REPRESENTATION SCHEMES OF FORMALLY SMOOTH ALGEBRAS

(Schemas von Darstellungen von formal glatten Algebren)

Dissertation
zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
an der Universität Wien
eingereicht von
Ali R. Assar

Wien, im Oktober 2000

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Erklärung

Im Herbst 1997 wurde ich zum Doktoratsstudium für Mathematik der Universität Wien unter der Betreuung von Prof. P. W. Michor zugelassen.

Die vorgelegte Dissertation entstand aus der Forschung, die ich in dem akademischen Jahr 1999 - 2000 in Form von Seminaren und unter Anleitung von Professor P. Michor und Professor A. Cap durchführte.

Diese Dissertation wurde Oktober 2000 der Universität Wien eingereicht.

A. R. Assar
oct. 2000

CURRICULUM VITAE

A. R. Assar, geboren am 17. August 1953, Teheran, Iran.

Nach Beendigung der Grundschule und des Gymnasiums in Teheran, wurde ich im Juni 1971 zum Studium der Physik an der Wisschenschaftsfakultät in Teheran zugelassen.

Im Juni 1975 erhielt ich mein B.Sc., 1st class honors, in Physik. Im Anschluss erhielt ich einen 2-jähriges Post graduate M.SC. Stipendium an die St. Andrews Universität in Schottland.

Im Herbst 1977 legte ich die Dissertation "Reduction of the principal series unitary irreducible representation of Poincarè group on an ahelian subgroup" vor und erhielt den Master of Science für theoretische Physik.

Danach wurde zu weiterführenden Studien in Theoretischer Physik zugelassen an der Institution in Triest (Italien), die heute den Namen SISSA trägt. Nach zweijährigem Studium (mit Prüfungen) ging in die USA, da seinerzeit in Italien ein Ph.D. nicht anerkannt war und zum gleichen Zeitpunkt die politischen Verhältnisse im Iran in Umbruch waren.

1981 kehrte ich nach Italien zurück und arbeitete in einer Forschungsgruppe für Hochenergie Physik in der Scuola Normale Superiore in Pisa mit.

Von 1981 - 1984 Zusammenarbeit mit Prof. R. Barbieri, öfter auch als Gastwissenschaftler im I.C.T.P., Triest, wo ich sehr von der Unterstützung und Ermunterung von Prof. A.Salam profitierte.

1984 erhielt ich vom iranischen Ministerium für Wissenschaft und Technolgie eine Einladung als ausserordentlicher Professor an die Universität Shahid Bahonar in Kermanan. Ich erteilte Kurse in: Quantum Mechanics, Quantum field theory, elementary particles general relativity, methods of Mathematical Physics, Group Theory.

1992 flüchtete ich nach Österreich. Zu diesem Zeitpunkt hatte ich zwei Jahre politischer Verfolgung, wahlloser Einkerkerung und Willkür hinter mir.

A. R. Assar

DANKSAGUNG

Ich danke für die Unterst"utzung, die mir während meiner Arbeit zuteil wurde. Besonderen Dank gebührt meinen Beratern, Professor P. W. Michor und Professor A. Cap für ihre Geduld und ihren Rat bei der Durchführung dieser Forschungsarbeit.

Auch meinem Bruder, Dr. G. R. Assar spreche ich hiermit meinen besonderen Dank aus für seine moralische und finanzielle Unterstützung während der drei Studienjahre. Ohne seine Hilfe hätte ich diese Arbeit nicht beginnen und beenden können.

Abschliessend danke ich auch Dr. M. Nikbakht-Tehrani und E. Brandstetter für ihre Freundschaft und Unterstützung in schwieriger Zeit.

A. R. Assar

EINFÜHRUNG

Die grundlegende Motivation zu dieser Arbeit resultiert aus einem Versuch von M. Kontsevich und A. Rosenberg, Modelle nichtkommunitativer glatter Räume zu konstruieren. In der üblichen (kommutativen) Geometrie, enthält die Algebra A aller stetigen Funktionen auf einer Mannigfaltigkeit (und noch allgemeiner auf einem topologischen Hausdorff Raum) alle geometrischen Informationen über den topologischen Raum selbst; Punkte des zugrundeliegenden Raumes entsprechen in ein-eindeutiger Weise dem maximalen Idealen der Algebra der Funktionen auf diesem Raum, welche kommutativ ist. Dieser algebraisch-geometrische Dualismus bricht zusammen, wenn der Versuch unternommen wird, das obige Bild auf den Fall zu übertragen, dass A eine nicht kommutative Algebra ist, denn in diesem Fall gibt es kein maximales Spektrum von A und deshalb gibt es keinen "Punkt" in dem möglicherweise deswegen nicht existierenden zugrundeliegenden Raum.

Es gab verschiedene Versuche, nicht kommutative Räume so zu konstruieren, dass doche eine unterliegende Punktmenge existiert. Solch ein Versuch wurde von den oben genannten Autoren unternommen, welche eine Methode zur Konstruktion von nicht kommutativen glatten Räumen vorschlugen. Hier ist glatt im Sinne der algebraischen Geometrie zu verstehen, als singularitätenfrei. Die hier interessierenden Algebras sind die sogenannten Quillen-glatten Algebren. Diese Algebren werden durch ihre Eigenschaft definiert, dass Homomorphismen über nilpotente Algebra-Erweiterungen liften. Man zeigt dann, dass die Darstellungs-Varietäten und allgemeiner die Darstellungs-Schemas die zu Quillen glatten Algebren assoziiert sind, glatte affine Varietäten sind.

Von den genannten Autoren wird vorgeschlagen, die disjunkte Vereinigung

$$\bigsqcup_{n \in \mathbb{N}^+} \operatorname{Rep}_n A$$

zu betrachten und dann eine Methode anzuwenden, die von Kapranov vorgeschlagen wurde in der Arbeit "Non-commutative geometry based on commutator expansions", math.ag/9802041 (1998), um eine Garbe von nichtkommutativen Algebren auf dieser Vereinigung zu erhalten und somit ein Modell einer nichtkommutativen affinen Varietät zu konstruieren.

In meinen Studien habe ich meine Aufmerksamkeit darauf beschränkt, eine verständliche und detaillierte Studie des Representations-Schemas Plans zu erstellen, welche, wie bereits ausgeführt, die Bausteine eines Modells der nichtkommutativen glatten Räume sind.

In Kapitel 1 werden die "formal glatten Algebren" untersucht. Das Kapitel beginnt mit einer kurzen Einführung in die Hochschild (co-)homology of associative Algebren. Das Konzept der Kähler Differentiale wird für den kommutativen und den nichtkommutativen Fall untersucht.

Das Problem der Algebra-Erweiterungen wird detailliert dargestellt und es zeigt sich, dass die 2-Zyklen welche äquivalente singuläre (oder abelsche) Erweiterungen $0 \to N \to B \to A \to 0$ beschreiben, dasselbe Element der Kohomologie-Gruppe $H^2(A, N)$ darstellen. Darüberhinaus wird der detaillierte Beweis des folgenden wohlbekannten Resultates gegeben: *Es sei A eine R-Algebra welches R-projectiv ist, und es sei ein A-Bimodul N gegeben. Dann besteht eine 1-1 Korrespondenz* *zwischen den Elementen in $H^2(A, N)$ und der Menge der Äquivalenzklassen von singulän Erweiterungen von A with kernel N.*

Konzepte, wie die homologische Dimension von Moduln and Algebren werden besprochen.

Es werden dann die grundlegenden Definitionen und Resultate über formal glatte Algebren präsentiert und in Theorem 11 werden äquivalente Bedingungen bewiesen, welche solche Algebras charakterisieren.

Danach wird eine bedeutende Klasse von formal glatten Algebren untersucht, nämlich die Pfad-Algebren von Köchern. Wir zeigen, dass die Algebra die zu einem orientierten Graphen assoziiert ist, formally glatt ist. Wir diskutieren viele weitere Eigenschaften von solchen Algebren. Die Konzepte von Darstellungen einer K"ocher-Algebra und von endlichdimensionalen Darstellungen eines K"ochers werden besprochen, und es zeigt sich, dass die Kategorien solcher Objecte are äquivalent sind. Abschliessend wird in Kapitel 1 das Konzept der Eulerform eines K"ochers Q vorgestellt und interessante Verbindungen zwischen der Eulerform von Q und den Dimensionen der Vectorräume $\mathrm{Hom}_{\mathbb{C}Q}(V, W)$ und $\mathrm{Ext}^1_{\mathbb{C}Q}(V, W)$ für gegebene Darstellungen V, W von Q.

In Kapitel 2 stellen wir die Konzepte der Halsband- und der Spur-Algebren vor. Diese Konzepte sind ursprünglich von C. Procesi in einem Papier über "invariant theory von $n \times n$ matrices under simultaneous conjugations" entwickelt worden. Wir beginnen mit der Untersuchung der GL_n -Orbits von $n \times n$ Matrizen und des entspechenden Quotientenraumes $M_n//GL_n$. Die Wirkung von GL_n am Polynomring $\mathbb{C}[M_n]$ wird betrachtet, und es zeigt sich, dass für den Teilring der invarianten Polynome gilt:

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)],$$

wo die σ_i die elementarsymmetrischen Funktionen sind, und wo X eine generische $n \times n$ Matrix ist.

In Theorem 6 zeigen wir eine alternative Beschreibung dieses Rings der Invariants, nämlich

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[\mathrm{tr}(X), \mathrm{tr}(X^2), \dots, \mathrm{tr}(X^n)].$$

In Sektion 3 des 2. Kapitels betrachten wir das Problem der simultanen Konjugiertheitsklassen von m-Tupeln von $n \times n$ Matrizen M_n^m. Die Klassifizierung der Orbits der GL_n-Wirkung durch Konjugation ist im allgemeinen ein unmögliches Unterfangen. Einige Beispiele im Falle von kleinen m und n werden betrachtet. In Sektion 4 stellen wir die Konzepte von Halsband- und Spur-Algebren vor. Es wird zuerst gezeigt, dass jede polynomiale Abbildung $f : M_n^m \to \mathbb{C}$ welche konstant auf den Orbits der glaichzeitigen Konjugation unter GL_n ist, ein Polynom ist in den Invarianten $\mathrm{tr}(X_{i_1} X_{i_2} \dots X_{i_l})$.

Dieser Ring der polynomialen Invarianten wird mit

$$\mathbb{N}_n^m := \mathbb{C}[M_n^m]^{GL_n}$$

bezeichnet und heißt die Halsband-Algebra. Dann betrachten wir alle GL_n-äquivarianten Abbildungen $M_n^m \to M_n$, welche unter punktweiser Multiplikation und Addition eine nichtkommutative Algebra bilden, welche mit

$$\mathbb{T}_n^m := \mathrm{Hom}_{\mathbb{C}}(M_n^m, M_n)^{GL_n} = M_n(\mathbb{C}[M_n^m])^{GL_n}$$

bezeichnet wird und die Spur-Algebra heißt. Klarerweise haben wir Einbettungen

$$\mathbb{N}_n^m := \mathbb{C}[M_n^m]^{GL_n} \xrightarrow{\cdot 1_n} \mathbb{T}_n^m := M_n(\mathbb{C}[M_n^m])^{GL_n} \to M_n(\mathbb{C}[M_n^m]),$$

und $\mathbb{N}_n^m$ wird dadurch mit dem Zentrum von $\mathbb{T}_n^m$ identifiziert. Es wird gezeigt, dass $\mathbb{T}_n^m$ als Algebra über dem Zentrum $\mathbb{N}_n^m$ durch die Elemente $X_1, \ldots, X_m$ erzeugt wird. Der Satz von Nagata-Higman wird benutzt um eine Schranke $l \leq 2^n - 1$ anzugeben für die Länge l von Monomen $X_{i_1} \ldots X_{i_l}$ welche $\mathbb{T}_n^m$ als Modul über $\mathbb{N}_n^m$ erzeugen. Später wird diese Schranke noch verbessert. Am Schluss von Kapitel 2 diskutieren wir die formalen Halsband- und Spur-Relationen und zeigen, wie diese Relationen aus den Cayley-Hamilton Relationen für $n \times n$-Matrizen folgen. Dabei verwenden wir die Methode der Polarisation oder der Multilinearisation.

Kapitel 3 ist eine Sammlung von Begriffen aus der algebraischen Geometrie, welche wir für die Überlegungen im Kapitel 4 benötigen. Wir diskutieren den Hilbert'schen Nullstellensatz, die Zariski Topologie, und erklären die Entsprechungen der Begriffe aus der Algebra und aus der Geometrie. In Abschnitt B sturieren wir das Konzept der affinen algebraischen Varietät in der allgemeinsten Form und spezifizieren die Kategorie dieser Objekte und ihrer Morphismen. In Abschnitt C diskutieren wir affine algebraische Varietäten aus funktorieller Sicht; dies ist am nötigsten für die Entwicklungen im Kapitel 4. Eine leichte Verallgemeinerung dieses Konzeptes führt zum Begriff des affinen Schemas. Im Abschnitt D betrachten wir die Konzepte des Tangentialkegels and des Tangentialraumes sowohl aus geometrischer Sicht, als auch aus funktorieller Sicht. Am Schluss werden einige Themen wie die Krull-Dimension einer Varietät and die intrinsische Definition des Tangentialkegels behandelt.

Kapitel 4 enthält eine detaillierte Studie Darstellungs-Varietät $\operatorname{Rep}_n A$ für eine assoziative unitale Algebra A. Wir zeigen, dass $\operatorname{Rep}_n A$ eine GL_n-Varietät ist und beschreiben die Orbits dieser Wirkung. Eine leichte Verallgemeinerung führt zum Darstellungs-Schema $\underline{\operatorname{Rep}_n} A$. In Sektion 3 besprechen wir in einem allgemeinen Rahmen den Ring der GL_n-Invarianten und die entsprechenden GL_n-Quotienten Varietäten, welche in Sektion 4 verwendet werden. Dort studieren wir die Invariantentheoretische Rekonstruktion von Cayley-Hamilton Algebren. Konzepte wie die einer Spur-Algebra oder die von Algebren welche durch eine Teilmenge Spur-erzeugt sind, werden besprochen. Die Spur-Abbildung wird in natürlicher Weise definiert. Dann wird gezeigt, dass eine Cayley-Hamilton Algebra A, welche durch m Elemente Spur-erzeugt ist, in kanonischer Weise durch ein natürlich gegebenes Ideal $N_A \triangleleft k[M_n^m]$ wie folgt rekonstruiert werden kann:

$$A = M_n\left(k[M_n^m]/N_A\right)^{GL_n}, \qquad \operatorname{tr}_A(A) = \left(k[M_n^m]/N_A\right)^{GL_n}.$$

Als Nächstes in Kapitel 4 wird eine geometrische Rekonstruktion von Cayley-Hamilton Algebren diskutiert, und die folgende Äquivalenz von Funktoren wird bewiesen:

$$\operatorname{Hom}^{\operatorname{tr}}_{k-\mathrm{alg}}(A, M_n(-)) \cong \operatorname{Hom}_{k-\mathrm{alg}}(k[M_n^m]/N_A, -).$$

Die GL_n-Orbit Quotienten Schemas $\underline{\mathrm{ISS}_n} A$ und $\underline{\mathrm{ISS}_n^{\mathrm{tr}}} A$ werden untersucht, und in Satz 13 wird gezeigt, dass der Funktor, welcher einer Cayley-Hamilton Algebra A

vom Grad n das GL_n-affine Schema $\underline{\mathrm{Rep}}_n^{tr}\, A$ aller Spur-erhaltenden n-dimensionalen Darstellungen von A zuordnet, eine Linksinverse besitzt. Diese Linksinverse ordnet einem GL_n-affinenSchema X die sogenannte Zeugen-Algebra $M_n(k[X])^{GL_n}$ zu, welche eine Cayley-Hamilton Algebra vom Grad n ist. In Abschnitt 6 betrachten wir den BEgriff der Cayley-Glattheit. Zuerst wird gezeigt, dass füer ein gegebenes affines Schema X der Koordinatenring $k[X]$ genau dann Grothendieck-glatt ist wenn X in allen geometrischen Punkten nichtsingulär ist; dies impliziert auch dass dann X eine affine Varietät ist. Ein analoges Resultat für Cayley-glatte Algebren wird in Satz 18 bewiesen. In Abschnitt 7 werden die Tangential- und Normal-Räume an die Darstellungs-Schemas $\underline{\mathrm{Rep}}_n\, A$ und $\underline{\mathrm{Rep}}_n^{tr}\, A$ im Detail untersucht. In Abschnitt 8 wird Luna's etaler Scheibensatzverwendet um um die etale lokale Struktur von $\underline{\mathrm{ISS}}_n A$ zu untersuchen. Im letzten Abschnitt untersuchen wir die Modul-Struktur des Normalraumes an die Darstellungsschemas und zeigen, wie markierte Köcher verwendet werden können um solche Strukturen zu beschreiben.

CHAPTER I

Formally smooth algebras

(1.1) Hochschild homology

Let A be an R-algebra where R is a commutative ring and let M be an A-bimodule. Let $C_n(A,M) := M \otimes_R A^{\otimes n}$, where $A^{\otimes n} = A \otimes \cdots \otimes A$ (n-fold). The Hochschild boundary map is an R-linear map $b : M \otimes_R A^{\otimes n} \longrightarrow M \otimes_R A^{\otimes n-1}$ given by the formula

$$b(m, a_1, \ldots, a_n) := (ma_1, a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n m, a_1, \ldots, a_{n-1}).$$

where for simplicity we have written $m \otimes a_1 \otimes \cdots \otimes a_n$ as $(m, a_1, \ldots, a_n)$. The formula makes sense because M is an A-bimodule.

To handle this map easily, let us introduce the maps $d_i : M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$ by

$$d_0(m, a_1, \ldots, a_n) := (ma_1, a_2, \ldots, a_n),$$

$$d_i(m, a_1, \ldots, a_n) := (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n), \quad 1 \le i < n$$

$$d_n(m, a_1, \ldots, a_n) := (a_n m, a_1, \ldots, a_{n-1}).$$

With this notation one has

$$\boxed{b = \sum_{i=0}^{n} (-1)^i d_i}$$

Lemma 1. $b \circ b = 0$.

Proof. It is immediate that

$$d_i \circ d_j = d_{j-1} \circ d_i \quad , \quad 0 \leq i \leq j \leq n$$

hold; from which it follows that $b \circ b = 0$. ∎

As a consequence of the above lemma, we get the Hochschild Complex

$$C_\bullet(A,M): \quad \cdots \longrightarrow M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M$$

where $C_n(A,M) := M \otimes A^{\otimes n}$.

In case $M = A$ one gets the following Hochschild complex

$$C_\bullet(A): \quad \cdots \longrightarrow A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \cdots \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A$$

which is sometimes called the cyclic bar complex in the literature.

Def. The n-th Hochschild homology group of a unital R-algebra A with coefficients in an A-bimodule M is the n-th homology group of the Hochschild complex $(C_\bullet(A,M), b)$. By definition $H_\bullet(A,M) := \bigoplus_{n \geq 0} H_n(A,M)$. ∎

In what follows we shall use the notation $HH_n(A)$ for the n-th Hochschild homology group $H_n(A,A)$.

(1.2) Kähler differentials.

In this section A denotes a commutative unital algebra over a commutative ring R.

The Kähler differentials of A over R is an A-module

denoted by $\Omega^1_{A|R}$ which has the following presentation:

for every $a \in A$ there is a generator of $\Omega^1_{A|R}$, denoted by da, subject to the following relations

(1) $d(r_1 a_1 + r_2 a_2) = r_1 da_1 + r_2 da_2$,

(2) $d(a_1 a_2) = a_1 da_2 + a_2 da_1$, $\quad \forall r_1, r_2 \in R, \ \forall a_1, a_2 \in A$

moreover we assume $d(r) = 0$ for all $r \in R$ (this is equivalent to the condition $d(1) = 0$). It follows that

$$\Omega^1_{A|R} = \langle da, \ \forall a \in A \mid (1) \text{ and } (2) \rangle_A$$

$$= \langle b\,da, \ \forall a,b \in A \mid (1), (2) \text{ and relations in } A \rangle_R .$$

Proposition 2. Let A be a commutative unital R-algebra. Then there is a canonical isomorphism

$$HH_1(A) \cong \Omega^1_{A|R} .$$

Moreover if M is a symmetric bimodule (i.e. $am = ma$, $\forall a \in A, \forall m \in M$), then $H_1(A,M) \cong M \otimes_A \Omega^1_{A|R}$.

Proof.

$$C_\bullet(A): \quad \cdots \xrightarrow{b} A \otimes A \otimes A \xrightarrow{b} A \otimes A \xrightarrow{b} A ,$$

$$b(a_1 \otimes a_2) = a_1 a_2 - a_2 a_1$$
$$= 0 \quad (\text{for } A \text{ is commutative}).$$

⇒ The last map b is trivial; hence $\operatorname{Ker} b = A \otimes A$.

The $\operatorname{Im}(A \otimes A \otimes A \xrightarrow{b} A \otimes A)$ is of the form

$$b(a_1 \otimes a_2 \otimes a_3) = a_1 a_2 \otimes a_3 - a_1 \otimes a_2 a_3 + a_3 a_1 \otimes a_2 .$$

Thus

$$HH_1(A) = A \otimes A \Big/ \langle a_1 a_2 \otimes a_3 - a_1 \otimes a_2 a_3 + a_3 a_1 \otimes a_2 \mid \forall a_1, a_2, a_3 \in A \rangle$$

where the denominator is the submodule of $A \otimes A$ generated by the relations of the form $a_1a_2\otimes a_3 - a_1\otimes a_2a_3 + a_3a_1\otimes a_2 = 0$.

We define a correspondence $\eta : \begin{cases} HH_1(A) \longrightarrow \Omega^1_{A|R} \\ a_1\otimes a_2 \longmapsto a_1\,da_2 \end{cases}$.

This is a well-defined mapping, for

$$\eta(a_1a_2\otimes a_3 - a_1\otimes a_2a_3 + a_3a_1\otimes a_2) = \eta(a_1a_2\otimes a_3) - \eta(a_1\otimes a_2a_3) +$$
$$+ \eta(a_3a_1\otimes a_2) = a_1a_2\,da_3 - a_1 d(a_2a_3) + a_3a_1\,da_2 =$$
$$= a_1a_2\,da_3 - a_1a_2\,da_3 - a_1a_3\,da_2 + a_3a_1\,da_2 =$$
$$= 0 \quad , \text{ for } A \text{ is commutative.}$$

It is also clear that η is a left A-module homomorphism.

Next, we define a correspondence in the opposite direction

$$\xi : \begin{cases} \Omega^1_{A|R} \longrightarrow HH_1(A) \\ a_1\,da_2 \longmapsto a_1\otimes a_2 \end{cases}$$

then $\xi(d(ab) - a\,db - b\,da) = 1\otimes ab - a\otimes b1 - 1b\otimes a$

$= 0$, using relations in $HH_1(A)$.

Hence ξ is a well-defined map (notice that A is unital) and it is a left A-module homomorphism. It is easy to see that $\xi\eta = I_{HH_1(A)}$ and $\eta\xi = I_{\Omega^1_{A|R}}$; so $\xi = \eta^{-1}$ and hence $HH_1(A) \cong \Omega^1_{A|R}$. ∎

(1.3) Derivations and differential forms

We continue with the assumption that A is a commutative unital algebra over a commutative ring R.

Def. Let M be an A-module. A derivation of A with values in M is an R-linear map $D : A \longrightarrow M$ which satisfies

$$\forall a,b \in A : \quad D(ab) = a(Db) + (Da)b .$$

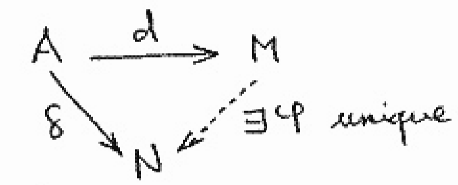
The R-module of all derivations of A in M is denoted

by $Der(A,M)$ or simply $Der(A)$ when $M = A$. ∎

Def. Universal derivation

A derivation $d : A \to M$ is said to be universal if for any other derivation $\delta : A \longrightarrow N$ there is a unique A-linear map (i.e. A-module homomorphism) $\varphi : M \longrightarrow N$ such that $\delta = \varphi\circ d$.

This is equivalent to the commutativity of the diagram



Therefore, a derivation $d : A \longrightarrow M$ is universal if it factorizes any other derivation $\delta : A \to N$ in a unique way. ∎

Construction of the universal derivation

Let $I = Ker(\mu : A\otimes A \longrightarrow A)$ where μ is the multiplication map. The algebra $A\otimes A$ (and hence $I = Ker(\mu) \subset A\otimes A$) is an A-bimodule for the multiplication on the left and on the right factors. Let us show that the bimodule I/I^2 is symmetric, i.e., the left and the right A-module structures coincide. As an A-module I is generated by elements $1\otimes x - x\otimes 1$, $x \in A$. The difference

$$a(1\otimes x - x\otimes 1) - (1\otimes x - x\otimes 1)a = (a\otimes x - ax\otimes 1) - (1\otimes xa - x\otimes a)$$
$$= -(1\otimes a - a\otimes 1)(1\otimes x - x\otimes 1) \in I^2$$
$$\Rightarrow \; a(1\otimes x - x\otimes 1) = (1\otimes x - x\otimes 1)a \quad (\text{mod } I^2) .$$

We now define a map

$$d : \begin{cases} A \longrightarrow I/I^2 \\ x \longmapsto (1\otimes x - x\otimes 1) \bmod I^2 \\ \qquad =: \overline{(1\otimes x - x\otimes 1)} \end{cases}$$

This is clearly a derivation, for

$$d(xy) = \overline{(1\otimes xy - xy\otimes 1)} = x\,\overline{(1\otimes y - y\otimes 1)} + \overline{(1\otimes x - x\otimes 1)}\,y$$
$$= x\,dy + (dx)\,y.$$

The derivation d is universal because given any other derivation $\delta : A \longrightarrow N$, there exists a unique A-linear map

$$\varphi : \begin{cases} I/I^2 \longrightarrow N \\ \overline{(1\otimes x - x\otimes 1)} \longmapsto \delta(x) \end{cases}$$

which makes the diagram $\begin{array}{ccc} A & \xrightarrow{d} & I/I^2 \\ & {\scriptstyle\delta}\searrow & \downarrow{\scriptstyle\varphi} \\ & & N \end{array}$ commutative.

(1.4) Module of Kähler differentials (A is unital & commutative)

We have already introduced the module of Kähler differentials $\Omega^1_{A|R}$ generated by the elements $a\,db$, $a,b \in A$. It easily follows that $d : A \longrightarrow \Omega^1_{A|R} : a \longmapsto da$ is the universal derivation for derivations from A into a bimodule:

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega^1_{A|R} \\ & {\scriptstyle\delta}\searrow & \downarrow{\scriptstyle\varphi} \\ & & N \end{array} \qquad \varphi(da) := \delta(a)$$

and $\varphi(d(ab)) = \delta(ab)$ as a simple calculation shows. It then follows, from the uniqueness up to isomorphism of the universal construction, that

$$I/I^2 \cong \Omega^1_{A|R}$$

where the isomorphism is given by $\overline{(1\otimes x - x\otimes 1)} \longmapsto dx$.

As a consequence of the universal property of $\Omega^1_{A|R}$ we can state the following

Proposition 3. The canonical A-linear map

$$\mathrm{Hom}_A(\Omega^1_{A|R}, M) \longrightarrow \mathrm{Der}(A,M)$$
$$f \longmapsto f\circ d$$

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega^1_{A|R} \\ & {\scriptstyle f\circ d}\searrow & \swarrow{\scriptstyle f} \\ & M & \end{array}$$

is an isomorphism. In other words the functor $\mathrm{Der}(-,M)$ is representable and is represented by Ω^1.

Proof. Easily follows from universality of $\Omega^1_{A|R}$. ∎

The module $\Omega^n_{A|R}$ of differential forms

By convention we put $\Omega^0_{A|R} = A$. The A-module of differential n-forms is, by definition, the exterior product

$$\Omega^n_{A|R} := \Lambda^n_A\, \Omega^1_{A|R}$$

Notice that the exterior product is over A and not over R. $\Omega^n_{A|R}$ is spanned, as an R-module, by the elements $a_0\,da_1 \wedge \dots \wedge da_n$, $a_i \in A$; we usually write this as $a_0\,da_1\,da_2 \cdots da_n$.

Example. Let V be a free R-module. Let $A = S(V)$ be the symmetric algebra of V (if V is finite dimensional with basis $x_1, \dots, x_n$, then clearly $S(V) = R[x_1, \dots, x_n]$, the polynomial algebra.) We will show that there is an isomorphism

$$S(V)\otimes V \cong \Omega^1_{S(V)|R} \qquad (*)$$
$$a\otimes v \longmapsto a\,dv$$

Any derivation D on $S(V)$ is completely determined by the value of D on V. So the map

$$S(V) \longrightarrow S(V)\otimes V$$
$$v_1 v_2 \cdots v_n \longmapsto \sum_i v_1 v_2 \cdots \underset{\substack{\uparrow \\ \text{omitted}}}{\hat{v}_i} \cdots v_n \otimes v_i$$

is a universal derivation. (Notice that $d(v_1v_2) = v_1 dv_2 + v_2 dv_1$ and under the identification $v_1 dv_2 \leftrightarrow v_1 \otimes v_2$ of proposition 2 we have $d(v_1v_2) = v_1\otimes v_2 + v_2 \otimes v_1$. Similarly for higher order terms.) Hence by proposition 3 the $S(V)$-map

$$\Omega^1_{S(V)|R} \longrightarrow S(V)\otimes V$$
$$d(v_1 \cdots v_n) \longmapsto \sum_i v_1 \cdots \hat{v}_i \cdots v_n \otimes v_i$$

is an isomorphism, as claimed. ∎

In particular $\Omega^1_{R[x_1,\dots,x_n]|R}$, as an $R[x_1,\dots,x_n]$-module is generated by $dx_1,\dots,dx_n$. It follows from (*) that

$$\Omega^n_{S(V)|R} \cong S(V)\otimes \Lambda^n V,$$

the module of differential n-forms on $S(V)$.

(1.5) A few words about Hochschild Cohomology of associative R-algebras.

In what follows A is an R-algebra (not necessarily commutative) and M is an A-bimodule.

Let us denote by $C^n(A,M)$ the R-module of n-linear maps $\underbrace{A\times A\times\cdots\times A}_{n\text{-fold}} \longrightarrow M$.

We define a map

$$\delta : C^n(A,M) \longrightarrow C^{n+1}(A,M)$$

by the following rule
$\forall x_1,\dots,x_{n+1}\in A$, $\forall f \in C^n(A,M)$,

$$\delta f(x_1,\dots,x_{n+1}) = x_1 f(x_2,\dots,x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1,\dots,x_{i-1}, x_i x_{i+1}, x_{i+2},\dots,x_{n+1}) + (-1)^{n+1} f(x_1,\dots,x_n)x_{n+1}. \qquad (1)$$

It is an easy exercise to show that $\delta^2 := \delta\circ\delta = 0$ (this follows from the fact that $(\delta f)(x) = f(bx)$.) Therefore,

$$0 \to C^0(A,M) \xrightarrow{\delta} C^1(A,M) \xrightarrow{\delta} C^2(A,M) \xrightarrow{\delta} \cdots$$

is a <u>cochain complex</u>, where $C^0(A,M) := M$.

Let

$$Z^n(A,M) := \operatorname{Ker}\left(\delta : C^n(A,M) \longrightarrow C^{n+1}(A,M)\right)$$
$$B^n(A,M) := \operatorname{Im}\left(\delta : C^{n-1}(A,M) \longrightarrow C^n(A,M)\right)$$

These are R-submodules of $C^n(A,M)$. Because $\delta^2=0$, we conclude that $Z^n(A,M) \supset B^n(A,M)$. The R-module

$$H^n(A,M) := Z^n(A,M)/B^n(A,M)$$

is called the <u>n-th Hochschild cohomology of A with coefficients in M</u>.

<u>Computation</u>

$\forall u \in M$: $\delta u \in C^1(A,M)$
$\therefore$ $(\delta u)(x) = xu - ux$, $\forall x \in A$.

$\Rightarrow Z^0(A,M) := \operatorname{Ker}(\delta) = \{u\in M \mid xu - ux = 0,\ \forall x\in A\}$.

This is a submodule of M. Clearly $H^0(A,M) = Z^0(A,M)$ for $B^0(A,M) = 0$.

Next, we compute $H^1(A,M)$:
for every $f\in C^1(A,M)$, $\delta f \in C^2(A,M)$ is given by
$\forall x_1,x_2\in A$: $\delta f(x_1,x_2) = x_1 f(x_2) - f(x_1x_2) + f(x_1)x_2$.

Now, $\delta f = 0 \iff f$ is an R-homomorphism $A \to M$ such that
$$f(x_1x_2) = x_1 f(x_2) + f(x_1)x_2 .$$
It is natural to call such a 1-cochain f a derivation of A into the A-A bimodule M. Therefore,
$$Z^1(A,M) = \mathrm{Der}(A,M).$$
If $u \in M$, u determines an inner derivation δu s.t.
$$(\delta u)(x) = xu - ux, \quad \forall x \in A.$$
We denote the set of all such derivations of A into M, by $\mathrm{Innder}(A,M)$. Clearly $\forall u \in M : \delta u \in \mathrm{Innder}(A,M)$.
$$\therefore \quad B^1(A,M) = \mathrm{Innder}(A,M).$$
We, therefore, have the following result
$$H^1(A,M) = \mathrm{Der}(A,M) / \mathrm{Innder}(A,M)$$

Next, we compute $H^2(A,M)$. Let $f \in C^2(A,M)$, i.e. f is a 2-cochain. Then $\delta f \in C^3(A,M)$ is given by
$$(\delta f)(x_1,x_2,x_3) = x_1 f(x_2,x_3) - f(x_1x_2,x_3) + f(x_1,x_2x_3) - f(x_1,x_2)x_3$$
$$\therefore \delta f = 0 \iff \forall x_1,x_2,x_3 \in A : x_1 f(x_2,x_3) + f(x_1,x_2x_3) = f(x_1x_2,x_3) + f(x_1,x_2)x_3 . \quad (*)$$
Therefore, $Z^2(A,M)$ is an R-submodule of the 2-cochains $C^2(A,M)$ such that every element $f \in Z^2(A,M)$ satisfies $(*)$. $Z^2(A,M)$ contains an R-submodule of $C^2(A,M)$ of maps of the form δg, $g \in C^1(A,M)$ (for $\delta(\delta g) = \delta^2 g = 0$). These 2-cochains are just the 2-boundaries $B^2(A,M) = \mathrm{Im}(\delta_1)$ (according to a previous computation, $(\delta g)(x_1,x_2) = x_1 g(x_2) - g(x_1x_2) + g(x_1)x_2$.) We then have the following quotient of R-

modules
$$H^2(A,M) = Z^2(A,M) / B^2(A,M) .$$
An interpretation of $H^2(A,M)$ appeared first in the literature by J.H.C. Whitehead and by Hochschild in connection to a classical structure theorem of finite dimensional algebras over a field: The so-called Wedderburn principal theorem. This is a problem in algebra extension that we shall consider now.

(1.6) Extension of algebras.

Let A be an R-algebra (not necessarily commutative) where R is a commutative unital ring such that A is R-projective, i.e. projective as an R-module. (e.g., A is a free R-module or A is an algebra over a field $\mathbb{F}$.)

Def. An extension B of A with kernel N is an exact sequence of algebras and algebra homomorphisms
$$0 \to N \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$$
We sometimes refer to this as the extension (B,β) of A where $\beta : B \to A$ is an algebra epimorphism. ∎

Given any extension (B,β) of A, A naturally carries a B-bimodule structure under the left and right multiplications by elements of B according to
$$\forall b \in B,\ \forall a \in A : \quad ba := \beta(b)a, \quad ab := a\beta(b) .$$
The kernel N of the epimorphism $\beta : B \to A$ is a 2-sided ideal in B and therefore it is also a B-bimodule

Def. If (B,β) is an extension of A such that there exists an algebra homomorphism $\sigma: A \to B$ s.t. $\beta\sigma = id_A$, then we say that A is segragated in (B,β) or that (B,β) is a cleft or an inessential extension of A. We usually refer to the algebra map σ as an algebra section ∎

Notice that when (B,β) is a cleft extension of A, then B contains a subalgebra mapped isomorphically onto A by B. A cleft extension is also referred to as a split extension.

Def. An extension (B,β) of A with kernel N is said to be singular (or abelian) if $N^2 = \{0\}$; i.e., to say if the product of every pair of elements in N is zero. ∎

I what follows we shall investigate the properties of singular extensions.

(i) When (B,β) is a singular extension of A with kernel N, we may give N the structure of an A-bimodule:

$\forall x \in N, \forall a \in A$: $\exists b \in B$ s.t. $\beta(b) = a$, for β is surjective; then we set

$ax := bx$, the algebra product of b and x in B

This is a well-defined action of A on N on the left, i.e., it does not depend on the choice of b for if $\beta(b') = a$ as well, then

$\beta(b) = \beta(b') \Rightarrow \beta(b - b') = 0 \Rightarrow b - b' \in N \Rightarrow (b - b')x = 0$, for $N^2 = 0$; $\Rightarrow bx = b'x$.

