta_{X'}\| \leq \|\eta\|$.

Let us suppose conversely that we have $f \in H(F(I),X)$ with $\iota_X \circ f \in F(X')'$. For any $Z \in \underline{K}$ we define

$(\theta f)_Z : F(I) \otimes Z \to X \otimes Z$ by

$(\theta f)_Z(\Sigma\, a_i \otimes z_i) = \Sigma\, f(a_i) \otimes z_i$.

$$\|(\theta f)_Z(\Sigma\, a_i \otimes z_i)\|_{X \hat{\otimes} Z} = \|\Sigma\, f(a_i) \otimes z_i\|_{X \hat{\otimes} Z}$$

$$= \sup_{h \in OH(Z,X')} |< \Sigma\, z_i \otimes f(a_i), h >|$$

$$= \sup_{h \in OH(Z,X')} |\Sigma < h(z_i),\ \iota_X \circ f(a_i) >|$$

$$= \sup_{h \in OH(Z,X')} |< \Sigma\, h(z_i) \otimes a_i,\ \iota_X \circ f >|$$

$$\leq \sup_{h \in OH(Z,X')} \|\Sigma\, F(\widehat{h(z_i)})a_i\|_{F(X')} \|\iota_X \circ f\|_{F(X')'}$$

$$= \sup_{h \in OH(Z,X')} \|F(h)\ \Sigma\, F(\hat{z}_i)a_i\|_{F(X')} \|\iota_X \circ f\|_{F(X')'}$$

$$\leq \|\Sigma\, F(\hat{z}_i)a_i\|_{F(Z)} \|\iota_X \circ f\|_{F(X')'}.$$

Thus $(\theta f)_Z$ extends to a continuous map : $F(Z) \to X \hat{\otimes} Z$ with $\|(\theta f)_Z\| \leq \|\iota_X \circ f\|_{F(X')'}$. By the naturality of the counit ϵ^F it is very easily seen that $((\theta f)_Z)$ is a natural transformation $\theta f\colon F \to X \hat{\otimes} .$. Clearly we have $(\theta f)_I = f$ and since $j^F\colon DF \to H(F(I),.)$ is natural, we are done.

2.6. <u>Proposition:</u> If $F\colon \underline{K} \to \mathrm{Ban}$ is of type Σ and if $X, X' \in \underline{K}$ and X' has the metric approximation property, then $DF(X') = F(X)'$ via j^F_X.

<u>Proof:</u> We first remark that $X' \hat{\otimes} . = (X \hat{\hat{\otimes}} .')'_e$ since X' has the metric approximation property. This is seen using the following commutative diagram

$$\begin{array}{ccccc}
X' \hat{\otimes} Y & & \xrightarrow{\quad a \quad} & & I_1(X'',Y) \\
\downarrow & & & & \downarrow c \\
I_1(X,Y'') & = & I_1(Y',X') & \xrightarrow{\ b\ } & I_1(Y',X''')
\end{array}$$

where a is isometric by II, 3.9. c), b and c are isometries by II, 2.9.

So we may compute

$$\begin{aligned}
DF(X') &= \operatorname{Nat}_{\mathrm{Ban}}(F, X' \hat{\otimes} .) \\
&= \operatorname{Nat}_{\mathrm{Ban}}(F, (X \hat{\hat{\otimes}} .')'_e) \\
&= \operatorname{Nat}_{\mathrm{Ban}}(F, (X \hat{\hat{\otimes}} .')') \quad \text{since } F \text{ is of type } \Sigma \text{ by IV, 1.12.} \\
&= ((X \hat{\hat{\otimes}} .') \mathop{\hat{\otimes}}_{\mathrm{Ban}} F)' \\
&= F(X)' \quad \text{by IV, 3.13. since } X \text{ has the metric}
\end{aligned}$$

approximation property (II, 3.11.).

2.7. The next two results belong to duality theory of contravariant functor which we do not develop; however, they are of independent interest.

Proposition: Let $G : \underline{K}^{op} \to Ban$ be a contravariant functor of type Σ. Then for any $X \in \underline{K}$ such that $X' \in \underline{K}$ we have

$$\mathrm{Nat}_{\underline{K}}(G, X \hat{\otimes} .') = \{f \in H(G(I),X) : \iota_X \circ f \in G(X)'\}$$

via j_X^G where $\iota_X : X \to X''$ is the canonical embedding and $\|f\|_{DF(X)} = \|\iota_X \circ f\|_{G(X)'}$.

The proof is the same as that of 2.5 with the obvious changes.

2.8. Proposition: Let $G : \underline{K}^{op} \to Ban$ be a contravariant functor of type Σ, let $X, X' \in \underline{K}$ and suppose that X'' has the metric approximation property. Then $\mathrm{Nat}_{\underline{K}}(G, X'' \hat{\otimes} .') = G(X)'$.

The proof is the same as that of 2.6 with the obvious changes.

2.9. We recall that a contractive morphism $f : X \to Y$ is said to be a weak retract (II, 1.9), if there exists a map $h : X' \to Y'$ with $\|h\| \leq 1$ such that $f' \circ h = 1_{X'}$.

The connection between weak retracts and duality is described by:

Proposition: If F is of type Σ and $f : X \to Y$ is a weak retract, then the following diagram is a pullback in Ban_1:

$$\begin{array}{ccc} DF(X) & \xrightarrow{\ j_X\ } & H(F(I),X) \\ {\scriptstyle DF(f)}\downarrow & & \downarrow{\scriptstyle H(F(I),f)} \\ DF(Y) & \xrightarrow{\ j_Y\ } & H(F(I),Y) \end{array}$$

where $j(\eta) = \eta_I$.

<u>Proof</u>: j is injective since F is of type Σ by IV, 1.12. $H(F(I),f)$ and $DF(f)$ are isometries, the latter by II, 1.9. Since $H(F(I),f)$ is isometric and j_Y is injective, the pullback of the half-diagram is $j_Y^{-1}(H(F(I),f)H(F(I),X)) =$

$= \{\eta \in Nat(F, Y \hat{\otimes} .) : \eta_I(F(I)) \subseteq f(X)\}$

$= Nat\ (F,\ X \hat{\otimes} .)$ since $DF(f)$ is isometric too, by I, 1.14.

<u>Remark</u>: The fact that F is of type Σ is only used to derive injectivity of j. If F is any functor with the property that $j : DF \to H_{F(I)}$ is injective, then the proposition remains valid.

2.10. <u>Corollary</u>: If $F : Ban \to Ban$ is of type Σ, then we have for $f \in H(F(I),X)$:

$f \in DF(X)$ iff $\iota_X \circ f \in DF(X'')$ and

$\|f\|_{DF(X)} = \|\iota_X \circ f\|_{DF(X'')}$.

<u>Proof</u>: use 2.11 and the fact that $\iota_X : X \to X''$ is a weak retract.

§ 3. Integral and nuclear maps.

As an application of our preceding considerations we now will study, as a special example, the functor of integral maps and its dual functor. This example will also show some close relations to the theory of operator ideals [64].

3.1. Definition: a) A map $f \in H(X,Y)$ is called integral, if it defines a bounded functional on $X \hat{\otimes} Y'$. (see II.2.9). $I_1(X,Y)$ denotes the space of integral maps from X into Y, $\|f\|_{I_1}$ the norm of the corresponding functional.

b) A map $f \in H(X,Y)$ is called nuclear, if it belongs to the canonical image of $X' \hat{\otimes} Y$. $N_1(X,Y)$ denotes the space of nuclear maps from X into Y, $\|f\|_{N_1}$ the quotient norm with respect to the canonical map $X' \hat{\otimes} Y \to N_1(X,Y)$. $L^1(X',Y)$ will denote the space of all maps $f \in H(X',Y)$ which belong to the canonical image of $X \hat{\otimes} Y$ and $\|f\|_{L^1}$ the corresponding quotient norm.

3.2. Remark: It follows immediately from the definitions that $I_1(X,Y)$, $N_1(X,Y)$ and $L^1(X',Y)$ are Banach spaces and that they define bifunctors in X and Y. By 2.5 $I_1(X,.)$ coincides with the dual functor of the inductive tensor product $X \hat{\hat{\otimes}} .$. $L^1(X',Y)$ is the associated total functor of $X \hat{\otimes} Y$ (IV.2.9). Since $I_1(X,Y)$ defines a reasonable norm on its essential part (IV.1.8), it follows from II.2.3 that $N_1(X,Y)$ is a linear subspace of $I_1(X,Y)$ and that $\|f\|_{I_1} \leq \|f\|_{N_1}$. But we have already seen (compare II.3.4 and

II.3.9) that $N_1(X,Y)$ need not be isometrically contained in $I_1(X,Y)$.

3.3. Definition: A map $f \in H(X,Y)$ is called weakly compact if OX is mapped onto a relatively weakly compact subset of Y. Since OX is w^*-dense in OX'' and f'' is $\sigma(X'',X') - \sigma(Y'',Y')$ - continuous, it follows that f is weakly compact iff f'' maps X'' into Y.

3.4. If $f \in N_1(X,Y)$ and $\epsilon > 0$ there exist by II.1.8 g) sequences $(x_n') \subset OX'$ and $(y_n) \subset OY$ and scalars (λ_n) such that $\Sigma|\lambda_n| < \|f\|_{N_1} + \epsilon$ and $f(x) = \Sigma\, \lambda_n < x,x_n' > y_n$ holds. We shall now give a representation for integral maps, which may be interpreted as a continuous analogon.

Proposition: A map $f \in H(X,Y)$ is integral if and only if there exists a measure $\mu \in M(OX' \times OY'')$ such that

$$< f(x),y' > = \int_{OX' \times OY''} < x,x' > < y',y'' > d\mu(x',y'') \text{ holds.}$$

(OX' and OY'' carry their w^*-topologies). In this case $\|f\|_{I_1} \leq \|\mu\|$ and μ may be chosen such that equality holds.

Proof: Let $j : X \to C(OX')$ be the canonical embedding defined by $x \to (x' \to < x,x' >)$. Analogously $k : Y' \to C(OY'')$. j and k are both isometries and consequently $X \hat{\hat{\otimes}} Y'$ is isometrically contained in $C(OX') \hat{\hat{\otimes}} C(OY'') = C(OX' \times OY'')$ (II.2.2 and II.2.4). If $f \in I_1(X,Y)$ it defines a continuous functional on $X \hat{\hat{\otimes}} Y'$ by $x \otimes y' \to < f(x),y' >$. If we extend this functional to $C(OX' \times OY'')$ we get a measure

$\mu \in M(OX' \times OY'') = C(OX' \times OY'')'$ which satisfies the above equation and has the same norm. Conversely, if f is given as above, it follows immediately from the definition of the inductive norm (II.2.2) that f is integral and satisfies $\|f\|_{I_1} \leq \|\mu\|$.

3.5. Corollary: A map $f \in H(X,Y)$ is integral if and only if there exists a compact space T, a measure $\mu \in M(T)$, maps $g \in H(X,C(T))$, $k \in H(M(T),Y'')$ such that $\iota_Y \circ f = k \circ h \circ g$ where $h \in H(C(T),M(T))$ is defined by $\varphi \in C(T) \to \varphi d\mu \in M(T)$.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\iota_Y} & Y'' \\ {\scriptstyle g}\downarrow & & & & \uparrow{\scriptstyle k} \\ C(T) & & \xrightarrow[h]{\qquad} & & M(T) \end{array}$$

The maps may be chosen such that $\|f\|_{I_1} = \|g\|\|\mu\|\|k\|$.

Proof: If one takes $T = OX' \times OY''$, g the embedding $X \to C(OX') \to C(T)$ k the adjoint of the corresponding embedding of Y', the necessity follows from the preceding proposition. Conversely $< \psi, h(\varphi) > = \int_T \varphi(t)\psi(t)d\mu(t)$ and this defines a continuous functional on $C(T) \hat{\otimes} C(T) = C(T \times T)$. By II.2.9,Lemma,h is integral,consequently $\iota_Y \circ f \in I_1(X,Y'')$ and again by II.2.9 $f \in I_1(X,Y)$.

3.6. Corollary: Any integral map is weakly compact.

Proof: It suffices to show this for the map h defined above. Take $v = |\mu|$. L^1_v is contained isometrically in M(T) via

$\varphi \to \varphi dv$. It follows that h takes its values actually in L^1_V. Since the weak topology is inherited by subspaces, it suffices again to consider the canonical map h_1 of $C(T)$ into L^1_V. Now $h_1(OC(T)) \subset OL^\infty_V$ which is $\sigma(L^\infty, L^1)$ compact. Consequently it is also $\sigma(L^\infty, L^\infty)$ and $\sigma(L^1, L^\infty)$ compact, since $\sigma(L^1, L^\infty)$ is a Hausdorff topology weaker than $\sigma(L^\infty, L^1)$.

3.7. In II.2.9 we have seen that for an integral map f the maps f' and $\iota_Y \circ f$ are also integral and the converse holds too. Nuclear maps do not behave as well. If f is nuclear, f' and $\iota_Y \circ f$ are again nuclear but the converse may fail. In a similar way, a map $f \in N_1(X', Y)$, which is $\sigma(X', X) - \sigma(Y, Y')$ continuous, need not belong to $L^1(X', Y)$. But everything becomes much easier if we consider spaces with the approximation property.

<u>Proposition</u>: Assume that $f \in N_1(X', Y)$ is $\sigma(X', X) - \sigma(Y, Y')$ continuous and that Y has the approximation property. Then $f \in L^1(X', Y)$ and $\|f\|_{N_1} = \|f\|_{L^1}$.

<u>Proof</u>: By II.1.9. $X \hat{\otimes} Y$ is isometrically contained in $X'' \hat{\otimes} Y$. By II.3.4 the map of $X'' \hat{\otimes} Y$ into $H(X', Y)$ is injective. Consequently the tensor $u \in X'' \hat{\otimes} Y$ which represents f is uniquely determined, all we have to show is that it belongs to $X \hat{\otimes} Y$. Assume that $g \in (X'' \hat{\otimes} Y)' = H(X'', Y')$ (II.1.7) is a functional which vanishes on $X \hat{\otimes} Y$. This means that the corresponding map vanishes on X. By II.1.10 $\langle u, g \rangle = tr((g \hat{\otimes} 1_Y)(u))$. An easy computation shows that f'' maps X''' into Y and that $(g \hat{\otimes} 1_Y)(u)$ defines a map f_1 which

satisfies $\iota_Y \circ f_1 = f'' \circ g'$. The $\sigma(X',X) - \sigma(Y,Y')$ continuity of f means that f' maps Y' into X. Consequently $f'' \circ g' = (g \circ f')' = 0$ and it follows again from II.3.4 that $tr((g \hat{\otimes} 1_Y)(u)) = 0$.

3.8. <u>Definition</u>: $f \in H(X',Y)$ is called a <u>Radon-Nikodym map</u> if $f \circ g \in N_1(Z,Y)$ for all $g \in I_1(Z,X')$ and all Banach spaces Z. RN(X',Y) denotes the space of Radon-Nikodym maps.

RN(X',Y) is a linear subspace of H(X',Y). According to 3.4 it suffices to consider Z = C(T), for T compact. Since M(T) has the metric approximation property (II.3.13), $N_1(C(T),Y)$ is an isometric subspace of $I_1(C(T),Y)$ (II.3.9). Consequently RN(X',Y) is closed in H(X',Y) and defines a bifunctor. The name Radon-Nikodym maps comes from the fact that $I_1(C(T),X)$ may be identified with certain X-valued vector measures on T and nuclear maps correspond to those measures having Radon-Nikodym derivatives.

3.9. <u>Theorem</u>: Any weakly compact map belongs to RN(X,Y)(X a dual space).

<u>Proof</u>: Assume that $f_1 \in I_1(Z,X)$. By 3.4 there exists a factorization:

$$\begin{array}{ccccc} Z & \xrightarrow{f_1} & X & \xrightarrow{\iota_X} & X'' \\ {\scriptstyle g}\downarrow & & & & \uparrow{\scriptstyle k} \\ C(T) & & \xrightarrow[h]{\qquad} & & M(T) \end{array}$$

where $T = \Theta Z' \times OX''$, $g(z)(z',x'') = \langle z,z' \rangle$, $\langle x',k(v) \rangle = \int \langle x',x'' \rangle dv(z',x'')$ and $h(\varphi) = \varphi d\mu$ for a certain fixed measure $\mu \in M(T)$.

a) We assume first that f has separable range. Replacing Y by the image of f we may as well assume that Y itself is separable. Since f is weakly compact, f'' maps X'' into Y. We define a function ψ on T by $\psi(z',x'') = f''(x'')$. For $y' \in Y'$ we have $y' \circ \psi(z',x'') = \langle f''(x''),y' \rangle = \langle f'(y'),x'' \rangle$. Consequently $y' \circ \psi$ is continuous on T with respect to the weak topology on Y. Since any norm-closed ball in Y is weakly closed and Y is separable, the function ψ is measurable. $\mu_1 = |\mu|$ is a finite measure on T and ψ is a bounded function, consequently ψ is also (Bochner) integrable. By II.1.8i) $L^1_{\mu_1}(T,Y) = L^1_{\mu_1}(T) \hat{\otimes} Y$. This means that $\psi = \sum_{n=1}^{\infty} \psi_n \otimes y_n$, where $\psi_n \in L^1_{\mu_1}(T)$, $y_n \in Y$ and $\sum \|\psi_n\|\|y_n\| < \infty$. For $\varphi \in C(T)$, $x' \in X'$ we have $\langle x',k \circ h(\varphi) \rangle = \int_T \langle x',x'' \rangle \varphi(z',x'')d\mu$. Therefore

$$\langle f'' \circ k \circ h(\varphi),y' \rangle = \langle f'(y'), k \circ h(\varphi) \rangle = \int_T \langle f'(y'),x'' \rangle \varphi d\mu =$$

$$= \int \langle \psi(t),y' \rangle \varphi(t)d\mu = \int \sum \langle y_n,y' \rangle \psi_n(t)\varphi(t)d\mu .$$

Now define $z_n' \in Z'$ by $\langle z,z_n' \rangle = \int \psi_n(t) \langle z,z' \rangle d\mu(z',x'')$. Then $\|z_n'\| \le \|\psi_n\|$ and for $z \in Z$ and $y' \in Y'$

$$\langle f \circ f_1(z),y' \rangle = \langle f'' \circ k \circ h \circ g(z),y' \rangle =$$

$$= \sum \langle y_n,y' \rangle \int \psi_n(t) \langle z,z' \rangle d\mu$$

$$= \sum \langle y_n,y' \rangle \langle z,z_n' \rangle$$ and this means that $\sum z_n' \otimes y_n$ represents $f \circ f_1$.

b) We will show now that our previous assumption was not really restrictive. We assert that $f'' \circ k \circ h$ is a compact map. Assume the contrary. Then $f'' \circ k \circ h(OC(T))$ is not precompact. But since a countable number of elements of $C(T)$ suffices to obtain this conclusion, there exists a separable subspace Z_1 of $C(T)$ such that $f'' \circ k \circ h(OZ_1)$ is not precompact. $j : Z_1 \to C(T)$ shall denote the isometric embedding. $h \circ j(Z_1)$ is again separable. Let Σ be the σ-algebra of Borel sets in T and φ_n be a dense subset of Z_1. Each φ_n may be approximated uniformly on T by a sequence of Σ-measurable step-functions. Let $(A_i)_{i=1}^{\infty} \subset \Sigma$ be those sets being used in this procedure. We consider the closed subspace V of $L^1_{\mu_1}$ which is generated by the characteristic functions c_{A_i}. By Lebesgue's theorem on dominated convergence $\Sigma_1 = \{A \in \Sigma : c_A \in V\}$ is a σ-algebra and V coincides with $L^1(T, \Sigma_1, \mu_1)$. Since $h(\varphi_n) \in V$, $h \circ j$ takes its values actually in V. If $v \in M(T)$ is a measure on Σ we consider its restriction to Σ_1, take the absolutely-continuous part and then the Radon-Nikodym derivative with respect to μ_1. In this way we get a projection $p : M(T) \to V$. The map $f'' \circ k \circ p$ is again weakly compact and has separable range, since V is separable. By 3.4 Cor.1 h is integral and from the first part of the proof it follows that $f'' \circ k \circ p \circ h$ is nuclear. Consequently $f'' \circ k \circ h \circ j = f'' \circ k \circ p \circ h \circ j$ is also nuclear. On the other hand, a nuclear transformation maps OX_1 into the closed convex hull of a null sequence and is therefore compact, which contradicts our assumption. We thus have shown that $f'' \circ k \circ h$ is compact and has therefore separable range. We remarked earlier that h takes its values in the subspace $L^1_{\mu_1}$ of $M(T)$ and since μ_1 is a Radon measure the image of $C(T)$ is dense in $L^1_{\mu_1}$. As before we get a projection $p: M(T) \to L^1_{\mu_1}$. $f'' \circ k \circ p$ has separable range and is weakly compact. By a) $f'' \circ k \circ h = f'' \circ k \circ p \circ h$ is nuclear.

3.10. Corollary: If one of the spaces X and Y is reflexive and either X' or Y' has the metric approximation property, then $(X \hat{\hat{\otimes}} Y)' = X' \hat{\otimes} Y'$ holds isometrically.

Proof: Assume that Y is reflexive. By II.2.9 $(X \hat{\hat{\otimes}} Y)' = I_1(X,Y')$. It follows from the metric approximation property and from II.3.9 that $X' \hat{\otimes} Y' = N_1(X,Y')$ is contained isometrically in $I_1(X,Y')$. Since Y is reflexive OY' is weakly compact. By the previous theorem the identity map of Y' belongs to $RN(Y',Y')$ and consequently $I_1(X,Y') = N_1(X,Y')$.

3.11. Definition: a) A series $\sum_{n=1}^{\infty} x_n$ is called weakly unconditionally convergent (w.u.c.) if $\sum_{n=1}^{\infty} |\langle x_n, x' \rangle| < \infty$ for all $x' \in X'$ (we do not require that the limit $\sum_{n=1}^{\infty} x_n$ exist).

b) A map $f \in H(X,Y)$ is called unconditionally converging if it transforms weakly unconditionally convergent into unconditionally convergent series, i.e. for every w.u.c. series $\Sigma\, x_n$ in X the series $\Sigma\, f(x_n)$ converges unconditionally.

3.12. Theorem: Any Radon Nikodym map is unconditionally converging.

Proof: Assume that $f \in RN(X,Y)$ and that $\Sigma\, x_n$ is w.u.c. in X.

a) We will show first that $(f(x_n))$ is a null sequence.

We consider maps $g_k: c_o \to X$ defined by $g_k((\lambda_n)) = \sum_{n=1}^{k} \lambda_n x_n$. According to the Banach-Steinhaus theorem these maps are uniformly bounded. Consequently $g((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n$ exists for $(\lambda_n) \in c_o$ and defines a bounded map $g: c_o \to X$. Now we consider the Rademacher functions r_n on $[0,1]$ defined by $r_n(t) = \operatorname{sgn} \sin 2^n \pi t$ $(n = 1,2,\ldots)$. It is easily seen that they form an orthonormal system in $L^2([0,1])$. $r_n \in L^\infty([0,1])$ and therefore $h(\psi) = (\langle \psi, r_n \rangle)$ defines a continuous map from $L^1([0,1])$ into l^∞. Since $L^2([0,1])$ is dense in $L^1([0,1])$ it follows that the image of h is contained in c_o and from now on we will consider h as a map from L^1 into c_o. Let $j: C([0,1]) \to L^1([0,1])$ be the canonical embedding. We assert that $h_1 = h \circ j$ is integral. Indeed, if $\varphi_i \in C([0,1])$ and $y_i = (y_{in}) \in l^1$ $(i = 1,\ldots,k)$ then $|\Sigma\langle h_1(\varphi_i), y_i\rangle| = |\sum_i \sum_n y_{in} \int_0^1 r_n(t)\varphi_i(t)dt| \leq$
$\leq \sup_{t \in [0,1]} \sum_n |\sum_i y_{in} \varphi_i(t)| = \|\sum_i y_i \varphi_i\|_{C([0,1],l^1)}$ and $C([0,1],l^1) = C([0,1]) \hat{\otimes} l^1$ by II. 2.4.

Consequently $f \circ g \circ h_1$ is nuclear, i.e. $f \circ g \circ h_1 = \sum_{i=1}^{\infty} \mu_i \otimes y_i$ where $\mu_i \in M([0,1])$, $y_i \in Y$ and $\Sigma \|\mu_i\|\|y_i\| < \infty$. Let m be the ordinary Lebesgue measure on $[0,1]$. We use the Lebesgue decomposition: $\mu_i = \mu_i' + \mu_i''$ where μ_i' is singular and μ_i'' absolutely continuous with respect to m. Furthermore $\|\mu_i'\|, \|\mu_i''\| \leq \|\mu_i\|$. For $y' \in Y'$ and $\varphi \in C([0,1])$ we have $\langle f \circ g \circ h_1(\varphi), y'\rangle = \Sigma \langle \varphi, \mu_i\rangle\langle y_i, y'\rangle =$
$= \langle \varphi, \Sigma \mu_i'\langle y_i, y'\rangle + \Sigma \mu_i'' \langle y_i, y'\rangle\rangle$. On the other hand

$< f \circ g \circ h_1(\varphi), y' > = \sum_n < f(x_n), y' > \int_0^1 r_n(t)\varphi(t)dt =$

$= < \varphi, \Sigma < f(x_n), y' > r_n \, dm >$. It follows that

$\sum_i < y_i, y' > \mu_i' + \sum_i < y_i, y' > \mu_i'' = \sum_n < f(x_n), y' > r_n \, dm$

Since $\Sigma < y_i y' > \mu_i'$ is again singular and $\sum_i < y_i, y' > \mu_i''$ is absolutely continuous, it follows that $\sum_i < y_i, y' > \mu_i' = 0$ for all $y' \in Y'$. Consequently $f \circ g \circ h_1 = \Sigma \mu_i'' \otimes y_i$ and we may assume without loss of generality that $\mu_i = \mu_i''$ is absolutely continuous with respect to m. By the theorem of Radon Nikodym $\mu_i = \psi_i \, dm$ with $\psi_i \in L^1([0,1])$. Now the above equation has the form $\sum_i < y_i, y' > \psi_i = \sum_n < f(x_n), y' > r_n$ almost everywhere (y' fixed). It follows that

$< f(x_n), y' > = \int_0^1 r_n(t) \sum_{n=1}^{\infty} < f(x_n), y' > r_n(t)dt =$

$= \int_0^1 r_n(t) \sum_{i=1}^{\infty} < y_i, y' > \psi_i(t)dt$ and that means

$f(x_n) = \sum_{i=1}^{\infty} y_i \int_0^1 r_n(t)\psi_i(t)dt$. To $\epsilon > 0$ there exists an index N such that $\sum_{i=N+1}^{\infty} \|\psi_i\| \|y_i\| = \sum_{i=N+1}^{\infty} \|\mu_i\| \|y_i\| < \epsilon$. Then $\|f(x_n)\| \leq$ $\leq \sum_{i=1}^{N} \|y_i\| \left| \int_0^1 r_n(t)\psi_i(t)dt \right| + \epsilon$ and the first term becomes smaller than ϵ for n large enough, since $(\int_0^1 r_n(t)\psi_i(t)dt)_{n=1}^{\infty}$ is a null sequence for $i = 1,..,N$.

b) Assume that $\sum_{n=1}^{\infty} f(x_n)$ does not converge unconditionally. Then there exists $\epsilon > 0$ and indices $1 = n_o < n_1 < n_2 < \ldots$ such that $\| \sum_{k=n_{i-1}}^{n_i - 1} f(x_k) \| > \epsilon$. Take $y_i = \sum_{k=n_{i-1}}^{n_i - 1} x_k$ $(i = 1,2,\ldots)$. Then $\sum_{i=1}^{\infty} y_i$ again is w.u.c. but $f(y_i)$ does not converge to zero contrary to a).