The element bx is of course in N since N is a 2-sided

ideal of B. In the same way we define

$$xa := xb ,$$

where b is any element of B s.t. $\beta(b) = a$. ∎

(ii) Conversely, let N be an A-bimodule and (B,β) be an extension of A. Suppose that there exists an R-module monomorphism $k: N \to B$ s.t.

$$0 \to N \xrightarrow{k} B \xrightarrow{\beta} A \to 0 \qquad \leftarrow ①$$

is an exact sequence of R-modules and that the following conditions are satisfied:

$$\begin{matrix} \forall x \in N \\ \forall b \in B \end{matrix} : \quad \begin{matrix} b\,k(x) = k(\beta(b)x) \\ k(x)\,b = k(x\beta(b)) \end{matrix} \qquad \leftarrow ②$$

Let us take $b = k(x')$ where x' is any element of N; then

$$\underset{=b}{k(x)\,k(x')} = k(x\beta(k(x'))) \quad , \text{ by } 2^{nd} \text{ equation in } ②,$$
$$= k(0) \quad , \text{ by exactness of } ① : \beta k = 0$$
$$= 0. \qquad \leftarrow ③$$

Therefore, if we identify N with its image under k, then ② shows that N is a 2-sided ideal of B and ③ shows that $N^2 = \{0\}$. $\Rightarrow$ ① gives us a singular extension of A with kernel N. ∎

To continue we need the following definition

Def. Two singular extensions (B,β) and (B',β') of A with kernel N are called equivalent if there exist an algebra homomorphism $\varphi: B \to B'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \xrightarrow{k} & B & \xrightarrow{\beta} & A & \to & 0 \\ & & \| {\scriptstyle id} & & \downarrow \varphi & & \| {\scriptstyle id} & & \\ 0 & \to & N & \xrightarrow{k'} & B' & \xrightarrow{\beta'} & A & \to & 0 \end{array} \quad \leftarrow (4)$$

is commutative, where k, k' are the monomorphisms from the kernel N to B and B' respectively.

It follows, as in the case of module extension (using the 5-lemma) that φ must be an isomorphism. ∎

(iii) We now give a "concrete" interpretation of the 2-dimensional cohomology module $H^2(A, N)$ of the algebra A with coefficients in a bimodule N.

Let A be an R-algebra which is R-projective and let N be an A-bimodule. Let (B, β) be a singular extension of A with kernel N. Thus we have an exact sequence of R-algebras

$$0 \to N \xrightarrow{k} B \xrightarrow{\beta} A \to 0$$

Since A is R-projective, this sequence as a sequence of R-modules splits, i.e., there is an R-homomorphism $\gamma : A \to B$ such that $\beta\gamma = id_A$. Therefore, as an R-module, B is isomorphic to $A \oplus N$.

With no loss of generality we may assume

$B = N \oplus A$ (R-module direct sum),

$$\forall x \in N, \forall a \in A : \begin{cases} k(x) = (x, 0) \\ \beta(x, a) = a \end{cases} \quad \leftarrow (5)$$

Using (2), (3) and (5) we immediately get

$$\begin{matrix} \forall x_1, x_2 \in N \\ \forall a_1, a_2 \in A \end{matrix} : \begin{cases} (x_1, 0)(x_2, 0) = (0, 0) \\ (x_1, 0)(0, a_2) = (x_1 a_2, 0) \\ (0, a_1)(x_2, 0) = (a_1 x_2, 0) \end{cases} \quad \leftarrow (6)$$

We must now only specify products like $(0, a_1)(0, a_2)$. Using the fact that β is an algebra map we can write

$$\beta[(0, a_1)(0, a_2)] = \beta(0, a_1)\, \beta(0, a_2) = a_1 a_2 .$$

Also because $\gamma : A \to B$ is only an R-linear mapping, its failure to be an algebra homomorphism is measured by

$$f(a_1, a_2) := \gamma(a_1 a_2) - \gamma(a_1)\gamma(a_2) . \quad \leftarrow (7)$$

Therefore, we can write

$$\begin{aligned} \beta(f(a_1, a_2)) &= \beta[\gamma(a_1 a_2) - \gamma(a_1)\gamma(a_2)] \\ &= \beta\gamma(a_1, a_2) - \beta(\gamma(a_1)\gamma(a_2)) \\ &= a_1 a_2 - \beta(\gamma(a_1))\, \beta(\gamma(a_2)) \\ &= a_1 a_2 - a_1 a_2 = 0 . \end{aligned}$$

$$\Rightarrow \quad f(a_1, a_2) \in N . \quad \leftarrow (8)$$

Therefore, one can generally write

$$(0, a_1)(0, a_2) = (f(a_1, a_2), a_1 a_2) \quad \leftarrow (9)$$

and since the algebra product on the left is linear in both a_1 and a_2, therefore, $f(a_1, a_2)$ is a bilinear mapping $f : A \times A \to N$. It follows that the multiplication in $B = N \oplus A$ (R-module direct sum) is given by

$$\boxed{\begin{matrix} \forall x_1, x_2 \in N \\ \forall a_1, a_2 \in A \end{matrix} : (x_1, a_1)(x_2, a_2) = (x_1 a_2 + a_1 x_2 + f(a_1, a_2), a_1 a_2)} \quad \leftarrow (10)$$

Notice that ⑥ and ⑨ are compatible with ⑩.

Because $f : A \times A \longrightarrow N$ is bilinear it is a Hochschild 2-cochain with values in N, i.e., $f \in C^2(A,N)$. We now show that <u>f is actually a 2-cocycle</u>. This follows by requiring the product ⑩ to be associative:

$$(0,0) = [((0,a_1)(0,a_2))(0,a_3) - (0,a_1)((0,a_2)(0,a_3))]$$
$$= (f(a_1,a_2), a_1a_2)(0,a_3) - (0,a_1)(f(a_2,a_3), a_2a_3)$$
$$= (f(a_1,a_2)a_3 + f(a_1a_2,a_3), a_1a_2a_3) -$$
$$- (a_1 f(a_2,a_3) + f(a_1,a_2a_3), a_1a_2a_3)$$
$$= (f(a_1,a_2)a_3 + f(a_1a_2,a_3) - a_1 f(a_2,a_3) + f(a_1,a_2a_3), 0)$$

$$\Longrightarrow f(a_1,a_2)a_3 + f(a_1a_2,a_3) - a_1 f(a_2,a_3) + f(a_1,a_2a_3) = 0$$

$$\Longrightarrow \delta f(a_1,a_2,a_3) = 0 \quad \Longrightarrow f \text{ is a 2-cocycle.}$$

<u>Thus, so far we have shown that every singular extension (B,β) of A with kernel N determines a 2-cocycle f of A with values in N.</u>

(iv) We now show that <u>2-cycles corresponding to equivalent singular extensions belong to the same coset of $Z^2(A,N)$ modulo $B^2(A,N)$, and hence determine the same element of the cohomology group $H^2(A,N)$.</u>

Suppose that (B,β) and (B',β') are equivalent singular extensions of A with kernel N. Referring to diagram ④ we see that

$$\forall x \in N: \quad \varphi(x,0) = \varphi k(x) = k'(x) = (x,0) \qquad (*)$$

and

$$\forall a \in A: \quad \beta' \varphi(0,a) = \beta(0,a) = a,$$

so there is a mapping $g : A \longrightarrow N$ s.t.

$$\forall a \in A: \quad \varphi(0,a) = (g(a), a). \qquad (**)$$

Because φ is an algebra morphism, the mapping g is clearly an R-homomorphism, i.e., <u>a 1-dimensional cochain</u> from A to N.. Using $(*)$ and $(**)$ we can write

$$\varphi(x,a) = \varphi[(x,0) + (0,a)]$$
$$= \varphi(x,0) + \varphi(0,a)$$
$$= (x,0) + (g(a),a) = (x + g(a), a).$$

Also

$$\varphi(x_1,a_1)\varphi(x_2,a_2) = (x_1 + g(a_1), a_1)(x_2 + g(a_2), a_2) =$$
$$= (x_1a_2 + g(a_1)a_2 + a_1x_2 + a_1 g(a_2) + f'(a_1,a_2), a_1a_2); \quad \leftarrow ⑪$$

where ⑩ is used in the second step, and where f' is a 2-cocycle determined by the extension (B',β').
On the other hand

$$\varphi[(x_1,a_1)(x_2,a_2)] = \varphi(x_1a_2 + a_1x_2 + f(a_1,a_2), a_1a_2) =$$
$$= (x_1a_2 + a_1x_2 + f(a_1,a_2) + g(a_1a_2), a_1a_2). \quad \leftarrow ⑫$$

It follows from ⑪ and ⑫ that

$$f'(a_1,a_2) = f(a_1,a_2) + g(a_1a_2) - a_1 g(a_2) - g(a_1)a_2 =$$
$$= f(a_1,a_2) - \delta g(a_1,a_2).$$

$$\therefore \quad f(a_1,a_2) - f'(a_1,a_2) = \delta g(a_1,a_2) \in B^2(A,N),$$

which proves the assertion.

(V) <u>We now show that every 2-cocycle $f : A \times A \longrightarrow N$ determines a singular extension of A with kernel N.</u>

Let $B = N \oplus A$, the external direct sum of R-modules. Define an operation of multiplication in B by means of (10). It is easy to verify that this multiplication is distributive over addition in B, and a simple computation, using the fact that f is a cocycle, shows that the multiplication is associative. To show that there is an identity element for multiplication, we notice that because f is a cocycle, for every element $a \in A$ we have:

$$0 = (\delta^2 f)(1_A, 1_A, a) = 1_A f(1_A, a) - f(1_A \cdot 1_A, a) + f(1_A, 1_A \cdot a) - f(1_A, 1_A) a =$$

$$= f(1_A, a) - f(1_A, a) + f(1_A, a) - f(1_A, 1_A) a .$$

$$\Rightarrow f(1_A, 1_A) a = f(1_A, a) .$$

Similarly one showes

$$a f(1_A, 1_A) = f(a, 1_A) . \qquad (*)$$

Then for every element $(x, a) \in B$ we have

$$(x, a)(-f(1_A, 1_A), 1_A) = (x 1_A - a f(1_A, 1_A) + f(a, 1_A), a 1_A) = (x, a) , \text{ where } (*) \text{ is used};$$

similarly $\quad (-f(1_A, 1_A), 1_A)(x, a) = (x, a)$.

$\therefore$ $(-f(1_A, 1A), 1_A)$ is an identity for multiplication in B. This proves that B is an algebra over R and we easily verify that if k and β are defined by (5) then the sequence (1) is exact and that the relations (2) are satisfied. Hence (B, β) is a singular extension of A with kernel N.

(vi) Finally, let g be a 1-cochain of A with coefficients in N. set $f' = f - \delta g$ and let (B', β') be

a singular extension of A with kernel N which corresponds to f' in the same way as (B, β) corresponds to f. Consider the mapping $\varphi: B \to B'$ defined by

$$\varphi(x, a) = (x + g(a), a) \quad , \quad \forall (x, a) \in B .$$

This makes the diagram (4) commutative and equations (11) and (12) show that φ is an algebra morphism.

$\therefore$ (B, β) and (B', β') are equivalent extions.

The preceding discussions establish the following

Theorem 4. Let A be an algebra over R which is R-projective and let N be an A-bimodule. Then there exists a one-to-one correspondence between $H^2(A, N)$ and the set of equivalent classes of singular extensions of A with kernel N. ∎

(1.7) Cleft (or split) singular extensions.

Let (B, β) be a cleft singular extension of A with kernel N and let f be a 2-cycle determined by this extension. Since the extension is split there exists an algebra section $\sigma: A \to B$, i.e., $\beta\sigma = id_A$. We can thus define a mapping $g: A \to N$ by setting

$$\forall a \in A : \quad \sigma(a) = (g(a), a) .$$

Because σ is an algebra morphism, g is, by def., a 1-cochain of A with values in N, i.e., $g \in C^1(A, N)$. Moreover we have

$$(g(a_1 a_2), a_1 a_2) = \sigma(a_1 a_2) = \sigma(a_1)\sigma(a_2) =$$

$= (g(a_1), a_1)(g(a_2), a_2)$
$= (a_1 g(a_2) + g(a_1)a_2 + f(a_1,a_2),\ a_1a_2)$.

$\Rightarrow g(a_1a_2) = a_1 g(a_2) + g(a_1)a_2 + f(a_1,a_2)$

$\therefore\ f(a_1,a_2) = -(a_1 g(a_2) - g(a_1a_2) + g(a_1)a_2)$
$= -\delta g(a_1,a_2)$. ← (13)

$\Rightarrow f \in B^2(A,N)$.

Thus the 2-cocycle f determines the zero element of $H^2(A,N)$. Conversely, it is clear (by reversing the steps in the argument) that if the 2-cocycle f in $Z^2(A,N)$ is actually a 2-coboundary (i.e., something like (13)) then the singular extension corresponding to f is cleft.

We have thus proved the complement of theorem 4 which is

Theorem 5. Let A be an algebra over a commutative ring R which is R-projective and N be an A-bimodule All split singular extensions of A with kernel N are equivalent. In the correspondence of theorem 4 the equivalence class of split singular extensions corresponds to the zero element of $H^2(A,N)$. ∎

Note. The semi-direct product $N \rtimes A$ is a cleft extension of A which corresponds to f=0. This is a direct sum of R-modules $N \oplus A$ on which a product is given by

$(n_1,a_1)(n_2,a_2) = (n_1a_2 + a_1n_2,\ a_1a_2)$.

What we have shown above is that every cleft extension is equivalent to a semi-direct product. ∎

The following result immediately follows from theorem 5.

Corollary 6. With the notation of theorem 2, if $H^2(A,N)=\{0\}$, then every singular extension of A with kernel N is cleft. ∎

To continue further we need some other definition and results which we shall discuss now.

Def. Homological dimension of a module.
Let M be a (left) module over a ring R. We say that M has finite homological dimension if M has a projective resolution $(P_\bullet, \varepsilon)$ for which $P_n = 0$ for all sufficiently large n. In this case the smallest integer n such that M has a projective resolution

$$\cdots 0 \to 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is called the homological dimension of M.

$\therefore$ M is projective $\Leftrightarrow$ hom. dim. M = 0. ∎

We have the following
Theorem. (See refrence Jacobson)
The following conditions on a module M are equivalent
(i) hom. dim. $M \le n$;
(ii) $Ext_R^{n+1}(M,N) = 0$ for all modules N;
(iii) given an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$$

in which every C_k, $k<n$ is projective, then C_n is projective. ∎

The concept of homological dimension of a module leads to the definition of homological dimension of a ring, and hence to that of an algebra.

Def. The left (right) global dimension of a ring R is the sup. hom. dim M for the left (right) modules of R. Thus the left (right) global dimension of R is zero iff every left (right) R-module is projective. ∎

Def. Let A be an R-algebra and let I be the set of natural numbers r s.t. for every A-bimodule M the cohomology group $H^r(A,M)$ is zero. If I is nonempty, it has a least element, say n+1. We then say that the homological dimension of A as an algebra over R is n and write $h.\dim_R A = n$. ∎

It follows from this definition that if $h.\dim_R A = -1$ then $H^0(A,M)=0$ for every A-bimodule M and in particular $H^0(A,A)=0$. However, we know that $H^0(A,A)=\{x\in A \mid xa=ax,\ \forall a\in A\}$ and hence $1_A \in H^0(A,A)=0 \Rightarrow A=\{0\}$. Conversely if $A=\{0\}$ then $H^r(A,A)=0$. Thus $h.\dim_R A=-1 \Leftrightarrow A=\{0\} \Leftrightarrow H^r(A,M)=0$ for all A-bimodule M and all natural numbers r.

Def. Let A be an algebra of finite dimension over a field F. A is said to be separable over F if for every field extension K of F the tensor product $K\otimes_F A$ is a semisimple algebra over K. ∎

If A is a separable algebra over F, then A itself, being isomorphic to $F\otimes_F A$ is semisimple. It is an established fact that if A is an algebra over F, then $h.\dim_F A = 0 \Leftrightarrow$ A is separable of finite dim. (as a vector space over F). For example $h.\dim_F M_n(F)=0$, so $M_n(F)$ is separable and finite dimensional.

Corollary 7. Let A be an R-algebra which is R-projective. If $\text{hom.}\dim_R A<2$ then every singular extension of A is cleft.

Proof. If $\text{hom.}\dim_R A<2$. then for every A-bimodule N we have $H^2(A,N)=0$ and so, according to Corollary 1, every singular extension of A with kernel N is cleft. ∎

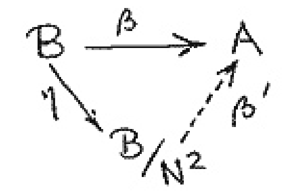
We shall now generalize Corollary 7 and this result includes the so-called Wedderburn principal theorem for algebras.

*Theorem 8. Let A be an algebra over a commutative ring R which is R-projective. If $\text{hom.}\dim_R A<2$ then every extension of A with nilpotent kernel is cleft.

Proof. Let $\text{hom.}\dim_R A<2$ and suppose (B,β) is an extension of A with nilpotent kernel N. If $N=\{0\}$ then

then B and A are isomorphic and hence the extension (B,β) is cleft. If $N^2=\{0\}$ but $N\neq\{0\}$ then (B,β) is a singular extension of A and hence, by corollary 7, it is cleft. We now proceed by induction. Suppose we have established for some natural number $m\geq 2$ that every extension of A whose kernel N_1 satisfies $N_1^m=\{0\}$ is cleft (this is our induction hypothesis).

Now, let (B,β) be an extension of A whose kernel N satisfies $N^{m+1}=\{0\}$, $N^m\neq\{0\}$. Then $N^2\subset N$; for certainly $N^2\subseteq N$, but $N^2=N$ would imply $\{0\}=N^{m+1}=N\neq\{0\}$, which is a contradiction; hence $N^2\subsetneq N$. Let $\eta: B\longrightarrow B/N^2$ be the canonical epimorphism. Because $\operatorname{Ker}(\eta)=N^2\subset N=\operatorname{Ker}(\beta)$ there exists an epimorphism $\beta': B/N^2\longrightarrow A$ such that the diagram



is commutative; i.e., $\beta'\eta=\beta$. Clearly $\operatorname{Ker}(\beta')=\eta(N)=N/N^2$ and since $(\eta(N))^2=\eta(N^2)=N^2/N^2=\{0\}$, we see that $(B/N^2,\beta')$ is a singular extension of A. By corollary 7 this extension is cleft which implies the existence of an algebra section $\sigma': A\longrightarrow B/N^2$, $\beta'\sigma'=\text{id}_A$. Let $B_1=\eta^{-1}(\sigma'(A))$; then B_1 is a subalgebra of B. If we set $\beta_1=\beta i$, where i is the inclusion monomorphism $B'\hookrightarrow B$, we see that (B_1,β_1) is an extension of A with kernel N^2. Since $(N^2)^m=\{0\}$, by induction hypothesis the extension (B_1,β_1) is cleft. Hence there exists an algebra section $\sigma_1 : A \longrightarrow B_1$, $\beta_1\sigma_1 = id_A$; then we can write

$\beta(i\sigma_1) = (\beta i)\sigma_1 = \beta_1\sigma_1 = id_A$,

which shows that β has a section $\sigma = i\sigma_1$; this implies that the original extension (B, β) is cleft. ∎

As a Corollary to this theorem we obtain the Wedderburn principal theorem. But first we need the following results (see reference Jacobson)

(i) If A is a finite dim. separable algebra over a field F, then $\text{hom. dim}_F A = 0$.

(ii) In a left Artinian ring R with identity the radical of R is the greatest element in the set of nilpotent 2-sided ideals of R.

Corollary 9. (Wedderburn principal theorem)

Let A be an algebra of finite dimension (as a vector space) over a field F. Let J be the radical of A and $\eta : A \longrightarrow A/J$ the canonical map. If A/J is a separable algebra over F then the extension (A, η) is cleft and A includes a subalgebra isomorphic to A/J.

Proof. Since A is finite dimensional over F it follows that A is an Artinian ring and hence the radical is nilpotent. By remark (i) above we have $\text{hom. dim}_F A/J = 0 < 2$. The result is now an immediate consequence of theorem 8. ∎

In the next section we shall consider the different enveloping algebra for an associative algebra A (not necessary commutative) and in the section following the

next we will introduce the concept of formal smoothness and investigate its characterizations.

(1.8) Differential envelope of an associative algebra.

We shall first generalize the Kähler differentials to the noncommutative case.

Def. Let A be an associative unital algebra over a commutative ring R. Let us define

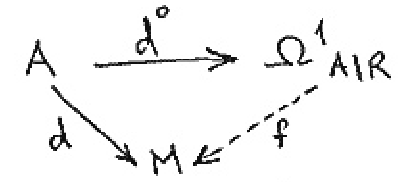
$$\Omega^0_{A|R} := A \quad , \quad \Omega^1_{A|R} := Ker(\mu : A \otimes A \longrightarrow A) \quad (1)$$

where $\mu(a \otimes b) = ab$, $\forall a, b \in A$, is the multiplication map of A. $\Omega^0_{A|R}$ and $\Omega^1_{A|R}$ are naturally A-A bimodules. We refer to $\Omega^1_{A|R}$ as the Kähler differential of A. ∎

Consider the mapping $d^0 : \begin{cases} A \longrightarrow \Omega^1_{A|R} \\ a \longmapsto 1 \otimes a - a \otimes 1 \end{cases}$

It easily follows that d^0 is a derivation from A to $\Omega^1_{A|R}$ i.e., $d^0(ab) = (d^0 a)b + a\, d^0 b$, $\forall a, b \in A$.

This derivation has the following universal property: given any A-A bimodule M and a derivation $d : A \longrightarrow M$, there exists a unique homomorphism of A-A bimodules $f : \Omega^1_{A|R} \longrightarrow M$ such that the diagram



is commutative, i.e., $d = f \circ d^0$.

Def. Let us write, for $n \geq 2$,

$$\Omega^n_{A|R} := \Omega^1_{A|R} \otimes_A \Omega^1_{A|R} \otimes_A \cdots \otimes_A \Omega^1_{A|R} \quad , \qquad \leftarrow (2)$$

the n-fold tensor product of the A-A bimodule $\Omega^1_{A|R}$ over A. $\Omega^n_{A|R}$ is an A-A bimodule by operation of A on the first and last factor $\Omega^1_{A|R}$ in $\Omega^n_{A|R}$ from the left and right respectively. The elements of $\Omega^n_{A|R}$ are called noncommutative differential forms of degree n on A. ∎

We now show that $\Omega^1_{A|R}$ is generated by d^0A as a left A-module (equally as a right A-module):

$\sum a_i \otimes b_i \in \Omega^1_{A|R}$ implies $\sum a_i \otimes b_i \in A \otimes A$ and $\sum a_i b_i = 0$

It follows that

$$\sum a_i \otimes b_i = \sum a_i (1 \otimes b_i - b_i \otimes 1) = \sum a_i \, d^0(b_i)$$

which shows that $\Omega^1_{A|R}$ is generated as a left A-module by $d^0(A) := \{ d^0(b_i) \mid b_i \in A \}$. similarly,

$$\sum a_i \otimes b_i = \sum (a_i \otimes 1 - 1 \otimes a_i) b_i = - \sum d^0(a_i)\, b_i$$

proving that $\Omega^1_{A|R}$ is generated as a right A-module by $d^0(A)$.

It follows that every element of $\Omega^n_{A|R}$, $n \geq 1$, can be written as a finite sum of elements of the form

$$a_0 d^0 a_1 \otimes d^0 a_2 \otimes \cdots \otimes d^0 a_n \quad , \quad a_0, a_1, \dots, a_n \in A$$

because in the def. of $\Omega^n_{A|R}$, given by (2) the n-fold tensor product is taken over A.

Let us define

$$\Omega_* A := \bigoplus_{n \geq 0} \Omega^n_{A|R} \qquad \leftarrow (3)$$

Clearly $\Omega_* A$ is a graded associative R-algebra in which

the multiplication is given by tensoring over A. It is also clear that $\Omega_* A$ is generated as an R-algebra by $A = \Omega^0_{A|R}$ and $d^0(A) \subset \Omega^1_{A|R}$.

To make $\Omega_* A$ into a differential graded algebra one needs to define a graded derivation $d : \Omega_* A \to \Omega_* A$ of degree +1 in a proper manner. This is achieved by the following lemma.

Lemma 10. There is a unique graded derivation $d : \Omega_* A \to \Omega_* A$, of degree +1, which extends $d^0 : \Omega^0_{A|R} \to \Omega^1_{A|R}$ and such that $d^2 = 0$. (Notice that by definition of a graded derivation we must have

$$d(\omega_p \cdot \omega_q) = (d\omega_p) \cdot \omega_q + (-1)^p \omega_p \cdot d\omega_q$$

for all $\omega_p \in \Omega^p_{A|R}$ and all $\omega_q \in \Omega^q_{A|R}$.)

Proof. Uniqueness of d: every element of $\Omega^n_{A|R}$ is an R-linear combination of elements of the form

$$a_0 d^0 a_1 \cdots d^0 a_n \quad , \quad a_0, a_1, \dots, a_n \in A.$$

Now, since $d^2 = 0$, and since d extends $d^0 : A = \Omega^0_{A|R} \to \Omega^1_{A|R}$, we must have

$$d(a_0 d^0 a_1 \cdots d^0 a_n) = d^0 a_0 \, d^0 a_1 \cdots d^0 a_n$$

which shows that d is uniquely determined by d^0.

Existence of d: we shall first construct $d^1 : \Omega^1_{A|R} \to \Omega^2_{A|R}$ such that $d^1 \circ d^0 = 0$ and such that

$$\forall a \in A, \ \forall \omega_1 \in \Omega^1_{A|R} : \begin{cases} d^1(a\omega_1) = (d^0 a)\omega_1 + a\, d^1\omega_1 \\ d^1(\omega_1 a) = (d^1 \omega_1) a - \omega_1 d^0 a \end{cases}$$

Let us define

$$d^1 : \begin{cases} A \otimes A \longrightarrow \Omega^1_{A|R} \otimes_A \Omega^1_{A|R} =: \Omega^2_{A|R} \\ \sum a_i \otimes b_i \longmapsto \sum d^0 a_i \otimes d^0 b_i \end{cases} \qquad (*)$$

and consider the restriction of d^1 to $\Omega^1_{A|R} \subset A\otimes A$. clearly $d^1: \Omega^1_{A|R} \longrightarrow \Omega^2_{A|R}$ is R-linear and

$$\forall a\in A: \quad d^1(d^0(a)) = d^1(1\otimes a - a\otimes 1)$$
$$= 0 \quad , \quad \text{for } d^0(1)=0. \quad \Rightarrow d^1\circ d^0 = 0.$$

Next, let $\omega_1 = \sum a_i\otimes b_i \in \Omega^1_{A|R}$; so $\sum a_i b_i = 0$. Then

$$d^1(a\omega_1) = d^1\left(\sum a a_i \otimes b_i\right) = \sum d^0(aa_i)\otimes d^0(b_i) =$$
$$= \sum \left(d^0(a)a_i + a\, d^0(a_i)\right)\otimes d^0 b_i = \sum d^0(a) a_i \otimes d^0 b_i + \sum a\, d^0(a_i)\otimes d^0(b_i)$$
$$= d^0(a)\otimes \sum a_i d^0(b_i) + a\, d^1\omega_1$$
$$= d^0(a)\,\omega_1 + a\, d^1\omega_1 \quad , \quad \text{where } \sum a_i\otimes b_i = \sum a_i d^0 b_i \text{ is used.}$$

Similarly,

$$d^1(\omega_1 a) = \sum d^0(a_i)\otimes d^0(b_i a)$$
$$= \sum d^0(a_i)\otimes d^0(b_i)\, a + \sum d^0(a_i)\otimes b_i\, d^0(a)$$
$$= (d^1\omega_1)\, a + \left(\sum d^0(a_i)\, b_i\right)\otimes d^0(a)$$
$$= (d^1\omega_1)\, a - \omega_1\, d^0 a \quad , \quad \text{where } \sum a_i\otimes b_i = -\sum d^0(a_i)\, b_i \text{ has been used.}$$

This proves that $d^1: \Omega^1_{A|R} \longrightarrow \Omega^2_{A|R}$ as given by (*) has the required property.

We must now establish the following

claim. For all $n\geq 1$ there exists $d^n: \Omega^n_{A|R} \longrightarrow \Omega^{n+1}_{A|R}$ s.t.

(i) $d^n\circ d^{n-1}=0$

(ii) $d^n(\omega_p\cdot\omega_q) = (d^p\omega_p)\cdot\omega_q + (-1)^p\omega_p\cdot d^q\omega_q$, for all $\omega_p\in\Omega^p_{A|R}$, $\omega_q\in\Omega^q_{A|R}$, s.t. $p+q=n$.

For $n=1$ we have just given the proof. We now proceed by induction. Assume that $n>1$ and that d^m has been constructed for $m<n$ satisfying properties (i) and (ii)

above. Define

$$\bar d^n: \begin{cases} \Omega^1_{A|R}\times \Omega^{n-1}_{A|R} \longrightarrow \Omega^{n+1}_{A|R} = \Omega^1_{A|R}\otimes_A \Omega^n_{A|R} \\ (\omega_1,\omega_{n-1}) \longmapsto d^1\omega_1\otimes\omega_{n-1} - \omega_1\otimes d^{n-1}\omega_{n-1} \end{cases} ,$$

notice that the two tensor signs in this definition appear in different positions in $\Omega^{n+1}_{A|R}$. $\bar d^n$ defines

$$d^n: \Omega^n_{A|R} = \Omega^1_{A|R}\otimes_A\Omega^{n-1}_{A|R} \longrightarrow \Omega^{n+1}_{A|R}$$

provided that

$$\bar d^n(\omega_1 a, \omega_{n-1}) = \bar d^n(\omega_1, a\omega_{n-1}) \quad , \quad \forall a\in A,\ \forall \omega_1\in\Omega^1_{A|R},\ \forall \omega_{n-1}\in\Omega^{n-1}_{A|R}.$$

But this is really the case for

$$\bar d^n(\omega_1 a,\omega_{n-1}) = d^1(\omega_1 a)\otimes\omega_{n-1} - (\omega_1 a)\otimes d^{n-1}\omega_{n-1}$$
$$= d^1\omega_1\otimes a\,\omega_{n-1} - \omega_1\otimes d^0a\otimes\omega_{n-1} - \omega_1\otimes a\, d^{n-1}\omega_{n-1}$$
$$= d^1\omega_1\otimes a\,\omega_{n-1} - \omega_1\otimes d^{n-1}(a\,\omega_{n-1})$$
$$= \bar d^n(\omega_1, a\,\omega_{n-1}).$$

Therefore,

$$d^n: \begin{cases} \Omega^n_{A|R} \longrightarrow \Omega^{n+1}_{A|R} \\ \omega_1\otimes\omega_{n-1} \longmapsto d^1\omega_1\otimes\omega_{n-1} - \omega_1\otimes d^{n-1}\omega_{n-1} \end{cases} \quad (**)$$

is well-defined. We now show that $d^n\circ d^{n-1}=0$.

Let $\omega_{n-1} = a\, d^0 b\otimes \omega_{n-2}\in\Omega^{n-1}_{A|R}$. Then

$$d^{n-1}\omega_{n-1} = d^1(a\, d^0 b)\otimes\omega_{n-2} - a\, d^0 b\otimes d^{n-2}\omega_{n-2}$$
$$= d^0 a\otimes d^0 b\otimes \omega_{n-2} - a\, d^0 b\otimes d^{n-2}\omega_{n-2} ,$$

where we have used the induction hypothesis that d^{n-1} satisfies (ii); thus, by (**)

$$d^n d^{n-1}\omega_{n-1} = -d^0a\otimes d^{n-1}(d^0 b\otimes\omega_{n-2}) - d^1(a\, d^0 b)\otimes d^{n-2}\omega_{n-2} + \underbrace{a\, d^0 b\otimes d^{n-1} d^{n-2}\omega_{n-2}}_{\to 0}$$
$$= -d^0a\otimes(\underbrace{d^1 d^0 b}_{\to 0}\otimes\omega_{n-2} - d^0 b\otimes d^{n-2}\omega_{n-2}) - d^0a\otimes d^0b\otimes d^{n-2}\omega_{n-2}$$
$$=0$$

where we have used the induction hypothesis $d^{n-1}d^{n-2}=0$; this establishes (i).

Finally, (ii) is satisfied for d^n in case $p=1$, $q=n-1$ by definition of d^n, given by (**), and it also follows for the case $p=0$, $q=n$ almost immediately. For $p>1$ ($q<n-1$) (ii) follows by induction hypothesis:

write $\omega_p \in \Omega^p_{A|R}$, $p \geqslant 2$ in the form

$\omega_p = \omega_1 \otimes \omega_{p-1} = \omega_1 \omega_{p-1}$. Then

$$d^n(\omega_p \omega_q) = d^n(\omega_1 \omega_{p-1} \omega_q) = d^1\omega_1 \otimes \omega_{p-1}\omega_q - \omega_1 \otimes d^{n-1}(\omega_{p-1}\omega_q) =$$

$$= (d^1\omega_1 \otimes \omega_{p-1}) \otimes \omega_q - \omega_1 \otimes (d^{p-1}\omega_{p-1}\cdot\omega_q + (-1)^{p-1}\omega_{p-1}d^q\omega_q)$$

$$= (d^1\omega_1 \otimes \omega_{p-1} - \omega_1 \otimes d^{p-1}\omega_{p-1}) \otimes \omega_q + (-1)^p \omega_p d^q \omega_q$$

$$= (d^p\omega_p)\omega_q + (-1)^p \omega_p d^q \omega_q \,, \quad \text{as claimed.} \quad \blacksquare$$

<u>Def.</u> Let A be an associative unital algebra over a commutative ring R. The differential graded algebra $\Omega(A) := (\Omega_*(A), d)$ is called <u>the differential envelope of A.</u> ∎

(1.9) Formally smooth algebras.

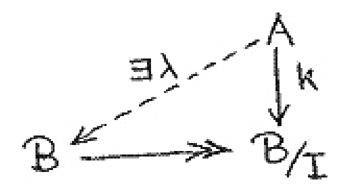
We want now to single out those algebras which would in the classical commutative case correspond to non-singular or smooth varieties. In classical algebraic geometry an algebra is smooth at a point if the Spec looks locally like an affine space. More

precisely there should be an analytic neighborhood where it is like an affine space; i.e., that the complete local ring at the point is a power series in the same number of variables as the dimension of the space. This says that the algebra has a certain freeness property which we shall discuss more fully now. We first generalize this smoothness property to the non-commutative case.

<u>Definition. Formal smoothness</u>

A $\mathbb{C}$-algebra A is said to be formally smooth if it satisfies the following lifting property:

Let B be a $\mathbb{C}$-algebra and $I \triangleleft B$ a 2-sided <u>nilpotent ideal of B.</u> If there is a $\mathbb{C}$-algebra morphism $A \xrightarrow{k} B/I$ then there exists a $\mathbb{C}$-algebra lift $\lambda : A \longrightarrow B$ which makes the diagram



commutative. ∎

This definition is too restrictive. In particular, a commutative smooth algebra does not have to satisfy this lifting property in the category of all $\mathbb{C}$-algebras.

<u>Example.</u> We shall demonstrate that the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ is formally smooth iff $n=1$!!

Consider the non-commutative affine $\mathbb{C}$-algebra

$$A = \mathbb{C}\langle x, y\rangle / (xy+yx, x^2, y^2)$$

which is just the Grassmann algebra generated by x and y. As a $\mathbb{C}$-vector space we have

$$A = \mathbb{C}1 \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy \qquad (*)$$

Consider the 2-sided ideal of A given by $I = \langle xy - yx \rangle_A$; that is

$$I = (xy - yx) + (xy + yx, x^2, y^2) / (xy + yx, x^2, y^2)$$

$$= (xy - yx,\ xy + yx, x^2, y^2) / (xy + yx, x^2, y^2) .$$

Therefore, $I^2 = 0$, and

$$A/I \cong \mathbb{C}\langle x, y\rangle / (xy - yx, xy + yx, x^2, y^2) \cong \mathbb{C}[x, y] / (x, y)^2$$

which is a 3-dimensional commutative algebra ($\cong \mathbb{C}1 \oplus \mathbb{C}x \oplus \mathbb{C}y$)

Consider the test object (A, I) given above and conjecture a lifting diagram

$$\begin{array}{ccc} & & \mathbb{C}[x_1, \dots, x_n] \\ & \overset{\exists \lambda ?}{\swarrow} & \downarrow k \\ A & \xrightarrow{\text{can.}} & A/I \end{array}$$

where

$$k : \begin{cases} x_1 \longmapsto x \\ x_2 \longmapsto y \\ x_i \longmapsto 0 \end{cases} , \ i \geq 3$$

λ must be of the form

$$\lambda(x_1) = x + \alpha \cdot xy \qquad \alpha, \beta \in \mathbb{C}$$
$$\lambda(x_2) = y + \beta \cdot xy$$

because λ must be $\mathbb{C}$-linear (we could add a term γyx to the first and δyx to the second, but by relations in A these are proportional to xy.) We then have

$$\lambda(x_1)\lambda(x_2) - \lambda(x_2)\lambda(x_1) = 2xy \neq 0 \qquad (\text{see } (*))$$

i.e., $[\lambda(x_1), \lambda(x_2)] = 2xy \neq 0$;

while $k(x_1)k(x_2) - k(x_2)k(x_1) = k([x_1, x_2]) = 0$, for

k is a morphism of commutative $\mathbb{C}$-algebras. Hence there is no lifting such as λ for $n \geq 2$. Clearly $\mathbb{C}[x]$. $\mathbb{C}\langle x\rangle$ is formally smooth in the category of all affine $\mathbb{C}$-algebras, as a subalgebra generated by one element is always commutative. ∎

It is now clear that formal smoothness is a categorical concept. A formally smooth $\mathbb{C}$-algebra in the category of all affine $\mathbb{C}$-algebras (i.e. finitely generated associative $\mathbb{C}$-algebras) will be referred to as a Quillen-smooth $\mathbb{C}$-algebra or a Q-smooth $\mathbb{C}$-algebra. A formally smooth affine $\mathbb{C}$-algebra in the category of commutative $\mathbb{C}$-algebras will be referred to as a Grothendieck smooth $\mathbb{C}$-algebra or a G-smooth $\mathbb{C}$-algebra. We shall later meet formally smooth algebras in another category of algebras, the **Cayley-Hamilton algebras**; those will be called Cayley-smooth algebras.

Before we give other important characterizations of Q-smooth algebras we will show that the definition of smoothness given before can be reduced to the case of singular (i.e. abelian) extension. Notice that our definition states that given any $\mathbb{C}$-algebra extension with nilpotent kernel, i.e., an exact sequence

$$0 \to I \to B \to B/I \to 0$$

of $\mathbb{C}$-algebras such that $I^n = 0$, then given any algebra map $A \xrightarrow{k} B/I$ there is an algebra map $\lambda : A \to B$ such that the diagram

$$\begin{array}{ccccccccc} & & & \exists\lambda \swarrow & A & & & & \\ & & & & \downarrow k & & & & (*) \\ 0 \to & I & \to & B & \to & B/I & \to 0 & & \end{array}$$

is commutative. Since I is nilpotent, I^2 is also nilpotent of lower degree and we have a nilpotent extension $0 \to I^2 \to B \to B/I^2 \to 0$. Suppose we are given an algebra morphism $k' : A \to B/I^2$; we then have an extended diagram

$$\begin{array}{ccccccccc} & & & & A & \searrow p_2 \circ k' = k & & & \ker(p_2) = I/I^2 \\ & & & & k' \downarrow & & & & \\ 0 \to I^2 \to & B & \overset{}{\underset{p_1}{\twoheadrightarrow}} & & B/I^2 & \underset{p_2}{\twoheadrightarrow} & B/I \to 0 & & (**) \end{array}$$

which gives a morphism $A \xrightarrow{k} B/I$ and hence by $(*)$ there exists an algebra map λ which makes the diagram

$$\begin{array}{ccc} & \exists\lambda \swarrow & A \\ & & \downarrow k \\ B & \underset{p_1}{\twoheadrightarrow} & B/I \end{array}$$

commutative. Therefore, from $(**)$, it follows that one has the following commutative triangle

$$\begin{array}{ccc} & \exists\lambda \swarrow & A \\ & & \downarrow k' \\ B & \twoheadrightarrow & B/I^2 \end{array}$$

However I^2 is nilpotent of lower degree of nilpotency than that of I. Repeating this process we get the following equivalent

Definition. A $\mathbb{C}$-algebra is formally smooth if it has the following lifting property for singular (i.e., Abelian) extensions:

For any algebra B and a 2-sided ideal $I \triangleleft B$ s.t. $I^2 = 0$, consider the singular extension

$$0 \to I \to B \to B/I \to 0$$

Then any algebra morphism $k : A \to B/I$ can be lifted

to an algebra morphism $\lambda : A \to B$ s.t. the diagram

$$\begin{array}{ccccccc} & & \lambda \swarrow & A & & & \\ & & & \downarrow k & & & \\ 0 \to & I \to & B & \to & B/I & \to 0 \end{array}$$

is commutative. ∎

The next theorem establishes equivalent conditions defining formal smoothness.

Theorem 11. Let A be an associative algebra over a field F. Then the following conditions are equivalent

(i) Lifting property for singular extensions; hence for Nilpotent extensions:

for any algebra B and a 2-sided ideal $N \triangleleft B$ s.t. $N^2 = \{0\}$, let

$$0 \to N \to B \to B/N \to 0$$

be the corresponding singular extension. Then any algebra morphism $k : A \to B/N$ can be lifted to an algebra homomorphism $\bar{k} : A \to B$; i.e., the following diagram is commutative

$$\begin{array}{ccccccc} & & \exists\bar{k} \swarrow & A & & & \\ & & & \downarrow k & & & \\ 0 \to & N \to & B & \to & B/N & \to 0 \end{array}$$

(ii) $\mathrm{Ext}^2_{A\text{-mod-}A}(A, M) = 0$ for any A-A bimodule M.

(iii) The A-bimodule $\Omega^1_A := \mathrm{Ker}(\mu : A \otimes A \to A)$ is a projective A^e-module (where $A^e = A \otimes A^{op}$).

Proof. (i) $\Rightarrow$ (ii) By (i) any singular extension $0 \to N \to B \xrightarrow{\varphi} A \to 0$ splits, for the existence of a section $\sigma : A \to B$ of epimorphism φ is ensured by the

Commutative diagram

$$\begin{array}{ccccccccc} & & & & & \exists\sigma \swarrow & A & & \\ & & & & & & \downarrow id_A & & \\ 0 & \to & N & \to & B & \xrightarrow{\varphi} & A & \to & 0 \end{array} \quad , \text{i.e.,} \quad \varphi\sigma = id_A .$$

Therefore, every singular extension of A is cleft. $\Rightarrow$

$\Rightarrow H^2(A,N)=0$. We recall that the Hochschild cohomology $H^i(A,N) = Ext^i_{A^e}(A,N)$ for any A-A bimodule N, which is clearly an A^e-module. It then follows that $Ext^2_{A^e}(A,N)=0$.

(ii) $\Rightarrow$ (iii) Notice that the multiplication map can also be written as $\mu(x\otimes y) = xy'$, where y' denotes the element of A corresponding to y in A^{op}. Thus we have the following exact sequence of A^e modules (or equivalently A-A bimodules, where $\Omega^1_A = Ker(\mu)$)

$$0 \to \Omega^1_A \to A^e \xrightarrow{\mu} A \to 0$$

This gives the following long exact sequence for Ext

$$0 = Ext^1_{A^e}(A^e,M) \to Ext^1_{A^e}(\Omega^1_A,M) \to Ext^2_{A^e}(A,M) \to$$
$$\to Ext^2_{A^e}(A^e,M) \to 0$$

where $Ext^1_{A^e}(A^e,M)=0$ because A^e is a free A^e-module and for similar reason $Ext^2_{A^e}(A^e,M)=0$.

$\therefore \quad Ext^1_{A^e}(\Omega^1_A,M) = Ext^2_{A^e}(A,M)$ for all A-A bimodules M. But by assumption $Ext^2_{A^e}(A,M) =$
$\Rightarrow Ext^1_{A^e}(\Omega^1_A,M)=0$ for all A^e-modules M, and this implies that Ω^1_A is a projective A^e-module.
(iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) easily follow by reversing the

arguments above. ∎

In two important papers J. Cuntz and D. Quillen have carried out a more careful study of smooth algebras (see the refrences) and have given other characterization of such algebras such as the existence of Yang-Mills derivation and the existence of a connection on the cotangent bundle. These matters won't be discussed here because they are unrelated to our line of study.

(1.10) Examples of Q-smooth algebras

Consider the commutative $\mathbb{C}$-algebra

$$C_k = \mathbb{C}[e_1,\dots,e_k] \Big/ \left(e_i^2 - e_i,\ e_ie_j,\ \sum_{i=1}^{k} e_i^2 - 1\right)$$

observe that as a $\mathbb{C}$-algebra,

$$C_k \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C} \qquad (k\text{-fold}),$$

i.e., the coordinate ring of k-distinct points.
C_k is the universal algebra in which 1 is decomposed into k orthogonal idempotents; i.e. if A is any $\mathbb{C}$-algebra s.t. $1 = a_1 + \cdots + a_k$, where $a_i \in A$ are idempotents and $a_ia_j = 0$, $i \neq j$, then there is an embedding $C_k \hookrightarrow A$, $e_i \mapsto a_i$.

Proposition 12. C_k is Q-smooth.
Using the universal embedding explained above this statement can be rephrased in the following form

If I is a nilpotent 2-sided ideal of a $\mathbb{C}$-algebra B and if $1=\bar{e}_1+\cdots+\bar{e}_k$ is a decomposition of 1 into orthogonal idempotents $\bar{e}_i \in B/I$ then we can lift this decomposition to $1=e_1+\cdots+e_k$ for orthogonal idempotents $e_i \in B$ s.t. $\pi(e_i)=\bar{e}_i$ where $\pi: B \twoheadrightarrow B/I$ is the canonical projection.

Proof. Suppose $I^l=0$; then for every $i\in I$, $1-i$ is a unit in B, for

$$(1-i)(1+i+i^2+\cdots+i^{l-1})=1-i^l=1.$$

Let $\bar{e}\in B/I$ be an idempotent and suppose that $\pi(x)=\bar{e}$ for some $x\in B$. Then $x-x^2\in I$, for $\pi(x-x^2)=$
$=\pi(x)-\pi(x)\pi(x)=\bar{e}-\bar{e}^2=\bar{e}-\bar{e}=0$.

$$\therefore\ 0=(x-x^2)^l=x^l-l\,x^{l+1}+\binom{l}{2}x^{l+2}+\cdots+(-1)^l x^{2l}$$

$\Rightarrow$ $x^l=ax^{l+1}$, where $a=l-\binom{l}{2}x+\cdots+(-1)^l x^{l-1}$.

We also have $ax=xa$.

Consider the element $e=(ax)^l\in B$, then

$$e^2=(ax)^{2l}=a^{2l}x^{2l}, \quad \text{for } ax=xa$$
$$=a^l(a^lx^{2l})$$
$$=a^lx^l, \quad \text{for } x^l=ax^{l+1}=a^2x^{l+2}=\cdots=a^lx^{2l} \quad (*)$$
$$=(ax)^l=e. \quad \Rightarrow e \text{ is an idempotent in B.}$$

We also have

$$\pi(e)=\pi(a^lx^l)=\pi(a)^l\pi(x)^l$$
$$=\pi(a)^l\pi(x)^{2l}, \quad \text{for } \pi(x)=\bar{e} \text{ is an idempotent}$$
$$=\pi(a^lx^{2l})=$$

$$=\pi(x^l), \quad \text{by } (*)$$
$$=(\pi(x))^l=\bar{e}^l=\bar{e}.$$

Next suppose $\bar{f}\in B/I$ is another idempotent s.t. $\bar{e}\bar{f}=0=\bar{f}\bar{e}$; then as above we can lift $\bar{f}$ to an idempotent $f'\in B$. Because

$\pi(f'e)=\pi(f')\pi(e)=\bar{f}\bar{e}=0$, then $f'e\in I$, i.e. $f'e$ is a nilpotent element, hence $1-f'e$ is a unit in B, so $(1-f'e)^{-1}$ exists and thus we can consider an element of the form

$$f=(1-e)(1-f'e)^{-1}f'(1-f'e).$$

Then one has

(i) f is an idempotent in B; for

$$f^2=(1-e)(1-f'e)^{-1}\underbrace{f'(1-f'e)}_{f'-f'^2e=f'-f'e=f'(1-e)}\cdot(1-e)(1-f'e)^{-1}f'(1-f'e)$$

$$=(1-e)(1-f'e)^{-1}f'(1-e)(1-e)(1-f'e)^{-1}f'(1-e)$$
$$=(1-e)(1-f'e)^{-1}f'(1-e)(1-f'e)^{-1}f'(1-e)$$
$$=(1-e)(1-f'e)^{-1}f'\underbrace{(1-f'e)(1-f'e)^{-1}}_{=1}f'(1-e)$$
$$=(1-e)(1-f'e)^{-1}f'^2(1-e)$$
$$=(1-e)(1-f'e)^{-1}f'(1-e)=(1-e)(1-f'e)^{-1}f'(1-f'e)=f.$$

(ii) $\pi(f)=\bar{f}$, for

$$\pi(f)=\pi(1-e)\left(\pi(1-f'e)\right)^{-1}\pi(f')\,\pi(1-f'e)$$
$$=(1-\pi(e))(1-\underbrace{\pi(f'e)}_{=0})^{-1}\pi(f')(1-\underbrace{\pi(f'e)}_{=0})$$
$$=(1-\bar{e})\bar{f}=\bar{f}-\bar{e}\bar{f}=\bar{f}, \quad \text{for } \bar{e}\bar{f}=0.$$

(iii) $e \cdot f = 0$, as $e(1-e) = e - e^2 = 0$.

Now, assume by induction that we have already lifted the pairwise orthogonal idempotents $\bar{e}_1, \dots, \bar{e}_{k-1} \in B/I$ to pairwise orthogonal idempotents $e_1, \dots, e_{k-1} \in B$. Then $e = e_1 + \dots + e_{k-1}$ is an idempotent of B s.t. $\overline{e e_k} = 0 = \bar{e}_k \bar{e}$. $\Rightarrow$ we can lift $\bar{e}_k$ to an idempotent $e_k \in B$ s.t. $e e_k = 0 = e_k e$. But then also

$$e_i e_k = (e_i e) e_k = 0 = e_k (e e_i) = e_k e_i ,$$

hence $e_1, \dots, e_{k-1}, e_k$ is a set of orthogonal idempotents in B. Finally, as $e_1 + \dots + e_{k-1} = i \in I$, we have that

$$e_1 + \dots + e_{k-1} = (e_1 + \dots + e_{k-1})^l = i^l = 0 ,$$

finishing the proof. ∎

Some important classes of Q-smooth algebras are

(1) The free algebra $k\langle x_1, \dots, x_n \rangle$ over a field k.

(2) The matrix algebra $\mathrm{Mat}(n \times n, k)$.

(3) The algebra of upper triangular matrices

$$A = \{ (a_{ij})_{1 \le i,j \le n} \mid a_{ij} \in k,\ a_{ij} = 0 \text{ for } i > j \} \subset \mathrm{Mat}(n \times n, k).$$

(4) $\mathcal{O}(C)$ the algebra of functions on a smooth affine curve over k.

(5) Path algebras of quivers which we shall stuo in details in the next section.

There also exist several constructions using which

one can construct new smooth algebras from old ones: let A and B be Q-smooth algebras over a field k. Then

(i) $A \oplus B$, the direct sum of Q-smooth algebras, is a Q-smooth algebra.

(ii) $A * B$, the free product of Q-smooth algebras is again Q-smooth.

(iii) $\mathrm{End}_{A\text{-mod}}(P)$, where P is a finitely generated projective A-module.

(iv) (Localization) If S is a subset of a formally smooth algebra A, then the algebra $A[S^{-1}]$ obtained by formally adjoining the inverses of S is formally smooth. In particular if $S \subset A$ is an arbitrary finitely generated multiplicative subset of A, then $S^{-1}A$ is smooth.

In the next chapter we shall study the representation space of formally smooth algebras and will discover that smoothness from algebraic point of view is equivalent to the (geometric) smoothness of the representation space considered as an affine variety.

(1.11) Path algebra of quivers

Def. A quiver is a directed graph, i.e., it is determined by the following data:

(i) a finite set Q_v of vertices denoted by $\{v_1, \dots, v_k\}$ for $k \in \mathbb{N}^+$;

(ii) a finite set Q_a of arrows which we shall denote by $\{a_1, \dots, a_n\}$, $n \in \mathbb{N}^+$.

Each arrow <u>starts</u> at a vertex $s(a)$ and <u>terminates</u> at a vertex $t(a)$, that is the arrows are completely determined by two maps

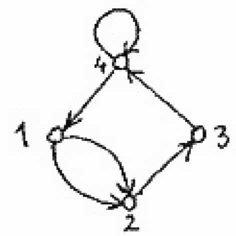
$$s, t : Q_a \longrightarrow Q_v \, .$$ ∎

It is clear that the given definition allows for multiple arrows between two vertices as well as loops at vertices.

To every quiver Q one can associate an integral $k \times k$ matrix, denoted by M_Q, called <u>the associated matrix of Q</u>, as follows

$$(M_Q)_{ij} = m_{ij} := \delta_{ij} - \text{the number of arrows from the vertex } v_i \text{ to } v_j .$$

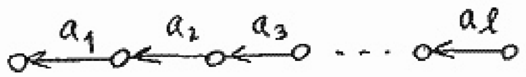
<u>Example</u>. If Q is given as



then

$$M_Q = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$ ∎

<u>Def</u>. A <u>nontrivial path</u> in a quiver Q is a string (or a word) $p = a_1 a_2 \cdots a_\ell$ in arrows, $\ell \geq 1$, s.t $t(a_{i+1}) = s(a_i)$ for all $1 \leq i \leq \ell$; that is the path starts at $s(p) = s(a_\ell)$ and terminates at $t(p) = t(a_1)$



For each vertex v_i of Q we also include a <u>trivial path</u>, denoted by e_i, which starts and terminates at v_i. ∎

<u>Def.</u> <u>The path algebra of a quiver</u>

The path algebra of a quiver over $\mathbb{C}$, denoted by $\mathbb{C}Q$, is a $\mathbb{C}$-algebra whose $\mathbb{C}$-basis is the set of all paths of the quiver Q (including the trivial paths e_i) and whose product is given by composition of paths; i.e., if p and q are two paths in Q, then

$$p \cdot q = \begin{cases} pq \, ; & \text{if } t(q) = s(p) \\ 0 \, , & \text{otherwise} \end{cases}$$

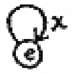
It is easily verified that this is an associative product bilinearity is trivial. ∎

<u>Examples</u>. (1) The path algebra $\mathbb{C}Q$ of the following quiver

is generated by the elements $\{e, f, x, y, u, v\}$ over $\mathbb{C}$ subject to the following relations

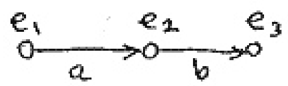
$e^2 = e$, $f^2 = f$, $e + f = 1$, $ef = 0 = fe$,
$e.x = x$, $e.y = y$, $e.u = u$, $e.v = 0$,
$x.e = x$, $y.e = y$, $u.e = 0$, $v.e = v$,
$f.x = 0$, $f.y = 0$, $f.u = 0$, $f.v = v$,
$x.f = 0$, $y.f = 0$, $u.f = u$, $u.f = 0$,
$x.v = 0$, $u.x = 0$,
$y.v = 0$, $u.y = 0$,
$u.u = 0$, $v.v = 0$.

(2) The free algebra $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ is the path algebra of the quiver

i.e., the path algebra of a quiver with one vertex (so $e=1$) and n oriented loops at this vertex. Obviously if $Q =$  then $\mathbb{C}Q = \mathbb{C}\langle x\rangle \cong \mathbb{C}[x]$. ∎

Path algebras of quivers have the following properties:

(1) $\mathbb{C}Q$ is finite dimensional iff the quiver Q has no loop or oriented cycles. For example if Q is given by



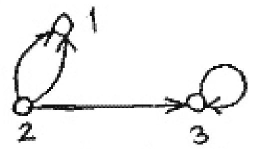
then $\mathbb{C}Q$ has a basis given by the paths $\{e_1, e_2, e_3, a, b, ba\}$ since $a \cdot b = 0$ whereas $b \cdot a = ba$; moreover

$$e_1 \cdot a = 0, \quad a \cdot e_1 = a, \quad e_2 \cdot b = 0, \quad b \cdot e_2 = b, \ \ldots \text{ etc.}$$

By just checking the multiplication table one obtains

$$\mathbb{C}Q \cong \begin{bmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{bmatrix}$$, the algebra of lower triangular matrices.

If there is more than one loop or oriented cycle, the algebra $\mathbb{C}Q$ will be very large, e.g., $Q =$ , $\mathbb{C}Q \cong \mathbb{C}\langle x, y\rangle$. However with one loop or oriented cycle, the algebra is still infinite dimensional but not as large as free algebras, e.g.

$Q =$ 

$$\mathbb{C}Q \cong \begin{bmatrix} \mathbb{C} & \mathbb{C}\oplus\mathbb{C} & 0 \\ 0 & \mathbb{C} & 0 \\ 0 & \mathbb{C}[x] & \mathbb{C}[x] \end{bmatrix}.$$

where x is the loop at vertex v_3.

(2) The set of trivial paths $\{e_1, \ldots, e_k\}$ is a complete set of orthogonal idempotents in $\mathbb{C}Q$, i.e.

$$e_i \cdot e_j = 0, \ i \neq j; \quad e_i^2 = e_i, \quad 1 = e_1 + \cdots + e_k.$$

(3) $\mathbb{C}Q$ has the following vector space direct sum

$$\mathbb{C}Q = \bigoplus_{i=1}^{k} \mathbb{C}Qe_i$$

where $\mathbb{C}Qe_i$ has as a basis the set of all paths starting at v_i; hence these subspaces are the projective left ideals of $\mathbb{C}Q$. Similarly $e_i\mathbb{C}Q$ has as a basis the set of all paths that terminate at v_i and since again one has

$$\mathbb{C}Q = \bigoplus_{i=1}^{k} e_i\mathbb{C}Q$$

these subspaces are the projective right ideals of $\mathbb{C}Q$. The subspace $e_j\mathbb{C}Qe_i$ has as a basis the paths starting at v_i and terminating at v_j. The subspace $\mathbb{C}Qe_i\mathbb{C}Q$ is the 2-sided ideal of $\mathbb{C}Q$ having as a basis all paths that pass through e_i.

(4) Let $0 \neq f \in \mathbb{C}Qe_i$ and $0 \neq g \in e_i\mathbb{C}Q$. It follows that $f \cdot g \neq 0$, for let p be a nontrivial path in f and q be a nontrivial path in g. Then the coefficient of $p \cdot q$ in $f \cdot g$ cannot be zero. Using this fact we can state the following

Proposition 13. The projective left ideals $\mathbb{C}Qe_i$ are indecomposable and are pairwise non-isomorphic.

Proof. If $\mathbb{C}Qe_i$ is not indecomposable then there exists a projection (an idempotent) $f \in \mathrm{Hom}_{\mathbb{C}Q}(\mathbb{C}Qe_i, \mathbb{C}Qe_i) \cong e_i\mathbb{C}Qe_i$. But then

$$f^2 = f = f \cdot e_i \quad \Rightarrow \quad f \cdot (f - e_i) = 0$$

contradicting the last point made above. Therefore, $\mathbb{C}Q$ is indecomposable.

For the second part let M be a $\mathbb{C}Q$-module. It follows that $\mathrm{Hom}_{\mathbb{C}Q}(\mathbb{C}Qe_i, M) \cong e_iM$. If $\mathbb{C}Qe_i \cong \mathbb{C}Qe_j$, then the isomorphism above gives elements $f \in e_i \mathbb{C}Q e_j$ and $g \in e_j \mathbb{C}Q e_i$ s.t. $f \cdot g = e_i$ and $g \cdot f = e_j$. But then $e_i \in \mathbb{C}Q e_j \mathbb{C}Q$, which is a contradiction unless $i = j$, for this subspace has as a basis all paths passing through v_j. ∎

(5) $\mathbb{C}Q$ is prime iff Q is strongly connected. That is, for all vertices v_i, v_j there is path from v_i to v_j.

Proof. By definition a ring R is prime if $I \cdot J \neq 0$ for every 2-sided ideals $I, J \neq 0$ of R. Equivalently $xRy \neq 0$ for every non-zero elements $x, y \in R$.

Suppose there is no path from v_i to v_j. Then for every $p \in \mathbb{C}Q$, $e_j p e_i = 0$. $\Rightarrow e_j \mathbb{C}Q e_i = 0$.

Conversely, suppose that there is a path between any two vertices. Let $x, y \in \mathbb{C}Q$, so each is a linear combination of paths in $\mathbb{C}Q$. Let p be a path appearing in the expression for y and q be a path appearing in the expression for x. Let $v_i = t(p)$ and $v_j = s(q)$. Then for every path α from v_i to v_j we have $q \alpha p \neq 0$. Hence $x \alpha y \neq 0$, implying that $\mathbb{C}Q$ is prime. ∎

(6) The radical of $\mathbb{C}Q$ has as a basis all paths from v_i to v_j for which there is no path from v_j to v_i.

(7) The center of $\mathbb{C}Q$ is of the form

$$\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{C}[x] \times \cdots \times \mathbb{C}[x]$$

with one factor for each connected component of Q (i.e., connected component for the underlying graph forgetting the orientations) and this factor is isomorphic to $\mathbb{C}[x]$ if the connected component is an oriented cycle

(8) $\mathbb{C}Q$ is left (resp. right) Noetherian (i.e., it satisfies

the ascending chain condition on the left (resp. right) ideals) iff for every vertex v_i belonging to an oriented cycle there is only one arrow starting at v_i (resp. only one arrow terminating at v_i). For example the path algebra of the quiver $\circ \longrightarrow \circlearrowleft$ is left, but not right, Noetherian; the path algebra of $\circlearrowleft \longrightarrow \circ$ is right but not left, Noetherian; and the path algebra of $\circlearrowleft \longrightarrow \circlearrowleft$ is neither left nor right Noetherian.

Finally we shall prove the most important fact, from our point of view, about path algebra of quivers.

Proposition 14. For every quiver Q the path algebra $\mathbb{C}Q$ is Quillen (or formally) smooth.

Proof. Let B be a $\mathbb{C}$-algebra and $I \triangleleft B$, a 2-sided nilpotent ideal of B. Conjecture the following commutative diagram

$$\begin{array}{ccc} & & \mathbb{C}Q \\ & \overset{\exists \tilde{\phi}?}{\swarrow} & \downarrow \phi \\ B & \xrightarrow{\ \pi\ } & B/I \end{array}$$

The decomposition of $1 = e_1 + \cdots + e_k$ into mutually orthogonal idempotents in $\mathbb{C}Q$ gives the decomposition of 1 into mutually orthogonal idempotents in B/I:

$$1 = \phi(e_1) + \cdots + \phi(e_k).$$

By proposition 12 this can be lifted to a decomposition

$$1 = \tilde{\phi}(e_1) + \cdots + \tilde{\phi}(e_k)$$

of mutually orthogonal idempotents in B (here $\tilde{\phi}$ is just a correspondence between the set of mutually orthogonal idempotents of $\mathbb{C}Q$ and B). We now take for any arrow $\overset{j}{\circ} \xleftarrow{a} \overset{i}{\circ}$ an arbitrary element $\tilde{\phi}(a) \in B$

given by

$$\tilde{\Phi}(a) \in \tilde{\Phi}(e_j)\,(\pi^{-1}(\phi(a)))\,\tilde{\Phi}(e_i) \;;$$

this correspondence extends to an algebra morphism $\mathbb{C}Q \xrightarrow{\tilde{\Phi}} B$ and we have

$$\pi(\tilde{\Phi}(a)) = \pi(\tilde{\Phi}(e_j))\,\pi(\pi^{-1}(\phi(a)))\,\pi(\tilde{\Phi}(e_i))$$
$$= \phi(e_j)\,\phi(a)\,\phi(e_i) = \phi(e_j a e_i) = \phi(a)$$

and hence the commutativity of the given diagram. ▮

(1.12) Representations of quivers and finite dimensional representations of the path algebras of quivers.

Def. A representation V of the quiver Q is given by
(i) a finite dimensional vector space V_i for each vertex $v_i \in Q_v$, and
(ii) a linear map $V_i \xrightarrow{V_a} V_j$ for every arrow $v_i \xrightarrow{a} v_j$ in Q.

If $\dim V_i = d_i$, the integral vector $(d_1,\dots,d_k) \in \mathbb{N}^k$ is called the dimension vector of V and will be denoted by $\dim V$. ▮

Def. A morphism $V \xrightarrow{\phi} W$ of representations V, W of Q is a set of linear maps $V_i \xrightarrow{\phi_i} W_i$, for all vertices $v_i \in Q_v$ s.t. the following diagram is commutative for every arrow $v_i \xrightarrow{a} v_j$ in Q

$$\begin{array}{ccc} V_i & \xrightarrow{V_a} & V_j \\ \phi_i \downarrow & & \downarrow \phi_j \\ W_i & \xrightarrow{W_a} & W_j \end{array}$$

clearly the composition of morphisms $V \xrightarrow{\phi} W \xrightarrow{\psi} X$ is given by the rule $(\psi \circ \phi)_i = \psi_i \circ \phi_i$. ▮

From these definitions it follows that the set of all finite dimensional representations of Q is a category that we shall denote by $\operatorname{Rep} Q$.

The reason that we consider the representations of quiver is the following important result.

Proposition 15. The category $\operatorname{Rep} Q$ is equivalent to the category of finite dimensional $\mathbb{C}Q$-representations (left-modules).

Proof. We must show that there exists a functor

$${}_{\mathbb{C}Q}\mathrm{Mod} \xrightarrow{F} \operatorname{Rep} Q$$

such that it has an inverse

$$\operatorname{Rep} Q \xrightarrow{F^{-1}} \mathrm{Mod}_{\mathbb{C}Q}$$

and this establishes the equivalence of the two categories.

Let $M \in {}_{\mathbb{C}Q}\mathrm{Mod}$ be an n-dim. representation of $\mathbb{C}Q$. We construct a representation V of Q by taking
(i) $V_i = e_i M$,
(ii) for every arrow $v_i \xrightarrow{a} v_j$ in Q we define $V_a : V_i \longrightarrow V_j$ by $V_a(x) = e_j a x$.

Observe that the dimension vector $\dim V = (d_1,\dots,d_k)$ satisfies $\sum_{i=1}^{k} d_i = n$ (for $e_1M + \dots + e_kM = (e_1+\dots+e_k)M = 1 \cdot M = M$, so that $\dim M := n = \sum_{i=1}^{k} \dim(e_i M)$).

If $\phi : M \longrightarrow N$ is a $\mathbb{C}Q$-linear map, that is ϕ is a morphism in the category ${}_{\mathbb{C}Q}\mathrm{Mod}$, then we have a linear map

$$\phi_i : \begin{cases} V_i = e_i M \longrightarrow V_i = e_i N \\ e_i m \longmapsto e_i \phi(m) = \phi(e_i m) \end{cases}$$

which clearly satisfies the compatibility condition:

$\forall x \in V_i$, $x = e_i m$ for some $m \in M$ and we have

$$(\phi_j \circ V_a)(x) = (\phi_j \circ V_a)(e_i m) = \phi_j(V_a(e_i m)) = \phi_j(e_j a e_i m)$$
$$= \phi(e_j a e_i m);$$

on the other hand

$$(W_a \circ \phi_i)(x) = W_a(\phi_i(e_i m)) = e_j a\, \phi_i(e_i m) = e_j a\, \phi(e_i m)$$
$$= \phi(e_j a e_i m),$$

and hence the required commutativity.

This procedure defines a functor ${}_{\mathbb{C}Q}\mathrm{Mod} \xrightarrow{F} \mathrm{Rep}\,Q$.

Conversely, let V be a representation of Q with dimension vector $\dim V = (d_1, \dots, d_k)$, so $V = \mathrm{Rep}\,Q$. Consider the $\sum d_i = n$-dimensional space $M = \oplus V_i$. We turn this into a $\mathbb{C}Q$-representation as follows:

Consider the canonical injection and projection maps

$$V_j \xhookrightarrow{i_j} M \xrightarrow{\pi_j} V_j .$$

We define an action of $\mathbb{C}Q$ on M by specifying the action of the generators of $\mathbb{C}Q$ as follows

$$\forall m \in M: \begin{cases} e_j m = i_j(\pi_j(m)) \\ a_l m = i_j(V_a(\pi_i(m))) \end{cases}, \quad V_i \xrightarrow{V_a} V_j$$

for all arrows $v_i \xrightarrow{a_l} v_j$.

For a morphism $V \xrightarrow{\phi} W$ of two representations of Q we define a mapping

$$\oplus_i V_i = M \xrightarrow{\tilde{\phi}} N = \oplus_j W_j$$
$$\tilde{\phi} = \sum_k i_k \circ \phi_k \circ \pi_k = \sum_k \tilde{\phi}_k$$

(i.e., $\underbrace{M \xrightarrow{\pi_k} V_k \xrightarrow{\phi_k} W_k \xrightarrow{i_k} N}_{\tilde{\phi}_k}$) which is clearly a morphism of left-$\mathbb{C}Q$ modules. This gives a functor $\mathrm{Rep}\,Q \xrightarrow{G} {}_{\mathbb{C}Q}\mathrm{Mod}$. It is straightforward to see that G and F are inverse to each other and this establishes the required equivalence. ∎

From now on we will identify representations of Q with left-$\mathbb{C}Q$ modules (i.e. representations of $\mathbb{C}Q$) via the above functors.

The associated matrix M_Q of the quiver Q can be used to define a bilinear map on $\mathbb{Z}^k$:

$$\langle \cdot, \cdot \rangle_Q : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z} \quad \text{by} \quad \langle \alpha, \beta \rangle_Q = \alpha \cdot M_Q \cdot \beta^t$$

for all row vectors $\alpha, \beta \in \mathbb{Z}^k$. This bilinear map is called the Euler form of Q. Its importance for the representation theory of Q becomes clear from the following result:

<u>Theorem 16.</u> For any two representations V and W of Q the following equality holds:

$$\boxed{\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}Q}(V,W) - \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathbb{C}Q}(V,W) = \langle \dim V, \dim W \rangle_Q}$$

<u>Proof.</u> We first establish that there is an exact sequence of $\mathbb{C}$-vectorspaces

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}Q}(V,W) \xrightarrow{\gamma} \bigoplus_{v_i \in Q_v} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i) \xrightarrow{\delta} \bigoplus_{a \in Q_a} \mathrm{Hom}_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) \longrightarrow$$
$$\xrightarrow{\varepsilon} \mathrm{Ext}^1_{\mathbb{C}Q}(V,W) \longrightarrow 0$$

where $\gamma(\phi) = (\phi_1, \dots, \phi_k)$, and

$$\delta(f_1, \dots, f_k) = \mu_a := f_{t(a)} \circ V_a - W_a \circ f_{s(a)}, \quad \forall (f_1, \dots, f_k) \in \bigoplus_{v_i \in Q_v} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i), \ \forall a \in Q_a$$

That is, μ_a is the obstruction to the commutativity of the following diagram

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{V_a} & V_{t(a)} \\ {\scriptstyle f_{s(a)}} \downarrow & \searrow^{\mu_a} & \downarrow {\scriptstyle f_{t(a)}} \\ W_{s(a)} & \xrightarrow{W_a} & W_{t(a)} \end{array}$$

By the definition of morphisms between representations of Q it is clear that

$$\mathrm{Ker}(\delta) = \mathrm{Im}(\gamma) \quad (= \mathrm{Hom}_{\mathbb{C}Q}(V,W)).$$

We define the map ε by sending a family of maps $(g_1, \dots, g_n) = (g_a)_{a \in Q_a}$ to the equivalence class of exact sequence

$$0 \longrightarrow W \xrightarrow{i} E \xrightarrow{p} V \longrightarrow 0$$

where for all $v_i \in Q_v$ we have $E_i = W_i \oplus V_i$ and the inclusion i and projection p are the obvious ones. For each generator $a \in Q_a$ the action of a on E is defined by the matrix

$$E_a = \begin{pmatrix} W_a & g_a \\ 0 & V_a \end{pmatrix} : E_{s(a)} = W_{s(a)} \oplus V_{s(a)} \longrightarrow W_{t(a)} \oplus V_{t(a)} = E_{t(a)}$$

This makes E into a left $\mathbb{C}Q$-module and it is now easily verified that the above short exact sequence is a sequence of left $\mathbb{C}Q$-modules. We must only show that the cokernel of δ can be identified with $\mathrm{Ext}^1_{\mathbb{C}Q}(V,W)$.

Suppose that $\{e_1, \dots, e_k, a_1, \dots, a_l\}$ is a set of generators of $\mathbb{C}Q$. A cycle is a linear map

$$\lambda : \mathbb{C}Q \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,W)$$

such that

$$\forall x, x' \in \mathbb{C}Q : \quad \lambda(xx') = \rho(x)\lambda(x') + \lambda(x)\sigma(x') \qquad (*)$$

where ρ determines the action of $\mathbb{C}Q$ on W and σ that on V. First consider an e_i; then the condition $(*)$ implies

$$\lambda(e_i^2) = \lambda(e_i) = p_i^W \lambda(e_i) + \lambda(e_i) p_i^V$$

whence, $\lambda(e_i) : V_i \longrightarrow W_i$. But then applying the condition $(*)$ once more, we get

$$\lambda(e_i) = \lambda(e_i^2) = p_i^W\left(p_i^W \lambda(e_i) + \lambda(e_i) p_i^V\right) + \left(p_i^W \lambda(e_i) + \lambda(e_i) p_i^V\right) p_i^V$$

$$= p_i^W \lambda(e_i) + \lambda(e_i) p_i^V + p_i^W \lambda(e_i) + \lambda(e_i) p_i^V$$

$$= 2\left(p_i^W \lambda(e_i) + \lambda(e_i) p_i^V\right) = 2\lambda(e_i). \quad \Longrightarrow \lambda(e_i) = 0.$$

(In the above computation the facts $(p_i^W)^2 = p_i^W$, $(p_i^V)^2 = p_i^V$, and $\lambda(e_i) p_i^V \in W_i$, so $p_i^W(\lambda(e_i) p_i^V) = \lambda(e_i) p_i^V$ are used.)