3.13. <u>Theorem</u>: If $X = C(T)$, then every unconditionally converging map $f \in H(X,Y)$ is weakly compact.

For the proof we need a technical argument:

<u>Lemma</u>: If $(\mu_n) \subset M(T)$ is a bounded sequence of measures and F_n are disjoint measurable subsets of T such that $|\mu_n(F_n)| > \delta > 0$ for all n, there exists a subsequence (ν_n) of (μ_n) and disjoint open sets G_n such that $|\nu_n(G_n)| > \delta/2$ for all n.

<u>Proof of the Lemma</u>: We may assume $\|\mu_n\| \leq 1$ and F_n compact (since μ_n is regular). By an induction process we shall define open sets G_r, measures ν_r $(r = 0,1,2,..)$, sequences of compact sets $F_n^{(r)}$ and sequences of measures $\nu_n^{(r)}$ $(r = 1,2,...)$ such that (1_r) $(\nu_n^{(r)})$ is a subsequence of $(\nu_n^{(r-1)})$

(2_r) $F_n^{(r)} \cap F_m^{(r)} = \emptyset$ for $n \neq m$

(3_r) $F_n^{(r)} \subset T \setminus \bigcup_{i=1}^{r} \overline{G_{i-1}}$

(4_r) $|\nu_n^{(r)}(F_n^{(r)})| > \delta_r = \delta - \sum_{i=1}^{r-1} \delta/2^{i+1}$

(5_r) $|\nu_r(G_r)| > \delta_{r+1} > \delta/2$ and $G_r \cap G_i = \emptyset$ for $i = 1,..,r-1$

We start with $\nu_n^{(1)} = \mu_n$, $F_n^{(1)} = F_n$, $G_o = \emptyset$, $\nu_o = 0$.

Suppose that $(\nu_n^{(r)})$, $(F_n^{(r)})$, G_{r-1}, ν_{r-1} have been defined in such a way that (1_r) - (4_r) and (5_{r-1}) are satisfied for $1 \leq r \leq k$. Let $N = [2^{k+2}/\delta] + 1$ and $F_o^{(k)} = \bigcup_{r=1}^{k-1} \overline{G_r}$. By (2_k) and (3_k), $F_i^{(k)} \cap F_j^{(k)} = \emptyset$ for $i \neq j$ $(i,j = 0,1,2,..)$. Hence there exist open sets O_i' such that $O_i' \supset F_i^{(k)}$ and

$O_i' \cap O_j' = \emptyset$ for $i \neq j$ $(i,j = 0,1,..,N)$. By the regularity of the measures $\nu_i^{(k)}$, one may find open sets O_i such that $O_i' \supset \overline{O_i} \supset F_i^{(k)}$ for $i = 0,1,..,N$ and $|\nu_i^{(k)}|(\overline{O_i} \setminus F_i^{(k)}) < \delta/2^{k+2}$ for $i = 1,2,..,N$. Let us set $A_i \doteq \{n > N : |\nu_n^{(k)}|(\overline{O}_i) < \delta/2^{k+2}\}$. Since the $\overline{O}_i$'s are mutually disjoint sets, $\sum_{i=1}^{N} |\nu_n^{(k)}|(\overline{O}_i) \leq$ $\leq \|\nu_n^{(k)}\| \leq 1$. Hence every index $n > N$ belongs to at least one of the sets A_i (by the construction of N). Therefore there is an index i_o such that the set A_{i_o} is infinite. Let us put $G_k = O_{i_o}$, $\nu_k = \nu_{i_o}^{(k)}$ and $(\nu_n^{(k+1)})$ the subsequence of $(\nu_n^{(k)})$ consisting of those elements whose indices belong to A_{i_o}.

Assume that $\nu_n^{(k+1)} = \nu_{j(n)}^{(k)}$.

Then $|\nu_{j(n)}^{(k)}(F_{j(n)}^{(k)} \cap T \setminus \bigcup_{r=1}^{k} \overline{G_r}))|$

$$\geq |\nu_{j(n)}^{(k)}(F_{j(n)}^{(k)})| - \sum_{r=1}^{k} |\nu_{j(n)}^{(k)}(F_{j(n)}^{(k)} \cap \overline{G_r})|$$

$$\geq \delta_k - |\nu_{j(n)}^{(k)}(F_{j(n)}^{(k)} \cap \overline{G_k})| \qquad \text{(by } (4_k) \text{ and } (3_k))$$

$$\geq \delta_k - |\nu_{j(n)}^{(k)}|(\overline{G_k}) \geq \delta_k - \delta/2^{k+2} > \delta_{k+1} .$$

Hence there exist compact subsets such that

$$F_n^{(k+1)} \subset F_{j(n)}^{(k)} \setminus \bigcup_{r=1}^{k} \overline{G_r} \text{ and } |\nu_n^{(k+1)}(F_n^{(k+1)})| > \delta_{k+1}.$$

Consequently $(1_{k+1}) - (4_{k+1})$ hold. Finally we have $|\nu_k(G_k)| = |\nu_{i_o}^{(k)}(O_{i_o})| \geq |\nu_{i_o}^{(k)}(F_{i_o}^{(k)})| - |\nu_{i_o}^{(k)}| (\overline{O_{i_o}} \setminus F_{i_o}^{(k)})$ $> \delta_k - \delta/2^{k+2} > \delta_{k+1}$ and $G_k \cap G_i \subset O_{i_o} \cap O_o = \emptyset$ for $i < k$. Thus condition (5_k) is satisfied too.

Obviously the sequences (ν_k) and (G_k) have the required properties.

<u>Proof</u> of the theorem:

f is weakly compact if and only if f' is. Assume that $f'(OY')$ is not weakly compact in $M(T)$. By a well known criterion (Dunford, Schwartz [21] p. 305) there exists a sequence $(\nu_n) \subset f'(OY')$ which satisfies the assumption of the lemma. Assume that the subsequence has already been chosen, i.e. there exist open, disjoint subsets G_n of T such that $|\nu_n(G_n)| > \delta > 0$ for all n. By the regularity of the ν_n there exist continuous functions $g_n \in OC(T)$ whose support is contained in G_n such that $|\langle g_n, \nu_n\rangle| > \delta$. Since the sets G_n are disjoint we have for any $\mu \in M(T)$: $\sum_{n=1}^{\infty} |\langle g_n, \mu\rangle| \leq \Sigma \langle |g_n|, |\mu|\rangle \leq \langle \Sigma |g_n|, \mu\rangle < \infty$, i.e. Σg_n is w.u.c. Consequently $\Sigma f(g_n)$ converges in Y, in particular $f(g_n)$ is a null-sequence in Y. $\nu_n = f'(y_n')$ with $y_n' \in OY'$ and it follows that $\langle g_n, \nu_n\rangle = \langle f(g_n), y_n'\rangle \to 0$, a contradiction.

3.14. We will now return to the study of duality of functors. First some remarks on theorem 1.11.: If $\wedge(A,.)$ is a subfunctor of $H(A,.)$, the map $f \to f'$ defines a subfunctor $\wedge'(.',A')$ of $H(.',A')$. This is just the situation of theorem 1.11. In addition we used the assumption that G is total on $\underline{K} \times \underline{K}$. An inspection of the proof shows that the description of $D^G \wedge(.',A)(X)$ remains valid if G is total on $\{\underline{X}\} \times \underline{K}$ only. If $\underline{K}$ is a subcategory of $\underline{L}$ and G is defined on $\underline{L} \times \underline{K}$, we may also define $D^G F(X) = \operatorname{Nat}_{\underline{K}} (F, G(X,.))$ for $X \in \underline{L}$ and if G is total on $\{\underline{X}\} \times \underline{K}$, theorem 1.11. remains also valid. In our case $G(X,Y) = X \hat{\otimes} Y$ and this bifunctor is defined on $\underline{Ban} \times \underline{Ban}$. It is total on $\{\underline{X}\} \times \underline{K}$ if $X \in \underline{A}$ or if $\underline{K} \subseteq \underline{A}$ (II.3.4).

In general, the associated total bifunctor coincides with $L^1(X',Y)$. If $\Lambda(A,.)$ is a subfunctor of $H(A,.)$, the dual assumes the form $D\,\Lambda(A,.)(X) = \{f \in H(A',X) : f \circ g' \in L^1(Y',X)$ $\forall g \in \Lambda(A,Y), Y \in \underline{K}\}$ and the norm is given by $\|f\|_{DG} =$ $= \sup\{\|f \circ g'\|_{L^1} : g \in O\,\Lambda(A,Y), Y \in \underline{K}\}$ (from now on we will only consider the cases $\underline{K} = \underline{A}$ or $\underline{K} = \underline{Ban}$ where this supremum is always finite). In particular, for $X = I$, which belongs to $\underline{A}$, we have $f \circ g' = g''(f)$ and thus $D\,\Lambda(A,.)(I) =$ $= \{f \in A'' : g''(f) \in Y\ \forall g \in \Lambda(A,Y),\ Y \in \underline{K}\}$. Since $\|g\|_\Lambda \leq \|g\|_H$ and equality holds for $Y = I$, $D\,\Lambda(A,.)(I)$ is an isometric subspace of A'' (this has already been shown in 1.13.).

3.15. Theorem: For the categories $\underline{K} = \underline{A}$ or $\underline{Ban}$ and an arbitrary Banach space X, $DI_1(A,.)(X)$ is isometrically contained in $H(A',X)$ and is a subspace of $RN(A',X)$. If $X \in \underline{A}$ then $DI_1(A,.)(X) = RN(A',X)$.

Proof: We will first treat the case $\underline{K} = \underline{A}$.

$I_1(A,.)$ is a subfunctor of $H(A,.)$ and $I_1(A,I) = A'$. Therefore we may apply 1.11. and 3.14. Taking $Y = I$ in the norm description of DI_1, one gets $\|f\|_{DI_1} \leq \|f\|_H$. On the other hand, since $Y \in \underline{A}$, $L^1(Y',X) = Y \hat{\otimes} X$ is isometrically contained in $I_1(Y',X)$ (II.3.9.). Consequently $\|f \circ g'\|_{L^1} = \|f \circ g'\|_{I_1} \leq \|f\|_H\ \|g'\|_{I_1} = \|f\|_H\ \|g\|_{I_1}$ (II.2.9.) and it follows that $\|f\|_{DI_1} = \|f\|_H$. Now assume that $Z \in \underline{Ban}$ and $h \in I_1(Z,A')$. We have an isometry $j: Z \to C(OZ')$. By II.2.9. $I_1(Z,A') = (Z \hat{\hat{\otimes}} A)'$ and $j \hat{\hat{\otimes}} 1_A$ is an isometry too. Consequently there exists $\overline{h} \in I_1(C(OZ'),A') = (C(OZ') \hat{\hat{\otimes}} A)'$ such that $\overline{h} \circ j = \overline{h} \cdot (j \hat{\hat{\otimes}} 1_A) = h$. By II. 2.9. $g = \overline{h}^t = \overline{h}' \circ \iota_A$ belongs to $I_1(A,C(OZ')')$ and $C(OZ')' \in \underline{A}$ by II. 3.13. If $f \in DI_1(A,.)(X)$ it follows that $f \circ g' \in L^1(C(OZ')",X) \subset$ $\subset N_1(C(OZ')",X)$. $g' = \iota_A' \circ \overline{h}"$ and $g' \circ \iota_{C(OZ')"} = \overline{h}$, consequently $f \circ h = f \circ \overline{h} \circ j \in N_1(Z,X)$ and this means that $f \in RN(A',X)$. Conversely if $X \in \underline{A}$, $f \in RN(A',X)$, $g \in I_1(A,Y)$ then $g' \in I_1(Y',A')$ (II. 2.9.) and therefore $f \circ g' \in N_1(Y',X)$. By 3.6. g is weakly compact and it follows that $(f \circ g')' =$ $= g" \circ f'$ maps X' into Y. Using 3.7. we see that $f \circ g' \in L^1(Y',X)$. Finally, consider the case $\underline{K} = \underline{Ban}$. If $X \in \underline{A}$, we may repeat the above proof. In the general case we associate to each natural transformation $\varphi \in \mathrm{Nat}_{\underline{Ban}}(I_1(A,.),X \hat{\hat{\otimes}} .)$ its restriction $\overline{\varphi}$ to $\underline{A}$. We assert that this restriction has the same norm: Assume that $Z \in \underline{Ban}$, $f \in I_1(X,Z)$. We consider the isometry $j: Z' \to Z_1 = C(OZ")$. $\iota_Z \circ f \in I_1(X,Z") = (X \hat{\hat{\otimes}} Z')'$ has an extension $g \in I_1(X,Z_1') =$ $= (X \hat{\hat{\otimes}} Z_1)'$, i.e. $\iota_Z \circ f = j' \circ g$. This extension may be chosen to be of the same norm, i.e. $\|g\|_{I_1} = \|\iota_Z \circ f\|_{I_1} = \|f\|$ (II. 2.9.). Using the naturality of φ and the fact that $1_X \hat{\otimes} \iota_Z$ is an isometry (II.1.9.) we get:

$$\|\varphi_Z(f)\|_{X \hat{\otimes} Z} = \|(1_X \hat{\otimes} \iota_Z) \circ \varphi_Z(f)\|_{X \hat{\otimes} Z"} = \|\varphi_{Z"}(\iota_Z \circ f)\|_{X \hat{\otimes} Z"}$$
$$= \|\varphi_{Z"}(j' \circ g)\|_{X \hat{\otimes} Z"} = \|(1_X \hat{\otimes} j') \circ \varphi_{Z_1'}(\varphi)\|_{X \hat{\otimes} Z"} \leq$$
$$\leq \|\varphi_{Z_1'}(g)\|_{X \hat{\otimes} Z_1'} \leq \|\overline{\varphi}\| \|g\|_{I_1} = \|\overline{\varphi}\| \|f\|_{I_1} \text{ since } Z_1' \in \underline{A}.$$

Remark: If X does not have the metric approximation property it may happen that $DI_1(A,.)(X) \neq RN(A',X)$.