Similarly using the condition $(*)$ for $a = e_{t(a)} a = a e_{s(a)}$ we deduce that $\lambda(a) : V_{s(a)} \longrightarrow W_{t(a)}$.

$$\therefore \quad \bigoplus_{a \in Q_a} \mathrm{Hom}_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) = Z(V,W).$$

Now, the image of δ gives rise to a family of morphisms $\lambda(a) = f_{t(a)} V_a - W_a f_{s(a)}$ for a given linear map

$$f = (f_i) : V \longrightarrow W, \qquad f \in \bigoplus_{v_i \in Q_v} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i),$$

which is just the subspace of boundaries, so

$$\mathrm{Im}(\delta) = B(V,W).$$

$$\therefore \quad \mathrm{Coker}(\delta) = Z(V,W)/B(V,W) = \mathrm{Ext}^1_{\mathbb{C}Q}(V,W)$$

and this establishes the exactness of the long sequence of vector spaces.

Using exactness of this long sequence we find the following condition on the dimensions of vector spaces appearing in this sequence:

let $\dim V = (r_1, \dots, r_k)$, $\dim W = (s_1, \dots, s_k)$, then by exactness we have

$$\dim \mathrm{Hom}_{\mathbb{C}}(V,W) - \dim \bigoplus_{v_i \in Q_v} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i) + \dim \bigoplus_{a \in Q_a} \mathrm{Hom}_{\mathbb{C}}(V_{s(a)}, W_{t(a)}) - \dim \mathrm{Ext}^1_{\mathbb{C}Q}(V,W) = 0$$

$$\Rightarrow \dim \operatorname{Hom}_{\mathbb{C}}(V,W) - \dim \operatorname{Ext}^1_{\mathbb{C}Q}(V,W) = \sum_{v_i \in Q_v} r_i s_i - \sum_{a \in Q_a} r_{s(a)} s_{t(a)}$$

$$= (r_1, \dots, r_k)\, M_Q\, (s_1, \dots, s_k)^t = \langle \dim V, \dim W \rangle_Q . \quad \blacksquare$$

This result allows us to reduce the (usually difficult) computation of Ext^1 spaces to that of the easier Hom spaces.

Example. Consider the quiver $\circ \longleftarrow \circ \longrightarrow \circ$ and representations V and W given by

$$V: \quad \mathbb{C} \xleftarrow{id} \mathbb{C} \xrightarrow{id} \mathbb{C}, \quad \text{so } \dim V = (1,1,1)$$

$$W: \quad \mathbb{C} \xleftarrow{id} \mathbb{C} \xrightarrow{0} 0, \quad \dim W = (1,1,0)$$

Then, $\operatorname{Hom}_{\mathbb{C}}(V,W) \cong \mathbb{C}$ as the following commutative diagram shows

$$\begin{array}{ccccc} \mathbb{C} & \xleftarrow{id} & \mathbb{C} & \xrightarrow{id} & \mathbb{C} \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow 0 \\ \mathbb{C} & \xleftarrow{id} & \mathbb{C} & \xrightarrow{0} & 0 \end{array}$$

(each linear map $\alpha: \mathbb{C} \to \mathbb{C}$ is given by multiplication by a complex number α, hence $\operatorname{Hom}(V,W) \cong \mathbb{C}$). Similarly the commutative diagram

$$\begin{array}{ccccc} \mathbb{C} & \xleftarrow{id} & \mathbb{C} & \xrightarrow{0} & 0 \\ \downarrow 0 & & \downarrow 0 & & \downarrow \alpha \\ \mathbb{C} & \xleftarrow{id} & \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

implies $\operatorname{Hom}_{\mathbb{C}}(W,V) \cong \mathbb{C}$.

The associated matrix of the quiver under consideration is

$$M_Q = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have

$\langle (1,1,1), (1,1,0) \rangle_Q = 1$, $\langle (1,1,0), (1,1,1) \rangle_Q = 0$. By the formula we found above one has

$$\dim \operatorname{Hom}_{\mathbb{C}}(V,W) - \dim \operatorname{Ext}^1_{\mathbb{C}Q}(V,W) = \langle (1,1,1), (1,1,0) \rangle_Q = 1$$

$$\Rightarrow 1 - \dim \operatorname{Ext}^1_{\mathbb{C}Q}(V,W) = 1 . \Rightarrow \operatorname{Ext}^1_{\mathbb{C}Q}(V,W) = 0 .$$

On the other hand

$$\dim \operatorname{Hom}_{\mathbb{C}}(W,V) - \dim \operatorname{Ext}^1_{\mathbb{C}Q}(W,V) = \langle (1,1,0), (1,1,1) \rangle_Q = 0$$

gives

$$1 - \dim \operatorname{Ext}^1_{\mathbb{C}Q}(W,V) = 0 . \Rightarrow \operatorname{Ext}^1_{\mathbb{C}Q}(W,V) \cong \mathbb{C} . \quad \blacksquare$$

Let us now fix a dimension vector $\alpha = (d_1, \dots, d_k) \in \mathbb{N}^k$ and consider the set of all representations V of Q s.t. $\dim V = \alpha$. This set will be denoted by $\operatorname{Rep}_\alpha Q$.

Because each representation V is now completely determined by linear maps

$$V_a : V_{s(a)} = \mathbb{C}^{\oplus d_{s(a)}} \longrightarrow \mathbb{C}^{\oplus d_{t(a)}} = V_{t(a)}$$

we see that $\operatorname{Rep}_\alpha Q$ can be identified with an affine space

$$\operatorname{Rep}_\alpha Q = \bigoplus_{a \in Q_a} M_{d_{s(a)} \times d_{t(a)}}(\mathbb{C}) \cong \mathbb{C}^r$$

where

$$r = \sum_{a \in Q_a} d_{s(a)} d_{t(a)} := \dim \operatorname{Rep}_\alpha Q$$

which also shows that $\operatorname{Rep}_\alpha Q$ is a smooth affine variety.

There is an action of the algebraic group

$$GL(\alpha) := GL_{d_1} \times \cdots \times GL_{d_k}$$

on this affine space by conjugation; that is, if $g = (g_1, \dots, g_k) \in GL(\alpha)$ and if $V = (V_a)_{a \in Q_a} \in \operatorname{Rep}_\alpha Q$ then $g \cdot V$ is given by the matrices

$$(g \cdot V)_a = g_{t(a)} V_a g_{s(a)}^{-1}, \quad \forall a \in Q_a .$$

Let $V, W \in \mathrm{Rep}_\alpha Q$. If V, W are isomorphic as representations of Q such an isomorphism is determined by invertible matrices $GL_{d_i} \ni g_i : V_i \longrightarrow W_i$ s.t. for every arrow $a \in Q_a$ we have the following commutative diagram

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{V_a} & V_{t(a)} \\ \downarrow g_{s(a)} & & \downarrow g_{t(a)} \\ W_{s(a)} & \xrightarrow{W_a} & W_{t(a)} \end{array}$$

i.e., $g_{t(a)} \circ V_a = W_a \circ g_{s(a)}$. Hence we see that two representations are isomorphic iff they belong to the same orbit under $GL(\alpha)$ action. In particular we see that

$$\mathrm{Stab}_{GL(\alpha)}(V) \cong \mathrm{Aut}_{\mathbb{C}Q}(V)$$

where $\mathrm{Aut}_{\mathbb{C}Q}(V)$ is the group of units of $\mathrm{Hom}_{\mathbb{C}Q}(V,V)$ and it is determined by the conditions $\det \neq 0$. Hence $\mathrm{Aut}_{\mathbb{C}Q}(V)$ is a Zariski open subvariety of $\mathrm{End}_{\mathbb{C}Q}(V)$ $(:= \mathrm{Hom}_{\mathbb{C}Q}(V,V))$ and hence $\dim \mathrm{Aut}_{\mathbb{C}Q}(V) = \dim \mathrm{End}_{\mathbb{C}Q}(V)$. Moreover, we know that the dimension of the orbit $\mathcal{O}(V)$ of V in $\mathrm{Rep}_\alpha Q$ is equal to

$$\dim \mathcal{O}(V) = \dim GL_\alpha - \dim \mathrm{Stab}_{GL(\alpha)}(V).$$

The dimension of GL_α is clearly $\sum_i d_i^2$. We can, therefore, get a geometric reformulation of the previous theorem in the following form.

Lemma 17. Let $V \in \mathrm{Rep}_\alpha Q$. Then

$$\dim \mathrm{Rep}_\alpha Q - \dim \mathcal{O}(V) = \dim \mathrm{End}_{\mathbb{C}Q}(V) - \langle\alpha,\alpha\rangle_Q = \dim \mathrm{Ext}^1_{\mathbb{C}Q}(V,V)$$

Proof. We can write

$$\dim \mathrm{Rep}_\alpha Q - \dim \mathcal{O}(V) = \sum_{a \in Q_a} d_{s(a)} d_{t(a)} - \left(\sum_i d_i^2 - \dim \mathrm{Aut}_{\mathbb{C}Q}(V)\right)$$

$$= \left(\sum_{a \in Q_a} d_{s(a)} d_{t(a)} - \sum_i d_i^2\right) + \dim \mathrm{End}_{\mathbb{C}Q}(V)$$

$$= -\langle\alpha,\alpha\rangle_Q + \dim \mathrm{End}_{\mathbb{C}Q}(V)$$

$$= \dim \mathrm{Ext}^1_{\mathbb{C}Q}(V,V),$$ where theorem 16 is used. ∎

References of chapter 1

The main references for background algebra are

(1) N. Jacobson ; "Basic algebra II"

(2) S. Lang ; "Algebra"

(3) P.M. COHN ; "Algebra II , III"

(4) Y. A. Drozd , V.V. Kirichenko ;
"Finite dimensional algebras."

The references for Kähler differentials and differential graded algebras are

(5) Jean-Louis LODAY ; "cyclic homology."

(6) Peter Seibt ; "cyclic homology of algebras."

The main references for smooth algebras are

(7) W. F. SCHELTER ,
"smooth algebras"
Journal of Algebra 103 , 677-685 , (1986).

(8) J. Cuntz & D. Quillen ,
"Algebra extension and nonsingularity"
Journal of the A.M.S. VoL. 8 , No. 2 , (APRIL 1995).

The main refence for quivers and all related problems discussed here is

(9) Lieven Le BRUYN
"Non-commutative geometry"
Expected to be published by A.M.S. in the mathematical surveys and Monograph series.

A general reference for all this work is

(10) M. Kontsevich , A. Rosenberg
"Non-commutative smooth spaces"

CHAPTER II

The necklace & the trace algebras

In this chapter we shall study the important results of C. Procesi on the invariant theory of $n \times n$ matrices under simultaneous conjugations. We will determine the ring of invariants for simultaneous conjugations and introduce the algebra of GL_n-equivariant maps from m-tuples of $n \times n$ matrices to the ring $M_n(\mathbb{C})$. We then apply the Nagata-Higman theorem on a specific quotient of this algebra to specify the length of the necklaces which generate the ring of invariants. Finally we formalize this argument and obtain an improved version of Nagata-Higman result in this special case.

(2.1) The GL_n-orbits of $n \times n$ matrices

In what follows M_n denotes the complex vector space $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$, $GL_n = \{A \in M_n \mid \det A \neq 0\}$ is the multiplicative group of invertible matrices.

$M_n(\mathbb{C})$ is an n^2-dimensional complex linear space and there is an action of GL_n on this space induced by conjugations, i.e.

$$\begin{cases} GL_n \times M_n \longrightarrow M_n \\ (g, A) \longmapsto gAg^{-1} \end{cases}$$

This action defines an equivalence relation on M_n

$$\forall A, B \in M_n : \quad A \sim B \iff \exists g \in GL_n : \; gAg^{-1} = B.$$

The equivalence class of $A \in M_n$ is the orbit of A under this action

$$\mathcal{O}(A) := \{gAg^{-1} \mid g \in GL_n\}$$

and we call this the conjugacy class of A. The set of all conjugacy class, denoted by M_n/GL_n, is therefore

$$M_n/GL_n := \{\mathcal{O}(A) \mid A \in M_n\}.$$

The Jordan normal form gives a particularly useful representative in each conjugacy class, i.e., any $A \in M_n$ can be conjugated to a block diagonal matrix of the form

$$\begin{bmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_k} \end{bmatrix}$$

such that each block has the following form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

and where $\lambda_1, \dots, \lambda_k$ are the eigenvalues of A; these are the roots of the characteristic polynomial of A

$$\chi_A(t) = \det(t\,1_n - A) \in \mathbb{C}[t].$$

The Jordan normal form is unique up to permutations of the blocks and hence gives a description of the set of orbits.

The n^2-dim. complex vector space M_n can be turned into a topological vector space by introducing a norm on it which can be done in several ways; let us choose the following norm

$$\forall A \in (a_{ij})_{i,j} \in M_n \;:\; \|A\| = \sum_{i,j=1}^{n} |a_{ij}|$$

and consider the topology induced by this norm on M_n. What we would like to do (if possible) is to turn the set of GL_n-orbits into a topological space such that the orbit map (or the quotient map)

$$\begin{cases} M_n \xrightarrow{\mathcal{O}} M_n/GL_n \\ A \longmapsto \mathcal{O}(A) \end{cases}$$

is continuous. If we require that M_n/GL_n has the Hausdorff property, i.e., its points be closed, then the continuity of $\mathcal{O}$ implies that for any $A \in M_n$ the orbit $\mathcal{O}(A) \hookrightarrow M_n$ is a closed set in M_n (for the inverse image of closed sets are closed under continuous maps). However, this cannot be the case for $n \geq 2$ as the following example shows: let

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

which according to Jordon normal form result belong to distinct GL_n-orbits. Let $\varepsilon \neq 0$ and consider

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} \lambda & \varepsilon^2 \\ 0 & \lambda \end{pmatrix} \in \mathcal{O}(A),$$

implying that B lies in the closure of $\mathcal{O}(A)$, as $\varepsilon \to 0$, and hence $\mathcal{O}(A)$ cannot be a closed orbit in M_2.

Therefore, the set of equivalence classes M_n/GL_n has a very unpleasant topological structure: it contains non-closed points. It is a general fact that the orbit $\mathcal{O}(A)$ is closed if and only if all Jordan blocks of A are 1-dimensional, i.e., when A is diagonalizable.

Therefore, one must try to find the best continuous approximation to this non-existent orbit space. Our aim is to construct a topological space, denoted by $M_n/\!\!/GL_n$ and a continuous surjection

$$M_n \xrightarrow{\pi} M_n/\!\!/GL_n$$

which has the following universal property:

any continuous function $f: M_n \longrightarrow \mathbb{C}$ which is constant along orbits factors through π, i.e., there exists a unique mapping $f': M_n /\!/ GL_n \longrightarrow \mathbb{C}$ which is continuous s.t. the diagram

$$\begin{array}{ccc} M_n & \xrightarrow{\pi} & M_n /\!/ GL_n \\ & {\scriptstyle f}\searrow & \swarrow {\scriptstyle f'} \\ & \mathbb{C} & \end{array}$$

is commutative. For this purpose we first need a good supply of functions which are constant on the orbits. If $A \sim B$, then A and B have the same set of eigen-values $\{\lambda_1, \cdots, \lambda_n\}$. Clearly this implies that any symmetric function in variables λ_i has the same values in A and B. In particular the elementary symmetric functions σ_ℓ, defined by,

$$\sigma_\ell(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < i_2 < \cdots < i_\ell} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_\ell}$$

have the desired property. These functions are also continuous on M_n; this can be seen from the characteristic equation of $A \in M_n$,

$$\chi_A(t) = \det(t \mathbb{1}_n - A) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(A)\, t^{n-1}$$
$$= \prod_{j=1}^{n} (t - \lambda_j) \quad \leftarrow ①$$

an taking into account that the determinant is a polynomial (and hence a continuous) function in entries of A; hence $\sigma_i(A)$ are also polynomial functions in entries of A and are continuous. Therefore we conclude that

$$\sigma_\ell : M_n \longrightarrow \mathbb{C}$$

are continuous functions that are constant along orbits.

Lemma 1. The continuous function

$$\pi : \begin{cases} M_n \longrightarrow \mathbb{C}^n \\ A \longmapsto (\sigma_1(A), \cdots, \sigma_n(A)) \end{cases}$$

is surjective.

Proof. We set up a correspondence between points of $\mathbb{C}^n$ and matrices $A \in M_n(\mathbb{C})$ in the following way

$$\mathbb{C}^n \ni a = (a_1, \cdots, a_n) \longleftrightarrow A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_n \\ -1 & 0 & 0 & \cdots & 0 & a_{n-1} \\ 0 & -1 & 0 & \cdots & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & O & & \ddots & 0 & a_2 \\ & & & & -1 & a_1 \end{bmatrix} \in M_n(\mathbb{C}) \quad \leftarrow ②$$

Expanding $\det(t \mathbb{1}_n - A)$ with respect to the first column (which need induction on the size n of A) gives

$$\det(t \mathbb{1}_n - A) = t^n - a_1 t^{n-1} + \cdots + (-1)^n a_n .$$

Comparing this expression with ① immediately yields

$$\pi(A) = (\sigma_1(A), \cdots, \sigma_n(A)) = (a_1, \cdots, a_n) . \quad ∎$$

Def. cyclic matrices.

A matrix $B \in M_n$ is said to be cyclic if there exists a column vector $v \in \mathbb{C}^n$ s.t. the set of vectors

$$\{v, Bv, B^2 v, \cdots, B^{n-1} v\}$$

form a basis of $\mathbb{C}^n$. In such a case v is called a cyclic vector for B. ∎

Any matrix $A \in M_n$ can be conjugated to the standard form ② iff A is cyclic. This can be seen as follows:

let $g \in GL_n$ be a base change in $\mathbb{C}^n$ from the standard basis to the basis given by the ordered n-tuple of vectors

$$(v, -Av, A^2 v, -A^3 v, \cdots, (-1)^{n-1} A^{n-1} v)$$

i.e., the column of g are these vectors in the given order.

If A is cyclic and v a cyclic vector for A then $(*)$ is a basis for $\mathbb{C}^n$ and A expressed in this new basis, i.e., gAg^{-1}, is equal to

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & a_n \\ -1 & 0 & 0 & \cdots 0 & a_{n-1} \\ 0 & -1 & 0 & \cdots 0 & a_{n-2} \\ & & \ddots & & \vdots \\ & 0 & & 0 & a_2 \\ & & & -1 & a_1 \end{bmatrix}$$

where $A^n v = (a_n, a_{n-1}, \dots, a_1)$. Conversely, any matrix in this form is cyclic.

Lemma 2. The cyclic matrices in M_n form a dense open subset.

Proof. Let $v = (x_1, \dots, x_n)^t \in \mathbb{C}^n$. The determinant of the $n \times n$ matrix

$$[v, Av, A^2v, \dots, A^{n-1}v] \qquad (*)$$

is a polynomial of degree n in x_i whose coefficients are functions $f_i(a_{11}, a_{12}, \dots, a_{nn})$ in entries a_{ij} of A. clearly

$$A \text{ is cyclic} \Leftrightarrow (*) \text{ is a basis for } \mathbb{C}^n \Leftrightarrow \det. \text{ of } (*) \text{ is non-zero.}$$

Therefore, A is cyclic iff at least one coefficient $f_j(a_{11}, \dots, a_{nn})$ is non-zero. Hence the non-cyclic matrices form the intersection of the hypersurfaces

$$V_i = \{ A = (a_{\ell k})_{\ell k} \in M_n \mid f_i(a_{11}, a_{12}, \dots, a_{nn}) = 0 \}$$

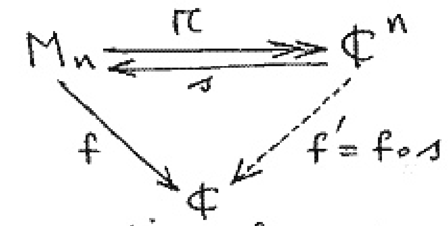
in the n^2-dimensional $\mathbb{C}$-vector space M_n. But the complement of a hypersurface is a dense open subset (being the inverse image of the dense subset $\mathbb{C} - \{0\}$ under a continuous function.) This finishes the proof. ∎

Theorem 3. The surjection

$$\pi : \begin{cases} M_n \longrightarrow \mathbb{C}^n \\ A \longmapsto (\sigma_1(A), \dots, \sigma_n(A)) \end{cases}$$

is the best continuous approximation to the orbit space problem.

Proof. Suppose $f : M_n \longrightarrow \mathbb{C}$ is a continuous function which is constant along conjugacy classes. Consider the diagram



where s is a section of π, i.e., $\pi \circ s = id_{\mathbb{C}^n}$, and it is defined by assigning $(a_1, \dots, a_n) = a \in \mathbb{C}^n$ to a matrix A in the standard form ②. Clearly s is continuous, so $f' = f \circ s$ is continuous. We must only show that the diagram is commutative, i.e., $f = f' \circ \pi$ holds. By continuity, it suffices to prove the equality on the dense open subset of cyclic matrices in M_n. But this is an immediate consequence of the following facts:

(i) any cyclic matrix lies in the orbit of a cyclic matrix in the standard form ②;

(ii) Given A in the standard form ②, $\pi(A) = (a_1, \dots, a_n)$, as shown in lemma ①, and hence

$$s \circ \pi(A) = s(a_1, \dots, a_n) = A \; ;$$

(iii) f is constant along the orbits. ∎

In the following example we will also see an instance of the following general result: every fiber $\pi^{-1}(a)$ contains exactly one closed GL_n-orbit; it may be contained in the closure of one or more orbits.

Example. GL_2-orbits in M_2

Any matrix $A \in M_2$ can be conjugated to an upper triangular form with the eigenvalues λ_1, λ_2 along the diagonal. The invariants are

$$\sigma_1 = tr(A) \quad , \quad \sigma_2 = \det A$$

and by what we established above the best approximation for the orbit space is given by the surjection

$$\pi : \begin{cases} M_2 \longrightarrow \mathbb{C}^2 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a+d \, , \, ad-bc) \end{cases}$$

From the characteristic equation, $t^2 - \sigma_1(A)t + \sigma_2(A) = 0$, of A it follows that A has equal eigenvalues iff $\sigma_1(A)^2 - 4\sigma_2(A) = 0$. This defines a closed curve C in $\mathbb{C}^2$

$$C = \{(x,y) \in \mathbb{C}^2 \mid x^2 - 4y = 0\} \quad , \quad \begin{cases} x_1 = \sigma_1(A) \\ x_2 = \sigma_2(A) \end{cases}$$

The description of the fibers $\pi^{-1}(a)$ is as follows:

(i) $a = (x,y) \in \mathbb{C}^2 - C$. In this case A has two distinct eigenvalues and hence $\pi^{-1}(a)$ consists of only one orbit which is then necessarily closed in M_2; namely that of the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\lambda_{1,2} = \frac{-x \pm \sqrt{x^2-4y}}{2}$.

(ii) $a \in (x,y) \in C$. In this case A has equal eigenvalues and this implies that $\pi^{-1}(a)$ consists of two orbits

$$\mathcal{O}\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right) \; , \; \mathcal{O}\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \; , \quad \lambda = \frac{x}{2}.$$

As noted earlier the second orbit (which is a closed orbit) lies in the closure of the first orbit; so $\pi^{-1}(a)$ has again a unique closed orbit lying in the closure of another orbit.

We can also described the fibers of π as closed subsets of M_2. For this purpose let us choose the following

linear basis for M_2

$$\alpha = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \beta = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then any $A \in M_2$ can be written as

$$A = \alpha u + \beta v + \gamma w + \delta z = \begin{pmatrix} \frac{u+v}{2} & w \\ z & \frac{u-v}{2} \end{pmatrix} \quad , \quad u, v, w, z \in \mathbb{C}$$

In terms of coordinate functions u, v, w, z we have

$$tr(A) = u = x \quad , \quad \det(A) = \frac{u^2 - v^2}{4} - wz = y$$

and hence the fibers $\pi^{-1}(a)$ of $a = (x,y) \in \mathbb{C}^2$ are the common zeros (in the 4-dim. space M_2) of the algebraic equations

$$\begin{cases} u = x \\ \frac{u^2 - v^2}{4} - wz = y \end{cases} \quad \text{or} \quad \begin{cases} u = x \\ v^2 + 4wz = x^2 - 4y \end{cases}$$

The first equation defines a 3-dim. subspace of M_2 in which the second equation is a quadric. If $a \notin C$, i.e. $x^2 - 4y \neq 0$, this quadric is non-degenerate and thus $\pi^{-1}(a)$ is a smooth 2-dimensional submanifold of M_2.

If $a \in C$ this quadric is a cone with its top being the point $\begin{pmatrix} x/2 & 0 \\ 0 & x/2 \end{pmatrix}$ and under the GL_2-action this point is fixed which gives us the closed orbit of dim. 0 (this is $\mathcal{O}\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$) and the other orbit is the cone minus this point and it is, therefore, a smooth (non-closed) submanifold of M_2. ∎

(2.2) Polynomial invariants of n×n matrices

Let x_{ij} be the coordinate functions on M_n, i.e.,

$$\forall A = (a_{ij})_{i,j} \in M_n \quad : \quad x_{ij}(A) = a_{ij} \in \mathbb{C}.$$

By pointwise addition and multiplication these functions generate a commutative algebra, namely the polynomial algebra

$$\mathbb{C}[M_n] := \mathbb{C}[x_{11}, x_{12}, \cdots, x_{nn}]$$

This is usually called the coordinate ring of M_n.

Every element $f \in \mathbb{C}[M_n]$ defines a complex valued function $f : M_n \longrightarrow \mathbb{C}$ by evaluation, i.e.

$\forall A \in M_n : \quad f(A) = f(a_{11}, a_{12}, \dots, a_{nn}) \in \mathbb{C}$;

such functions are called <u>regular functions on M_n</u> ; thus $\mathbb{C}[M_n]$ is the ring (or the algebra) of regular functions on M_n

An $n \times n$ matrix X defined by

$$X := (x_{ij})_{i,j} \in M_n(\mathbb{C}[M_n])$$

is called <u>the generic $n \times n$ matrix</u> and if we expand $\det(t\mathbb{1}_n - X)$, we see that the functions $\sigma_i \in \mathbb{C}[M_n]$. Viewing M_n as an n^2-dimensional affine variety with coordinate ring $\mathbb{C}[M_n]$ we shall now introduce a much coarser topology on M_n than the usual $\mathbb{C}$-topology, namely the <u>Zariski topology</u>.

Let $\mathbb{C}^n$ be the n-dim. complex vector space and y_i, $i = 1, \dots, n$, be the coordinate functions corresponding to the standard basis vectors. The coordinate ring of $\mathbb{C}^n$ is $\mathbb{C}[y_1, \dots, y_n]$ and again every element $f \in \mathbb{C}[y_1, \dots, y_n]$ defines a function on $\mathbb{C}^n$ by evaluation. Let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, such that $f(a) = f(a_1, \dots, a_n) = 0$. The <u>zero set of f</u>, i.e.,

$$\mathbb{Z}(f) = \{a \in \mathbb{C}^n \mid f(a) = 0\}$$

is a closed subset of $\mathbb{C}^n$. More generally for any subset $S \subset \mathbb{C}[y_1, \dots, y_n]$ we define its zero set by

$$\mathbb{Z}(S) = \{a \in \mathbb{C}^n \mid f(a) = 0 \ , \ \forall f \in S\} .$$

Clearly

$$\mathbb{Z}(S) = \bigcap_{f \in S} \mathbb{Z}(f)$$

and because each $\mathbb{Z}(f)$ is closed and arbitrary intersection of closed sets is closed, we conclude that $\mathbb{Z}(S)$ is a closed subset of $\mathbb{C}^n$. If a is a zero of f and f', it is a zero of $f + f'$ and of $f \cdot g$ for any $g \in \mathbb{C}[y_1, \dots, y_n]$. Thus

$$\mathbb{Z}(S) = \mathbb{Z}(I)$$

where $I = \langle S \rangle$ is the ideal generated by $S \subset \mathbb{C}[y_1, \dots, y_n]$. By Hilbert's basis theorem every ideal in $\mathbb{C}[y_1, \dots, y_n]$ is finitely generated. Hence, $I = (f_1, \dots, f_\ell)$ for some $f_i \in \mathbb{C}[y_1, \dots, y_n]$, $i = 1, \dots, \ell$. Hence

$$\mathbb{Z}(S) = \mathbb{Z}(I) = \mathbb{Z}(f_1, \dots, f_\ell) .$$

<u>Def.</u> The Zariski topology on $\mathbb{C}^n$ is the topology s.t. its closed sets are the zero sets $\mathbb{Z}(I)$ where I runs over ideals of $\mathbb{C}[y_1, \dots, y_n]$. ∎

The following lemma justifies this definition.

<u>Lemma 4.</u> Let $I, J, (I_j)_j$ be ideals of $\mathbb{C}[y_1, \dots, y_n]$.

(i) $I \subset J \Rightarrow \mathbb{Z}(I) \supset \mathbb{Z}(J)$

(ii) $\bigcap_j \mathbb{Z}(I_j) = \mathbb{Z}(\sum_j I_j)$

(iii) $\mathbb{Z}(I) \cup \mathbb{Z}(J) = \mathbb{Z}(I \cap J) = \mathbb{Z}(I \cdot J)$.

<u>Proof.</u> Easy verification (see the next chapter). ∎

This lemma justifies the following facts:

by (i) we define $\mathbb{Z}(\{0\}) = \mathbb{C}^n$; hence $\mathbb{C}^n$ is a closed set. By (ii) arbitrary intersection of closed sets is closed and by (iii) finite union of closed sets is closed.

Thus the zero sets $\mathbb{Z}(I)$ satisfy the axioms of a topology on $\mathbb{C}^n$ by closed sets; the Zariski topology.

Consider the Zariski topology on $\mathbb{C}^1$. This is a topology in which <u>closed sets are finite sets</u>, for every polynomial function of degree n in $\mathbb{C}[x]$ has n roots. The <u>Zariski open sets</u> are the complements of such finite subsets of $\mathbb{C}^1$. Therefore, the Zariski open sets are huge, indicating that the Zariski topology is far from being Hausdorff. It is, therefore, clear that the $\mathbb{C}$-topology on $\mathbb{C}^n$ is finer than Zariski topology.

Let $\mathbb{D}(I) := \mathbb{C}^n - Z(I)$ be a Zariski open set, where I is a proper ideal.

claim. $I \neq \{0\} \Rightarrow \mathbb{D}(I)$ is a dense (open) set in $\mathbb{C}^n$

Proof. Suppose that $\mathbb{D}(I)$ is not a dense open subset of $\mathbb{C}^n$, i.e., $\overline{\mathbb{D}(I)} \subsetneq \mathbb{C}^n$. Then $\mathbb{D}(I) \subset \overline{\mathbb{D}(I)} = Z(J)$ for some ideal $J \lhd \mathbb{C}[y_1, \dots, y_n]$. It then follows that $\forall f \in I, \forall g \in I : \mathbb{D}(f) \subset Z(g)$ (for $\mathbb{D}(f) \subset \mathbb{D}(I)$, $\forall f \in I$; and $Z(J) \subset Z(g)$, $\forall g \in J$). However, this is a contradiction because $f \cdot g$ is a non-zero polynomial but the above argument implies

$$\forall a \in \mathbb{C}^n : \quad (f \cdot g)(a) = f(a)\, g(a) = 0$$

which is only possible if $f \cdot g$ is the zero polynomial. ∎

We will now show that the set of cyclic matrices is a Zariski open subset of M_n. Let $X = (x_{ij})_{i,j}$ be a generic $n \times n$ matrix and $v = (v_1, \dots, v_n)^t$ a generic column vector. Consider the square matrix

$$[v, Xv, X^2 v, \dots, X^{n-1} v] \in M_n\left(\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}; v_1, \dots, v_n]\right)$$

The determinant of such a matrix can be written as

$$d(X, v) = \sum_{k=1}^{m} f_k(x_{ij})\, g_k(v_\ell) \qquad (*)$$

for some $m \in \mathbb{N}^+$, where f_k and g_k are polynomial functions. We want to show that the set of cyclic matrices in M_n is $\mathbb{D}(f_1, \dots, f_m)$. Let $A \in M_n$ s.t. at least one of $f_k(A) \neq 0$. Then the polynomial

$$d(A, v) = \sum_{k=1}^{m} f_k(A)\, g_k(v_\ell) \neq 0$$

and hence there exists $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ s.t. $d(A, a) \neq 0$.

$\Rightarrow a^t$ is a cyclic vector for A (i.e. $\{a^t, Aa^t, \dots, A^{n-1}a^t\}$ is a basis for $\mathbb{C}^n$). Conversely, if A is cyclic there

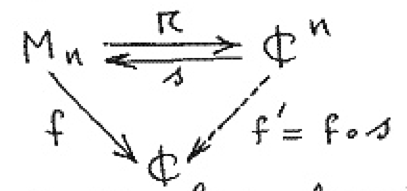
exists a cyclic vector c for A and hence the determinant of $[c, Ac, \dots, A^{n-1}c]$ is non-zero. Therefore at least one of the polynomials f_k in $(*)$ is non-zero. ∎

The following theorem can be used to specify the ring of polynomial invariants of M_n under GL_n-action.

Theorem 5. Let $f : M_n \longrightarrow \mathbb{C}$ be a regular function on M_n which is constant on the GL_n-orbits (i.e., on the conjugacy classes). Then

$$f \in \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)].$$

Proof. By theorem 3 there exists a commutative diagram



where f' is a regular function on $\mathbb{C}^n$ and hence it is a polynomial in the coordinate function of $\mathbb{C}^n$ (which are $\sigma_i(X)$, $i = 1, \dots, n$), so

$$f' \in \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)] \hookrightarrow \mathbb{C}[M_n].$$

Moreover, f and f' are equal on a Zariski open (dense) subset of M_n, which is the set of cyclic matrices, therefore, they are equal as polynomials in $\mathbb{C}[M_n]$. ∎

The ring of invariants. The group GL_n acts by automorphisms on the polynomial ring $\mathbb{C}[M_n]$. The action of GL_n on $\mathbb{C}[M_n]$ is, therefore, given by

$$\begin{array}{l} \forall g \in GL_n \\ \forall A \in M_n \\ \forall f \in \mathbb{C}[M_n] \end{array} : \quad \phi_g(f)(A) = f(g^{-1} A g) \qquad \leftarrow ③$$

The algebra of polynomials which are invariant under this action is called the ring of polynomial invariants, and is usually written as

$$\mathbb{C}[M_n]^{GL_n} := \{ f \in \mathbb{C}[M_n] \mid \Phi_g(f) = f, \ \forall g \in GL_n \}. \quad \leftarrow (4)$$

clearly this is the same as the ring of polynomial functions on M_n which are constant on orbits. Thus, by previous theorem we have

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[\sigma_1(X), \dots, \sigma_n(X)]. \quad \leftarrow (5)$$

There is an equivalent description of this ring which we shall explain now. Consider the polynomial

$$f_n(t) := \prod_{i=1}^{n} (t - \lambda_i) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i t^{n-i}$$

where σ_i are the elementary symmetric polynomials in variables $\lambda_1, \dots, \lambda_n$. We know that these polynomials are algebraically independent and generate the ring of symmetric polynomials in λ_i, $i=1,\dots,n$; i.e.,

$$\mathbb{C}[\sigma_1, \dots, \sigma_n] = \mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n} \quad \leftarrow (6)$$

where S_n is the symmetric group of degree n, and its acts by automorphism on the polynomial ring $\mathbb{C}[\lambda_1, \dots, \lambda_n]$ according to $\pi(\lambda_i) = \lambda_{\pi(i)}$. The subalgebra of polynomials fixed under these automorphisms is precisely the symmetric polynomials in λ_i's. We show that the Newton s-functions $s_i = \lambda_1^i + \dots + \lambda_n^i$, $i = 1, \dots, n$, constitute another generating set for the ring of symmetric polynomials, i.e.

$$\mathbb{C}[\sigma_1, \dots, \sigma_n] = \mathbb{C}[s_1, \dots, s_n]. \quad \leftarrow (7)$$

The proof follows if we can show that each σ_i can be expressed as a polynomial in s_j; but this is

a consequence of the identities

$$s_j - \sigma_1 s_{j-1} + \sigma_2 s_{j-2} - \dots + (-1)^{j-1} \sigma_{j-1} s_1 + (-1)^j j \cdot \sigma_j = 0. \quad \leftarrow (8)$$

∎

Next, suppose that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix A; then A can be conjugated to an upper triangular matrix A' with diagonal entries $(\lambda_1, \dots, \lambda_n)$. Therefore,

$$tr(A) = tr(A') = \lambda_1 + \dots + \lambda_n = s_1,$$

and in general A^i is conjugated to the upper triangular form A'^i with diagonal entries $(\lambda_1^i, \dots, \lambda_n^i)$; so

$$tr(A^i) = tr(A'^i) = \lambda_1^i + \dots + \lambda_n^i = s_i, \quad 1 \le i \le n.$$

This arguments imply the following

Theorem 6. Let $X = (x_{ij})_{i,j}$ be the generic matrix of coordinate functions of M_n. Consider the GL_n-action on M_n by conjugation. Then the ring of polynomial invariants is generated by the traces of powers of X, i.e.,

$$\mathbb{C}[M_n]^{GL_n} = \mathbb{C}[tr(X), tr(X^2), \dots, tr(X^n)]. \quad \leftarrow (9)$$

Proof. Follows immediately from (8), (7) and (5). ∎

(2.3) Simultaneous conjugacy classes

Later in this chapter we shall be concerned with representation theory of affine $\mathbb{C}$-algebras (i.e., finitely generated, not necessarily commutative algebras over $\mathbb{C}$). For that purpose it will be crucial to extend what we have done for conjugacy classes of matrices to simultaneous conjugacy classes of m-tuples of matrices.

Let us define

$$M_n^m := M_n \oplus \cdots \oplus M_n \quad , \quad (m\text{-fold direct sum}) .$$

As noted earlier, M_n is an n^2- dim. complex vector space, hence M_n^m is an mn^2- dim. complex vector space. There is an action of GL_n on M_n^m as follows:

$$\forall (A_1, A_2, \dots, A_m) \in M_n^m, \quad \forall g \in GL_n :$$
$$g \cdot (A_1, \dots, A_m) := (gA_1g^{-1}, \dots, gA_mg^{-1}) \in M_n^m . \qquad (10)$$

Unfortunately, as it is clear from this definition there is no result parallel to that of Jordan normal form in this case. This shows that the classification of the orbits of m- tuples of matrices under simultaneous conjugations is in general an impossible task; however, for small m and n one can find the GL_n orbits by ad hoc methods.

Example. GL_2-orbits in $M_2^2 = M_2 \oplus M_2$.

Our approach will be as the one employed in the case of conjugacy classes problem, i.e., we will try to approximate the orbit space via polynomial functions on M_2^2 which are constant on orbits.

Let $(A, B) \in M_2^2$. Clearly $tr(A)$, $det(A)$, $tr(B)$, $det(B)$ and $tr(AB)$ are constant on the orbits of GL_2 action given by (10). In the next section it will be shown that these five functions generate all polynomial functions which are constant on orbits. We now establish that the mapping

$$\pi : \begin{cases} M_2^2 \longrightarrow \mathbb{C}^5 \\ (A, B) \longmapsto (tr(A), det(A), tr(B), det(B), tr(AB)) \end{cases}$$

is surjective such that each fiber contains only one closed

orbit.

surjectivity of π. Let $(x_1, \dots, x_5) \in \mathbb{C}^5$; we will show that there exists $(A, B) \in M_2^2$ s.t. $\pi(A, B) = (x_1, \dots, x_5)$. Consider the open set $x_1^2 \neq 4x_2$. We have seen before that this characterizes those $A \in M_2$ s.t. A has distinct eigenvalues and hence is diagonalizable. Thus we can choose as a representative of the orbit $\mathcal{O}(A, B)$ a pair

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right) \in M_2^2 \quad , \quad \lambda \neq \mu .$$

It remains to show that the set of equations

$$\begin{cases} x_3 = c_1 + c_4 \\ x_4 = c_1c_4 - c_2c_3 \\ x_5 = \lambda c_1 + \mu c_2 \end{cases}$$

has a solution. Since $\lambda \neq \mu$, the first and the last equations uniquely determine c_1 and c_4; using these in the second equation gives us c_2c_3.

In a similar manner, points $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5$ belonging to the open set $x_3^2 \neq 4x_4$ lie in the image of π. Finally, consider the complement of these open sets, that is those $(x_1, \dots, x_5) \in \mathbb{C}^5$ s.t. $x_1^2 = 4x_2$ and $x_3^2 = 4x_4$; in this case consider

$$(A, B) = \left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \mu & 0 \\ c & \mu \end{bmatrix} \right)$$

where $\lambda = \frac{1}{2}x_1$, $\mu = \frac{1}{2}x_3$ (so both $x_1^2 = 4x_2$ and $x_3^2 = 4x_4$ are satisfied). The remaining equation $x_5 = tr(AB) = 2\lambda\mu + c$ uniquely determines c. Hence π is surjective.

The fibers of π.

Let $(A, B) \in M_2^2$; and assume that A and B have a common eigenvector v. Let $w \in \mathbb{C}^2$ be linearly independent of v and let $g \in GL_n$ be the matrix of base-change from the standard basis in $\mathbb{C}^2$ to the basis $\{v, w\}$. It follows that $\mathcal{O}(A,B)$ contains a pair of upper triangular matrices

$$\left(\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}\right)$$

that we choose as a representant of $\mathcal{O}(A,B)$, with π image $(x_1, \dots, x_5)$. The coordinates x_1, x_2 determine the eigenvalues of A only as an unordered set $\{a_1, a_2\}$ and similarly x_3, x_4 determine the set $\{b_1, b_2\}$. Hence

$$tr(AB) = a_1 b_1 + a_3 b_3 \quad \text{or} \quad tr(AB) = a_1 b_3 + a_3 b_1$$

and, therefore, it satisfies the equation

$$(tr(AB) - a_1 b_1 - a_3 b_3)(tr(AB) - a_1 b_3 - a_3 b_1) = 0.$$

Using $x_1 = tr(A) = a_1 + a_3$, $x_2 = \det(A) = a_1 a_3$, $x_3 = tr(B) = b_1 + b_3$, $x_4 = \det(B) = b_1 b_3$ and $x_5 = tr(AB)$, this gives the equation of a hypersurface $H \hookrightarrow \mathbb{C}^5$,

$$H: \quad x_5^2 - x_1 x_3 x_5 + x_1^2 x_4 + x_3^2 x_2 - 4 x_2 x_4 = 0.$$

Therefore, H is a 4-dim. subset of $\mathbb{C}^5$ with singularities given by the set of common zeroes of

$$\frac{\partial f}{\partial x_i} = 0, \quad 0 \leq i \leq 5.$$

These singularities form a 2-dim. submanifold S which can be parametrized as $(2a, a^2, 2b, b^2, 2ab)$, $a, b \in \mathbb{C}$.

It can be shown (see LeBruyn) that the smooth submanifolds of $\mathbb{C}^5$:

$$\mathbb{C}^5 - H, \quad H - S, \quad S$$

exhaust all possible types of fiber behavior. These are

(i) $p \in \mathbb{C}^5 - H$ (i.e., $p \notin H$). In this case $\pi^{-1}(p)$ is a unique orbit and hence it must be closed.

(ii) $p \in H - S$ and $(A,B) \in \pi^{-1}(p)$. Then A and B are simultaneously upper triangularizable, with a_1, a_2 b_1, b_2 as

as their eigenvalues respectively. Then either $a_1 \neq a_2$ or $b_1 \neq b_2$ for otherwise $p \in S$. Suppose $a_1 \neq a_2$; it can be shown that there are only three distinct orbits

$$\mathcal{O}\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}\right), \quad \mathcal{O}\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\right), \quad \mathcal{O}\left(\begin{bmatrix} a_2 & 0 \\ 0 & a_1 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}\right).$$

The case of $b_1 \neq b_2$ can be handled similarly. Clearly the second is the unique closed orbit in $\pi^{-1}(p)$ which lies in the closure of the first and third orbits (so it is contained in the intersection of closures of the 1st and 3rd orbits).

(iii) $p \in S$. In this case $\pi^{-1}(p)$ consists of an infinite family of 2-dim. orbits together with the orbit

$$\mathcal{O}\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}\right)$$

which consist of one point (and hence is closed in M_2^2) and lies in the closure of each of the 2-dimensional orbits. ∎

(2.4) The necklace and the trace algebras

We would like to specify the ring of all polynomial maps

$$M_n^m := \underbrace{M_n \oplus \cdots \oplus M_n}_{m\text{-fold}} \xrightarrow{f} \mathbb{C}$$

which are constant on the orbits of the GL_n-action on M_n^m by simultaneous conjugation, i.e.

$$\begin{matrix} \forall (A_1, \dots, A_m) \in M_n^m \\ \forall g \in GL_n \end{matrix} : \quad g \cdot (A_1, \dots, A_m) := (g A_1 g^{-1}, \dots, g A_m g^{-1}).$$

Corresponding to this action on M_n^m there is an induced GL_n action on the polynomial maps on M_n^m. The coordinate algebra of the $m \cdot n^2$-dimensional complex vector space M_n^m, denoted by $\mathbb{C}[M_n^m]$ is a polynomial ring in mn^2 variables

$$x_{ij}(k), \quad 1 \leq k \leq m, \quad 1 \leq i, j \leq n.$$

The action of GL_n on the generic matrices

$$X_k = (x_{ij}(k))_{i,j} \in M_n(\mathbb{C}[M_n^m])$$

is defined by

$$g \cdot x_{ij}(k) := (g^{-1} X_k g)_{ij} \quad \leftarrow (11)$$

which in turn defines the required GL_n-action on the polynomial maps $f \in \mathbb{C}[M_n^m]$.

The GL_n-action given by (11) preserves the subspaces spanned by the entries of each generic matrix. We can, therefore, define a graduation on $\mathbb{C}[M_n^m]$ by attributing a degree to each variable

$$\deg(x_{ij}(k)) = (0,0,\dots,1,0,\dots,0) \quad , \ 1 \text{ at place } k \ ;$$

which actually means assignment of degree e_k to each generic matrix X_k. It then follows that

$$\mathbb{C}[M_n^m] = \bigoplus_{(d_1,\dots,d_m)\in\mathbb{N}^m} \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)}$$

where $\mathbb{C}[M_n^m]_{(d_1,\dots,d_m)}$ is the subspace of all multi-homogeneous forms in $x_{ij}(k)$ of degree $(d_1,\dots,d_m)$, i.e., in each monomial term of f there are exactly d_k factors coming from the generic matrix X_k, $1 \le k \le m$. Each of these subspaces is invariant under GL_n-action

$$\forall f \in \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)} \ \forall g \in GL_n \ : \ g\cdot f \in \mathbb{C}[M_n^m]_{(d_1,\dots,d_m)} .$$

clearly, if $f \in \mathbb{C}[M_n^m]$ is constant on the GL_n-orbits i.e., if $f \in \mathbb{C}[M_n^m]^{GL_n}$, the ring of invariants, then each of its multihomogeneous components is also an invariant. Therefore, to determine $\mathbb{C}[M_n^m]^{GL_n}$ it suffices to determine all multihomogeneous invariants.

In an important paper C. Procesi has proved many interesting results on the rings of invariants (see the references). We shall review some of them here.

<u>Theorem 7.</u> (C. Procesi)

Any polynomial function $f: M_n^m \longrightarrow \mathbb{C}$ which is constant on orbits under the action of GL_n by conjugation is a polynomial in the invariants

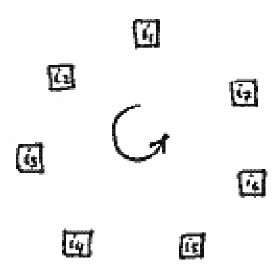
$$tr(X_{i_1} X_{i_2} \cdots X_{i_\ell})$$

where $X_{i_1},\dots,X_{i_\ell}$ run over all possible choices of generic matrices from the set $\{X_1,\dots,X_m\}$. ∎

This ring of polynomial invariant is denoted by $\mathbb{N}_n^m$ and is called the <u>necklace algebra</u>.

$$\therefore \ \mathbb{N}_n^m := \mathbb{C}[M_n^m]^{GL_n} \hookrightarrow \mathbb{C}[M_n^m] \quad , \ \text{subalgebra inclusion} .$$

The terminology "necklace" is justified by the observation that the generators $tr(X_{i_1} \cdots X_{i_\ell})$ of $\mathbb{N}_n^m$ are only determined up to cyclic permutations of the generic matrices X_i ; hence each generator of $\mathbb{N}_n^m$ can be specified graphically by a <u>necklace word</u> such as



where in each box one inserts a chosen generic matrix from the set $\{X_1,\dots,X_m\}$, multiplies them in a cylic order and takes the trace of the monomial obtained this way to obtain a generator of $\mathbb{N}_n^m$.

To make the result announced in theorem 7 more specific and practically more useful, we must put a bound on the length of the necklaces necessary to generate N_n^m. We follow closely C. Procesi's paper for this purpose.

Consider the <u>GL_n-equivariant polynomial maps</u> from M_n^m to M_n; i.e., polynomial maps $M_n^m \xrightarrow{f} M_n$ such that the following diagram is commutative

$$\begin{array}{ccc} M_n^m & \xrightarrow{\quad f \quad} & M_n \\ {\scriptstyle g\cdot g^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle g\cdot g^{-1}} \\ M_n^m & \xrightarrow{\quad f \quad} & M_n \end{array}$$

where $g\cdot g^{-1}$ indicates the GL_n-action by simultaneous conjugations. Under pointwise addition and multiplication in the target space M_n, the set of all such maps form a non-commutative algebra over $\mathbb{C}$, denoted by $\mathbb{T}_n^m$ and called the <u>trace algebra</u>. This is a subalgebra of the algebra of all polynomial maps $f: M_n^m \longrightarrow M_n$ (which we can identify with $M_n(\mathbb{C}[M_n^m])$) So there is a subalgebra embedding

$$\mathbb{T}_n^m \hookrightarrow M_n(\mathbb{C}[M_n^m]).$$

Using the diagonal embedding $\mathbb{C} \hookrightarrow M_n$, any invariant polynomial map on M_n^m (i.e., an element of N_n^m) determines a GL_n-equivariant map. Equivalently one can use the diagonal embedding

$$\mathbb{C}[M_n^m] \hookrightarrow M_n(\mathbb{C}[M_n^m])$$

to embed the necklace algebra N_n^m into the trace

algebra $\mathbb{T}_n^m$

$$N_n^m := \mathbb{C}[M_n^m]^{GL_n} \hookrightarrow \mathbb{T}_n^m := M_n(\mathbb{C}[M_n^m])^{GL_n},$$

(diagonal embedding)

in this way N_n^m can be identified with the center of $\mathbb{T}_n^m$ and one can consider $\mathbb{T}_n^m$ as an algebra over the commutative ring (or algebra) N_n^m.

It it clear that a basic source of GL_n-equivariant maps $M_n^m \longrightarrow M_n$ are the coordinate maps

$$X_i : \begin{cases} M_n^m \longrightarrow M_n \\ (A_1, \dots, A_m) \longmapsto A_i \end{cases}$$

and each coordinate map is represented by a generic matrix $X_i \in M_n(\mathbb{C}[M_n^m])$.

<u>Proposition 8.</u> As an algebra over the necklace algebra N_n^m, the trace algebra $\mathbb{T}_n^m$ is generated by the elements $\{X_1, \dots, X_m\}$.

<u>Proof.</u> To any GL_n-equivariant map $f: M_n^m \longrightarrow M_n$ we can associate a map

$$tr(f X_{m+1}) : \begin{cases} M_n^{m+1} := M_n^m \oplus M_n \longrightarrow \mathbb{C} \\ (A_1, \dots, A_m, A_{m+1}) \longmapsto tr(f(A_1, \dots, A_m)\cdot A_{m+1}) \end{cases}$$

By equivariance of f we have that

$$\forall g \in GL_n : \ f(gA_1g^{-1}, \dots, gA_mg^{-1}) = g\, f(A_1, \dots, A_m)\, g^{-1},$$

therefore,

$$\begin{aligned} tr\big(f(gA_1g^{-1}, \dots, gA_mg^{-1})\cdot gA_{m+1}g^{-1}\big) &= \\ &= tr\big(g\, f(A_1, \dots, A_m) g^{-1} \cdot g A_{m+1} g^{-1}\big) \\ &= tr\big(g\, f(A_1, \dots, A_m)\cdot A_{m+1}\, g^{-1}\big) \\ &= tr\big(f(A_1, \dots, A_m)\cdot A_{m+1}\big) \ ; \end{aligned}$$

hence $tr(f X_{m+1})$ is an invariant polynomial on M_n^{m+1} and it is clearly linear in X_{m+1}. Using theorem 7, we can write

$$tr(f X_{m+1}) = \sum g_{i_1 \cdots i_\ell} \, tr(X_{i_1} \cdots X_{i_\ell} X_{m+1})$$

where $g_{i_1 \cdots i_\ell} \in \mathbb{N}_n^m$, and where we have used the necklace property to cyclically permute the terms in the trace sign such that X_{m+1} appears as the last factor. But then

$$tr(f X_{m+1}) = tr(g X_{m+1}) \qquad (*)$$

where

$$g = \sum g_{i_1 \cdots i_\ell} X_{i_1} \cdots X_{i_\ell}.$$

Finally, using the fact that the trace map is non-degenerate on M_n, i.e., $tr(AC) = tr(BC)$ for all $C \in M_n \Rightarrow A = B$, $(*)$ implies $f = g$. ∎

Assigning to each generic matrix X_i a grade one, we see that the trace algebra $\mathbb{T}_n^m$ is a positively graded algebra over $\mathbb{C}$:

$$\mathbb{T}_n^m = \mathbb{T}_0 \oplus \mathbb{T}_1 \oplus \mathbb{T}_2 \oplus \cdots, \qquad \mathbb{T}_0 \cong \mathbb{C}.$$

Our aim is now to find a bound on the length of the monomials in X_i necessary to generate $\mathbb{T}_n^m$ as a module over the necklace algebra $\mathbb{N}_n^m$. For this purpose we need the following result from non-commutative nil algebras.

Theorem 9. (Nagata-Higman) Let A be an associative algebra without unit over $\mathbb{C}$. Suppose $\exists n \in \mathbb{N}^+$ s.t. $a^n = 0$, $\forall a \in A$. Then

$A^{2^n-1} = 0$; i.e., every word of length $2^n - 1$ is zero:

$$a_1 a_2 \cdots a_{2^n-1} = 0, \qquad \forall a_i \in A, \; i = 1, \ldots, 2^n - 1.$$

Proof. The proof is by induction on n, the case $n=1$ being obvious. Let us define

$$\forall a, b \in A: \quad f(a,b) = ba^{n-1} + aba^{n-2} + a^2ba^{n-3} + \cdots + a^{n-2}ba + a^{n-1}b.$$

For all $\alpha \in \mathbb{C}$ we must have

$$0 = (\alpha a + b)^n = \alpha^n a^n + \alpha^{n-1} f(a,b) + \cdots + b^n$$

which implies that all coefficients of α^i, $1 \le i \le n$, must be zero and in particular $f(a,b) = 0$. It then follows that

$$\forall a,b,c \in A: \quad 0 = f(a,c)b^{n-1} + f(a,cb)b^{n-2} + f(a,cb^2)b^{n-3} + \cdots + f(a,cb^{n-1}) =$$
$$= na^{n-1}cb^{n-1} + c\,f(b,a^{n-1}) + ac\,f(b,a^{n-2}) + a^2c\,f(b,a^{n-3}) + \cdots + a^{n-2}c\,f(b,a)$$

This immediately implies that $a^{n-1}cb^{n-1} = 0$.

Let $I \triangleleft A$ be the 2-sided ideal of A generated by all elements a^{n-1}, $a \in A$. Then $I \cdot A \cdot I = 0$. Consider the quotient algebra $\bar{A} = A/I$; clearly

$$\forall \bar{a} \in \bar{A}: \quad \bar{a}^{n-1} = 0.$$

By induction hypothesis (applied for the algebra $\bar{A}$) we may assume that $\bar{A}^{2^{n-1}-1} = 0$, or equivalently $A^{2^{n-1}-1} \subseteq I$. Then we have

$$A^{2^n-1} = A^{2(2^{n-1}-1)+1} = A^{2^{n-1}-1} \cdot A \cdot A^{2^{n-1}-1} \subseteq I \cdot A \cdot I = 0$$

which finishes the proof. ∎

Proposition 10. The trace algebra $\mathbb{T}_n^m$ is generated as a module over the necklace algebra by all monomials in the generic matrices $X_{i_1} X_{i_2} \cdots X_{i_\ell}$ of length $\ell \le 2^n - 1$.

Proof. Using the diagonal embedding $N_n^m \hookrightarrow M_n(\mathbb{C}[M_n^m])$ it is clear that N_n^m commutes with any generic matrix X_i. Let $\mathbb{T}_+$ and $\mathbb{N}_+$ be the parts of strictly positive degrees of $\mathbb{T}_n^m$ and $\mathbb{N}_n^m$ respectively, and consider the graded associative algebra (without unit)

$$A = \mathbb{T}_+ / \mathbb{N}_+ \cdot \mathbb{T}_+$$

Any element $t \in \mathbb{T}_+$ satisfies an equation of the form

$$t^n + c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n = 0 \quad , \quad c_i \in \mathbb{N}_+ , \; i = 1, \dots, n$$

This follows from the observation that all coefficients of the characteristic polynomial of a matrix can be expressed as polynomials in the traces of powers of the matrix. We, therefore, conclude that

$$\forall a \in A : \quad a^n = 0 .$$

Applying the Nagata-Higman theorem we get $A^{2^n-1} = 0$.

Let $\mathbb{T}'$ be the graded $\mathbb{N}_n^m$-submodule of $\mathbb{T}_n^m$ spanned by all monomials in generic matrices X_i of degree at most $2^n - 1$; then by what has been said above we can write

$$\mathbb{T}_n^m = \mathbb{T}' + \mathbb{N}_+ \cdot \mathbb{T}_n^m$$

We claim $\mathbb{T}_n^m = \mathbb{T}'$. Assume the contrary, i.e. $\exists t \in \mathbb{T}_n^m$, a homogeneous polynomial of minimal degree d which is not contained in $\mathbb{T}'$ but we still have

$$t = t' + c_1 \cdot t_1 + c_2 \cdot t_2 + \cdots + c_k \cdot t_k$$

with t', c_i, t_i being homogeneous elements. But then $\deg(t_i) < d$ and this implies $t_i \in \mathbb{T}'$ for all i. This in turn implies $t' \in \mathbb{T}'$, a contradiction. ∎

Using this result we can now put a bound on the length of the necklaces which generate the algebra $\mathbb{N}_n^m$.

Theorem 11. The necklace algebra $\mathbb{N}_n^m$ is generated by all elements of the form

$$tr(X_{i_1} X_{i_2} \cdots X_{i_\ell})$$

where $\ell \leq 2^n$.

Proof. Let $\mathbb{T}$ be the $\mathbb{C}$-subalgebra of $\mathbb{T}_n^m$ generated by the generic matrices X_i. Clearly $tr(\mathbb{T}_+)$ generates the ideal $\mathbb{N}_+$. Let S be the set of all monomial in X_i of degree at most $2^n - 1$. By the previous proposition we have an embedding

$$\mathbb{T} \hookrightarrow \mathbb{T}_n^m \hookrightarrow \mathbb{N}_n^m \cdot S .$$

The trace map $tr : \mathbb{T}_n^m \longrightarrow \mathbb{N}_n^m$ is $\mathbb{N}_n^m$-linear, and because $\mathbb{T}_+ = \mathbb{T} \cdot (\mathbb{C}X_1 + \cdots + \mathbb{C}X_m)$, we have

$$tr(\mathbb{T}_+) \subset tr(\mathbb{N}_n^m \cdot S(\mathbb{C}X_1 + \cdots + \mathbb{C}X_m)) \subset \mathbb{N}_n^m \cdot tr(S')$$

where S' is the set of monomials in the X_i of degree at most 2^n. Let $\mathbb{N}'$ be the $\mathbb{C}$-subalgebra of $\mathbb{N}_n^m$ generated by all $tr(S')$; then $tr(\mathbb{T}_+) \subset \mathbb{N}_n^m \cdot \mathbb{N}'_+$. This implies

$$\mathbb{N}_+ := \mathbb{N}_n^m \, tr(\mathbb{T}_+) \subset \mathbb{N}_n^m \cdot \mathbb{N}'_+$$

$$\therefore \quad \mathbb{N}_n^m = \mathbb{N}' + \mathbb{N}_n^m \, \mathbb{N}'_+ .$$

It then follows, by a similar argument as in proposition 10, or lemma of Nakayama, that $\mathbb{N}_n^m = \mathbb{N}'$. ∎

Example. The algebras $\mathbb{N}_2^2$ and $\mathbb{T}_2^2$.

In the special case of 2x2 matrices we have the following useful matrix identities

$$\forall A,B\in M_2 : \begin{cases} 0 = A^2 - tr(A)A + \det(A) \\ AB+BA = tr(AB) - tr(A)tr(B) + tr(A)B + tr(B)A \end{cases} \quad \leftarrow (12)$$

We shall refer to these relations as the <u>identities of 2x2 matrices</u>.

Let N' be the subalgebra of N_2^2 generated by $tr(X_1)$, $tr(X_2)$, $\det(X_1)$, $\det(X_2)$ and $tr(X_1X_2)$. Using (12) and N_2^2-linearity of the trace on $\mathbb{T}_2^2$ we see that the trace of any monomial in X_1 and X_2 of degree $d \geq 3$ can be expressed in elements of N' and traces of monomials of degree $\leq d-1$. Hence

$$N_2^2 = \mathbb{C}[tr(X_1), tr(X_2), \det(X_1), \det(X_2), tr(X_1X_2)] \quad \leftarrow (13)$$

As we have shown earlier the mapping

$$\pi : \begin{cases} M_2^2 \longrightarrow \mathbb{C}^5 \\ (A,B) \longmapsto (tr(A), \det(A), tr(B), \det(B), tr(AB)) \end{cases}$$

is surjective which proves that there can be no algebraic relations among the generators in (13).

Next, notice that, using (12), any monomial in X_1, X_2 of degree $d \geq 3$ can be written as a linear combination of $1, X_1, X_2, X_1X_2$ with coefficients in N_2^2. We will show that these four elements are linearly independent over N_2^2. Assume otherwise; so there exists a relation

$$X_1X_2 = \alpha 1_2 + \beta X_1 + \gamma X_2 \qquad (*)$$

$\alpha, \beta, \gamma \in N_2^2$. As (*) is a relation in generic matrices, it must hold for all possible choices $(A,B) \in M_2^2$. But taking $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we have $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and this contradicts (*). Thus $\{1, X_1, X_2, X_1X_2\}$ is

linearly independent over N_2^2 and generate $\mathbb{T}_2^2$:

$$\mathbb{T}_2^2 = N_2^2 \cdot 1_2 \oplus N_2^2 X_1 \oplus N_2^2 X_2 \oplus N_2^2 X_1X_2 \,. \quad \leftarrow (14)$$

We could, of course, have chosen the generators $tr(X_i^2)$ instead of $\det(X_i)$, $i=1,2$, because using the characteristic equation of X_i,

$$X_i^2 - tr(X_i)X_i + \det(X_i) = 0$$

and taking trace of this yields

$$2\det(X_i) = (tr(X_i))^2 - tr(X_i^2)$$

$$\Rightarrow \det(X_i) = \tfrac{1}{2}\left[(tr(X_i))^2 - tr(X_i^2)\right]. \qquad \blacksquare$$

(2.5) The necklace and the trace relations

In this section we shall explain how the relations among the elements of the necklace algebra can be obtained as formal consequences of the Cayley-Hamilton equations for $n \times n$ matrices. Here we shall only describe the results and methods; for the proofs and detailed discussions we refer to the refrences of this chapter.

Let $S = \{x_1, x_2, \dots, x_i, \dots\}$ be an infinite set of non-commutative variables. We introduce an equivalence relation in the set of all possible words in these variables as follows:

two words of the same length l

$$\omega_1 = x_{i_1}x_{i_2}\cdots x_{i_l}, \qquad \omega_2 = x_{j_1}x_{j_2}\cdots x_{j_l}$$

are said to be <u>equivalent</u> if ω_2 is obtained from ω_1 by <u>a cyclic permutation</u>.

To each equivalence class of words we assign a formal variable $t(x_{i_1}x_{i_2}\cdots x_{i_\ell})$; clearly

$$t(x_{i_1}x_{i_2}\cdots x_{i_\ell}) = t(x_{\sigma(i_1)}x_{\sigma(i_2)}\cdots x_{\sigma(i_\ell)})$$

for every cyclic permutation σ.

Def. The formal necklace algebra, denoted by $\mathbb{N}^\infty$, is the polynomial algebra over $\mathbb{C}$ in these infinitely many variables $t(x_{i_1}\cdots x_{i_\ell})$. Notice that $\mathbb{N}^\infty$ is a commutative algebra. ∎

Def. The formal trace algebra $\mathbb{T}^\infty$ is

$$\mathbb{T}^\infty := \mathbb{N}^\infty \otimes_{\mathbb{C}} \mathbb{C}\langle x_1, x_2, \ldots, x_i, \ldots\rangle, \qquad \leftarrow (15)$$

i.e., $\mathbb{T}^\infty$ is the free algebra on the non-commutative variables x_i with coefficients in the polynomial algebra $\mathbb{N}^\infty$. ∎

Def. The formal trace map is an $\mathbb{N}^\infty$-linear map defined by

$$Tr : \begin{cases} \mathbb{T}^\infty \longrightarrow \mathbb{N}^\infty \\ \sum a_{i_1\cdots i_\ell} x_{i_1}\cdots x_{i_\ell} \longmapsto \sum a_{i_1\cdots i_\ell} t(x_{i_1}\cdots x_{i_\ell}) \end{cases} \qquad \leftarrow (16)$$

where $a_{i_1\cdots i_\ell} \in \mathbb{N}^\infty$. ∎

Analogously we define the infinite necklace and infinite trace algebras. Let

$$M_n^\infty := \bigoplus_{i=1}^{\infty} M_n^{(i)},$$

the infinite direct sum of copies of M_n. Each element of M_n^∞ is an infinite sequence $(A_1, A_2, \ldots, A_i, \ldots)$ of $n\times n$ complex matrices with finitely many non-zero members. There is an action of GL_n on M_n^∞ by simultaneous conjugations.

Def. The infinite necklace algebra $\mathbb{N}_n^\infty$ is the algebra of polynomial functions $f : M_n^\infty \longrightarrow \mathbb{C}$, which are

constant on the GL_n-orbits. That is

$$\mathbb{N}_n^\infty := \mathbb{C}[M_n^\infty]^{GL_n}. \qquad ∎$$

It is clear that $\mathbb{N}_n^\infty$ is generated as a $\mathbb{C}$-algebra by invariants $tr(\omega)$ where ω runs over all words in generic matrices $X_k = (x_{ij}(k))_{i,j}$ which belong to the k-component of M_n^∞.

Def. The infinite trace algebra $\mathbb{T}_n^\infty$ is the algebra of GL_n-equivariant maps $M_n^\infty \longrightarrow M_n$; i.e.,

$$\mathbb{T}_n^\infty := M_n(\mathbb{C}[M_n^\infty])^{GL_n}.$$

Clearly $\mathbb{T}_n^\infty$ is the $\mathbb{C}$-algebra generated by $\mathbb{N}_n^\infty$ and the generic matrices X_k, $1 \le k \le \infty$. ∎

We also have the embeddings

$$\mathbb{N}_n^\infty \xhookrightarrow{\text{diagonal}} \mathbb{T}_n^\infty \hookrightarrow M_n(\mathbb{C}[M_n^\infty]). \qquad \leftarrow (17)$$

which shows that there is an obvious trace map on $\mathbb{T}_n^\infty$ and

$$tr(\mathbb{T}_n^\infty) = \mathbb{N}_n^\infty.$$

The following observation is crucial for what follows next. Using the freeness property of the formal trace algebra, (15), we conclude that there exists a (natural) algebra epimorphism

$$\mathbb{T}^\infty \xrightarrow{\ \tau\ } \mathbb{T}_n^\infty \qquad \leftarrow (18)$$

which induces, using the embedding (17), an epimorphism

$$\mathbb{N}^\infty \xrightarrow{\ \nu\ } \mathbb{N}_n^\infty. \qquad \leftarrow (19)$$

Clearly these algebra maps are given by

$$\begin{cases} \tau(t(x_{i_1}\cdots x_{i_\ell})) = \nu(t(x_{i_1}\cdots x_{i_\ell})) = tr(X_{i_1}\cdots X_{i_\ell}) \\ \tau(x_i) = X_i, \quad \forall i, \end{cases} \qquad \leftarrow (20)$$

and ν, τ are compatible with the trace maps on $\mathbb{T}^{\infty}$ and $\mathbb{T}_n^{\infty}$, i.e., the following diagram is commutative

$$\begin{array}{ccc} \mathbb{T}^{\infty} & \xrightarrow{\ \tau\ } & \mathbb{T}_n^{\infty} \\ \mathrm{Tr}\downarrow & & \downarrow \mathrm{tr} \\ \mathbb{N}^{\infty} & \xrightarrow{\ \nu\ } & \mathbb{N}_n^{\infty} \end{array}$$

Our aim is to find the *necklace relations*, i.e. $\mathrm{Ker}(\nu)$, and the *trace relations*, i.e., $\mathrm{Ker}(\tau)$. Once these are found, we can obtain the relations among the necklaces in $\mathbb{N}_n^m$ by setting $x_i = 0$, $\forall i > m$ and $t(x_{i_1} \cdots x_{i_\ell}) = 0$, $\forall i_j > m$.

Claim. $\mathrm{Ker}(\nu)$ and $\mathrm{Ker}(\tau)$ are formal consequences of the Cayley-Hamilton polynomial for $n \times n$ matrices.

Proof. See the references. ∎

We shall explain what is meant by this statement. Let $X \in M_n(A)$ be a matrix with coefficients in a commutative $\mathbb{Q}$-algebra A. Its characteristic polynomial is

$$\chi_X(t) := \det(t\,\mathbb{1}_n - X) \in A[t] \qquad \leftarrow (21)$$

and by Cayley-Hamilton theorem we know that $\chi_X(X) = 0$.

The coefficients of the characteristic polynomial can be expressed as polynomial functions in the $\mathrm{tr}(X^i)$ for $1 \le i \le n$. For example when $n=2$,

$$\chi_X(t) = t^2 - \mathrm{tr}(X)\, t + \tfrac{1}{2}\left(\mathrm{tr}(X)^2 - \mathrm{tr}(X^2)\right).$$

For general n one uses a formal recursive algorithm to express the elementary symmetric functions in terms of Newton symmetric functions as follows:

given $f(t) = \prod_{i=1}^{n} (t - \lambda_i) = t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$

$\left(\equiv t^n + \sum_{i=1}^{n} (-1)^i \sigma_i t^{n-i}\right)$

we have

$$\frac{f'(t)}{f(t)} = \frac{d \ln f(t)}{dt} = \sum_{i=1}^{n} \frac{1}{t - \lambda_i} = \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} \sum_{i=1}^{n} \lambda_i^k$$

$$\therefore \quad f'(t) = f(t) \cdot \left(\sum_{k=0}^{\infty} \frac{1}{t^{k+1}} \sum_{i=1}^{n} \lambda_i^k \right) \qquad \leftarrow (22)$$

Equating coefficients of equal powers of t on both sides gives the required expressions of the elementary symmetric functions σ_j in terms of Newton symmetric functions $\sum_{i=1}^{n} \lambda_i^k$.

If λ_i, $i = 1, \ldots, n$ are the eigenvalues of $X \in M_n$, then $f(t) = \chi_X(t)$ and $\sum_{i=1}^{n} \lambda_i^k = \mathrm{tr}(X^k)$. One can, therefore, use the above algorithm to express $f(t)$ in terms of $\sum_{i=1}^{n} \lambda_i^k$ and then replace $\sum_{i=1}^{n} \lambda_i^k$ by $\mathrm{tr}(X^k)$ to get the characteristic polynomial of X in the desired form.

This procedure allows one to construct a *formal Cayley-Hamilton polynomial* $\chi_x(x) \in \mathbb{T}^{\infty}$ for an element $x \in \mathbb{T}^{\infty}$ by the following replacements in the above characteristic polynomial

$$\begin{cases} \mathrm{tr}(X^k) \longrightarrow t(x^k) \\ t^{\ell} \longrightarrow x^{\ell} \end{cases} \qquad \leftarrow (23)$$

If x is one of the variables $x_i \in S$, then $\chi_x(x) \in \mathbb{T}^{\infty}$ is a homogeneous element of degree n. *Moreover, by the Cayley-Hamilton theorem it follows immediately that $\chi_x(x)$ is a trace relation.*

Once $\chi_x(x)$, $x \in \mathbb{T}^{\infty}$, is in hand, we can apply the process of polarization (which is equivalent to multilinearization) with respect to variables

$$y_1, y_2, \dots, y_n \in \mathbb{T}^\infty,$$

whose first step is to replace $\chi_x(x)$ by

$$\chi_{y_1+y_2}(y_1+y_2) - \chi_{y_1}(y_1) - \chi_{y_2}(y_2)$$

and continue to get the full polarized form $G(y_1, \dots y_n)$ which is multilinear in the variables $y_1, \dots, y_n$.