3.16. Special cases

a) If A is reflexive and has the metric approximation property, it follows from 3.10 that $I_1(A,X) = A' \hat{\otimes} X$. Therefore $DI_1(A,.)(X) = H(A',X)$.

b) $A' = C(T)$ (e.g. $A = l^1$, then $A' = l^\infty = C(\beta\mathbb{N})$ where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$). By 3.13 any member of $DI_1(A,.)(X)$ is weakly compact as a map from A' to X. Now it follows from the description given in 3.14 that $D^2I_1(A,.)(I) = A''$. It is easily seen that the canonical transformation of $I_1(A,.)$ into D^2I_1 induces the canonical embedding of A into A''. Consequently $I_1(A,.)$ is not reflexive. By 2.4. $I_1(A,X) = D(A \hat{\hat{\otimes}} .)(X)$. This gives an example of a dual functor which is not reflexive. (The equation $I_1(A,X) = D(A \hat{\hat{\otimes}} .)(X)$ holds also for the case $\underline{K} = \underline{A}$. This follows by a slight modification of the proof of 2.5: Since $A \hat{\hat{\otimes}} .$ preserves isometric embeddings, it suffices to evaluate the given natural transformation at all finite dimensional subspaces of X'.)

Exercises

The following exercises show some properties of the duality $D^{\hat{}}$ defined by the inductive tensor product.

1) If F is of type Σ, then $D^{\hat{}}F = H_{F(I)}$.

2) $D^{\hat{}}F$ is always isometrically contained in $H_{F(I)}$.

3) If n is a sequence space, then
$D^{\hat{}}n(X) = H_{l^\infty}(n, l^1 \hat{\otimes} X)$

4) Develop a concept of duality for contravariant functors which is similar to that of § 1 by using contra-contravariant bifunctors.

CHAPTER VI

Kan extensions

§ 1. General remarks on Kan extensions

1.1. First we collect definitions and elementary properties. Let $\underline{L}$, $\underline{K}$ be full subcategories of Ban, let $S : \underline{K} \to \underline{L}$ be a functor. Then we have by IV, 3.9

$$FS(.) = H(..,S(.)) \hat{\otimes}_{(..)\in \underline{L}} F(..) = \operatorname{Nat}_{(..)\in\underline{L}} (H(S(.),..),F(..))$$

$$GS(.) = G \hat{\otimes}_{(..)\in\underline{L}} H(S(.),..) = \operatorname{Nat}_{(..)\in\underline{L}} (H(..,S(.)),G(..))$$

for $F \in Ban^{\underline{L}}$ and $G \in Ban^{\underline{L}^{op}}$.

Lan_S and $Ran_S : Ban^{\underline{K}} \to Ban^{\underline{L}}$ are respectively left and right adjoint to the functor $Ban^S : F \to F \circ S$; likewise Lan_S and $Ran_S : Ban^{\underline{K}^{op}} \to Ban^{\underline{L}^{op}}$ are respectively left and right adjoint to the functor $Ban^S : : G \to G \circ S$ from $Ban^{\underline{L}^{op}} \to Ban^{\underline{K}^{op}}$.

They are given by

$$Lan_S F_1 = H(S(..),.) \hat{\otimes}_{(..)\in\underline{K}} F_1(..) ,$$

$$Ran_S F_1 = \operatorname{Nat}_{(..)\in\underline{K}} (H(.,S(..)),F_1(..)) , \quad F_1 \in Ban^{\underline{K}}$$

$$Lan_S G_1 = G_1(..) \hat{\otimes}_{(..)\in\underline{K}} H(.,S(..))$$

$$Ran_S G_1 = \operatorname{Nat}_{(..)\in\underline{K}} (H(S(..),.), G_1(..)), \quad G_1 \in Ban^{\underline{K}^{op}}$$

1.2. Now let $\underline{K}$ be a full subcategory of Ban and consider the embedding functor $\underline{K} \to$ Ban. We denote it also by $\underline{K}$ for short, so we write $Lan_{\underline{K}}$ and $Ran_{\underline{K}}$ for the associated Kan-extensions.

Definition: A functor $F : \mathrm{Ban} \to \mathrm{Ban}$ is called $\underline{K}$-computable if $F = \mathrm{Lan}_{\underline{K}}\,(F|\underline{K})$ via the counit $\mathrm{Lan}_{\underline{K}}(F|\underline{K}) \to F$ of the adjunction (4.1), i.e. the map $H\, \hat{\otimes}_{\underline{K}}(F|\underline{K}) \to F$, $f \otimes \xi \to F(f)\xi$. (Analogously for contravariant functors).

A functor F is called $\underline{K}$ - complete, if $F = \mathrm{Ran}_{\underline{K}}\,(F|\underline{K})$ via the unit of the adjunction (4.1), i.e. the map

$\tau_X : F(X) \to \mathrm{Nat}_{\underline{L}}(H(X,.), (F|\underline{K}))$,

$(\tau_X \xi)_K\,(f) = F(f)\xi$, $K \in \underline{K}$, $f \in H(X,K)$, $\xi \in F(X)$.

(Analogously for contravariant functors).

1.3. Proposition: a) If $F : \mathrm{Ban} \to \mathrm{Ban}$ is $\underline{K}$-computable, then $\mathrm{Nat}_{\mathrm{Ban}}(F,F_1) = \mathrm{Nat}_{\underline{K}}(F|\underline{K},F_1|\underline{K})$ for all functors $F_1 : \mathrm{Ban} \to \mathrm{Ban}$.

If $G : \mathrm{Ban}^{op} \to \mathrm{Ban}$ is $\underline{K}$-computable, then $\mathrm{Nat}_{\mathrm{Ban}}(G,G_1) = \mathrm{Nat}_{\underline{K}}(G|\underline{K},G_1|\underline{K})$ for all functors $G_1 : \mathrm{Ban}^{op} \to \mathrm{Ban}$.

b) If $F : \mathrm{Ban} \to \mathrm{Ban}$ is $\underline{K}$-complete, then $\mathrm{Nat}_{\mathrm{Ban}}(F_1,F) = \mathrm{Nat}_{\underline{K}}(F_1|\underline{K},F|\underline{K})$ for all functors $F_1 : \mathrm{Ban} \to \mathrm{Ban}$.

If $G : \mathrm{Ban}^{op} \to \mathrm{Ban}$ is $\underline{K}$-complete, then $\mathrm{Nat}_{\mathrm{Ban}}(G_1,G) = \mathrm{Nat}_{\underline{K}}(G_1|\underline{K},G|\underline{K})$ for all functors $G_1 : \mathrm{Ban}^{op} \to \mathrm{Ban}$.

Proof: a) $\mathrm{Nat}_{\mathrm{Ban}}(F,F_1) = \mathrm{Nat}_{\mathrm{Ban}}(\mathrm{Lan}_{\underline{K}}\,(F|\underline{K}),F_1) = \mathrm{Nat}_{\underline{K}}(F|\underline{K}, F_1|\underline{K})$ by 1.1 .

b) $\mathrm{Nat}_{\mathrm{Ban}}(F_1,F) = \mathrm{Nat}_{\mathrm{Ban}}(F_1, \mathrm{Ran}_{\underline{K}}(F|\underline{K})) = \mathrm{Nat}_{\underline{K}}(F_1|\underline{K}, F|K)$.

Likewise for contravariant functors.

1.4. First we fix some notation: if $F\colon \mathrm{Ban} \to \mathrm{Ban}$ is a functor, then we write

$$F^{\underline{K}} = \mathrm{Lan}_{\underline{K}}(F|\underline{K}) : \mathrm{Ban} \to \mathrm{Ban},$$

$$F_{\underline{K}} = \mathrm{Ran}_{\underline{K}}(F|\underline{K}) : \mathrm{Ban} \to \mathrm{Ban};$$

we call $F^{\underline{K}}$ the $\underline{K}$-computable part of F and $F_{\underline{K}}$ the $\underline{K}$-completion of F. Via unit and counit (1.1.) we have canonical natural transformations

$$F^{\underline{K}} \to F \to F_{\underline{K}} \quad \text{(cf 1.2.)}$$

If $\underline{K} = \mathrm{Fin}$, then we omit the prefix Fin for short: Hence, by the completion of a functor F, for example, we mean just the Fin-completion F_{Fin}.

We will use the analogous notation for contravariant functors.

1.5. <u>Proposition:</u> Let $F\colon \mathrm{Ban} \to \mathrm{Ban}$ and $G\colon \mathrm{Ban}^{op} \to \mathrm{Ban}$ be functors and $X \in \mathrm{Ban}$. Then we have by (IV, 3.11.) and 1.1.:

$$F^{\underline{K}}(X) = \mathrm{Lan}_{\underline{K}}\ (F|\underline{K})(X) = H(\,.\,,X) \mathbin{\hat{\otimes}}_{\underline{K}} (F|\underline{K}) =$$
$$= \mathrm{Lan}_{\underline{K}}\ (H^X|\underline{K}) \mathbin{\underset{\mathrm{Ban}}{\hat{\otimes}}} F = (H^X)^{\underline{K}} \mathbin{\underset{\mathrm{Ban}}{\hat{\otimes}}} F.$$

$$F_{\underline{K}}(X) = \mathrm{Ran}_{\underline{K}}\ (F|\underline{K})(X) = \underset{\underline{K}}{\mathrm{Nat}}\ (H_X, F|\underline{K}) =$$
$$= \underset{\mathrm{Ban}}{\mathrm{Nat}}\ (\mathrm{Lan}_{\underline{K}}\ (H_X|K), F) = \underset{\mathrm{Ban}}{\mathrm{Nat}}\ ((H_X)^{\underline{K}}, F).$$

$$G^{\underline{K}}(X) = \mathrm{Lan}_{\underline{K}}\ (G|\underline{K})(X) = (G|\underline{K}) \mathbin{\underset{\underline{K}}{\hat{\otimes}}} H(X,\,.\,) =$$
$$= G \mathbin{\underset{\mathrm{Ban}}{\hat{\otimes}}} \mathrm{Lan}_{\underline{K}}\ (H_X|\underline{K}) = G \mathbin{\underset{\mathrm{Ban}}{\overset{\underline{K}}{\hat{\otimes}}}} (H_X)^{\underline{K}}.$$

$$G_{\underline{K}}(X) = \mathrm{Ran}_{\underline{K}}\ (G|\underline{K})(X) = \underset{\underline{K}}{\mathrm{Nat}}\ (H^X, G|\underline{K}) =$$
$$= \underset{\mathrm{Ban}}{\mathrm{Nat}}\ (\mathrm{Lan}_{\underline{K}}\ (H^X|\underline{K}), G) = \underset{\mathrm{Ban}}{\mathrm{Nat}}\ ((H^X)^{\underline{K}}, G)$$

1.6. To justify the notation 1.4 we show:

Lemma: $(F^{\underline{K}})^{\underline{K}} = F^{\underline{K}}$, $(F_{\underline{K}})_{\underline{K}} = F_{\underline{K}}$ and likewise for contravariant functors.

Proof: $F^{\underline{K}}|\underline{K} = F|\underline{K}$ since for $X \in \underline{K}$ we have $F^{\underline{K}}(X) =$ $= H(.,X) \hat{\otimes}_{\underline{K}}(F|\underline{K}) = F(X)$ by IV, 3.8. So $(F^{\underline{K}})^{\underline{K}} =$ $= \mathrm{Lan}_{\underline{K}} (F^{\underline{K}}|\underline{K}) = \mathrm{Lan}_{\underline{K}} (F|\underline{K}) = F^{\underline{K}}$. All other arguments are similar.

1.7. Proposition: Let $\underline{K}$ be a full subcategory of Ban which contains the one-dimensional space I.

a) If $F : \underline{K} \to \mathrm{Ban}$ is of type Σ on $\underline{K}$, then $\mathrm{Lan}_{\underline{K}} F$ is of type Σ on Ban.

b) If $F : \underline{K} \to \mathrm{Ban}$ is total on $\underline{K}$ (cf IV, 1.17) then $\mathrm{Ran}_{\underline{K}} F$ is total on Ban.

Likewise for contravariant functors.

Proof: a) $\mathrm{Lan}_{\underline{K}} F(X) = H(.,X) \hat{\otimes}_{\underline{K}} F = H(I,X) \overline{\otimes}_{\alpha} F(I)$ for a reasonable crossnorm α by IV, 3.12.

b) Let $0 \neq \eta \in \mathrm{Ran}_{\underline{K}} F(X) = \mathrm{Nat}_{\underline{K}} (H_X,F)$.

Then there is $K \in \underline{K}$, $f \in H(X,K)$ with $\eta_K(f) \neq 0$, so that, by hypothesis, there is $k' \in K'$ such that $F(k')\eta_K(f) \neq 0$ in $F(I)$. But then $\mathrm{Ran}_{\underline{K}} F(f'(k'))(\eta) \neq 0$ since

$[\mathrm{Ran}_{\underline{K}} F(f'(k'))(\eta)]_I(1) = [\mathrm{Nat}_{\underline{K}} (H(k'\circ f,.),F)(\eta)]_I (1)$

$= [\eta \circ H(k' \circ f,.)]_I (1_I) = \eta_I(k' \circ f) = \eta_I \circ H(X,k')(f)$

$= F(k')\eta_K(f) \neq 0$.