For example, in case $n=2$, for every $x \in \mathbb{T}^\infty$ we have

$$\chi_x(x) = x^2 - t(x)x + \tfrac{1}{2}\left(t(x)^2 - t(x^2)\right)$$

Polarizing with respect to variables y_1 and y_2 gives

$$G(y_1, y_2) := y_1 y_2 + y_2 y_1 - t(y_1) y_2 - t(y_2) y_1 + t(y_1) t(y_2) - t(y_1 y_2).$$

Multiplying this expression by an extra variable, say y_3, and apply on the resulting expression the formal trace Tr, we get

$$F(y_1, y_2, y_3) := t(y_1 y_2 y_3) + t(y_2 y_1 y_3) - t(y_1)\, t(y_2 y_3) + - t(y_2) t(y_1 y_3) + t(y_1) t(y_2) t(y_3) - t(y_1 y_2) t(y_3).$$

Similarly for higher n.

We now give the main result.

Theorem 12. The necklace relations $\mathrm{Ker}(\nu)$ is the ideal of $\mathbb{N}^\infty$ generated by all elements

$$F(y_1, y_2, \dots, y_{n+1})$$

where y_i run over all words in the variables $\{x_1, x_2, \dots, x_i, \dots\}$.

The trace relations $\mathrm{Ker}(\tau)$ is the 2-sided ideal of the formal trace algebra $\mathbb{T}^\infty$, generated by all elements

$$F(y_1, \dots, y_{n+1}) \quad , \quad G(y_1, \dots, y_n)$$

where y_i run over all words in variables $\{x_1, \dots, x_i, \dots\}$. ∎

It has been shown (e.g., see Le Bruyn) that the necklace and trace relations can be used to improve the bound $2^n - 1$ in Nagata – Higman problem to n^2. We only give the result.

Theorem 13. The necklace algebra $\mathbb{N}_n^m$ is generated as a $\mathbb{C}$-algebra by elements of the form

$$tr(X_{i_1} X_{i_2} \cdots X_{i_\ell})$$

where $\ell \leq n^2+1$. The trace algebra $\mathbb{T}_n^m$ is generated, as a module over $\mathbb{N}_n^m$, by words

$$X_{i_1} X_{i_2} \cdots X_{i_\ell}$$

in generic matrices of length $\ell \leq n^2$. ∎

References.

(1) C. Procesi. The invariant theory of $n \times n$ matrices. Adv. in Math. 19 (1976) 306-381.

(2) Lieven Le Bruyn. As in chapter 1.

CHAPTER III
Some Algebraic Geometry

In this chapter k stands for an algebraically closed field of characteristic zero, unless otherwise explicitly stated. In particular $k = \mathbb{C}$.

Section A. Algebra - Geometry correspondences

(A.1) Algebraic subsets in k^n.

Let k^n be an n-dim. k-vector space and denote the coordinate functions corresponding to the i-th standard basis vector by x_i. The coordinate algebra of k^n (or the coordinate ring of k^n) is the polynomial ring generated by these coordinate functions, i.e., $k[x_1,\dots,x_n]$.

Any polynomial $f \in k[x_1,\dots,x_n]$ determines a k-valued function on k^n by evaluation. The set of points $a=(a_1,\dots,a_n)\in k^n$ s.t. $f(a) = f(a_1,\dots,a_n)=0$ is called the zero set of f and is denoted by $\mathbb{Z}(f;k)$ or simply by $\mathbb{Z}(f)$. More generally for any subset $S\subset k[x_1,\dots,x_n]$ the zero set of S is defined to be

$$\mathbb{Z}(S) := \{a\in k^n \mid f(a)=0,\ \forall f\in S\}.$$

Such subsets of k^n are called algebraic subsets of k^n.

Because $\mathbb{Z}(f) = f^{-1}(\{0\})$ and $\{0\}\subset k$ is a closed set in the usual topology of k, and because f, being a polynomial function, is continuous, the inverse image $\mathbb{Z}(f)$ of $\{0\}$ is a closed subset of k^n in the usual topology of k^n. Using the fact that

$$\mathbb{Z}(S) = \bigcap_{f\in S} \mathbb{Z}(f)$$

we conclude that $\mathbb{Z}(S)$ is a closed subset of k^n in its usual topology.

If $a\in k^n$ is a zero of $f_1, f_2\in k[x_1,\dots,x_n]$, then it is also a zero of f_1+f_2 and of $f_1\cdot g$ for every $g\in k[x_1,\dots,x_n]$.

Thus

$$\mathbb{Z}(S) = \mathbb{Z}(I)$$

where $I=(S)$ is the ideal of $k[x_1,\dots,x_n]$ generated by S. By Hilbert basis theorem every ideal of $k[x_1,\dots,x_n]$ is finitely generated. It follows that $I = (f_1,\dots,f_r)$, for some $f_i\in k[x_1,\dots,x_n]$, and thus we have

$$\mathbb{Z}(S) = \mathbb{Z}(I) = \mathbb{Z}(f_1,\dots,f_r).$$

This shows that the algebraic sets can also be described as sets $\mathbb{Z}(I)$ where I is an ideal in $k[x_1,\dots,x_n]$.

Lemma 1. Let $I, J, (I_j)_j$ be ideals in $k[x_1,\dots,x_n]$.

(1) $I\subset J \Rightarrow \mathbb{Z}(I)\supset\mathbb{Z}(J)$,

(2) $\bigcap_j \mathbb{Z}(I_j) = \mathbb{Z}(\sum_j I_j)$,

(3) $\mathbb{Z}(I)\cup\mathbb{Z}(J) = \mathbb{Z}(I\cap J) = \mathbb{Z}(I\cdot J)$.

Proof. (1) $\forall a\in\mathbb{Z}(J)$: $f(a)=0$, $\forall f\in J$.
$\Rightarrow f(a)=0$, $\forall f\in I$, because $I\subset J$.
$\Rightarrow a\in\mathbb{Z}(I)$. $\Rightarrow \mathbb{Z}(J)\subset\mathbb{Z}(I)$.

(2) $a\in\mathbb{Z}(\sum_j I_j) \Rightarrow a\in\mathbb{Z}(I_j)$, $\forall j$; for $I_j\subset\sum_j I_j$ and by (1) $\mathbb{Z}(I_j)\supset\mathbb{Z}(\sum_j I_j)$.

$\Rightarrow a\in\bigcap_j\mathbb{Z}(I_j)$. $\Rightarrow \mathbb{Z}(\sum_j I_j)\subset\bigcap_j\mathbb{Z}(I_j)$.

On the other hand, let $a\in\bigcap_j\mathbb{Z}(I_j)$. For every $h\in\sum_j I_j$, we have $h = f_1+f_2+\dots+f_j+\dots$, $f_j\in I_j$. But $f_j(a)=0$, for $a\in\mathbb{Z}(I_j)$ for all j. $\Rightarrow h(a)=0$. $\Rightarrow a\in\mathbb{Z}(\sum_j I_j)$. $\Rightarrow \bigcap_j\mathbb{Z}(I_j)\subset\mathbb{Z}(\sum_j I_j)$; and hence the required equality.

(3) $I\cdot J\subset I\cap J\subset I$ and J,

$\therefore\ \mathbb{Z}(I,J) := \mathbb{Z}(I) \cup \mathbb{Z}(J) \subset \mathbb{Z}(I \cap J)$, by (1),
$\subset \mathbb{Z}(I \cdot J)$, by (1) .

Next, suppose
$a \notin \mathbb{Z}(I) \cup \mathbb{Z}(J)$. $\Rightarrow \exists f \in I, \exists g \in J : f(a) \neq 0 \neq g(a)$.
$\Rightarrow (fg)(a) \neq 0 \Rightarrow a \notin \mathbb{Z}(I \cdot J)$.
$\therefore\ a \notin \mathbb{Z}(I) \cup \mathbb{Z}(J) \Rightarrow a \notin \mathbb{Z}(I \cdot J)$
The contrapositive of this implication is
$$a \in \mathbb{Z}(I \cdot J) \Rightarrow a \in \mathbb{Z}(I) \cup \mathbb{Z}(J).$$
This implies $\mathbb{Z}(I \cdot J) \subset \mathbb{Z}(I) \cup \mathbb{Z}(J)$, and hence the required equality. ∎

This lemma justifies:
(i) by (1) we have $\mathbb{Z}(\{0\}) = k^n$; hence k^n is a closed set; (ii) by (2) arbitrary intersection of closed sets is a closed set; (iii) by (3) finite union of closed sets. is closed. These motivate the following definition.

Definition. The Zariski topology on k^n is the topology with closed sets given by the zero sets $\mathbb{Z}(I)$, where I runs over the ideals of $k[x_1, \cdots, x_n]$. ∎

Zariski open sets in k^n. Let us consider the Zariski topology of k. This is a topology in which closed sets are finite sets, for every polynomial function of degree n in $k[x]$ has n roots in k. The Zariski open sets are just the complements of such finite subsets of k. Therefore, the open sets are huge and hence the Zariski topology is far from being Hausdorff. It is also clear that the k-topology of k^n is much finer than the Zariski topology of k^n.

(Reminder. A subset S of a topological space X is said to be dense if $\overline{S} = X$; i.e., if the closure of S is X. It can be easily shown that S is dense iff every open subset of X contains some point of S.)

Let us denote the Zariski open set $k^n - \mathbb{Z}(I)$ by $\mathbb{D}(I)$, where I is a proper ideal of $k[x_1, \cdots, x_n]$.

Proposition 2. $I \neq \{0\} \Rightarrow \mathbb{D}(I)$ is a dense (open) subset of k^n.

Proof. Suppose $\mathbb{D}(I)$ is not a dense open subset of k^n, i.e., $\overline{\mathbb{D}(I)} \subsetneq k^n$. Now, $\mathbb{D}(I) \subset \overline{\mathbb{D}(I)}$ and the closed set $\overline{\mathbb{D}(I)}$ must be of the form $\mathbb{Z}(J)$ for some proper ideal $J \lhd k[x_1, \cdots, x_n]$. But then if $f \in I$, $g \in J$, we have $\mathbb{D}(f) \subset \mathbb{Z}(g)$ (for $\mathbb{D}(f) \subset \mathbb{D}(I)$, $\forall f \in I$; and $\mathbb{Z}(J) \subset \mathbb{Z}(g)$ $\forall g \in J$.) But this results in a contradiction for fg is a non-zero polynomial, but
$$\forall a \in k^n : \quad (fg)(a) = f(a)g(a) = 0$$
which is only possible if fg is a zero polynomial. ∎

(A.2) Hilbert's Nullstellensatz

The Nullstellensatz is the heart of classical algebraic geometry. It establishes a relation between the algebraic subsets of k^n and the ideals of $k[x_1, \cdots, x_n]$. Let us ask the following question: When are the equations
$$g(x_1, \cdots, x_n) = 0, \quad g \in I \lhd k[x_1, \cdots, x_n]$$
consistent? Put it another way: when does a set of polynomial equations have a common zero. Obviously, the equations
$$g_i(x_1, \cdots, x_n) = 0, \quad i = 1, \cdots, m$$
are inconsistent if there exist polynomial $f_i \in k[x_1, \cdots, x_n]$

such that
$$\sum_{i=1}^{m} f_i g_i = 1 \quad ;$$
i.e., if $1 \in (g_1, \dots, g_m)$, which implies
$$(g_1, \dots, g_m) = k[x_1, \dots, x_n] .$$
The following theorem provides the converse of this.

Theorem 3. Hilbert Nullstellensatz (weak form).
Every proper ideal $I \triangleleft k[x_1, \dots, x_n]$ has a zero in k^n.

Proof. To every point $a \in k^n$ we associate a k-algebra homomorphism ev_a (evaluate at a):
$$ev_a : \begin{cases} k[x_1, \dots, x_n] \longrightarrow k \\ f(x_1, \dots, x_n) \longmapsto f(a_1, \dots, a_n) . \end{cases}$$
clearly one has
$$a \in \mathbb{Z}(I) \Longleftrightarrow I \subset \mathrm{Ker}\, ev_a .$$
conversely, every unital k-algebra homomorphism $\varphi : k[x_1, \dots, x_n] \to k$ is determined by its values on the generators x_i, $i = 1, \dots, n$.
If $I \subset \mathrm{Ker}(\varphi)$, then
$$k^n \ni a = (a_1, \dots, a_n) := (\varphi(x_1), \dots, \varphi(x_n)) \in \mathbb{Z}(I) ,$$
for $\varphi(g(x_1, \dots, x_n)) = g(\varphi(x_1), \dots, \varphi(x_n))$, because φ is a homomorphism of k-algebras.

Thus to prove the theorem, we show that there exists a k-algebra homomorphism $\varphi : k[x_1, \dots, x_n]/_I \longrightarrow k$; for then clearly $I \subset \mathrm{Ker}(\varphi)$.

Now, since every proper ideal $I \triangleleft k[x_1, \dots, x_n]$ is contained in a maximal ideal $\mathfrak{m}$, then
$$I \subset \mathfrak{m} \Rightarrow \mathbb{Z}(I) \supset \mathbb{Z}(\mathfrak{m})$$
and therefore, it suffices to prove the theorem for a maximal $\mathfrak{m}$. Then $A := k[x_1, \dots, x_n]/\mathfrak{m}$ is a field and it is finitely generated as an algebra over k by

$$\bar{x}_i = x_i + \mathfrak{m} \quad , \quad i = 1, \dots, n .$$
The theorem is proved if we show $A = k$. But this is just the consequence of Zariski's lemma (see the remarks below). ∎

Remarks. (i) Zariski's lemma: Let $k \subset K$ be fields (k not necessarily algebraically closed). If K is finitely generated as an algebra over k, then K is algebraic over k. (Hence $K = k$ if k is algebraically closed.)

(ii) Let A be an integral domain, and let K be a field containing A. An element $\alpha \in K$ is said to be integral over A if it is a root of a monic polynomial with coefficients in A, i.e., if it satisfies an equation
$$\alpha^n + a_1 \alpha^{n-1} + \dots + a_n = 0 \quad , \quad a_i \in A .$$
The elements of K which are integral over A form a ring, called the integral closure of A in K.

(iii) There is a generalization of Zariski's lemma, called the Noether normalization theorem:

Let k be a field and $A \neq 0$ be a finitely generated k-algebra. Then there exist elements $y_1, \dots, y_r \in A$ which are algebraically independent over k and s.t. A is integral over $k[y_1, \dots, y_r]$. ∎

We shall now discuss the correspondence between algebraic sets and ideals. For a subset $W \subset k^n$ we shall write $\mathbb{I}(W)$ for the set of polynomials which are zero on W, i.e.
$$\mathbb{I}(W) = \{ f \in k[x_1, \dots, x_n] \mid f(a) = 0 , \ \forall a \in W \} .$$

This is clearly an ideal of $k[x_1,\dots,x_n]$ and the following obvious relations hold:

(i) $V \subset W \implies \mathbb{I}(V) \supset \mathbb{I}(W)$,
(ii) $\mathbb{I}(\emptyset) = k[x_1,\dots,x_n]$; $\mathbb{I}(k^n) = 0$,
(iii) $\mathbb{I}(\bigcup_i W_i) = \bigcap_i \mathbb{I}(W_i)$.

As an example let $a = (a_1,\dots,a_n) \in k^n$. Then clearly $\mathbb{I}(a) \supset (x_1-a_1,\dots,x_n-a_n)$; for $(x_1-a_1,\dots,x_n-a_n)$ is a maximal ideal, because "evaluation at a" defines an isomorphism

$$k[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n) \xrightarrow{\simeq} k .$$

As $\mathbb{I}(a) \neq k[x_1,\dots,x_n]$, we must have $\mathbb{I}(a) = (x_1-a_1,\dots,x_n-a_n)$.

Definition. rad(I)

The radical of an ideal $I \lhd k[x_1,\dots,x_n]$, denoted by $\mathrm{rad}(I)$ or $\sqrt{I}$, is defined to be the following subset of $k[x_1,\dots,x_n]$:

$$\mathrm{rad}(I) := \{ f \in k[x_1,\dots,x_n] \mid f^m \in I \text{, for some } m \in \mathbb{N}^+ \}$$

It is clear from this definition that
(i) $\mathrm{rad}(I)$ is an ideal of $k[x_1,\dots,x_n]$,
(ii) $I \subset \mathrm{rad}(I)$,
(iii) $\mathrm{rad}(\mathrm{rad}(I)) = \mathrm{rad}(I)$.

Definition. Radical ideal

An ideal $I \lhd k[x_1,\dots,x_n]$ is said to be radical if it equals its radical, i.e., if $I = \mathrm{rad}(I)$. Clearly this means $f^m \in I \implies f \in I$. ∎

Equivalently, I is radical iff $k[x_1,\dots,x_n]/I$ is a reduced ring, i.e., a ring without nilpotent elements.
Now, since

$$k[x_1,\dots,x_n]/I = \text{An integral domain} \iff I \text{ is prime}$$

and because every integral domain is reduced, it follows that every prime ideal (& a fortiori every maximal ideal) is radical. We also have the following results:

(i) If I and J are radical, $I \cap J$ is radical but $I+J$ need not be so. For example, let $I = (x^2-y)$ and $J = (x^2+y)$, which are prime ideals in $k[x,y]$ and hence are radical ideals. However, $x^2 \in I+J$ but $x \notin I+J$, which shows that $I+J$ is not radical.

(ii) Let $a = (a_1,\dots,a_n) \in \mathbb{I}(W) \subset k^n$. As $f^m(a) = f(a)^m$, then $f^m(a) = 0$ whenever $f(a) = 0$. $\implies (f^m \in \mathbb{I}(W) \implies f \in \mathbb{I}(W)) \implies$
$\implies \mathbb{I}(W)$ is a radical ideal.

In particular suppose $f \in \mathrm{rad}(I)$. $\implies f^m \in I$, for some $m \in \mathbb{N}^+$. $\implies f^m|_{\mathbb{Z}(I)} = 0 \implies f^m \in \mathbb{I}(\mathbb{Z}(I))$
$\implies f \in \mathbb{I}(\mathbb{Z}(I))$, for $\mathbb{I}(\mathbb{Z}(I))$ is radical.

$$\therefore \quad \mathbb{I}(\mathbb{Z}(I)) \supset \mathrm{rad}(I).$$

The following theorem states that these two ideals are actually equal.

Theorem 4. Hilbert Nullstellensatz (Strong form)

(a) The ideal $\mathbb{I}\mathbb{Z}(I) = \mathrm{rad}\, I$; and in particular $\mathbb{I}\mathbb{Z}(I) = I$ if I is a radical ideal.
(b) The set $\mathbb{Z}\mathbb{I}(W)$ is the smallest algebraic subset of k^n containing W; in particular $\mathbb{Z}\mathbb{I}(W) = W$ if W is an algebraic set.

Proof. (a) We have already noted that $\mathbb{I}\mathbb{Z}(I) \supset \mathrm{rad}(I)$. For the reverse inclusion let $h \in \mathbb{I}\mathbb{Z}(I)$, i.e., h is identically zero on $\mathbb{Z}(I)$. We must show that $h^m \in I$ for some $m \in \mathbb{N}^+$ (and this implies $h \in \mathrm{rad}(I)$ and hence $\mathbb{I}\mathbb{Z}(I) \subset \mathrm{rad}(I)$). We may assume $h \neq 0$, for $0 \in I$. Let $I = (g_1, \dots, g_l)$ and consider the system of $l+1$ equations in $n+1$ variables $x_1, \dots, x_n, y$,

$$\begin{cases} g_i(x_1, \dots, x_n) = 0, & i = 1, \dots, l, \\ 1 - y\, h(x_1, \dots, x_n) = 0. \end{cases}$$

If $(a_1, \dots, a_n, b)$ satisfies the first l equations, then $(a_1, \dots, a_n) \in \mathbb{Z}(I)$, and this implies $h(a_1, \dots, a_n) = 0$; therefore $(a_1, \dots, a_n, b)$ does not satisfy the last equation. It follows from the Nullstellensatz (weak form) that there exist $f_i \in k[x_1, \dots, x_n, y]$, $i = 1, \dots, l+1$, s.t.

$$1 = \sum_{i=1}^{l} f_i g_i + f_{l+1} \cdot (1 - y h).$$

Regarding this as an identity in the field $k(x_1, \dots, x_n, y)$, which is the field of fractions of the integral domain $k[x_1, \dots, x_n, y]$, and substituting $\frac{1}{h}$ for y we obtain the identity

$$1 = \sum_{i=1}^{l} f_i(x_1, \dots, x_n, \tfrac{1}{h}) \cdot g_i(x_1, \dots, x_n) \qquad (*)$$

in $k(x_1, \dots, x_n)$. Clearly

$$f_i(x_1, \dots, x_n, \tfrac{1}{h}) = \frac{\text{polynomial in } x_1, \dots, x_n}{h^{m_i}}$$

for some m_i. Let m be the largest of the m_i. On multiplying the identity $(*)$ by h^m, we obtain an equation

$$h^m = \sum_i (\text{polynomial in } x_1, \dots, x_n) \cdot g_i(x_1, \dots, x_n) \in I.$$

(b) Let V be an algebraic set containing $W \subset k^n$, and write $V = \mathbb{Z}(I)$. Then $I \subset \mathbb{I}(W)$ and so $\mathbb{Z}(I) \supset \mathbb{Z}\mathbb{I}(W)$ which implies the required result. If W is an algebraic set, then $W = \mathbb{Z}(I)$ for some $I \lhd k[x_1, \dots, x_n]$; it follows that

$$\begin{aligned} \mathbb{Z}\mathbb{I}(W) &= \mathbb{Z}\mathbb{I}(\mathbb{Z}(I)) \\ &= \mathbb{Z}(\mathrm{rad}(I)), && \text{for } \mathbb{I}(\mathbb{Z}(I)) = \mathrm{rad}(I), \\ &\subset \mathbb{Z}(I), && \text{for } I \subset \mathrm{rad}(I). \end{aligned}$$

This together with the previous inclusion gives

$$W := \mathbb{Z}(I) = \mathbb{Z}\mathbb{I}(W). \quad \blacksquare$$

The maps $\mathbb{Z}$ and $\mathbb{I}$ are related to each other according to the following

Corollary 5. The map $I \xmapsto{\mathbb{Z}} \mathbb{Z}(I)$ defines a 1:1 correspondence between the set of radical ideals in $k[x_1, \dots, x_n]$ and the set of algebraic subsets of k^n; its inverse is $\mathbb{I}$.

Proof. We know that $\mathbb{I}\mathbb{Z}(I) = I$ if I is radical, and $\mathbb{Z}\mathbb{I}(W) = W$ if W is an algebraic set. $\Rightarrow \mathbb{Z} = \mathbb{I}^{-1}$. $\blacksquare$

Remarks. (a) Notice that $\mathbb{Z}(0) = k^n$ and so

$$\mathbb{I}(k^n) = \mathbb{I}\mathbb{Z}(0) = \mathrm{rad}(0) = 0$$

as we had claimed earlier.

(b) The 1:1 correspondence in the Corollary 5 is order reversing. Therefore the maximal proper radical ideals correspond to the minimal nonempty algebraic sets. But the maximal proper radical ideals are simply the maximal ideals in $k[x_1, \dots, x_n]$ and the minimal nonempty algebraic sets are the one-point sets. As

$$\mathbb{I}((a_1, \dots, a_n)) = (x_1 - a_1, \dots, x_n - a_n),$$

this shows that the maximal ideals of $k[x_1, \dots, x_n]$ are precisely the ideals of the form $(x_1 - a_1, \dots, x_n - a_n)$. Notice that this is in conformity with the fact that

$$k[x_1,\dots,x_n]\Big/(x_1-a_1,\dots,x_n-a_n) \cong k$$

is a field.

(c) The algebraic set $\mathbb{Z}(I)$ is empty iff $I = k[x_1,\dots,x_n]$; for

$$\mathbb{Z}(I)=\emptyset \Rightarrow \mathbb{I}\,\mathbb{Z}(I) \;(=\mathrm{rad}(I)) = k[x_1,\dots,x_n]$$
$$\Rightarrow 1\in \mathrm{rad}(I) \Rightarrow 1 \in I.$$

(d) Let W and W' be algebraic sets. Then $W\cap W'$ is the largest algebraic subset contained in both W and W'.
$\Rightarrow$ $\mathbb{I}(W\cap W')$ is the smallest radical ideal containing both $\mathbb{I}(W)$ and $\mathbb{I}(W')$. Hence

$$\mathbb{I}(W\cap W') = \mathrm{rad}\,(\mathbb{I}(W)+\mathbb{I}(W')).$$ ∎

(A.3) Zariski topology and Coordinate ring of an algebraic set.

We now want to examine the Zariski topology on the algebraic subsets of k^n. We have seen that the Zariski topology of k^n is much coarser than the k-topology of k^n. Part (b) of theorem 4 says that for any subset $W\subset k^n$, $\mathbb{Z}\,\mathbb{I}(W)$ is the closure of W (i.e., the smallest Zariski closed subset of k^n containing W) and Corollary 5 says that there is a 1:1 correspondence between the algebraic subsets (i.e. closed subsets) of k^n and the radical ideals of $k[x_1,\dots,x_n]$.

Let V be an algebraic subset of k^n, and let $\mathbb{I}(V) = \mathcal{J} \lhd k[x_1,\dots,x_n]$. Then the algebraic subsets of V correspond to the radical ideals of $k[x_1,\dots,x_n]$ which contain $\mathcal{J}$.

Proposition 6. Let $V\subset k^n$ be an algebraic subset.
(a) The points of V are closed for the Zariski topology (thus V is a T_1-space).
(b) Every descending chain of closed subsets of V becomes constant, i.e., given

$$V_1\supset V_2\supset V_3\supset \cdots \qquad \text{(closed subsets of } V)$$

eventually $V_N = V_{N+1} = \cdots$, for some N. Alternatively every ascending chain of open sets becomes constant.
(c) Every open covering of V has a finite subcovering.

Proof. (a) We have already seen that $\{(a_1,\dots,a_n)\}$ is the algebraic set defined by the ideal $(x_1-a_1,\dots,x_n-a_n)$. Hence every point is a Zariski closed subset of V.
(b) A sequence $V_1\supset V_2\supset\cdots$ of closed subsets of V gives rise to the sequence of radical ideals $\mathbb{I}(V_1)\subset\mathbb{I}(V_2)\subset\cdots$ which eventually becomes constant, because $k[x_1,\dots,x_n]$ is a Noetherian ring.
(c) Let $V=\bigcup_{\alpha\in S} U_\alpha$ with U_α open. Choose an $\alpha_0\in S$; if $U_{\alpha_0}\neq V$, then $\exists\,\alpha_1\in S$ s.t. $U_{\alpha_0}\subsetneq U_{\alpha_0}\cup U_{\alpha_1}$. If $U_{\alpha_0}\cup U_{\alpha_1}\neq V$ then there exists $\alpha_2\in S$ etc. In this way we get a sequence of open subsets of V

$$U_{\alpha_0}\subset U_{\alpha_0}\cup U_{\alpha_1}\subset U_{\alpha_0}\cup U_{\alpha_1}\cup U_{\alpha_2}\subset\cdots$$

and by taking complements, this gives rise to a sequence of closed subsets of V:

$$U^c_{\alpha_0}\supset U^c_{\alpha_0}\cap U^c_{\alpha_1}\supset U^c_{\alpha_0}\cap U^c_{\alpha_1}\cap U^c_{\alpha_2}\supset\cdots$$

By (b) this sequence must eventually stop at $\emptyset$; i.e. $\exists\,\alpha_n$ s.t.

$$U^c_{\alpha_0}\cap U^c_{\alpha_1}\cap\cdots\cap U^c_{\alpha_n}=\emptyset$$

$$\Rightarrow\; V=\emptyset^c = U_{\alpha_0}\cup U_{\alpha_1}\cup\cdots\cup U_{\alpha_n}.$$ ∎

A topological space having property (b) is said to be Noetherian. The condition is equivalent to the following: every nonempty set of closed subsets of V has a minimal element. A space having property (c) is said to be quasi-compact. (It would have been called compact if V were Hausdorff, which is not the case here).

Now, let $V \subset k^n$ be an algebraic subset and let $\mathbb{I}(V) = \mathcal{J}$. An element $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ defines a mapping $k^n \longrightarrow k$, $a = (a_1, \dots, a_n) \longmapsto f(a) = f(a_1, \dots, a_n) \in k$, whose restriction to V depends only on the coset $\bar{f} := f + \mathcal{J}$ in the quotient ring

$$k[V] := k[x_1, \dots, x_n]/\mathcal{J} := k[\bar{x}_1, \dots, \bar{x}_n]$$

where $\bar{x}_i := x_i \bmod (\mathcal{J})$. Moreover, two polynomials $f_1(x_1, \dots, x_n)$ and $f_2(x_1, \dots, x_n)$ restrict to the same function on V only if they define the same element of $k[V]$. Thus $k[V]$ can be identified with the ring of functions $V \longrightarrow k$. We call $k[V]$ the ring of regular functions on V or the coordinate ring of V. It is a finitely generated reduced k-algebra (for $\mathcal{J} = \mathbb{I}(V)$ is a radical ideal), but it is not generally an integral domain, for being reduced only implies $k[\bar{x}_1, \dots, \bar{x}_n]$ is nilpotent free.

Let J be an ideal in $k[V]$; we set

$$\mathbb{Z}(J) = \{a \in V \mid f(a) = 0 \ , \ \forall f \in J\}.$$

Let $W = \mathbb{Z}(J)$. The maps

$$k[x_1, \dots, x_n] \xrightarrow{\pi} k[V] = k[x_1, \dots, x_n]/\mathcal{J} \longrightarrow k[W] := k[V]/J$$

should be regarded as restricting a function from k^n to V, and then restricting that function to W. Then $J \longmapsto \pi^{-1}(J)$

is a bijection from the set of ideals of $k[V]$ to the set of ideals of $k[x_1, \dots, x_n]$ containing $\mathcal{J}$, under which radical, prime, and maximal ideals correspond to radical, prime and maximal ideals respectively (each of these conditions can be checked on the quotient ring, and the result follows by $k[x_1, \dots, x_n]/\pi^{-1}(J) \cong k[V]/J$). Clearly $\mathbb{Z}(\pi^{-1}(J)) = \mathbb{Z}(J)$ and so it follows that $J \longmapsto \mathbb{Z}(J)$ gives a bijection between the set of radical ideals of $k[V]$ and the algebraic subsets (i.e., the Zariski closed subsets) of V.

For $h \in k[V]$, we write

$$\mathbb{D}(h) := \{a \in V \mid h(a) \neq 0\}.$$

This is an open subset of V, because it is the complement of the Zariski closed subset $\mathbb{Z}((h))$, i.e., $\mathbb{D}(h) = V - \mathbb{Z}((h))$.

Proposition 7. (a) The points of V are in one-to-one correspondence with the maximal ideals of $k[V]$.
(b) The closed subsets of V are in one-to-one correspondence with the radical ideals of $k[V]$.
(c) The sets $\mathbb{D}(h)$, $h \in k[V]$, form a basis for the topology of V, i.e., each $\mathbb{D}(h)$ is open, and each open set is a union (in fact a finite union) of $\mathbb{D}(h)$'s.

Proof. (a) and (b) are obvious from the above discussion.
(c) We have already observed that $\mathbb{D}(h)$ is open. Any other open set $U \subset V$ is the complement of a set of the form $\mathbb{Z}(J)$, J a radical ideal in $k[V]$. If $J = (f_1, \dots, f_m)$, then clearly $U = \bigcup_{i=1}^{m} \mathbb{D}(f_i)$ (for $\mathbb{Z}(J) = \bigcap_{i=1}^{m} \mathbb{Z}(f_i)$. $\Rightarrow U := \mathbb{Z}(J)^c = \bigcup_{i=1}^{m} \mathbb{Z}(f_i)^c = \bigcup_{i=1}^{m} \mathbb{D}(f_i)$.) ∎

The $\mathbb{D}(h)$ are called the basic or principal open subsets of V

Notice that

$$\mathcal{D}(h)\subset\mathcal{D}(h') \Leftrightarrow Z(h)\supset Z(h') \Leftrightarrow \mathrm{rad}((h))\subset \mathrm{rad}((h')) \Leftrightarrow$$
$$\Leftrightarrow h^r\in (h') \text{ for some } r\in\mathbb{N}^+$$
$$\Leftrightarrow h^r = h'g \text{ for some } g\in k[V].$$

(A.4) Irreducible algebraic sets.

A nonempty subset W of a topological space V is said to be irreducible if it satisfies any of the following equivalent conditions:

(a) W is not the union of two proper closed subsets;
(b) any two non-empty open subsets of W have a nonempty intersection;
(c) any nonempty open subset of W is dense.

The equivalences (a)$\Leftrightarrow$(b) and (b)$\Leftrightarrow$(c) are obvious. Also, one sees that if W is irreducible, and $W = W_1\cup\cdots\cup W_r$ with each W_i closed, then $W = W_i$ for some i.

This notion is not useful for Hausdorff topological spaces because such a space is irreducible only if it consists of a single point, otherwise any two points have disjoint open neighborhoods, and so (b) fails.

Proposition 8. An algebraic set W is irreducible iff $\mathbb{I}(W)$ is prime.

Proof. Suppose $fg\in\mathbb{I}(W)$. At each point of W either f or g is zero, and so

$$W\subset Z(f)\cup Z(g).$$

Hence
$$W = (W\cap Z(f))\cup(W\cap Z(g)).$$
As W is irreducible, one of these sets, say $W\cup Z(f)$, must equal W. But then $W\subset Z(f)$ and hence $\mathbb{I}(W)\supset\mathbb{I}Z(f) = \mathrm{rad}((f))$. $\Rightarrow f\in\mathbb{I}(W)$. Thus $\mathbb{I}(W)$ is prime.

Conversely, suppose $W = Z(I)\cup Z(J)$ where I and J are radical ideals (i.e., W is the union of two closed sets.) We have to show that $W = Z(I)$ or $Z(J)$. Recall that $W := Z(I)\cup Z(J) = Z(I\cap J)$ and $I\cap J$ is radical; hence $\mathbb{I}(W) = \mathbb{I}Z(I\cap J) = I\cap J$, for $I\cap J$ is radical. If $W\neq Z(I)$, then $\exists f\in I$ s.t. $f\notin\mathbb{I}(W)$. But $fg\in I\cap J = \mathbb{I}(W)$ for all $g\in J$, and, because $\mathbb{I}(W)$ is prime and $f\notin\mathbb{I}(W)$, this implies $g\in\mathbb{I}(W)$. $\Rightarrow J\subset\mathbb{I}(W)\Rightarrow Z(J)\supset W$, which implies $Z(J)=W$. ∎

Thus, so far we have established the following one to one correspondences

radical ideals $\overset{1:1}{\longleftrightarrow}$ algebraic subsets.
prime ideals $\overset{1:1}{\longleftrightarrow}$ irreducible algebraic subsets.
maximal ideals $\overset{1:1}{\longleftrightarrow}$ one-point sets.

These correspondences are valid whether we mean ideals in $k[x_1,\ldots,x_n]$ and algebraic subsets of k^n, or ideals in $k[V]$ and algebraic subsets of V.

Notice that the last correspondence implies that the maximal ideals in $k[V]$ are of the form $(\bar{x}_1-a_1,\ldots,\bar{x}_n-a_n)$, $a=(a_1,\ldots,a_n)\in V$.

Example. Let $f\in k[x_1,\ldots,x_n]$. As we know that $k[x_1,\ldots,x_n]$ is a UFD, then

$$(f) \text{ is prime} \Leftrightarrow f \text{ is irreducible}.$$

Thus,
$$Z(f) \text{ is irreducible} \Leftrightarrow f \text{ is irreducible}.$$
On the other hand suppose f factors as $f=\prod_i f_i^{m_i}$,

with f_i distinct irreducible polynomials. Then

$$(f) = \bigcap_i (f_i^{m_i}),$$

$$\mathrm{rad}((f)) = (\prod_i f_i) = \bigcap_i (f_i),$$

and $\mathbb{Z}(f) = \bigcup_i \mathbb{Z}(f_i)$, with $\mathbb{Z}(f_i)$ irreducible. ∎

Proposition 9. Let V be a Noetherian topological space. Then V is a finite union of irreducible closed subsets, $V = V_1 \cup V_2 \cup \cdots \cup V_m$. Moreover, if the decomposition is irredundant, i.e., if there are no inclusions among the V_i, then V_i are uniquely determined up to order.

Proof. Suppose the first assertion is false. Then, because V is Noetherian, there will be a closed subset W of V that is minimal among those that cannot be written as a finite union of irreducible closed subsets. But then such a W cannot itself be irreducible, and so $W = W_1 \cup W_2$ with each W_i a proper closed subset of W. From the minimality of W it follows that each W_i is a finite union of IRR closed subsets, and so therefore is W. We have thus arrived at a contradiction.