Likewise for contravariant functors.

1.8. <u>Proposition</u>: Let $F : \mathrm{Fin} \to \mathrm{Ban}$ and $G : \mathrm{Fin}^{op} \to \mathrm{Ban}$ be functors and $X \in \mathrm{Ban}$. Then we have:

$$\mathrm{Lan}_{\mathrm{Fin}}\, F(X) = \varinjlim \{F(E),\ E \subseteq X,\ E \in \mathrm{Fin}\} .$$

$$\mathrm{Ran}_{\mathrm{Fin}}\, F(X) = \varprojlim \{F(X/M)\ ,\ M \subseteq X,\ X/M \in \mathrm{Fin}\} .$$

$$\mathrm{Lan}_{\mathrm{Fin}}\, G(X) = \varinjlim \{G(X/M),\ M \subseteq X,\ X/M \in \mathrm{Fin}\}.$$

$$\mathrm{Ran}_{\mathrm{Fin}}\, G(X) = \varprojlim \{G(E),\ E \subseteq X,\ E \in \mathrm{Fin}\} .$$

The limits are always in Ban, E runs through all finite dimensional subspaces of X and M through all finite codimensional subspaces. Compare I, 1.16 and I, 1.20.

<u>Proof</u>: Write $LF(X) = \varinjlim \{F(E),\ E \subseteq X,\ E \in \mathrm{Fin}\}$. Then by the universal property of the colimit it is readily seen that $X \to LF(X)$ is the object transformation of a functor $LF : \mathrm{Ban} \to \mathrm{Ban}$, whose action on morphisms is given by the universal property of the colimit, that $L : \mathrm{Ban}^{\mathrm{Fin}} \to \mathrm{Ban}^{\mathrm{Ban}}$ is a functor too and that L is left adjoint to the restriction functor $F_1 \to F_1|\mathrm{Fin}$ from $\mathrm{Ban}^{\mathrm{Ban}}$ to $\mathrm{Ban}^{\mathrm{Fin}}$. Now $\mathrm{Lan}_{\mathrm{Fin}}$ is left adjoint to that restriction functor (1.1) and since any two left adjoints of the same functor are naturally equivalent, we see that $L = \mathrm{Lan}_{\mathrm{Fin}}$. The other claims can be proved in a similar manner. qed.

1.9. Examples: Let $A \in Ban$.

a) $A \hat{\otimes}\, .$ is computable, since it commutes with any colimits in Ban_1 (II, 1.8c) and since $X = \varinjlim \{E,\ E \subseteq X,\ E \in Fin\}$ by I, 1.20.

b) $A \hat{\hat{\otimes}}\, .$ is computable: the inductive tensor product commutes with isometries (II, 2.2), so $\{E \hat{\hat{\otimes}} E_1 : E \subseteq A,\ E_1 \subseteq X;\ E, E_1 \in Fin\}$ is a cofinal subcategory of $\{E : E \subseteq A \hat{\hat{\otimes}} X,\ E \in Fin\}$. Hence $A \hat{\hat{\otimes}} X = \varinjlim \{E \hat{\hat{\otimes}} E_1 : E \subseteq A,\ E_1 \subseteq X;\ E, E_1 \in Fin\}$

$$= \varinjlim \{\varinjlim \{E \hat{\hat{\otimes}} E_1,\ E \subseteq A,\ E \in Fin\} : E_1 \subseteq X,\ E_1 \in Fin\}$$

$$= \lim \{A \hat{\hat{\otimes}} E_1,\ E_1 \subseteq X,\ E_1 \in Fin\} .$$

c) The functor $L^1(A',.)$ of V, 3.1b) is not computable if A does not have the approximation property, since it coincides with the computable functor $A \hat{\otimes}.$ on Fin and so $L^1(A',.)^{Fin} =$ $= A \hat{\otimes}\, .$ by 1.6.

d) $(H_A)^{Fin} = A' \hat{\hat{\otimes}}\, .$ since they agree on Fin and the latter is computable.

e) Let $I_1(A,.)$ be the functor of II, 2.9 or V, 3.1 a). Then $I_1(A,.)^{Fin} = A' \hat{\otimes}\, .$ since they agree on Fin.

f) $A \hat{\otimes}\, .'$ is computable since

$$(A \hat{\otimes} .')^{Fin}(X) = (A \hat{\otimes} .') \underset{Ban}{\hat{\otimes}} (H_X)^{Fin} \quad \text{by } 1.5$$

$$= (A \hat{\otimes} .') \underset{Ban}{\hat{\otimes}} (X' \hat{\otimes} .) \quad \text{by d)}$$

$$= A \hat{\otimes} X' \quad \text{by remark 1) in IV, 3.11.}$$

If $A = I$, then we may conclude by 1.8:

$$X' = (.')^{Fin}(X) = \underrightarrow{\lim}\{(X/M)', M \subseteq X, X/M \in Fin\}$$

$$= \underrightarrow{\lim}\{M^{\perp}, M \subseteq X, X/M \in Fin\}.$$

But this is clear since each finite dimensional subspace of X' is of the form $M^{\perp}$, $X/M \in Fin$.

g) $A \hat{\hat{\otimes}} .'$ is computable:

$$A \hat{\hat{\otimes}} X' = A \hat{\hat{\otimes}} (\underrightarrow{\lim}\{E, E \subseteq X', E \in Fin\})$$

$$= \underrightarrow{\lim}\{A \hat{\hat{\otimes}} E, E \subseteq X', E \in Fin\} \quad \text{by b)}$$

$$= \underrightarrow{\lim}\{A \hat{\hat{\otimes}} (X/M)', M \subseteq X, X/M \in Fin\} \text{ by the}$$

remark in f),

$$= (A \hat{\hat{\otimes}} .')^{Fin}(X) \quad \text{by } 1.8.$$

h) $$(A \hat{\otimes} .)_{Fin}(X) = \underset{Ban}{Nat}((H_X)^{Fin}, A \hat{\otimes} .) \quad \text{by } 1.5$$

$$= \underset{Ban}{Nat}(X' \hat{\otimes} ., A \hat{\otimes} .) \quad \text{by d)}$$

$$= D(X' \hat{\otimes} .)(A)$$

$$= I_1(X', A) \quad \text{by V, 2.5 and II, 2.9}$$

i) $$(A \hat{\hat{\otimes}} .)_{Fin}(X) = \underset{Ban}{Nat}((H_X)^{Fin}, A \hat{\hat{\otimes}} .) \quad \text{by } 1.5$$

$$= \underset{Ban}{Nat}(X' \hat{\hat{\otimes}} ., A \hat{\hat{\otimes}} .) \quad \text{by d)}$$

$$= H(X', A).$$

j) $(H_A)_{Fin}(X) = \underset{Ban}{Nat}((H_X)^{Fin}, H_A)$ by 1.5

$= \underset{Ban}{Nat}(X' \hat{\hat{\otimes}} ., A' \hat{\hat{\otimes}} .)$ by d) and IV, 1.12

$= H(X',A') = H(A,X'')$.

k) $I_1(A,.)_{Fin} = (A' \hat{\otimes} .)_{Fin} = I_1(.',A') = I_1(A,.'')$

since $I_1(A,.)$ and $A' \hat{\otimes} .$ coincide on Fin.

l) As special cases we get:

$(Id)_{Fin} = ''$; i.e. $X'' = \lim \{X/M, M \subseteq X, X/M \in Fin\}$

by 1.8, i.e. the result we derived in I, 1.20.

m) $(A \hat{\otimes} .')_{Fin}(X) = \underset{Ban}{Nat}((H^X)^{Fin}, A \hat{\otimes} .')$ by 1.5

$= \underset{Ban}{Nat}(X \hat{\hat{\otimes}} .', A \hat{\otimes} .')$ by g)

$= I_1(X,A)$ by a contravariant analogon to V, 2.5.

n) $(A \hat{\hat{\otimes}} .')_{Fin}(X) = \underset{Ban}{Nat}((H^X)^{Fin}, A \hat{\hat{\otimes}} .')$ by 1.5

$= \underset{Ban}{Nat}(X \hat{\hat{\otimes}} .', A \hat{\hat{\otimes}} .')$

$= H(X,A)$.

o) $H(.,A)_{Fin}(X) = (H(.,A)_e)_{Fin}(X)$ by 1.8

$= (A \hat{\hat{\otimes}} .')_{Fin}(X) = H(X,A)$.

1.10. The counterexample in 1.9 c) is the typical one, as is shown by:

j) $(H_A)_{Fin}(X) = \underset{Ban}{\mathrm{Nat}}((H_X)^{Fin}, H_A)$ by 1.5

$= \underset{Ban}{\mathrm{Nat}}(X' \hat{\hat{\otimes}} ., A' \hat{\hat{\otimes}} .)$ by d) and IV, 1.12

$= H(X',A') = H(A,X'')$.

k) $I_1(A,.)_{Fin} = (A' \hat{\otimes} .)_{Fin} = I_1(.',A') = I_1(A,.'')$ since $I_1(A,.)$ and $A' \hat{\otimes}$. coincide on Fin.

l) As special cases we get:

$(Id)_{Fin} = ''$; i.e. $X'' = \lim \{X/M,\ M \subseteq X,\ X/M \in \mathrm{Fin}\}$

by 1.8, i.e. the result we derived in I, 1.20.

m) $(A \hat{\otimes} .')_{Fin}(X) = \underset{Ban}{\mathrm{Nat}}((H^X)^{Fin}, A \hat{\otimes} .')$ by 1.5

$= \underset{Ban}{\mathrm{Nat}}(X \hat{\hat{\otimes}} .', A \hat{\otimes} .')$ by g)

$= I_1(X,A)$ by a contravariant analogon to V, 2.5.

n) $(A \hat{\hat{\otimes}} .')_{Fin}(X) = \underset{Ban}{\mathrm{Nat}}((H^X)^{Fin}, A \hat{\hat{\otimes}} .')$ by 1.5

$= \underset{Ban}{\mathrm{Nat}}(X \hat{\hat{\otimes}} .', A \hat{\hat{\otimes}} .')$

$= H(X,A)$.

o) $H(.,A)_{Fin}(X) = (H(.,A)_e)_{Fin}(X)$ by 1.8

$= (A \hat{\hat{\otimes}} .')_{Fin}(X) = H(X,A)$.

1.10. The counterexample in 1.9 c) is the typical one, as is shown by:

Proposition: a) If $F : \mathrm{Ban} \to \mathrm{Ban}$ is any functor of type Σ and if X has the metric approximation property, then $F(X) = F^{\mathrm{Fin}}(X)$ and the canonical mapping $\tau_X^F : F(X) \to F_{\mathrm{Fin}}(X)$ (cf 1.2, 1.4) is isometric.

b) If $G : \mathrm{Ban}^{op} \to \mathrm{Ban}$ is any functor of type Σ and if X' has the metric approximation property, then $G(X) = G^{\mathrm{Fin}}(X)$ and the canonical mapping $\tau_X^G : G(X) \to G_{\mathrm{Fin}}(X)$ is isometric.

Proof: a) $F^{\mathrm{Fin}}(X) = (.' \hat{\hat{\otimes}} X) \underset{\mathrm{Ban}}{\hat{\otimes}} F$ by 1.5 and 1.9 g)

$= F_e(X)$ by IV, 3.13.

$= F(X)$.

$\tau_X^F(\xi)_Z(f) = F(f)\xi$, $\xi \in F(X)$, $f \in X' \hat{\hat{\otimes}} Z$, where

$F_{\mathrm{Fin}}(X) = \underset{\mathrm{Ban}}{\mathrm{Nat}}(X' \hat{\hat{\otimes}} .,F)$ by 1.5 and 1.9 d).

By hypothesis $X' \hat{\hat{\otimes}} X = K(X,X)$ has a left approximate unit (u_j) bounded by 1 (cf II, 3.10) and $F(X) = F_e(X)$ is an essential left Banach $K(X,X)$-module. Thus for all $\xi \in F(X)$ we have $\|\tau_X^F(\xi)_X(u_j) - \xi\| = \|F(u_j)\xi - \xi\| \to 0$ and so $\|\tau_X^F(\xi)\| = \|\xi\|$.

b) $G^{\mathrm{Fin}}(X) = G \underset{\mathrm{Ban}}{\hat{\otimes}} (X' \hat{\hat{\otimes}} .)$ by 1.5 and 1.9 d)

$= G_e(X)$ by IV, 3.13.

The proof of the second assertion is similar to that in a), using II, 3.11, Corollary.

1.11. We now want to study the connections between dual and computable functors.

Proposition: If F: Ban $\to$ Ban is computable, then

$DF(X') = F(X)'$ for all $X \in$ Ban.

Proof:
$$
\begin{aligned}
DF(X') &= \underset{\text{Ban}}{\text{Nat}}\,(F, X'\hat{\otimes}\,.) \\
&= \underset{\text{Fin}}{\text{Nat}}\,(F, X'\hat{\otimes}\,.) \text{ by 1.3. a)} \\
&= \underset{\text{Fin}}{\text{Nat}}\,(F, (X\hat{\otimes}\,.\,')') \text{ by II, 2.8.} \\
&= ((X\hat{\otimes}\,.\,') \underset{\text{Fin}}{\hat{\otimes}} F)' \\
&= F(X)'.
\end{aligned}
$$

1.12. Proposition: If G: $\text{Ban}^{\text{op}} \to$ Ban is computable, then

$\underset{\text{Ban}}{\text{Nat}}\,(G, X''\hat{\otimes}\,.\,') = G(X)'$ for all $X \in$ Ban.

Proof:
$$
\begin{aligned}
\underset{\text{Ban}}{\text{Nat}}\,(G, X''\hat{\otimes}\,.\,') &= \underset{\text{Fin}}{\text{Nat}}\,(G, X''\hat{\otimes}\,.\,') \text{ by 1.3. a)} \\
&= \underset{\text{Fin}}{\text{Nat}}\,(G, (X'\hat{\otimes}\,.)') \text{ by II, 2.8.} \\
&= (G \underset{\text{Fin}}{\hat{\otimes}} (X'\hat{\otimes}\,.))' = G(X)'.
\end{aligned}
$$

1.13. Proposition: A covariant computable functor F transforms weak retracts into weak retracts and weak sections into weak sections. A contravariant computable functor G transforms weak retracts into weak sections and weak sections into weak retracts.

Proof: Let $f: X \to Y$ be a weak retract (V, 2.9., II, 1.9.). Then there is $h: X' \to Y'$, $\|h\| \le 1$ with $f' \circ h = 1_{X'}$.

So $F(f)' \circ DF(h) = DF(f') \circ DF(h)$ by 1.11

$$= DF(f' \circ h)$$

$$= DF(1_{X'}) = 1_{DF(X')} = 1_{F(X)'} ,$$

i.e. $F(f)$ is a weak retract.

$$\underset{Ban}{Nat}(G, G' \hat{\otimes} .') \circ G(f)' =$$

$$= \underset{Ban}{Nat}(G, h' \hat{\otimes} .') \circ \underset{Ban}{Nat}(G, f'' \hat{\otimes} .') \quad \text{by 1.12}$$

$$= \underset{Ban}{Nat}(G, (h' \circ f'') \hat{\otimes} .')$$

$$= \underset{Ban}{Nat}(G, (1_{X'})' \hat{\otimes} .') = 1_{G(X)'},$$

i.e. $G(f)$ is a weak section.

If on the other hand f is a weak section, i.e. there is $h : Y' \to X'$ with $h \circ f' = 1_{Y'}$, then

$DF(h) \circ F(f)' = DF(h) \circ DF(f')$ by 1.11

$$= DF(h \circ f') = DF(1_{Y'}) = 1_{DF(Y')}$$

$= 1_{F(Y)'}$, so $F(f)$ is a weak section,

and $G(f)' \circ \underset{Ban}{Nat}(G, h' \hat{\otimes} .') =$

$$= \underset{Ban}{Nat}(G, f'' \hat{\otimes} .') \circ \underset{Ban}{Nat}(G, h' \hat{\otimes} .') \quad \text{by 1.12}$$

$$= \underset{Ban}{Nat}(G, (f'' \circ h') \hat{\otimes} .') = 1_{G(Y)'} ,$$

so $G(f)$ is a weak retract.

§ 2. Extensions from the categories $\underline{L}^p$

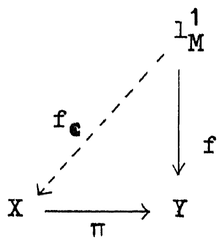
In this section we use the general set-up of § 1 to carry over some of Grothendieck's considerations for $\otimes$-norms [33] to arbitrary functors. For a $\otimes$-norm α (i.e. a tensor product, IV, 2.10., which is computable in each variable) he defines norms $/\alpha$, $\backslash\alpha$, etc which turn out to be $\underline{l}^1$ - and $\underline{l}^\infty$ - computable functors. We treat arbitrary functors and $\underline{l}^p$, $\underline{L}^p$, $\underline{l}^p_{fin}$ Kan extensions over them for $1 \leq p \leq \infty$.

2.1. Definition: $\underline{l}^p$ denotes the full subcategory of $\underline{Ban}$, consisting of all spaces l^p_M, M a set, $1 \leq p \leq \infty$. $\underline{l}^p_{fin}$ consists of all finite dimensional spaces in $\underline{l}^p$. $\underline{L}^p$ consists of all spaces $L^p(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is a measure space.

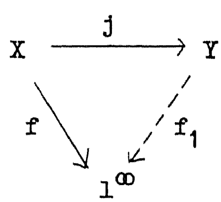
If $X \in \underline{Ban}$, we denote by $\{l^1_1(X) \xleftarrow{f_X} l^1_2(X)\}$ the spectral family, where $l^1_1(X) = l^1_{OX}$, π_X: $l^1_{OX} \to X$ is the canonical quotient map (cf. I. 1.11.), $l^1_2(X) = l^1_{O(\ker \pi_X)}$ and f_X is the composition of the canonical mappings $l^1_{OX} \leftarrow \ker \pi_X \leftarrow l^1_{O(\ker \pi_X)}$. It follows that $X = \varinjlim \{l^1_1(X) \xleftarrow{f_X} l^1_2(X)\}$.
If $f \in OH(X,Y)$, we may define $l^1_1(f) : l^1_1(X) \to l^1_1(Y)$ by $l^1_1(f)(e_x) = e_{f(x)}$ for $x \in OX$, where e_x denotes the unit vector corresponding to x. It is easily seen that $l^1_1(f)$

induces a map $l_2^1(f) : l_2^1(X) \to l_2^1(Y)$. Thus $l_1^1(X)$ and $l_2^1(X)$ become functors in the usual categorical sense, but the assignment for morphisms is not linear. Analogously we define $\{l_1^\infty(X) \xrightarrow{g_X} l_2^\infty(X)\}$, where $l_1^\infty(X) = l_{0X'}^\infty$, $i_X : X \to l_{0X'}^\infty$ denotes the canonical isometry, $l_2^\infty(X) = l_{0(\mathrm{coker}\ i_X)'}^\infty$. This time $X = \underleftarrow{\lim} \{l_1^\infty(X) \xrightarrow{g_X} l_2^\infty(X)\}$.

2.2. <u>Lemma</u>: a) If $\pi : X \to Y$ is a quotient map, $f \in H(l_M^1, Y)$, $\epsilon > 0$, there exists a map $f_\epsilon \in H(l_M^1, X)$, such that $\|f_\epsilon\| \leq \|f\|(1+\epsilon)$ and the following diagram commutes:



b) If $j : X \to Y$ is an isometry and $f \in H(X, l_M^\infty)$, there exists a map $f_1 \in H(Y, l_M^\infty)$, such that $\|f_1\| = \|f\|$ and the following diagram commutes:



Proof: a) Let $\{e_m\}_{m \in M}$ be the canonical base of l^1_M.

Since π is a quotient map, there exist elements $x_m \in X$ such that $\pi(x_m) = f(e_m)$ and $\|x_m\| \le \|f(e_m)\|(1+\epsilon)$. Now define $f_\epsilon\,(\alpha_m) = \Sigma\, \alpha_m x_m$.

b) Let $\{e'_m\}_{m \in M}$ be the coordinate functionals of l^∞_M. Since j is an isometry, $f'(e'_m)$ can be extended to a functional $y'_m \in Y'$ with equal norm. Define $f_1(y) = (\langle y, y'_m \rangle)$.

2.3. Proposition: If $F : \underline{l}^1 \to \underline{\text{Ban}}$ is covariant, then

$$F^{\underline{l}^1}(X) = \varinjlim \{F(l^1_1(X)) \leftarrow F(l^1_2(X))\} .$$

If $G : \underline{l}^1 \to \underline{\text{Ban}}$ is contravariant, then

$$G_{\underline{l}^1}(X) = \varprojlim \{G(l^1_1(X)) \to G(l^1_2(X))\} .$$

Proof: We will start with the contravariant case:

By 1.5 $G_{\underline{l}^1}(X) = \text{Nat}_{\underline{l}^1}\,(H(.,X), G(.))$.

If $\varphi \in \text{Nat}_{\underline{l}^1}\,(H(.,X), G(.))$, then $\varphi_o = \varphi_{l^1_1(X)}(\pi_X) \in G(l^1_1(X))$ and $G(f_X)\varphi_o = \varphi_{l^1_2(X)}(\pi_X \circ f_X) = 0$, i.e. $\varphi_o \in \ker G(f_X)$. $\|\varphi_o\| \le \|\varphi\|$ and the assignment $\varphi \mapsto \varphi_o$ is clearly linear. Conversely, assume that $\varphi_o \in \ker G(f_X)$. $\pi_X : l^1_1(X) \to X$ is a quotient map. If $h \in H(l^1,X)$, there exists by 2.2 a map $\bar{h} \in H(l^1, l^1_1(X))$ such that $h = \pi_X \circ \bar{h}$. Now define $\varphi_{l^1}(h) = G(\bar{h})\varphi_o$. This definition is inevitable in order to get $\varphi_{l_1^1(X)}(\pi_X) = \varphi_o$. We will show that φ_{l^1} is well defined and natural. Assume that $\bar{\bar{h}}$ is another map which satisfies $h = \pi_X \circ \bar{\bar{h}}$. Then im $(\bar{h} - \bar{\bar{h}}) \subset \ker \pi_X$. Using the previous lemma again, it follows that $\bar{h} - \bar{\bar{h}} = f_X \circ h_o$ where $h_o \in H(l^1, l_2^1(X))$. Now we need the assumption $\varphi_o \in \ker f_X$ to get $G(\bar{h})\varphi_o = G(\bar{\bar{h}})\varphi_o + G(h_o)G(f_X)\varphi_o = G(\bar{\bar{h}})\varphi_o$ and this means that φ_{l^1} is well defined. It follows easily that φ_{l^1} is linear and the norm estimate for $\bar{h}$ shows that $\|\varphi_{l^1}\| \le \|\varphi_o\|$. Finally take $f \in H(l_M^1, l_N^1)$ and $h \in H(l_N^1, X)$, $\bar{h} \in H(l_N^1, l_1^1(X))$ as before. Then $\pi_X \circ \bar{h} \circ f = h \circ f$ and consequently $\varphi_{l_M^1}(h \circ f) = G(\bar{h} \circ f)\varphi_o = G(f)G(\bar{h})\varphi_o = G(f)\varphi_{l_N^1}(h)$. If $g \in OH(Y,X)$, then $(G_{\underline{l}^1}(g)\varphi)(\pi_Y) = \varphi_{l_1^1(X)}(g \circ \pi_Y) = \varphi_{l_1^1(X)}(\pi_X \circ l_1^1(g)) = G(l_1^1(g))\varphi_o$.

Thus the assignment $\varphi \to \varphi_o$ defines a natural and isometric isomorphism.

Now the covariant case: $F^{\underline{l}^1}(X) = H(.,X) \hat{\otimes}_{l^1} F(.)$. We define a map $\rho : F(l_1^1(X)) \to F^{\underline{l}^1}(X)$ by $f \in F(l_1^1(X)) \to \pi_X \otimes f$.

The dual map $\rho' : \mathrm{Nat}_{\underline{l}^1}(H(.,X), F'(.)) \to F'(l_1^1(X))$ is exactly the one studied before (with $G(X) = F'(X)$). If follows that ρ' is an isometric isomorphism onto the subspace $\ker F'(f_X)$ of $F'(l_1^1(X))$. Consequently ρ is a quotient map, whose kernel coincides with $(\ker F'(f_X))^{\perp} = \overline{\operatorname{im} F(f_X)}$.

2.4. <u>Theorem</u>: $F : \mathrm{Ban} \to \mathrm{Ban}$ is $\underline{l}^1$-computable if and only if F transforms cokernels of strict morphisms into cokernels.

$G : \mathrm{Ban}^{op} \to \mathrm{Ban}$ is $\underline{l}^1$-complete if and only if G transforms cokernels of strict morphisms into kernels.

<u>Proof</u>: The map $f_X : l^1_2(X) \to l^1_1(X)$ is a strict morphism (I.1.6). Consequently, if F commutes with cokernels of strict morphisms then $F^{\underline{l}^1}(X) = \varinjlim \{F(l^1_1(X)) \leftarrow F(l^1_2(X))\} = F(X)$ by 2.3. The same proof works for G.

Conversely let $f : X \to Y$ be a strict morphism and $\gamma = \operatorname{coker} f$ $\gamma : Y \to Z$. Since f is strict, it follows that $l^{\infty}_s(\gamma) : l^{\infty}_s(Y) \to l^{\infty}_s(Z)$ is the cokernel of $l^{\infty}_s(f)$. This means $H(l^1_s,\gamma) = \operatorname{coker} H(l^1_s,f)$ and so the natural transformation $H(.,\gamma) : H(.,Y) \to H(.,Z)$ on $\underline{l}^1$ is the cokernel of the natural transformation $H(.,f) : H(.,X) \to H(.,Y)$. By the adjointness relation $H(G \mathbin{\hat{\otimes}_{\underline{l}^1}} F,U) = \mathrm{Nat}_{\underline{l}^1}(G,H(F,U))$ we see that the functor $. \mathbin{\hat{\otimes}_{\underline{l}^1}} F : \mathrm{Ban}^{\underline{l}^{1op}} \to \mathrm{Ban}$ is left adjoint to $H(F,.)$: $\mathrm{Ban} \to \mathrm{Ban}^{\underline{l}^{1op}}$. Hence it commutes with cokernels and is $H(.,\gamma) \mathbin{\hat{\otimes}_{\underline{l}^1}} F = \operatorname{coker} H(.,f) \mathbin{\hat{\otimes}_{\underline{l}^1}} F$, i.e. $F^{\underline{l}^1}(\gamma) = \operatorname{coker} F^{\underline{l}^1}(f)$.

If G is $\underline{l}^1$-complete, then $G(X) = G_{\underline{l}^1}(X) = \mathrm{Nat}_{\underline{l}^1}(H(.,X),G)$.

Once again, let $Y = \operatorname{coker} f$ in Ban_1, $X \xrightarrow{f} Y \xrightarrow{\gamma} Z$. Then $H(.,\gamma) = \operatorname{coker} H(.,f)$ on $\underline{l}^1$. $H(U, \operatorname{Nat}(G_1, G)) = $ $= \operatorname{Nat}_{\underline{l}^1}(U \hat{\otimes} G_1, G) = \operatorname{Nat}_{\underline{l}^1}(G_1, H(U,G))$ shows that $\operatorname{Nat}_{\underline{l}^1}(.,G)\colon Ban^{\underline{l}^{1\,op}} \to Ban$ is adjoint on the right to $H(.,G)\colon Ban \to Ban^{\underline{l}^{1\,op}}$, so it transforms colimits into limits, in particular, cokernels into kernels and we conclude $\operatorname{Nat}_{\underline{l}^1}(H(.,\gamma),G) = \ker \operatorname{Nat}_{\underline{l}^1}(H(.,f),G)$, i.e. $G_{\underline{l}^1}(\gamma) = \ker G_{\underline{l}^1}(f)$.