Suppose that
$$V = V_1 \cup V_2 \cup \cdots \cup V_m = W_1 \cup W_2 \cup \cdots \cup W_n$$
are two irredundant decompositions. Then $V_i = \bigcup_j (V_i \cap W_j)$, and so because V_i is irreducible, $V_i \subset V_i \cap W_j$ for some j. Therefore, there is a function $f: \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$ s.t. $V_i \subset W_{f(i)}$ for some i. Similarly there is a function $g: \{1, \dots, n\} \longrightarrow \{1, \dots, m\}$ s.t. $W_j \subset V_{g(j)}$. Since $V_i \subset W_{f(i)} \subset V_{gf(i)}$, and because each decomposition is irredundant, we have $gf(i) = i$ and $V_i = W_{f(i)}$. Similarly $fg = id$. Thus f and g are bijective and the

decompositions differ only in the numbering of index sets. ∎

The V_i are called the irreducible components of V. They are the minimal closed irreducible subsets of V. In the previous example $\mathbb{Z}(f_i)$ are the irreducible components of $\mathbb{Z}(f)$.

A Hausdorff space is Noetherian iff it is finite, in which case the irreducible components are the one-point sets.

Corollary 10. A radical ideal $I \lhd k[x_1, \dots, x_n]$ is a finite intersection of prime ideals
$$I = P_1 \cap \cdots \cap P_n$$
and if there are no inclusions among P_is, then they are uniquely determined up to order.

Proof. Write $\mathbb{Z}(I) = \bigcup_i V_i$, V_i are irreducible closed subsets of k^n, and take $P_i = \mathbb{I}(V_i)$. ∎

Remarks. (a) An ideal q in a ring R is called primary if $q \neq R$ and
$$xy \in q \Rightarrow \text{either } x \in q \text{ or } y^n \in q \text{ for some } n \in \mathbb{N}^+$$
In other words
$$q \text{ is primary} \Leftrightarrow R/q \neq \{0\} \text{ and every zero divisor in } R/q \text{ is nilpotent}.$$

A prime ideal in a ring R is in some sense a generalization of a prime number. The corresponding generalization of a power of a prime is a primary ideal.

In a Noetherian ring every ideal I has a decomposition into primary ideals $I = \bigcap_i q_i$. For radical ideals, this becomes a much simpler decomposition into prime ideals as in the above Corollary.

(b) In $k[x]$, $(f(x))$ is radical iff f is square free in which case f is a product of distinct irreducible polynomials, $f = p_1 \cdots p_r$ and $(f) = (p_1) \cap \cdots \cap (p_r)$ (a polynomial is divisible by f iff it is divisible by each p_i). ∎

Section B Affine algebraic varieties

(B.1) Ringed spaces and their morphisms.

Definition. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, called the structure sheaf of the ringed space. ∎

Definition. A morphism of ringed space $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (ϕ, f) where $\phi : X \to Y$ is a continuous mapping and $f : \mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheaves given by:

to each open subset $U \subset Y$, $f(U)$ is a ring homomorphism

$$f(U) : \Gamma(U, \mathcal{O}_Y) \longrightarrow \Gamma(\phi^{-1}(U), \mathcal{O}_X)$$

which is compatible with the restriction homomorphisms, i.e., whenever $U \supseteq V$ are open in Y the diagram

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_Y) & \xrightarrow{f(U)} & \Gamma(\phi^{-1}(U), \mathcal{O}_X) \\ \downarrow \rho^U_V & & \downarrow \rho^{\phi^{-1}(U)}_{\phi^{-1}(V)} \\ \Gamma(V, \mathcal{O}_Y) & \xrightarrow{f(V)} & \Gamma(\phi^{-1}(V), \mathcal{O}_X) \end{array}$$

is commutative. ∎

For each $x \in X$, f induces a homomorphism of the stalks:

$$f_x : \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$$

by taking the direct limit.

The definition of a ringed space can be slightly generalized by taking $\mathcal{O}_X$ to be a sheaf of R-algebras over X. Sometimes, for simplicity of notations $\Gamma(U, \mathcal{O}_X)$ will be denoted by $\mathcal{O}_X(U)$. Then

$$\mathcal{O}_x \; (= \mathcal{O}_{X,x}) := \varinjlim \mathcal{O}_X(U)$$

where the direct limit is over open neighborhoods U of x. In all interesting cases $\mathcal{O}_x$ is a local ring with maximal ideal the set of germs which vanish at x.

A particularly simple, but important, case of a ringed space is when $\mathcal{O}_X$, the structure sheaf, is a k-algebra of functions on X. Then, a morphism $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of such ringed spaces will be simply denoted by ϕ, and this is a continuous mapping $\phi : X \longrightarrow Y$ such that

$$\forall \text{ open } U \subset Y : \quad f \in \mathcal{O}_Y(U) \implies f \circ \phi \in \mathcal{O}_X(\phi^{-1}(U))$$

as we have mentioned in general form earlier. The map induced on stalks in this case is given by

$$\mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$$
$$(f, U) \longmapsto (f \circ \phi, \phi^{-1}(U))$$

where (f, U) stands for a germ of functions (the equivalence class of functions that agree on U).

(B.2) Review of the ring of fractions

Let R be a commutative unital ring and S a multiplicative subset of R, i.e.

$$S \subset R : \quad 1 \in S \quad \text{and} \quad ab \in S \text{ for all } a, b \in S.$$

(Thus S is a submonoid of the multiplicative monoid of R.)

Define on the product set $R \times S$ a relation according to

$$(a, s) \sim (b, t) \iff \exists\, u \in S : \; u(at - bs) = 0. \qquad (1)$$

This is an equivalence relation. Let us denote by a/s the equivalence class of the element (a, s). On $R \times S/\sim$

define addition and multiplication according to

$$a/s + b/t = \frac{at + bs}{st}, \qquad a/s \cdot b/t = ab/st.$$

These operations are well-defined and make $R \times S/\sim$ into a ring, denoted by $S^{-1}R$; i.e.

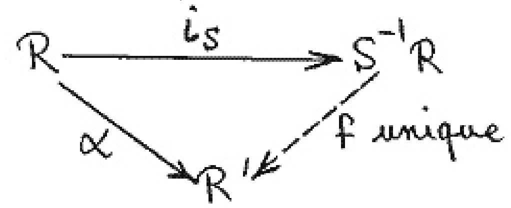
$$S^{-1}R = \{ a/s \mid a \in R, s \in S \}$$

together with the above defined operations. There is a canonical ring homomorphism

$$i_S : \begin{cases} R \longrightarrow S^{-1}R \\ a \longmapsto a/1 \end{cases}.$$

Note that i_S is not necessarily injective; e.g. if $0 \in S$, $S^{-1}R$ is the zero ring (see (1)).

The homomorphism i_S has the following universal property: every $s \in S$ is mapped to a unit in $S^{-1}R$, and any other homomorphism $\alpha : R \longrightarrow R'$ of rings with the property $\alpha(S) \subset R'^*$ (where R'^* is the set of units of R'), factors uniquely through $S^{-1}R$: this means



is commutative. Explicitly f is given by

$$f : \begin{cases} S^{-1}R \longrightarrow R' \\ a/s \longmapsto \alpha(a) \cdot \alpha(s)^{-1} \end{cases}.$$

It can be easily checked that this is well-defined and it is indeed a ring homomorphism.

As usual, this universal property determines the pair $(S^{-1}R, i_S)$ uniquely up to an isomorphism. ∎

In the special where R is an integral domain, we can form the field of fractions $\mathbb{F} = S^{-1}R$, where $S = R - \{0\}$.

Then for any other multiplicative subset $S \subset R$ s.t. $0 \notin S$ $S^{-1}R$ can be identified with $\{a/s \in \mathbb{F} \mid a \in R, s \in S\}$, and this is a subring of $\mathbb{F}$.

The following two examples are particularly important:

(1) Let $h \in R$. Then $S_h := \{1, h, h^2, \dots\}$, the multiplicative monoid generated by h is a multiplicative subset of R and we shall write $R_h = S_h^{-1}R$. Thus every element of R_h can be written in the form a/h^m, $a \in R$, and

$$a/h^m = b/h^n \iff h^N(ah^n - bh^m) = 0 \quad \text{for some } N \in \mathbb{N}^+.$$

In case R is an integral domain with field of fractions $\mathbb{F}$, R_h is a subring of $\mathbb{F}$, whose elements are of the form a/h^m, $a \in R$, $m \in \mathbb{N}^+$.

(2) Let p be a prime ideal in R. Then $S_p := R - p$ is a multiplicative subset: $\forall a, b \in S_p$: $ab \notin S_p \Rightarrow ab \in p$ $\Rightarrow a$ or $b \in p \Rightarrow a$ or $b \notin S_p$, a contradiction.

Let us write $R_p := S_p^{-1}R$. Thus as a set

$$R_p = \{a/c \mid a \in R, c \notin p\},$$

and $a/c = b/d \iff s(ad - bc) = 0$, for some $s \notin p$.

Let us consider the following subset of R_p:

$$m_p = \{a/s \mid a \in p, s \notin p\}.$$

(i) m_p is an ideal of R_p:

$\forall b/c \in R_p$, $\forall a/s \in m_p$: $b/c \cdot a/s = ba/cs \in m_p$,

for $ba \in p$ and if $cs \in p$ then c or $s \in p$, a contradiction. So $cs \notin p$. $\Rightarrow ba/cs \in m_p$.

(ii) m_p is a maximal ideal in R_p:

because the elements of R_p which are not in m_p are units.

(iii) m_p is the unique maximal ideal of R_p.

Therefore, R_p is a local ring and

$R_p/m_p =: k(p)$ is called the residue field at p. ■

We shall also make use of the following important lemma.

Lemma 1. For any unital ring R and any $h \in R$ the map

$$\sum a_i \bar{x}^i \longmapsto \sum a_i/h^i$$

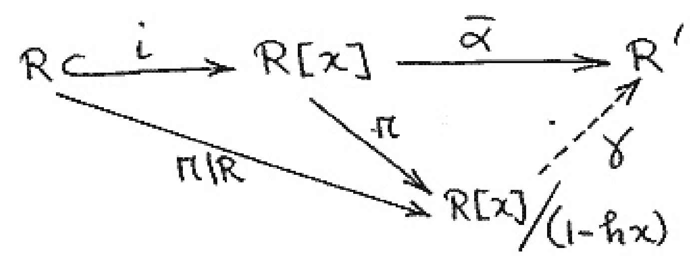
defines an isomorphism

$$R[\bar{x}] := R[x]/(1-hx) \xrightarrow{\simeq} R_h.$$

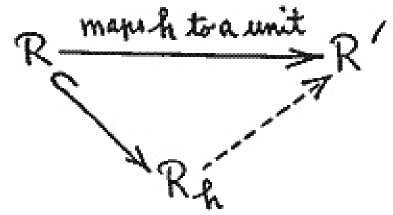
Proof. In the ring $R[\bar{x}] = R[x]/(1-hx)$, $1 = hx$; so h is a unit. Consider a ring homomorphism $\alpha : R \to R'$ s.t. $\alpha(h)$ is a unit in R'. Then α extends to a ring homomorphism

$$\bar{\alpha} : \begin{cases} R[x] \longrightarrow R' \\ \sum a_i x^i \longmapsto \sum \alpha(a_i)\,\alpha(h)^{-i} \end{cases}$$

Under this homomorphism $1 - hx \overset{\bar{\alpha}}{\longmapsto} 1 - \alpha(h)\alpha(h)^{-1} = 0$, i.e. $\mathrm{Ker}(\pi) \subset \mathrm{Ker}(\bar{\alpha})$, where $\pi : R[x] \longrightarrow R[x]/(1-hx)$ is the canonical homomorphism. Therefore, $\bar{\alpha}$ factors through the quotient ring $R[\bar{x}]$:



$\therefore$ $\bar{\alpha} = \gamma \circ \pi$. $\Rightarrow$ $\alpha = \gamma \circ (\pi|R)$, and γ is unique with this property. Therefore, $R[\bar{x}] := R[x]/(1-hx)$ has the same universal property as R_h:



and hence the two are isomorphic by the map that makes h^{-1} correspond to x. ■

(B.3) Ringed space structure of an algebraic set

In what follows k is a an arbitrary field.

Let $V \subset k^n$ be an algebraic subset. An element $h \in k[V]$ defines a functions $V \longrightarrow k$, and h^{-1} defines a function

$$\mathbb{D}(h) \longrightarrow k \quad , \quad a \longmapsto \frac{1}{h(a)} \quad , \quad \forall a \in \mathbb{D}(h) \subset k^n.$$

Therefore, any pair of functions $g, h \in k[V]$ with $h \neq 0$ defines a function

$$\begin{cases} \mathbb{D}(h) \longrightarrow k \\ a \longmapsto \dfrac{g(a)}{h(a)} \end{cases} \qquad (*)$$

<u>Definition.</u> A function $f : U \longrightarrow k$ on an open subset $U \subset V$ is said to be <u>regular</u> if it is of the form $(*)$ in a neighborhood of each point in U. This means

$\forall a \in U$, $\exists g, h \in k[V]$ with $h(a) \neq 0$
s.t. the functions f and g/h agree
in a neighborhood of a.

Let us write <u>$\mathcal{O}_V(U)$ for the set of regular functions on U.</u>

For example, if $V = k^n$, then a function $f : U \longrightarrow k$ is regular at a point $a \in U$ if there are polynomials $g(x_1, \cdots, x_n)$, $h(x_1, \cdots, x_n) \in k[x_1, \cdots, x_n] := k[V]$ with $h(a) \neq 0$ and $f(b) = g(b)/h(b)$ for all b such that the right hand side is defined.

<u>Proposition 2.</u> The mapping $U \longmapsto \mathcal{O}_V(U)$ defines a sheaf of k-algebras on V, denoted by $\mathcal{O}_V$.

<u>Proof.</u> (i) obviously a constant function is regular. Suppose f, f' are regular on U and let $a \in U$. Then $\exists g, g', h, h' \in k[V]$ s.t. $h(a) \neq 0 \neq h'(a)$ and that f and f' agree with g/h and g'/h' respectively in a neighborhood of a. Then ff' agrees with gg'/hh' near a. $\Rightarrow$ ff' is regular on U. Similarly $f \pm f'$ are regular on U. $\Rightarrow$ $\mathcal{O}_V(U)$ is a k-algebra.

(ii) It follows from definition of a regular function that the restriction of a regular function to an open subset is again regular.

(iii) The condition for f to be regular is clearly local.

This establishes that $\mathcal{O}_V$ is a sheaf of k-algebras on V. ■

The next proposition is of fundamental importance; to state and prove it we first need the following result.

<u>Lemma 3.</u> The element $g/h^m \in k[V]_h$ defines the zero function on $\mathbb{D}(h)$ iff $gh = 0$ in $k[V]$ (and hence $g/h^m = 0$ in $k[V]_h$).

<u>Proof.</u> If g/h^m is zero on $\mathbb{D}(h)$, then since h is zero on the complement of $\mathbb{D}(h)$ it follows that $gh = 0$ on V, i.e., gh is zero in $k[V]$.

Conversely, if $gh = 0$, then $g(a)h(a) = 0$, $\forall a \in V \subset k^n$; $\Rightarrow g(a) = 0$, $\forall a \in \mathbb{D}(h)$ (because $h(a) \neq 0$ on $\mathbb{D}(h)$). ■

<u>Proposition 4.</u>

(a) The canonical map $k[V]_h \longrightarrow \mathcal{O}_V(\mathbb{D}(h))$ is an isomorphism.

(b) For every $a \in V$ there exists a canonical isomorphism

$\mathcal{O}_a \longrightarrow k[V]_{m_a}$, where m_a is the maximal ideal $(\bar{x}_1 - a_1, \ldots, \bar{x}_n - a_n)$.

Proof. (a) Lemma 3 shows that the kernel of the map $k[V]_h \longrightarrow \mathcal{O}_V(\mathbb{D}(h))$ is zero and hence this map is injective. We show it is surjective, i.e., we show that every regular function f on $\mathbb{D}(h)$ (i.e., $f \in \mathcal{O}_V(\mathbb{D}(h))$) has a representation $f = \frac{r}{h^\nu}$ on all of $\mathbb{D}(h)$, where $\nu \in \mathbb{N}^+$ and $r \in k[V]$:

We know that V is Noetherian, i.e., it has ascending chain condition on the open subsets. This implies that any open subset of V is also Noetherian and hence it is quasi-compact. Therefore, $\mathbb{D}(h)$ is quasi-compact and we can write

$$\mathbb{D}(h) = \bigcup_{i=1}^{n} \mathbb{D}(h_i)$$

(for $\mathbb{D}(h)$ is a finite union of open subsets and $\mathbb{D}(h_i)$ form a basis for the open sets in V). Then f has a representation $f = g_i/h_i$ on $\mathbb{D}(h_i)$, $i = 1, \ldots, n$ (i.e., $f|\mathbb{D}(h_i) = \frac{g_i}{h_i}$).

On the open set $\mathbb{D}(h_i) \cap \mathbb{D}(h_j) = \mathbb{D}(h_i h_j)$ we have

$$\frac{g_i}{h_i} = \frac{g_j}{h_j} \quad \Longrightarrow \quad \frac{g_i h_j - g_j h_i}{h_i h_j} = 0 \quad \text{on } \mathbb{D}(h_i h_j).$$

$\Longrightarrow$ $h_i h_j (g_i h_j - g_j h_i) = 0$ in $k[V]$, (by lemma 3), and hence $h_i h_j (g_i h_j - g_j h_i) = 0$ on all of V ($i, j \in [1, n]$)

Writing $f = \frac{g_i h_i}{h_i^2}$ on $\mathbb{D}(h_i)$, we can without restriction assume that

$$g_i h_j - g_j h_i = 0 \quad \text{on all of V},$$

i.e., $g_i h_j - g_j h_i = 0$ in $k[V]$.

Now, from $\mathbb{D}(h) = \bigcup_{i=1}^{n} \mathbb{D}(h_i)$ it follows that $h \in \text{Rad}(h_1, \ldots, h_n)$. $\Longrightarrow h^\nu = \sum_{i=1}^{n} k_i h_i$, for some

$k_i \in k[V]$, $i = 1, \ldots, n$. Now, define

$$r := \sum_{i=1}^{n} k_i g_i \quad ;$$

we get

$$h^\nu g_j = \sum_{i=1}^{n} (k_i h_i) g_j = \sum_{i=1}^{n} (k_i g_i) h_j = r h_j, \quad \forall j = 1, \ldots, n.$$

$$\therefore \quad f|\mathbb{D}(h_j) := \frac{g_j}{h_j} = \frac{r}{h^\nu}, \text{ for all } j = 1, \ldots, n;$$

$$\Longrightarrow \quad f = \frac{r}{h^\nu} \text{ on all of } \mathbb{D}(h).$$

This proves the assertion.

(b) First notice that in the definition of the stalk of a sheaf at $a \in V$, it suffices to consider pairs (f, U) with U lying in a fixed basis for the neighborhood of a. Thus each element of $\mathcal{O}_a$ is represented by a pair $(f, \mathbb{D}(h))$ where $h(a) \neq 0$ and $f \in k[V]_h$; and two pairs $(f_1, \mathbb{D}(h_1))$ and $(f_2, \mathbb{D}(h_2))$ represent the same element of $\mathcal{O}_a$ iff f_1 and f_2 restrict to the same function on $\mathbb{D}(h)$, where

$$a \in \mathbb{D}(h) \subset \mathbb{D}(h_1 h_2) = \mathbb{D}(h_1) \cap \mathbb{D}(h_2).$$

Now, for each $h \notin m_a$ (i.e., $h(a) \neq 0$) there exists a canonical homomorphism

$$\alpha_h : k[V]_h \longrightarrow k[V]_{m_a}.$$

Now, we map the element of $\mathcal{O}_a$ represented by $(f, \mathbb{D}(h))$ to $\alpha_h(f)$. It is easily checked that this map is well-defined, injective and surjective. ∎

Corollary 5. If $V \subset k^n$ is an algebraic set, then $\mathcal{O}(V)$ is isomorphic to $k[V]$ as a k-algebra.

Proof. Since $V = \mathbb{D}(1)$, $1 \in k[V]$, the result follows immediately from proposition 4. ∎

The proposition gives us an explicit description of the value of $\mathcal{O}_V$, the sheaf of regular functions on V, on any basic open set, and of the ring of germs at any point $a \in V$. When V is irreducible, this becomes a little simpler because all the rings are subrings of $k(V)$, the field of fractions of the integral domain $k[V]$. We have:

$$\begin{cases} \Gamma(\mathbb{D}(h), \mathcal{O}_V) = \{ g/h^N \in k(V) \mid g \in k[V],\ N \in \mathbb{N}^+ \}, \\ \mathcal{O}_a = \{ g/h \in k(V) \mid h(a) \neq 0 \}, \\ \Gamma(U, \mathcal{O}_V) = \bigcap \mathcal{O}_a, \quad \text{intersection over all } a \in U \\ \qquad = \bigcap \Gamma(\mathbb{D}(h_i), \mathcal{O}_V), \quad \text{if } U = \bigcup \mathbb{D}(h_i). \end{cases}$$

Notice that every nonzero element of $k(V)$ defines a function on some open subset of V; we call such an object a rational function on V. The last equality then says that the regular functions on U are the rational functions on V that are defined at each point of U.

Examples.

(1) Let $V = k^n$. The ring of regular functions on V, $\Gamma(V, \mathcal{O}_V)$, is $k[x_1, \dots, x_n]$. For any non-zero polynomial $h(x_1, \dots, x_n)$ the ring of regular functions on $\mathbb{D}(h)$ is

$$\{ g/h^N \in k(x_1, \dots, x_n) \mid g, h \in k[x_1, \dots, x_n] \}.$$

For a point $a = (a_1, \dots, a_n) \in k^n$, the ring of germs of functions at a is

$$\mathcal{O}_a = \{ g/h \in k(x_1, \dots, x_n) \mid h(a) \neq 0 \}$$
$$= k[x_1, \dots, x_n]_{(x_1 - a_1, \dots, x_n - a_n)}$$

which is a local ring with the maximal ideal consisting of those elements g/h with $g(a) = 0$.

(2) Let $U = \{ (a,b) \in k^2 \mid (a,b) \neq 0 \}$. This is an open subset of k^2. However, it is not a basic open subset of k^2, because its complement $\{(0,0)\}$ has dimension zero, and therefore cannot be of the form $\mathbb{Z}(f)$. Since $U = \mathbb{D}(x) \cup \mathbb{D}(y)$, the ring of regular functions on U is

$$\Gamma(\mathbb{D}(x), \mathcal{O}_U) \cap \Gamma(\mathbb{D}(y), \mathcal{O}_U) = k[x,y]_x \cap k[x,y]_y ;$$

$\therefore$ any regular function on U (as an element of $k(x,y)$) can be written as $f = \dfrac{g(x,y)}{x^N} = \dfrac{h(x,y)}{y^M}$. Since $k[x,y]$ is a UFD, we can assume that the fractions are in their lowest terms. On multiplying through by $x^N y^M$, we find that $g(x,y)\, y^M = h(x,y)\, x^N$. Because x does not divide the LHS, it cannot divide the RHS, hence $N = 0$. Therefore, $f \in k[x,y]$, and every regular function extends to k^2: $\Gamma(U, \mathcal{O}_U) = k[x,y]$. ∎

(B.4) Affine algebraic varieties

We have just seen that every algebraic set V gives rise to a ringed space $(V, \mathcal{O}_V)$; $\mathcal{O}_V$ = the sheaf of regular functions on V.

Definition. An affine algebraic variety over k is a ringed space which is isomorphic to a ringed space of the form $(V, \mathcal{O}_V)$. ∎

A morphism of affine algebraic varieties is a morphism of ringed spaces. We often call it a regular map $V \longrightarrow W$ or a morphism $V \longrightarrow W$, and write $\mathrm{Mor}(V,W)$

for the set of all such morphisms.

With these definitions, the affine algebraic varieties over k become a category.

In particular, every algebraic set has a natural structure of an affine variety. We usually write $\mathbb{A}^n$ for k^n regarded as an affine variety.

We also note that the affine varieties we have constructed so far have all been embedded in $\mathbb{A}^n$. We shall now see how to construct "unembedded" affine varieties.

A reduced finitely generated k-algebra is called an affine k-algebra. Thus for such an algebra A, there exist $\bar{x}_i \in A$ (not necessarily algebraically independent), such that $A = k[\bar{x}_1, \cdots, \bar{x}_n]$ and the kernel of the homomorphism

$$\begin{cases} k[x_1, \cdots, x_n] \longrightarrow A \\ x_i \longmapsto \bar{x}_i, \quad i = 1, \cdots, n, \end{cases}$$

is a radical ideal, I_A. For every maximal ideal $\mathfrak{m} \lhd A$ one can identify $A/\mathfrak{m}$ with k (recall the Zariski's lemma). For $f \in A$, we write $f(\mathfrak{m})$ for the image of f in $A/\mathfrak{m} = k$; i.e., $f(\mathfrak{m}) = f \bmod (\mathfrak{m})$.

Now, we can associate with any affine k-algebra A a ringed space $(V, \mathcal{O}_V)$ as follows:

we take V to be the set of maximal ideals in A. Let $h \in A$, $h \neq 0$; define

$$D(h) = \{\mathfrak{m} \mid h(\mathfrak{m}) \neq 0, \text{ i.e. } h \notin \mathfrak{m}\}$$

and endow V with a topology for which $D(h)$ are a basis (i.e., the topology generated by these open sets). A pair of elements $g, h \in A$, $h \neq 0$, define a function

$$\begin{cases} D(h) \longrightarrow k \\ \mathfrak{m} \longmapsto g(\mathfrak{m})/h(\mathfrak{m}) \end{cases}$$

and we define a function $f : U \longrightarrow k$ on an open subset $U \subset V$ to be regular if it is of this form on each point of U. We shall write $\mathcal{O}_V(U)$ $(= \Gamma(U, \mathcal{O}_V))$ for the set of regular functions on U. Note that $\mathcal{O}_V$ is a sheaf over V since it is defined by a local condition.

Proposition 6. The pair $(V, \mathcal{O}_V)$ is an affine variety with $\Gamma(V, \mathcal{O}_V) = A$.

Proof. We must show that $(V, \mathcal{O}_V)$ is isomorphic to a ringed space associated with an algebraic set. Present A as a quotient

$$k[x_1, \cdots, x_n]/I_A = k[\bar{x}_1, \cdots, \bar{x}_n].$$

Then the map

$$\varphi : \begin{cases} Z(I_A) \longrightarrow V \\ (a_1, \cdots, a_n) \longmapsto (\bar{x}_1 - a_1, \cdots, \bar{x}_n - a_n) \lhd A \end{cases}$$

is a bijection with the inverse

$$\varphi^{-1} : \begin{cases} V \longrightarrow Z(I_A) \subset k^n \\ \mathfrak{m} \longmapsto (\bar{x}_1(\mathfrak{m}), \cdots, \bar{x}_n(\mathfrak{m})). \end{cases}$$

It is easy to check that φ is a homeomorphism and that a function f on an open subset of V is regular (according to the above given definition) iff $f \circ \varphi$ is regular. ∎

We shall write $\mathrm{specm}(A)$ for the topological space V, and $\mathrm{Specm}(A)$ for the ringed space $(V, \mathcal{O}_V)$.

If we start with an affine variety V and let $A = \Gamma(V, \mathcal{O}_V)$, then $\mathrm{Specm}(A) \cong (V, \mathcal{O}_V)$ canonically. We again write $k[V]$ for $\Gamma(V, \mathcal{O}_V)$, the ring of regular functions on the whole of V (the global section).

Thus for each affine k-algebra A, we have an affine variety Specm(A), and conversely, for each affine variety $(V, \mathcal{O}_V)$ we have an affine k-algebra $\Gamma(V, \mathcal{O}_V)$. We shall make this correspondence into an equivalence of categories.

Remark. A radical ideal $I \lhd k[x_1, \dots, x_n]$ is the intersection of the maximal ideals containing it. Indeed the maximal ideals in $k[x_1, \dots, x_n]$ are all of the form $m_a = (x_1 - a_1, \dots, x_n - a_n)$, and $f \in m_a \Leftrightarrow f(a) = 0$. Thus $m_a \supset I \Leftrightarrow a \in \mathbb{Z}(I)$, and if $f \in m_a$ for all $a \in \mathbb{Z}(I)$, then f is zero on $\mathbb{Z}(I)$, i.e., $f \in \mathbb{I}(\mathbb{Z}(I)) = I$.

This remark implies that, for any affine k-algebra A, the intersection of the maximal ideals of A is zero, because A is isomorphic to a k-algebra $k[x_1, \dots, x_n]/I$ and we can apply the remark to I. Therefore, the map that associates with $f \in A$ the map

$$\varphi_f : \begin{cases} \mathrm{specm}(A) \longrightarrow k \\ m \longmapsto f(m) \end{cases}$$

is injective (i.e. $f \xmapsto{\;\xi\;} \varphi_f$ is injective, for
$\mathrm{Ker}\,\xi = \{f \in A \mid \varphi_f = 0\} = \{f \in A \mid f(m) = 0,\ \forall m \in \mathrm{specm}A\}$
$= \{f \in A \mid f \in m,\ \forall m \in \mathrm{specm}A\} = \{f \in A \mid f \in \bigcap_{\mathrm{specm}(A)} m\}$
$= \mathrm{rad}\,A = 0$.)

Therefore, A can be identified with the ring of function on specm(A). ▮

(B.5) The category of affine algebraic varieties.

Let $\eta : A \to B$ be a homomorphism of affine k-algebras. For any $h \in A$, $\eta(h)$ is invertible in

in $B_{\eta(h)}$ and hence the homomorphism $A \to B \to B_{\eta(h)}$ extends to a homomorphism

$$\begin{cases} A_h \longrightarrow B_{\eta(h)} \\ g/h^m \longmapsto \eta(g)/\eta(h)^m \end{cases}$$

by using the universal universal property

$$\begin{array}{ccc} A & \longrightarrow & A_h \\ & \searrow & \swarrow \\ & B_{\eta(h)} & \end{array}$$

(Notice that by assumption the algebras are unital.)

For any maximal ideal $n \lhd B$, $m := \eta^{-1}(n)$ is a maximal ideal in A because $A/m \to B/n = k$ is an injective map of k-algebras and this implies $A/m = k$. Thus η defines a map

$$\varphi : \begin{cases} \mathrm{specm}\,B \longrightarrow \mathrm{specm}\,A \\ n \longmapsto \eta^{-1}(n) = m \end{cases}$$

We want to show that φ is continuous. Let $m = \eta^{-1}(n) = \varphi(n)$; we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & B \\ \downarrow & & \downarrow \\ A/m & \xrightarrow{\ =\ } & B/n \end{array}$$

Recall that the image of an element $f \in A$ in $A/m = k$ is denoted by $f(m)$. Therefore, the commutativity of the diagram implies

$$f(m) = f(\varphi(n)) = (\eta(f))(n). \qquad \Rightarrow \quad f \circ \varphi = \eta(f). \quad (*)$$

Now, since obviously we have, for an open set

$$\mathcal{D}(f) := \{m \in \mathrm{specm}(A) \mid f(m) \neq 0\}$$

of specm(A)

$$\varphi^{-1}\mathcal{D}(f) = \mathcal{D}(f \circ \varphi)$$

it follows from (*) that

$$\varphi^{-1} D(f) = D(\eta(f))$$

which is an open set; this implies that φ is continuous.

Next, let f be a regular function on $D(h)$ and write $f = g/h^m$, $g \in A$. Then from (*) we see that $f \circ \varphi$ is a function on $D(\eta(h))$ defined by $\eta(g)/\eta(h)^m$. In particular, it is regular, and so $f \longmapsto f \circ \varphi$ maps regular functions on $D(h)$ to regular functions on $D(\eta(h))$. It follows that $f \longmapsto f \circ \varphi$ sends regular functions on any open subset of specm (A) to regular functions on the inverse image of the open subset. Thus η is a morphism Specm (B) $\longrightarrow$ Specm (A).

Conversely, by definition, a morphism

$$\varphi : (V, \mathcal{O}_V) \longrightarrow (W, \mathcal{O}_W)$$

of affine algebraic varieties defines a homomorphism of the associated affine k-algebras $k[W] \;(= \Gamma(W, \mathcal{O}_W)) \longrightarrow k[V] \;(= \Gamma(V, \mathcal{O}_V))$. Since these maps are inverse of each other, we have thus established the following

Proposition 7. For any affine algebras A and B,

$$\mathrm{Hom}_{k\text{-alg.}}(A, B) \xrightarrow{\simeq} \mathrm{Mor}(\mathrm{Specm}(B), \mathrm{Specm}(A))$$

and for any affine varieties V and W,

$$\mathrm{Mor}(V, W) \xrightarrow{\simeq} \mathrm{Hom}_{k\text{-alg.}}(k[W], k[V]). \quad \blacksquare$$

Recall that a functor $F : \mathcal{K} \longrightarrow \mathcal{K}'$ of categories $\mathcal{K}$ and $\mathcal{K}'$ is said to be an equivalence of these categories if there exists a functor $G : \mathcal{K}' \longrightarrow \mathcal{K}$ s.t. $GF \cong 1_{\mathcal{K}}$ and $FG \cong 1_{\mathcal{K}'}$, where $1_{\mathcal{K}}$ is the identity functor on $\mathcal{K}$.

We can put proposition 7 in the form of a statement about categories as follows:

The functor $A \longmapsto$ Specm(A) is a (contravariant) equivalence from the category of affine k-algebras $\mathcal{A}lg_k$ to that of affine varieties over k, Aff/k, with the inverse

$$(V, \mathcal{O}_V) \longmapsto \Gamma(V, \mathcal{O}_V).$$

(B.6) Explicit form of the morphisms of affine varieties

Proposition 8. Let $V = Z(I) \subset k^m$, $W = Z(J) \subset k^n$. The following conditions on a continuous map $\varphi : V \longrightarrow W$ are equivalent:

(i) φ is regular;

(ii) the components $\varphi_1, \dots, \varphi_n$ of φ are regular;

(iii) $f \in k[W] \Rightarrow f \circ \varphi \in k[V]$.

Proof. (i) ⇒ (ii) By def. $\varphi_i = y_i \circ \varphi$ where y_i is the coordinate function $(b_1, \dots, b_n) \longmapsto b_i : W \longrightarrow k$. Hence this implication follows directly from the definition of a regular map.

(ii) ⇒ (iii) The map $f \longmapsto f \circ \varphi$ is a k-algebra hom. from the ring of all functions $W \longrightarrow k$ to the ring of all functions $V \longrightarrow k$, and (ii) says that the map sends the coordinate functions y_i on W into $k[V]$. Since y_i's generate $k[W]$ as a k-algebra, this implies that this map sends $k[W]$ into $k[V]$.

(iii) ⇒ (i) The map $\eta : \begin{cases} k[W] \longrightarrow k[V] \\ f \longmapsto f \circ \varphi \end{cases}$ is a homomorphism of k-algebras. It therefore defines a map specm $k[V] \longrightarrow$ specm $k[W]$, and it remains to show that this coincides with φ when we identify specm $k[V]$ with V and specm $k[W]$ with W. Let $a \in V$ and $b = \varphi(a) \in W$, and let $\mathfrak{m}_a$, $\mathfrak{m}_b$ be the ideals of elements of $k[V]$ and $k[W]$ that are zero at a and b respectively. Then, for $f \in k[W]$,

$$\eta(f) \in \mathfrak{m}_a \iff f(\varphi(a)) = 0 \iff f(b) = 0 \iff f \in \mathfrak{m}_b.$$

Therefore $\eta^{-1}(\mathfrak{m}_a) = \mathfrak{m}_b$, which is what we needed to show. (see the def. of φ in subsection (5)). ∎

Remark. For all $a \in V$, $f \longmapsto f\circ\varphi$ maps germs of regular functions at $\varphi(a)$ to germs of regular functions at a in fact, it induces a local homomorphism $\mathcal{O}_{W,\varphi(a)} \longrightarrow \mathcal{O}_{V,a}$. ∎

Now, in the settings of proposition 8, consider the equations

$$y_i = P_i(x_1, \dots, x_m), \quad i = 1, \dots, n;$$

where P_i's are polynomials in $x_1, \dots, x_m$. On one hand they define a mapping

$$\varphi : \begin{cases} k^m \longrightarrow k^n \\ (a_1, \dots, a_m) = a \longmapsto (P_1(a_1, \dots, a_m), \dots, P_n(a_1, \dots, a_m)) \end{cases}$$

on the other hand, they define a homomorphism of k-algebra

$$\eta : \begin{cases} k[y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_m] \\ y_i \longmapsto P_i(x_1, \dots, x_m). \end{cases}$$

This map coincides with $f \longmapsto f\circ\varphi$, because

$$(\eta(f))(a) = f(\dots, P_i(a), \dots) = f(\varphi(a)).$$

Now, consider closed subsets $Z(I) \subset k^m$, $Z(J) \subset k^n$, where I and J are radical ideals. We show that

φ maps $Z(I)$ into $Z(J)$ iff $\eta(J) \subset I$.