2.5. <u>Example:</u> By 2.4. the functor $H(X,.)$ is $\underline{l}^1$-computable if and only if X has the lifting property, i.e. if $f \in H(X,Y)$ and $\gamma\colon Z \to Y$ is a quotient map, there exists $g \in H(X,Z)$ with $\|g\| \le \|f\|(1+\epsilon)$ such that $f = \gamma \circ g$ (compare 2.2.a). It follows from a result of Grothendieck (see [76 p. 487]) that such spaces are already isometrically isomorphic to l^1_M for some index set M.

2.6. <u>Proposition:</u> If $F\colon \underline{l}^\infty \to \underline{Ban}$ is covariant, then

$$F_{\underline{l}^\infty} = \lim_{\leftarrow} \{F(l_1^\infty(X)) \to F(l_2^\infty(X))\}.$$

If $G\colon \underline{l}^\infty \to \underline{Ban}$ is contravariant, then

$$G^{\underline{l}^\infty} = \lim_{\rightarrow} \{G(l_1^\infty(X)) \leftarrow G(l_2^\infty(X))\}.$$

<u>Proof:</u> By 1.5. $F_{\underline{l}^\infty}(X) = \operatorname{Nat}(H(X,.),F)$. If $\varphi \in \operatorname{Nat}(H(X,.),F)$, then $\varphi_0 = \varphi_{l_1^\infty(X)}(i_X) \in \ker F(g_X) \subset F(l_1^\infty(X))$. Conversely if

$\varphi_o \in \ker F(g_X)$, $h \in H(X,l^\infty)$ there exists a map $\bar{h} \in H(l_1^\infty(X),l^\infty)$ such that $h = \bar{h} \circ i_X$ (2.2), and we define $\varphi_X(h) = F(\bar{h})\varphi_o$. By the same methods as in 2.3 it follows that this correspondence $\varphi \longleftrightarrow \varphi_o$ defines a natural isometric isomorphism between $F_{\underline{l}^\infty}(X)$ and $\ker F(g_X)$. If G is contravariant, $G^{\underline{l}^\infty}(X) = G \hat{\otimes}_{\underline{l}^\infty} H(X,.)$. We define $\rho : G(l_1^\infty(X)) \to G^{\underline{l}^\infty}(X)$ by $g \to g \otimes i_X$. The dual map $\rho' : \mathrm{Nat}_{\underline{l}^\infty}(H(X,.),G'(.)) \to G'(l_1^\infty(X))$ coincides with the isometric embedding studied before. Therefore ρ is a quotient map with kernel im $\overline{G(g_X)}$.

2.7. <u>Theorem</u>: $F : \underline{\mathrm{Ban}} \to \underline{\mathrm{Ban}}$ is l^∞-complete if and only if F transforms kernels of strict morphisms into kernels.

$G : \underline{\mathrm{Ban}}^{op} \to \underline{\mathrm{Ban}}$ is $\underline{l}^\infty$-computable if and only if G transforms kernels of strict morphisms into cokernels.

<u>Proof</u>: The map $g_X : l_1^\infty(X) \to l_2^\infty(X)$ is a strict morphism. Consequently, if F commutes with kernels of strict morphisms, then $F_{\underline{l}^\infty}(X) = \varprojlim \{F(l_1^\infty(X)) \to F(l_2^\infty(X))\} = F(X)$ by 2.6. The same proof works for G.

The converse follows also as in 2.4: Let f be a strict morphism, $k = \ker f$, $Z \xrightarrow{k} X \xrightarrow{f} Y$. Since $H(X,l^\infty) = H(l^1,X')$ (I.2.13) and $k' = \mathrm{coker}\, f'$, it follows from the proof of 2.4 that $\mathrm{coker}\, H(f,.) = H(k,.)$ on $\underline{l}^\infty$. The equation $H(U, \mathrm{Nat}_{\underline{l}^\infty}(F_1,F)) = \mathrm{Nat}_{\underline{l}^\infty}(U \hat{\otimes} F_1,F) = \mathrm{Nat}_{\underline{l}^\infty}(F_1,H(U,F))$

shows that the functor $\mathrm{Nat}(.,F) : \mathrm{Ban}^{\underline{l}^{\infty}\,op} \to \mathrm{Ban}$ has a left adjoint. Consequently $\mathrm{Nat}\ (\mathrm{coker}\ H(f,.),F) = \ker\ \mathrm{Nat}\ (H(f,.),F)$. For a contravariant functor G we have $H(G \hat{\otimes}_{\underline{l}^{\infty}} F,U) = \mathrm{Nat}_{\underline{l}^{\infty}}\ (F,H(G,U))$ i.e. $G \hat{\otimes}_{\underline{l}^{\infty}} .$ has a right adjoint and consequently $\mathrm{coker}\ (G \hat{\otimes}_{\underline{l}^{\infty}} H(f,.)) = G \hat{\otimes}_{\underline{l}^{\infty}} H(k,.)$.

2.8. Remark: By 2.7 the functor $H(.,X)'$ is $\underline{l}^{\infty}$-complete if and only if X has the $(1+\epsilon)$-extension property for any $\epsilon > 0$, i.e. for $f \in H(Y,X)$ and an isometry $j : Y \to Z$ there exists $g \in H(Z,X)$ with $\|g\| \le \|f\|(1+\epsilon)$ such that $g \circ j = f$. It follows from a result of Lindenstrauss [45,p.82] that such a space already has the isometric extension property (i.e. $1+\epsilon$ replaced by 1). If we denote as $\underline{P}_1$ the category of all spaces with the isometric extension property, then 2.6 and 2.7 remain valid if the category $\underline{l}^{\infty}$ is replaced by $\underline{P}_1$. (compare this with 2.5).

2.9. Lemma: Let M be a finite dimensional subspace of L^p $(1 \le p \le \infty)$, $\epsilon > 0$. Then there exists a finite dimensional subspace $N \supset M$ and an isomorphism $j : N \to l_n^p$ $(n = \dim N)$ such that $\|j\|\|j^{-1}\| \le 1+\epsilon$.

Proof: Let $(x_i)_{i=1}^m$ be a basis for M. There exist constants $K > L > 0$ such that $L \max |\lambda_i| \le \|\sum_{i=1}^m \lambda_i x_i\| \le K \max |\lambda_i|$ for all $(\lambda_i) \subset I$. We approximate each x_i by some element

$y_i \in L^p$, which takes only a finite number of values, such that $\|x_i - y_i\| < \min(L/2m, \epsilon/(2mK))$. Then $\|\Sigma \lambda_i y_i\| \leq$ $\leq \|\Sigma \lambda_i x_i\| + \Sigma|\lambda_i|\|x_i - y_i\| \leq \max|\lambda_i|(K + m\cdot\frac{K}{2m}) < 2K \max|\lambda_i|$ and analogously $\|\Sigma \lambda_i y_i\| \geq \frac{L}{2} \max|\lambda_i|$. By the Hahn-Banach theorem, there are $y_i' \in (L^p)'$ such that $< y_i, y_j' > = \delta_{ij}$ and $\|y_i'\| \leq 2K$. Each element y_i has a representation of the form $\sum_j a_{ij} c_{A_{ij}}$, where $(A_{ij})_{j=1}^{n_i}$ is a finite partition into measurable sets, and the α_{ij} are distinct scalars. If we take all finite intersections of the A_{ij} $(i = 1, \ldots, m,$ $j = 1,\ldots,n_i)$ we get another partition $(B_j)_{j=1}^{n}$. The subspace $N_o = \{\sum_j \alpha_j c_{B_j} : \alpha_j \in I\}$ is clearly isometrically isomorphic to l_n^p and furthermore $y_i \in N_o$ for $i = 1,\ldots,m$. We have $\|\sum_{i=1}^{m} y_i' \otimes (x_i - y_i)\| \leq m \,.\, 2K \,.\, \epsilon/(2m\, K) = \epsilon$. Consequently if we define $f : L^p \to L^p$ by $f(x) = x +$ $+ \Sigma < x, y_i' > (x_i - y_i)$, then $(1-\epsilon)\|x\| \leq \|f(x)\| \leq (1+\epsilon)\|x\|$ and $f(y_i) = x_i$. Now it is easily seen that the subspace $N = f(N_o)$ meets our requirements with $\epsilon' = (1+\epsilon)/(1-\epsilon) - 1$.

2.10. Proposition: Let $F, F_1 : \underline{L}^p \to \underline{Ban}$ be covariant and $G, G_1 : \underline{L}^p \to \underline{Ban}$ be contravariant functors $(1 \leq p \leq \infty)$, F, G of type Σ.

Then $\mathrm{Nat}_{\underline{L}^p}(F,F_1) = \mathrm{Nat}_{\underline{l}^p}(F,F_1) = \mathrm{Nat}_{\underline{l}^p_{fin}}(F,F_1)$

and $\mathrm{Nat}_{\underline{L}^p}(G,G_1) = \mathrm{Nat}_{\underline{l}^p}(G,G_1) = \mathrm{Nat}_{\underline{l}^p_{fin}}(G,G_1)$.

Proof: We restrict ourselves to the covariant case, the proof for G is similar. By IV. 1. 12 we may also assume that F_1 is essential, i.e. $F(L^p) = L^p \underset{\alpha}{\otimes} X$, $F_1(L^p) = L^p \underset{\beta}{\otimes} Y$ for reasonable norms α and β (IV.1.8). If $\varphi \in \mathrm{Nat}_{\underline{L}^p}(F,F_1)$ its restrictions define natural transformations on $\underline{l}^p$ and $\underline{l}^p_{fin}$. Conversely assume that $\varphi \in \mathrm{Nat}_{\underline{l}^p_{fin}}(F,F_1)$ and define $\varphi_{L^p} : L^p \otimes X \to L^p \underset{\beta}{\otimes} Y$ by $\varphi_{L^p} = 1_{L^p} \otimes \varphi_I$. It is easily seen that φ_{L^p} agrees with the original map on l^p_{fin} and is uniquely determined by this condition. We will show that φ_{L^p} is continuous on $L^p \underset{\alpha}{\otimes} X$. Take an approximate identity (v_ι) for $(L^p)' \hat{\otimes} L^p$ (II.3.10) with $\|v_\iota\| \le 1$, $v_\iota \in (L^p)' \otimes L^p$. Let M_ι be the range of v_ι. To $\epsilon > 0$ there exist, by the preceding lemma, finite dimensional subspaces $N_\iota \supset M_\iota$ and isomorphisms $j_\iota : N_\iota \to l^p_{n_\iota}$ such that $\|j_\iota\|\|j_\iota^{-1}\| \le 1+\epsilon$. Denote by $\bar{v}_\iota : L^p \to N_\iota$ and $k_\iota : l^p_{n_\iota} \to L^p$ the maps that coincide with v_ι resp. j_ι^{-1}. Assume that $u \in L^p \otimes X$, then by IV.1.13 : $\|(1 \otimes \varphi_I)(u)\| = \lim_\iota \|(v_\iota \otimes \varphi_I)(u)\| =$
$= \lim \|(k_\iota \circ j_\iota \circ \bar{v}_\iota \otimes \varphi_I)(u)\| = \lim_\iota \|(k_\iota \otimes 1_Y)\circ\varphi_{l^p_{n_\iota}} \circ$
$\circ (j_\iota \circ \bar{v}_\iota \otimes 1_X)(u)\| \le \lim_\iota \|k_\iota\|\|\varphi_{l^p_{n_\iota}}\|\|j_\iota \circ \bar{v}\|\|u\|_\alpha \le (1+\epsilon)\|\varphi\|\|u\|_\alpha.$

2.11. Theorem: Let $F : \mathrm{Ban} \to \mathrm{Ban}$ be a covariant functor.

Then $(F^{\underline{L}^p})^{Fin} = (F^{\underline{l}^p})^{Fin} = F^{\underline{l}^p_{fin}}$

$$(F_{\underline{L}^p})_{Fin} = (F_{\underline{l}^p})_{Fin} = F_{\underline{l}^p_{fin}} \quad . \quad (1 \le p \le \infty).$$

Let $j : .' \hat{\otimes} X \to H(.,X)$ be the canonical embedding and let $k : . \hat{\otimes} X' \to H(.,X)'$ be given by $\langle u, k(y \otimes x') \rangle =$ $= \langle u(y), x' \rangle$ $(u \in H(Y,X))$. Now $\langle y' \otimes x, j' \circ k(y \otimes x') \rangle =$ $= \langle y, y' \rangle \langle x, x' \rangle = \langle y' \otimes x, y \otimes x' \rangle$, i.e. $j' \circ k = 1$. Consequently k is an isometry. ρ is given by $\rho(\varphi) = j' \circ \varphi$ and it follows that ρ is a quotient map.

2.13. Proposition: A (Fin-) computable, covariant functor $F : Ban \to Ban$ is $\underline{l}^1$-computable, if and only if it commutes with quotient maps.

A (Fin-) complete, covariant functor $F : Ban \to Ban$ is $\underline{l}^\infty$-complete, if and only if it commutes with isometries.

In both cases it suffices if the condition holds for finite dimensional spaces.

Proof: Assume that F commutes with isometries on Fin. Take $M \in Fin$, $\varepsilon > 0$. By the same method as in II.2.8 one finds spaces M_1, $l_n^\infty \in Fin$, an isomorphism $\varphi : M \to M_1$ such that $\|\varphi\| \|\varphi^{-1}\| \leq 1 + \varepsilon$ and an isometry $j : M_1 \to l_n^\infty$. Since all spaces are finite dimensional, we have algebraically $F(M) = M \otimes F(I)$, $F(M_1) = M_1 \otimes F(I)$ and $F(j) = j \otimes F(I)$. Clearly $\operatorname{im} F(j) = \operatorname{im} j \otimes F(I)$. According to 2.7 $F_{\underline{l}^\infty}$ transforms kernels of strict morphisms into kernels, and so $F_{\underline{l}^\infty}(M_1) =$ $\operatorname{im} j \otimes F(I)$. But since $F(j)$ is an isometry, the latter space coincides with $F(M_1)$, i.e. $F(M_1) = F_{\underline{l}^\infty}(M_1)$ isometrically.

$\|F(\varphi)\|\|F_{\underline{l}^\infty}(\varphi^{-1})\| \leq 1+\epsilon$ and so the norms on $M \otimes F(I)$, which are induced by F and $F_{\underline{l}^\infty}$ differ only by $(1+\epsilon)^2$. Since ϵ was arbitrary, $F(M) = F_{\underline{l}^\infty}(M)$ holds too, i.e. $F = F_{\underline{l}^\infty}$ on Fin. Since by 2.11 Cor. 2 both functors are computable, they have to agree everywhere. The proof for $\underline{l}^1$ is similar, using dual arguments.

Exercises

1) Let F be Fin-complete. Then there exists a natural transformation from $F_1(.', F(I))$ into F.

2) If F is Fin-complete and $X \in \underline{A}$, then $DF(X) = (0)$. (Use the fact that $DF(X) \subseteq H(X', DF(I))$.

3) Show that $(.'\hat{\hat{\otimes}} X)_{\underline{l}^1} = K(., X)$.

Dually one gets $(. \hat{\hat{\otimes}} X)_{\underline{l}^\infty}(Y') = K(Y,X)$ and

$I_1(.', X')^{\underline{l}^1} = K(., X)$ for $X \in$ Ban.

It follows that Prop. 2.12 is not valid for p = 1. (Take a space X which has the approximation property but not the metric approximation property and use II.3.9).

4) Show that $(F_{\underline{Fin}})_e(X) = F_e(X)$ for $X \in \underline{A}$.

List of Symbols

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References

[1] M. ALTMAN, Contractors, approximate identities and factorization in Banach algebras, Pac. J. Math 48 (1973), 323 - 334.

[2] I. AMEMIYA, K. SHIGA, On tensor products of Banach spaces, Kodai Math. Sem. Rep. 9 (1957) 161 - 178.

[3] G.F. BACHELIS, J.E. GILBERT, Banach spaces of compact multipliers and their dual spaces, Math. Z. 125 (1972) 285 - 297.

[4] H. BUCHWALTER, Espaces de Banach et dualité, Publ. Dép. Math. Lyon 3 - 2 (1966), 2 - 61.

[5] H. BUCHWALTER, Topologies, bornologies et compactologies, Thêse Doc. Sc. Math. Fac. Sc. Lyon 1968.

[6] M.C. BUNGE, Relative Functor Categories and Categories of Algebras, Journal of Algebra 11 (1969), 64 - 101.

[7] S.D. CHATTERJI, Martingale convergence and the Radon Nikodym theorem in Banach spaces, Math. Scand. 22 (1968) 21 - 41.

[8] S. CHEVET, Sur certains produits tensoriels topologiques d'espaces de Banach, Z. Wahrscheinlichkeitstheorie verw. Geb. 11, 120 - 138 (1969); C.R. Acad. Sci. Paris, Sèr A, 266 (1968).

[9] J. CIGLER, Funktoren auf Kategorien von Banachräumen,Mh. für Mathematik 78 (1974) 15 - 24.

[10] J. CIGLER, Duality for functors on Banach spaces, preprint, Vienna 1973.

[11] J. CIGLER, Funktoren auf Kategorien von Banachräumen, Lecture Notes, University of Vienna, 1974.

[12] J. CIGLER, Tensor products of functors on categories of Banach spaces, Springer Lecture Notes 540, 164 - 187.

[13] J. CIGLER, Zur Dualität von Funktoren, die durch Funktionenräume definiert sind, Mh für Mathematik 82, 117 - 123 (1976)

[14] G. CROFTS, Generating classes of perfect Banach sequence spaces, Proc. AMS 36 (1972) 137 - 143.

[15] A.M. DAVIE, The approximation problem for Banach spaces, Bull. London Math. Soc. 5 (1973) 261 - 266.

[16] D.W. DEAN, The equation $L(E,X^{**}) = L(E,X)^{**}$ and the principle of local reflexivity, Proc. AMS 40 (1973) 146 - 148.

[17] J. DIESTEL, The Radon-Nikodym property and the coincidence of integral and nuclear operators, Rev. Roum. Math pures et appl. 42 (1972), 1611 - 1620

[18] N. DINCULEANU, Vector measures, Pergamon Press 1967.

[19] J. DIXMIER, Sur un théorème de Banach, Duke Math. J. 15 (1948), 1057 - 1071.

[20] E.J. DUBUC, Kan extensions in enriched category theory, Springer Lecture Notes 145 (1970).

[21] N. DUNFORD, J.T. SCHWARTZ, Linear Operators I, General Theory, Interscience Publishers, New York 1958.

[22] A. DVORETZKY, C.A. ROGERS, Absolute and Unconditional convergence in normed linear spaces, Proc. Nat. Acad. Sci (USA) 36 (1950), p. 192 - 197.

[23] P. ENFLO, A counterexample to the approximation problem, Acta Math. 130 (1973) 309 - 317.

[24] J. EPEMA, Functors on the category of Hilbert spaces, thesis, Rijksuniversiteit Groningen 1973.

[25] J. EPEMA, H.C. MEIJER, W. OOSTENBRINK, Normed left ideals in B(H), Math. Institute Rijksuniv. Groningen, Report - ZW - 71 - 02.

[26] T. FIGIEL, W.B. JOHNSON, The approximation property does not imply the bounded approximation property, Proc. AMS 41 (1973), p 197 - 200.

[27] T. FIGIEL, A. PELCZYNSKI, On Enflo's method of construnction of Banach spaces without the approximation property (russ.), Usp. Mat. Nauk SSSR 28 - 6 (1973), p. 95 - 108.

[28] J. FISHER-PALMQUIST, D.C. NEWELL, Triples on functor categories, J. Algebra 25 (1973) 226 - 258.

[29] I.C. GOHBERG, M.G. KREIN, Introduction to the theory of linear non-selfadjoint operators, Transl. of Math. Monographs 18, AMS (1969).

[30] Y. GORDON, D.R. LEWIS, J.R. RETHERFORD, Banach ideals of operators with applications, J. Functional Analysis 14 (1973) 85 - 129.

[31] M. GROSSER, Bidualräume und Vervollständigungen von Banachmoduln. Dissertation Wien 1976.

[32] A. GROTHENDIECK, Produits tensoriels topologiques et espaces nucléaires, Mem. AMS 16 (1955).

[33] A. GROTHENDIECK, Résumé de la théorie metrique des produits tensoriels topologiques, Bol. Soc. Matem. Sao Paulo 7 (1952).

[34] S.L. GULICK, T.S. LIN, A.C.M. van ROOIJ, Group algebra modules I, Can. J. Math. 19 (1966) 133 - 150.

[35] C. HERZ, J. WICK PELLETIER, Dual functors and integral operators in the category of Banach spaces, preprint 1974.

[36] J.R. HOLUB, Compactness in topological tensor products and operator spaces, PAMS 36.

[37] W.B. JOHNSON, A complementably universal conjugate Banach space and its relation to the approximation problem, Isr. J. of Math. 13 (1972) p. 301 - 310.

[38] B.E. JOHNSON, An introduction to the theory of centralizers, Proc. London Math. Soc. 14 (1964) 294 - 320.

[39] B.E. JOHNSON, Cohomology in Banach algebras. Mem. A.M.S. 127 (1972).

[40] KONECNY, Tensorprodukte und Funktoren auf Kategorien von Banachräumen, Dissertation Wien 1976.

[41] I.V. KUSNEZOV, A.S. SHVARTS, Duality of functors and duality of categories (russ.) Sibirskij Mat. J. 9 (1968) 840 - 856.

[42] S. KWAPIEN, On Enflo's example of a Banach Space without the approximation property, Sem. Goulaouic - Schwartz 1972 - 73, Exp. VIII.

[43] M. LEINERT, A commutative Banach algebra which factorizes but has no approximate units, Proc. AMS 55 (1976), 345 - 346.

[44] V.L. LEVIN, Tensor products and functors in categories of Banach spaces defined by KB-lineals. Transl. Moscow Math. Soc. 20 (1969) 41 - 77 (AMS).

[45] J. LINDENSTRAUSS, Extension of compact operators, Mem. AMS 48 (1964).

[46] F.E.J. LINTON, Autonomous categories and duality of functors, J. Algebra 2 (1965), 315 - 349.

[47] F.E.J. LINTON, On a pullback lemma for Banach spaces and the functorial semantics of double dualization, preliminary report (1970).

[48] V. LOSERT, Dualität von Funktoren und Operatorenideale, Math. Nachr. to appear.

[49] S. Mac LANE, Categories for the Working Mathematician, Springer (1972) GTM 5.

[50] H.C. MEIJER, Normed left ideals, Thesis Rijksuniversiteit Groningen 1973.

[51] P. MICHOR, Funktoren auf Kategorien von Banachräumen, Dissertation, Wien 1973.

[52] P. MICHOR, Zum Tensorprodukt von Funktoren auf Kategorien von Banachräumen, Mh. für Mathematik 78 (1974) 117 - 130.

[53] P. MICHOR, Funktoren zwischen Kategorien von Banach- und Waelbroeckräumen, Sitzungsberichte österr. Akad. Wiss. Abt II, 182, (1973) 43 - 65.

[54] P. MICHOR, Duality for contravariant functors on Banach spaces, Mh für Mathematik 82, 177 - 186 (1976).

[55] P. MICHOR, Functors and categories of Banach spaces (Tensor products, Operator ideals and functors on categories of Banach Spaces). Lecture Notes in Math. 651, Springer 1978.

[56] P. MICHOR, Banach Semikategorien I, II, III, Sitzungsber. österr. Akad. Wiss. Abt. II, 185, 181 - 238 (1976).

[57] B.S. MITIAGIN, A.S. SHVARTS, Functors on categories of Banach spaces, Russian Math. Surveys 19 (1964) 65 - 127.

[58] J.W. NEGREPONTIS, Duality of functors on categories of Banach spaces, J. pure appl. Algebra 3 (1973) 119 - 131.

[59] W. OOSTENBRINK, Normed ideals, thesis Rijksuniversiteit Groningen 1973.

[60] W.L. PASCHKE, A factorable Banach algebra without approximate unit, Pac. J. Math. 46 (1973), 249 - 251.

[61] J.W. PELLETIER, Dual functors and the Radon-Nikodym Property in the Category of Banach spaces, Preprint York University.

[62] A. PIETSCH, Ideale von S_p-Operatoren in Banachräumen. Studia Math. 38 (1970) 59 - 69.

[63] A. PIETSCH, Adjungierte normierte Operatorenideale Math. Nachrichten 48 (1971) 189 - 212.

[64] A. PIETSCH, Theorie der Operatorenideale, Jena 1972.

[65] H.R. PITT, A note on bilinear forms, J. London Math. Soc. 11 (1936), 171 - 174.

[66] R.S. POKASEJEWA, A.S. SHVARTS, Functors in categories of Banach spaces (russ), Uspechij 19 (1964) 65 - 130.

[67] K.L. POTHOVEN, Compact functors and their duals in categories of Banach spaces, Trans. AMS 155 (1971) 148 - 159.

[68] G. RACHER, Funktoren auf Kategorien von Banachmoduln, Dissertation Wien 1974.

[69] H. REITER, L^1-algebras and Segal algebras, Springer, Lecture Notes 231.

[70] M.A. RIEFFEL, Induced Banach representations of Banach algebras and locally compact groups, J. Functional Analysis 1 (1967) 443 - 491.

[71] M.A. RIEFFEL, Multipliers and tensor products of L^p-spaces of locally compact groups, Studia Mathematica 33 (1969) 71 - 82.

[72] P. SAPHAR, Produits tensoriels d'espaces de Banach et classes d'applications linéaires, Studia Math. 38 (1970) 71 - 100.

[73] H.H. SCHAEFER, Topological Vector Spaces, Springer Verlag, New York - Heidelberg - Berlin 1971.

[74] R. SCHATTEN, A theory of cross spaces Ann. of Math. Studies no 26, Princeton 1950.

[75] R. SCHATTEN, Norm ideals of completely continuous operators, Springer Verlag, Berlin - Göttingen - Heidelberg 1960.

[76] Z. SEMADENI, Banach spaces of continuous functions I, MM 55, Warschau 1971.

[77] Z. SEMADENI, Monads and Their Eilenberg-Moore Algebras in Functional Analysis, Queen's Papers in Pure and Applied Mathematics No. 33, Kingston, Ontario 1973.

[78] Z. SEMADENI, A. WIWEGER, A theorem of Eilenberg-Watts type for tensor products of Banach spaces, Studia Math. 38 (1970), 235 - 242.

[79] A.S. SHVARTS, Duality of functors, Dokl. Akad. Nauk 148 (1963) 288 - 291.

[80] A.S. SHVARTS, Functors in categories of Banach spaces, Dokl. Akad. Nauk 149 (1963) 44 - 47.

[81] H. SPITZER, Funktoren und Tensorprodukte von Funktoren auf Kategorien von Banachräumen, Dissertation, Wien 1977.

[82] L. WAELBROECK, Some theorems about bounded structures, J. Functional Analysis 1 (1967), 392 - 408.

[83] J. WICHMANN, Bounded approximate units and bounded approximate identities, PAMS 41 (1973), 547 - 550.

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