Suppose $\varphi(Z(I)) \subset Z(J)$ and let $f \in J$. For every $a \in Z(I)$ we can write

$$(\eta(f))(a) = f(\varphi(a)) = 0, \quad \text{for } \varphi(a) \in Z(J).$$

$$\therefore \eta(f) \in \mathbb{I}(Z(I)) = I.$$

Conversely, suppose $\eta(J) \subset I$ and let $a \in Z(I)$. For every $f \in J$ we can write

$$f(\varphi(a)) = (\eta(f))(a) = 0, \quad \text{for } \eta(f) \in I.$$

$\therefore \varphi(a) \in Z(J)$, for all $a \in Z(I)$. $\Rightarrow \varphi(Z(I)) \subset Z(J)$.

When these conditions hold, φ is the morphism of affine varieties $Z(I) \longrightarrow Z(J)$ corresponding to the homomorphism $k[y_1, \dots, y_n]/J \longrightarrow k[x_1, \dots, x_m]/I$ defined by η. Therefore, the morphisms $Z(I) \to Z(J)$ are all of the form

$$a \longmapsto (P_1(a), \dots, P_n(a)), \quad P_i \in k[x_1, \dots, x_m].$$ ∎

(B.7) Examples.

(a) Consider a k-algebra K. From a k-algebra homomorphism $\eta : k[x] \longrightarrow K$ we obtain an element $\eta(x) \in K$ and $\eta(x)$ determines η completely. Moreover $\eta(x)$ can be taken to be any element of K. Thus

$$\begin{cases} \mathrm{Hom}_{k\text{-alg}}(k[x], K) \xrightarrow{\simeq} K \\ \eta \longmapsto \eta(x). \end{cases}$$

Now, according to proposition 7

$$\mathrm{Mor}(V, \mathbb{A}^1) = \mathrm{Hom}_{k\text{-alg}}(k[x], k[V]) \cong k[V].$$

Thus the regular map $V \longrightarrow \mathbb{A}^1$ are simply regular functions on V (as we expected).

(b) Let us denote by $\mathbb{A}^0$ the ringed space $(V_0, \mathcal{O}_{V_0})$ with V_0 consisting of a single point, and $\Gamma(V_0, \mathcal{O}_{V_0}) = k$. Equivalently $\mathbb{A}^0 = \operatorname{Specm} k$. Then, for any affine variety V we have

$$\operatorname{Mor}(\mathbb{A}^0, V) \cong \operatorname{Hom}_{k\text{-alg}}(k[V], k) \cong V$$

where the last map sends $\eta \in \operatorname{Hom}_{k\text{-alg}}(k[V], k)$ to the point corresponding to the maximal ideal $\operatorname{Ker}(\eta)$.

(c) Consider $t \longmapsto (t^2, t^3) : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$. This is bijective onto its image, the variety $V: y^2 = x^3$; but it is not isomorphism onto its image, i.e., the inverse map is not a morphism: Because of proposition 7, it suffices to show that it does not induce an isomorphism on the ring of regular functions. We have $k[\mathbb{A}^1] = k[t]$ and $k[V] = k[x,y]/(y^2 - x^3) =: k[\bar{x}, \bar{y}]$.

The map on the rings is

$$\begin{cases} k[\bar{x}, \bar{y}] \longrightarrow k[t] \\ \bar{x} \longmapsto t^2, \quad \bar{y} \longmapsto t^3 \end{cases}$$

which is injective, but the image is $k[t^2, t^3] \neq k[t]$, so it is not surjective. In fact $k[\bar{x}, \bar{y}]$ is not integrally closed:

$$\left(\frac{\bar{y}}{\bar{x}}\right)^2 - \bar{x} = 0 \;\Rightarrow\; \frac{\bar{y}}{\bar{x}} \text{ is integral over } k[\bar{x}, \bar{y}];$$

however, $\bar{y}/\bar{x} \notin k[\bar{x}, \bar{y}]$.

(Actually $\bar{y}/\bar{x}$ maps to t under the inclusion $k(\bar{x}, \bar{y}) \hookrightarrow k(t)$). ∎

(B.8) Subvarieties.

Let A be an affine k-algebra and I be an ideal in A. We define

$$\mathbb{Z}(I) = \{ P \in \operatorname{specm}(A) \mid f(P) = 0 \text{ for all } f \in I \}$$
$$= \{ \mathfrak{m}, \text{ the maximal ideals in } A \mid I \subset \mathfrak{m} \}.$$

This is a closed subset of $\operatorname{specm}(A)$, and every closed subset is of this form.

Now, assume I is radical, so that A/I is again reduced. Corresponding to the homomorphism $A \longrightarrow A/I$, we get a regular map

$$\operatorname{Specm}(A/I) \longrightarrow \operatorname{Specm}(A).$$

The image is $\mathbb{Z}(I)$, and $\operatorname{specm}(A/I) \longrightarrow \mathbb{Z}(I)$ is a homeomorphism. Thus every closed subset of $\operatorname{specm}(A)$ has a natural ringed space structure, making it into an affine algebraic variety. We call $\mathbb{Z}(I)$ with this structure a closed subvariety of $V := \operatorname{specm}(A)$. ∎

Remark. If $(V, \mathcal{O}_V)$ is a ringed space and Z is a closed subvariety of V, we can define a ringed space structure on Z as follows: Let U be an open subset of Z, and let f be a function $U \longrightarrow k$; then $f \in \Gamma(U, \mathcal{O}_Z)$ if for each $p \in U$ there is a germ (U', f') of a function at p (regarded as a point of V) s.t. $f'|_{Z \cap U'} = f$. One can check that when this construction is applied to $Z = \mathbb{Z}(I)$, the ringed space structure is that given in the above paragraph. ∎

<u>Proposition 9.</u> Let $(V, \mathcal{O}_V)$ be an affine variety and let $h \in k[V]$, $h \neq 0$. Then $(\mathbb{D}(h), \mathcal{O}_V|\mathbb{D}(h))$ is an affine variety; in fact if $V = \text{specm}(A)$, then $\mathbb{D}(h) = \text{specm}(A_h)$. More explicitly, if $V = \mathbb{Z}(I)$, then

$$\begin{cases} \mathbb{D}(h) \longrightarrow k^{n+1} \\ (a_1, \dots, a_n) \longmapsto (a_1, \dots, a_n, h(a_1, \dots, a_n)^{-1}) \end{cases}$$

defines an isomorphism of $\mathbb{D}(h)$ onto $\mathbb{Z}(I, 1 - hx_{n+1})$.

<u>Proof.</u> The map $A \longrightarrow A_h$ defines a morphism $\text{specm}(A_h) \longrightarrow \text{specm}(A)$. The image is $\mathbb{D}(h)$, and it is routine (using lemma 1) to verify the rest of the statement. ∎

For example, there is an isomorphism of affine varieties

$$\mathbb{A}^1 - \{0\} \longrightarrow V \subset \mathbb{A}^2$$
$$\bar{x} \longmapsto (\bar{x}, \tfrac{1}{\bar{x}})$$

where V is the subvariety $xy = 1$ of $\mathbb{A}^2$. ∎

<u>Important remark.</u> We have seen that all <u>closed subsets</u> of an affine variety V, and all <u>basic open subsets</u>, are again affine varieties, <u>but it need not be true that $(U, \mathcal{O}_V|U)$ is an affine variety, U open in V.</u> Note that if a ringed space $(U, \mathcal{O}_U)$ is an affine variety, then we must have $(U, \mathcal{O}_U) \cong \text{Specm}(A)$, $A = \Gamma(U, \mathcal{O}_U)$. In particular the map

$$P \longmapsto m_P = \{f \in A \mid f(P) = 0\}$$

will be a bijection from U onto $\text{specm}(A)$. Consider

$$U = \mathbb{A}^2 - \{(0,0)\} \subset \mathbb{A}^2.$$

Clearly $U = \mathbb{D}(x) \cup \mathbb{D}(y)$, and we have seen in example 2

of page 30 that $\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) = k[x,y]$. However, $U \longrightarrow \text{specm}(k[x,y])$ is not a bijection, because the ideal $(x,y) = (x-0, y-0)$ is not in the image. In any case, U is clearly a union of affine algebraic varieties; this is an example of a more general object; the so-called <u>"algebraic variety"</u>. ∎

Finally we consider the relation between homomorphism of affine k-algebras and the induced mapping between the corresponding varieties.

<u>Proposition 10.</u> Let $\eta : A \longrightarrow B$ be a homomorphism of affine k-algebras, and let $\varphi : \text{Specm}(B) \longrightarrow \text{Specm}(A)$ be the corresponding morphism of the affine varieties (so that $\eta(f) = \varphi \circ f$).

(i) <u>The image of φ is dense iff η is injective.</u>

(ii) <u>φ defines an isomorphism of $\text{Specm}(B)$ onto a closed subvariety of $\text{Specm}(A)$ iff η is surjective.</u>

<u>Proof.</u> (i) Let $f \in A$. If the image of φ is dense, then

$$f \circ \varphi = 0 \implies f = 0;$$

i.e., $\eta(f) = 0 \implies f = 0$. Hence η is injective. Conversely, if the image of φ is not dense, there will be a non-zero function $f \in A$ that is zero on the image of φ, i.e., $f \circ \varphi = 0$, which is $\eta(f) = 0$.

$\therefore \exists f \neq 0 : \eta(f) = 0. \implies \eta$ is not injective.

The contrapositive statement is

$$\eta \text{ is injective} \implies \text{Im}(\varphi) \text{ is dense}.$$

(ii) If η is surjective, then it defines an isomorphism $A/I \longrightarrow B$ where $I = \text{Ker}(\eta)$. This induces an isomorphism of $\text{Specm}(B)$ with its image in $\text{Specm}(A)$. ∎

Definition. A regular map $\varphi : V \longrightarrow W$ of affine algebraic varieties is said to be <u>dominant</u> if $\mathrm{Im}(\varphi)$ is dense in W. The proposition then says that

$$\varphi \text{ is dominant} \iff \eta : f \longmapsto f\circ\varphi \;:\; \Gamma(W,\mathcal{O}_W) \longrightarrow \Gamma(V,\mathcal{O}_V)$$

is injective. ∎

Section C — Affine algebraic varieties: Functor point of view.

In this section we consider an alternative approach in which an affine variety is defined to be a functor from the category of affine k-algebras Alg_k to the category of sets, Sets. All we did in sections A and B can be done in this setting as we shall see.

(C.1) System of algebraic equations

Every subset $S \subset k[x_1,\dots,x_n]$ defines a system of algebraic equations $\{ f(x_1,\dots,x_n)=0 \mid f \in S\}$; we shall denote this system with the same symbol S.

Let K be a k-algebra (in particular K can be a field extension of k). A <u>solution of S in K</u> or a <u>(common) zero of S in K</u> is an n-tuple $(a_1,\dots,a_n)\in K^n$ s.t. $\forall f \in S: \; f(a_1,\dots,a_n)=0$.

The set of solutions of S in K will be denoted by $\mathbb{Z}(S;K)$. Letting K vary, one gets different sets of solutions of S, each a subset of K^n. These solution sets $\mathbb{Z}(S;K)$ can be related in the following way:

Let $\varphi : K \longrightarrow K'$ be a homomorphism of k-algebras. This naturally extends to a homomorphism $\varphi^{\oplus^n} : K^n \longrightarrow K'^n$ of the direct products. Therefore,

$$\forall a = (a_1,\dots,a_n)\in \mathbb{Z}(S;K) : \quad \varphi^{\oplus^n}(a) = (\varphi(a_1),\dots,\varphi(a_n)) \in \mathbb{Z}(S;K')$$

which can be easily checked using the def. of a k-algebra homomorphism. Therefore, φ induces a map, denoted by $\mathbb{Z}(S;\varphi)$, of the solution sets

$$\mathbb{Z}(S;\varphi) : \mathbb{Z}(S;K) \longrightarrow \mathbb{Z}(S;K')$$

which satisfies the following properties:

(i) $\mathbb{Z}(S; id_K) = id_{\mathbb{Z}(S;K)}$;

(ii) for k-algebra homomorphisms $K \xrightarrow{\varphi} K' \xrightarrow{\psi} K''$ one has

$$\mathbb{Z}(S; \psi \circ \varphi) = \mathbb{Z}(S; \psi) \circ \mathbb{Z}(S; \varphi).$$

Therefore, the correspondence

$$K \longmapsto \mathbb{Z}(S; K) \;, \quad \varphi \longmapsto \mathbb{Z}(S; \varphi)$$

defines a <u>covariant functor</u> $X := \mathbb{Z}(S; -) : Alg_k \longrightarrow Sets$, from the category of k-algebras to the category of sets.

<u>Definition.</u> An affine algebraic variety over a field k is a functor

$$X : Alg_k \longrightarrow Sets$$

given by $X(K) = \mathbb{Z}(S; K)$, $X(\varphi) = \mathbb{Z}(S; \varphi)$, for a given system of algebraic equations S over k. ∎

<u>Examples.</u> (1) The functor $K \longrightarrow K^n$ is an affine algebraic variety corresponding to the system of equations $0 = 0$, where 0 on the RHS is the zero element of the k-algebra K an the 0 on the LHS is the zero of the polynomial algebra $k[x_1, \cdots, x_n]$.

(2) The functor $K \longrightarrow \emptyset \subset K^n$ is an affine algebraic variety corresponding to the system of equations $1 = 0$. It is called the empty affine algebraic variety over k and is denoted by $\emptyset_k$. ∎

Now, to every solution $a = (a_1, \cdots, a_n) \in \mathbb{Z}(S; K)$ we can associate a mapping

$$ev_{S,a} : \begin{cases} k[x_1, \cdots, x_n] \longrightarrow K \\ x_i \longmapsto a_i \end{cases}, \quad i = 1, \cdots, n;$$

which extends to a k-algebra homomorphism which we shall denote by the same symbol $ev_{S,a}$. Clearly one

has

$$a \in \mathbb{Z}(S; K) \iff ev_{S,a} S = \{0\}.$$

Denoting by (S) the ideal generated by the set S, it follows that $(S) \subseteq Ker(ev_{S,a})$, hence one gets the commutative diagram

$$\begin{array}{ccc} k[x_1, \cdots, x_n] & \xrightarrow{ev_{S,a}} & K \\ & \searrow & \nearrow \text{unique} \\ & k[x_1, \cdots, x_n]/(S) & \end{array}$$

i.e., $ev_{S,a}$ factors through $k[x_1, \cdots, x_n]/(S)$.

Conversely, any k-algebra homomorphism $k[x_1, \cdots, x_n]/(S) \longrightarrow K$ composed with the canonical surjection $k[x_1, \cdots, x_n] \longrightarrow k[x_1, \cdots, x_n]/(S)$ defines a k-algebra homomorphism $k[x_1, \cdots, x_n] \longrightarrow K$. Let $x_i \longmapsto a_i$ under this homomorphism; then $(a_1, \cdots, a_n)$ defines a solution of S, because if $f \in S$ then the image $f(a)$ of f is zero. This argument establishes the following natural bijection

$$\boxed{\mathbb{Z}(S; K) \overset{1:1}{\longleftrightarrow} Hom_{k\text{-}alg}\left(k[x_1, \cdots, x_n]/(S), K\right)} \qquad (*)$$

The following proposition gives the condition under which two systems of algebraic equations $S, S' \subset k[x_1, \cdots, x_n]$ define the same variety.

<u>Proposition 1.</u> Two systems of algebraic equations given by $S, S' \subset k[x_1, \cdots, x_n]$ define the same affine algebraic variety iff $(S) = (S')$.

<u>Proof.</u> Let $(S) = (S')$. Given $f \in S$ we can express f as a linear combination of polynomials $G \in S'$ with coefficients in $k[x_1, \cdots, x_n]$. $\Rightarrow \mathbb{Z}(S'; K) \subset \mathbb{Z}(S; K)$. The inverse

inclusion is proved similarly. Thus

$$(S)=(S') \Rightarrow \mathbb{Z}(S;K)=\mathbb{Z}(S';K).$$

To prove the inverse of this implication we use the bijection (*). Take $K = k[x_1,\dots,x_n]/(S)$ and $a=(\bar{x}_1,\dots,\bar{x}_n)$ where $\bar{x}_i = x_i \bmod (S)$, $i=1,\dots,n$. For every $f\in S$ we have

$$f(a)=f(\bar{x}_1,\dots,\bar{x}_n) \equiv f(x_1,\dots,x_n) \bmod (S) = 0. \Rightarrow a\in \mathbb{Z}(S;K).$$

Now, since $\mathbb{Z}(S;K)=\mathbb{Z}(S';K)$, then for every $f\in(S')$ we have $f(a)=0$ in K; i.e. $f\in(S)$. $\Rightarrow (S')\subset(S)$. Similarly one proves $(S)\subset(S')$ and hence the result follows. ∎

A direct consequence of this proposition is the following: Let X be the affine variety defined by a system of algebraic equations S over k. Then the ideal (S) depends only on X and is <u>called the defining ideal of X</u>. It is denoted by $I(X)$.

The Hilbert's basis theorem shows that every ideal $I \lhd k[x_1,\dots,x_n]$ is finitely generated and hence one can always restrict to a finite system of algebraic equations. For any ideal $I=(S)\lhd k[x_1,\dots,x_n]$, we define a functor

$$\tilde{I}: Alg_k \longrightarrow Sets$$

by

$$\tilde{I}(K)=\{x\in K^n \mid f(x)=0,\ \forall f\in I\}$$
$$:= \mathbb{Z}(S;K) \ ;$$

so this is an affine algebraic variety defined by any set of generators S of I. Conversely, every X is equal to $\widetilde{I(X)}$. Therefore, every affine variety can be defined by a finite system of algebraic equations. It follows from proposition 1, that the correspondence $I \longmapsto \tilde{I}$ is a 1:1 correspondence

$$\{\text{polynomial ideals}\} \xleftrightarrow{1:1} \{\text{affine algebraic varieties}\}.$$ ∎

A subvariety of an algebraic variety can also be defined using the language of functors. Let $\mathbb{C}$ be a category and $F, G : \mathbb{C} \longrightarrow Sets$ be functors. G is said to be a <u>subfunctor</u> of F, written $G\subset F$, if the following conditions hold:

(i) $G(A)\subset F(A)$, $\forall A\in Ob(\mathbb{C})$,

(ii) for every morphism $\varphi \in Mor_{\mathbb{C}}(A,B)$ the map $G(\varphi): G(A)\longrightarrow G(B)$ is equal to the restriction of the map $F(\varphi): F(A)\longrightarrow F(B)$ to the subset $G(A)\subset F(A)$.

<u>Definition</u>. An affine algebraic variety X' over k is said to be a subvariety of an affine algebraic variety X over k if X' is a subfunctor of X. ∎

Clearly every affine algebraic variety over k is a subvariety of some n-dim. affine space $\mathbb{A}^n_k$ over k. Using <u>proposition 1</u> we can state the following

<u>Proposition 2</u>. An affine algebraic variety X' is a subvariety of an affine algebraic variety X iff $I(X)\subset I(X')$. ∎

(C.2) Affine algebraic sets.

Let X be an affine algebraic variety over k. For different k-algebras K the set of K-points $X(K)$ could be quite different. It is, for example, possible that $X(K)=\emptyset$ while $X\neq\emptyset_k$. In case K is taken to be an algebraically closed field then $X(K)$ is nonempty unless $X=\emptyset_k$. This is a consequence of the Hilbert's Nullstellensatz, as we shall point out in what follows.

Definition. Let K be an algebraically closed field containing the field k. A subset $V \subset K^n$ is said to be an affine algebraic k-set if there exists an affine variety X over k s.t. $V = X(K)$. ∎

We refer to k as the ground field or the field of definition of V. Since every polynomial with coefficients in k can be considered as a polynomial with coefficients in K, we may consider an affine algebraic k-set as an affine algebraic K-set. This is often done when we do not want to specify to which field the coefficients belong. In this case we call V simply as an affine algebraic set.

Let us now see when two different systems of equations define the same algebraic set. We recall that if $I \lhd k[x_1,\dots,x_n]$, then $rad(I) = \{f \in k[x_1,\dots,x_n] \mid f^n \in I$, for some $n \in \mathbb{N}^+\}$; then $I \subset rad(I) \lhd k[x_1,\dots,x_n]$. In the present settings Hilbert's Nullstellensatz takes the following form:

Let K be an algebraically closed field and S, S' be two systems of algebraic equations in the same number of variables over a subfield k of K (i.e. $S, S' \subset k[x_1,\dots,x_n]$) Then

$$\mathbb{Z}(S;K) = \mathbb{Z}(S';K) \iff rad((S)) = rad((S')).$$

An immediate consequence of this result is the following:

Let X be an affine algebraic variety over a field k and let K be an algebraically closed extension of k. Then

$$X(K) = \emptyset \iff 1 \in I(X).$$ ∎

We recall that when $I \lhd R$ is a radical ideal, then R/I is a reduced ring, i.e., it has no nilpotent elements.

The algebra-geometry correspondence of section 1 will in the present settings read as follows:

Let K be an algebraically closed extension of k. The correspondence

$$V \longmapsto I(V) := \{f \in k[x_1,\dots,x_n] \mid f(a) = 0,\ \forall a \in V\}$$
$$I \longmapsto V(I) := \{a \in K^n \mid f(a) = 0,\ \forall f \in I\}$$

define a bijective map

$$\left\{\begin{matrix}\text{affine algebraic k-sets}\\ \text{in } K^n\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{radical ideals}\\ \text{in } k[x_1,\dots,x_n]\end{matrix}\right\}.$$

It also follows directly from the definition of affine algebraic k-sets that

(i) the intersection $\bigcap_j V_j$ of any family of affine algebraic k-sets and the union $\bigcup_j V_j$ of any finite family of affine algebraic k-sets is an affine algebraic k-set in K^n.

(ii) $\emptyset$ and K^n are affine algebraic k-sets.

Therefore, one can define a topology on K^n for which the closed sets are just the affine algebraic subsets of K^n. This topology is not Hausdorff; however it satisfies the T_1-property. ∎

(C.3) Morphisms of affine algebraic varieties.

We have defined two systems of algebraic equations to be equivalent if they have the same set of solutions. This is quite familiar from the theory of linear equations; however this is a too strong notion to work with. We can succeed in solving a system of equations if we are able to find a bijection from the set of solutions to the set of solutions of another system of equations which can be solved explicitly. This is the basic idea behind the notion of a morphism between affine algebraic varieties.

<u>Definition</u>. A morphism $f: X \to Y$ of affine algebraic varieties over a field k is a <u>set of maps</u> $f(K): X(K) \to Y(K)$ where K runs over the set of k-algebras, s.t. for every k-algebra homomorphism $\phi: K \to K'$ the following diagram is commutative:

$$\begin{array}{ccc} X(K) & \xrightarrow{X(\phi)} & X(K') \\ {\scriptstyle f(K)}\downarrow & & \downarrow{\scriptstyle f(K')} \\ Y(K) & \xrightarrow{Y(\phi)} & Y(K') \end{array}$$

∎

This definition enables us to introduce the category of affine algebraic varieties over a field k, denoted by Aff/k.

Recall that a morphism (or a natural transformation) $f: F \to F'$ between two functors $F, F': \mathcal{C} \to \mathcal{C}'$ is a collection of morphisms $f(A): F(A) \to F'(A)$, $A \in Ob(\mathcal{C})$, in $\mathcal{C}'$ s.t. for every morphism $(\phi: A \to B) \in Mor_{\mathcal{C}}(A,B)$ the following diagram is commutative

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\phi)} & F(B) \\ {\scriptstyle f(A)}\downarrow & & \downarrow{\scriptstyle f(B)} \\ F'(A) & \xrightarrow{F'(\phi)} & F'(B) \end{array}$$

One defines in a straighforward manner the composition of functor morphisms and also the category of functors $FCT(\mathcal{C}, \mathcal{C}')$. Two functors $F, F' \in Ob\, FCT(\mathcal{C}, \mathcal{C}')$ are called <u>isomorphic</u> (or <u>equivalent</u>) if there are morphisms $f: F \to F'$, $g: F' \to F$ s.t. for every $A \in Ob(\mathcal{C})$ the map $f(A), g(A)$ are inverse to each other.

The definition given above is a special case of morphism between functors; it can now be restated as follows:

A morphism from an affine variety X to an affine variety Y is a morphism of the corresponding functors $Alg_k \to Sets$. Two affine varieties are isomorphic if the corresponding functors are isomorphic. The category Aff/k is a subcategory of the functor category $FCT(Alg_k, Sets)$. ∎

Let X be an affine variety. We know that for every k-algebra K there is a natural bijection

$$X(K) \longrightarrow Hom_k\left(k[x_1,\dots,x_n]/(S),\ K\right)$$

where $X(K) = Z(S; K)$. It is immediately checked that for any homomorphism of k-algebras $f: K \to K'$ we have the commutative diagram

$$\begin{array}{ccc} X(K) & \longrightarrow & Hom_k\left(k[x_1,\dots,x_n]/(S),\ K\right) \\ {\scriptstyle X(f)}\downarrow & & \downarrow{\scriptstyle f\circ} \\ X(K') & \longrightarrow & Hom_k\left(k[x_1,\dots,x_n]/(S),\ K'\right) \end{array}$$

where $f\circ$ is composition with f:

$$\begin{array}{ccc} k[x_1,\dots,x_n]/(S) & \longrightarrow & K \\ & \searrow & \downarrow{\scriptstyle f} \\ & & K' \end{array}$$

Therefore, the <u>functor X is isomorphic to</u> the functor

$$Hom_k\left(k[x_1,\dots,x_n]/(S),\ -\right): Alg_k \longrightarrow Sets.$$

which assigns to a k-algebra K the set of homomorphisms $Hom_k\left(k[x_1,\dots,x_n]/(S),\ K\right)$.

In what follows, we shall denote the factor k-algebra $k[x_1,\dots,x_n]/(S)$ by $\mathcal{O}(X)$ and call it <u>the coordinate ring (or algebra) of X</u>. The elements of this algebra can be viewed as functions on the set of points of X. To see this, let $a \in X(K)$ be a K-point; for an element $\varphi \in \mathcal{O}(X)$ we find a polynomial $P \in k[x_1,\dots,x_n]$ representing φ and put

$$\varphi(a) = P(a).$$

Obviously this definition is independent of the choice of representative. Another way to see this is to view the point $a \in X(K)$ as a homomorphism

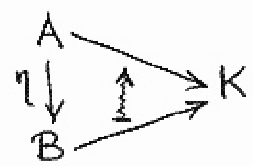
$$ev_{X,a} : \mathcal{O}(X) \longrightarrow K \;;$$

then $\varphi(a) = ev_{X,a}(\varphi)$. It is also clear that the range of φ depends on the argument: if a is a K-point, then $\varphi(a) \in K$. ∎

Let $\eta : A \longrightarrow B$ be a homomorphism of k-algebras. For every k-algebra K there is a natural map of sets

$$\mathrm{Hom}_k(B,K) \longrightarrow \mathrm{Hom}_k(A,K)$$

which is obtained by composing a map $B \longrightarrow K$ with η:



It easily follows that this defines a morphism of functors

$$\mathrm{Hom}_k(B,-) \longrightarrow \mathrm{Hom}_k(A,-) .$$

It, therefore, follows that there exist a natural map

$$\boxed{\mathrm{Hom}_k\big(\mathcal{O}(Y), \mathcal{O}(X)\big) \longrightarrow \mathrm{Mor}_{Aff/k}(X,Y)}$$

Let us first specify explicitly this correspondence:

take a K-point $a = (a_1, \dots, a_n) \in X(K)$; it defines a homomorphism

$$ev_{X,a} : \begin{cases} \mathcal{O}(X) := k[x_1,\dots,x_n]/(S) \longrightarrow K \\ \quad x_i \longmapsto a_i \quad , \; i = 1, \dots, n . \end{cases}$$

Composing this homomorphism with a given homomorphism $\eta : \mathcal{O}(Y) := k[x'_1,\dots,x'_m]/(S') \longrightarrow \mathcal{O}(X)$, we get a homomorphism $ev_{X,a} \circ \eta : \mathcal{O}(Y) \longrightarrow K$. Let $b = (b_1, \dots, b_m)$

where $b_i = ev_{X,a} \circ \eta(x'_i)$, $i = 1, \dots, m$. This defines a K-point of Y. Varying K, we obtain a morphism $X \longrightarrow Y$ which corresponds to the homomorphism $\eta : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$.

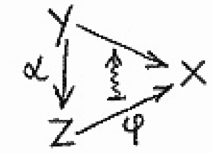
<u>We shall prove that this correspondence is bijective.</u>
This is, however, a special case of a general result called the "Yoneda lemma" that we shall discuss now.

Let $\mathcal{C}$ be a category; let us associate to every object $X \in Ob(\mathcal{C})$ a functor $h_X : \mathcal{C} \longrightarrow Sets$ defined by

$$h_X(Z) = \mathrm{Mor}_{\mathcal{C}}(Z,X)$$

$$h_X(\alpha : Y \to Z) : h_X(Z) \longrightarrow h_X(Y) \quad , \text{ given by}$$

$$\varphi \longmapsto \varphi \circ \alpha \;;$$



If $\psi : X \longrightarrow Y$ is a morphism in $\mathcal{C}$ we can define a natural transformation

$$h(\psi) : \begin{cases} h_X \longrightarrow h_Y \\ (\alpha : Z \to X) \longmapsto (\psi \circ \alpha : Z \longrightarrow Y) \end{cases} .$$

In this way we <u>obtain a functor</u>

$$h : \mathcal{C} \longrightarrow \bar{\mathcal{C}} := \mathrm{FCT}(\mathcal{C}^{op}, Sets)$$

where $\mathcal{C}^{op}$ is the opposite or dual category of $\mathcal{C}$.

<u>Lemma 2. (Yoneda).</u> For any two objects $X, Y \in ob(\mathcal{C})$ the map

$$h : \begin{cases} \mathrm{Mor}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Mor}_{\bar{\mathcal{C}}}(h_X, h_Y) \\ \quad \psi \longmapsto h(\psi) \end{cases}$$

is bijective.

<u>Proof.</u> We must show that h has an inverse. Let $f : h_X \to h_Y$; for every $Z \in ob(\mathcal{C})$, h defines a map

$f(Z): h_X(Z) \longrightarrow h_Y(Z)$. Put $Z = X$; then $f(X)(id_X)$ is a map $\psi: X \to Y$ belonging to the set $h_Y(X)$. Let us define a mapping

$$u: \begin{cases} Mor_{\overline{\mathcal{C}}}(h_X, h_Y) \longrightarrow Mor_{\mathcal{C}}(X,Y) \\ f \longmapsto \psi = f(X)(id_X), \end{cases}$$

We claim that $u = h^{-1}$. To see this let $\psi: X \to Y$; then

$$u \circ h(\psi) = u(h(\psi)) = h(\psi)(X)(id_X)$$
$$= id_X \circ \psi = \psi,$$

so u is a left inverse to h. Next, for any $f: h_X \to h_Y$ and $(\alpha: Z \to X) \in h_X(Z)$ we have the following commutative diagram (arising from the def. of morphisms of functors)

$$\begin{array}{ccc} h_X(X) & \xrightarrow{\quad f(X) \quad} & h_Y(X) \\ {\scriptstyle h_X(\alpha)}\downarrow & & \downarrow{\scriptstyle h_Y(\alpha)} \\ h_X(Z) & \xrightarrow{\quad f(Z) \quad} & h_Y(Z) \end{array}$$

$f(X)$ sends id_X to $u(f): X \to Y$; $h_Y(\alpha)$ sends $u(f)$ to $u(f) \circ \alpha$. On the other hand $h_X(\alpha)$ sends id_X to $id_X \circ \alpha = \alpha$.
Hence $f(Z)(\alpha) = u(f) \circ \alpha$, by commutativity of the diagram,
$= h(u(f))(\alpha)$, by def. of h.

Therefore, $f = h(u(f))$ and hence u is also a right inverse of h. ∎

Recall that two objects A, B of a category are said to be isomorphic if there are morphisms $f: A \to B$, $g: B \to A$ s.t. $g \circ f = id_A$, $f \circ g = id_B$; we then write $A \cong B$.

Corollary 3. Let $A, B \in ob(\mathcal{C})$. Then,

$$A \cong B \text{ in } \mathcal{C} \iff h_A \cong h_B \text{ in } \overline{\mathcal{C}}. \quad ∎$$

To apply this lemma we take $\mathcal{C} = Alg_k^T$, the dual category of k-algebras. For any k-algebra K, we have

$$h_{\mathcal{O}(X)}(K) = Hom_k(\mathcal{O}(X), K).$$

Thus the Yoneda lemma implies that there is a 1:1 correspondence between the set of morphisms from an affine algebraic variety X to another such variety Y and homomorphisms of their coordinate algebras $\phi: \mathcal{O}(Y) \to \mathcal{O}(X)$. Its corollary says that X and Y are isomorphic iff their coordinate algebras are isomorphic.

Let $\phi: \mathcal{O}(Y) \to \mathcal{O}(X)$ be a homomorphism of the coordinate algebras of two affine algebraic varieties given by a system $S \subset k[x_1, \dots, x_n]$ defining X and a system $S' \subset k[x'_1, \dots, x'_m]$ defining Y. $\Rightarrow$ $\mathcal{O}(Y) = k[x'_1, \dots, x'_m]/(S')$, $\mathcal{O}(X) = k[x_1, \dots, x_n]/(S)$.
Therefore ϕ is defined by assigning to each x'_i an element $\varphi_i \in \mathcal{O}(X)$, which is a coset of a polynomial $P_i(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$.
$\Rightarrow$ ϕ is determined by a collection of polynomials

$$(P_1(x_1, \dots, x_n), \dots, P_m(x_1, \dots, x_n)).$$

Since the homomorphism $\begin{cases} k[x'_1, \dots, x'_m] \longrightarrow \mathcal{O}(X) \\ x'_i \longmapsto P_i(x_1, \dots, x_n) + (X) \end{cases}$ factors through the ideal (Y), then

$$\forall F \in (Y): \quad F(P_1(x_1, \dots, x_n), \dots, P_m(x_1, \dots, x_n)) \in (X). \quad (*)$$

Note that it suffices to check the condition $(*)$ only for the generators of (Y); e.g. the polynomials in the system S' defining Y. In terms of polynomials $(P_1(x_1, \dots, x_n), \dots, P_m(x_1, \dots, x_n))$ satisfying $(*)$, the morphism $f: X \to Y$ is given as follows:

$$f(K)(a) = (P_1(a), \dots, P_m(a)) \in Y(K), \quad \forall a \in X(K).$$

It follows from what we said above that a morphism ϕ

given by polynomials $(P_1(x_1,\dots,x_n),\dots,P_m(x_1,\dots,x_n))$ satisfying (*), is an isomorphism iff there exists polynomials $(Q_1(x'_1,\dots,x'_m),\dots,Q_n(x'_1,\dots,x'_m))$ s.t.

$$\begin{cases} G(Q_1(x'_1,\dots,x'_m),\dots,Q_n(x'_1,\dots,x'_m)) \in (Y), & \forall G \in (X), \\ P_i(Q_1(x'_1,\dots,x'_m),\dots,Q_n(x'_1,\dots,x'_m)) \equiv x'_i \mod (Y), & i=1,\dots,m, \\ Q_j(P_1(x_1,\dots,x_n),\dots,P_m(x_1,\dots,x_n)) \equiv x_j \mod (X), & j=1,\dots,n. \end{cases}$$

Remark. The main problem of (affine) algebraic geometry is to classify affine algebraic varieties up to isomorphism. Of course, this is a hopelessly difficult problem. ∎

Examples. (1) Let Y be given by the equation $x_1'^2 - x_2'^3 = 0$, and $X = \mathbb{A}^1_k$ with $\mathcal{O}(X) = k[x]$. A morphism $f: X \to Y$ is given by a pair of polynomials (x^3, x^2). Given a k-algebra K, we have

$$f(K)(a) = (a^3, a^2) \in Y(K), \quad \forall a \in X(K) = K.$$

The affine algebraic varieties X and Y are not isomorphic since their coordinate rings are not isomorphic: the quotient field of the algebra $\mathcal{O}(Y) = k[x'_1,x'_2]/(x_1'^2 - x_2'^3)$ contains an element x'_1/x'_2 which does not belong to $\mathcal{O}(Y)$ but its square is an element of $\mathcal{O}(Y)$ (= coset of x'_2); but we know that the polynomial ring, and hence $k[x]$, does not have this property.

(2) The circle $X = \{x_1^2 + x_2^2 - 1 = 0\}$ is isomorphic to the hyperbola $Y = \{x'_1 x'_2 - 1 = 0\}$ provided the field k contains a square root of -1 and $\mathrm{Char}(k) \neq 2$. ($x'_1 \mapsto x_1 + i x_2$, $x'_2 \mapsto x_1 - i x_2$ sends hyperbola to circle). ∎

Consider the special case of morphism $f: X \to Y$ where $Y = \mathbb{A}^1_k$ (the affine line). Then f is defined by a homomorphism of the corresponding coordinate algebras

$\mathcal{O}(Y) = k[x'] \longrightarrow \mathcal{O}(X)$. Every such homomorphism is determined by its value at x', i.e., by an element of $\mathcal{O}(X)$. This gives us one more interpretation of the elements of the coordinate algebra $\mathcal{O}(X)$. This time as a morphisms from X to $\mathbb{A}^1_k$ and hence again can be thought of as functions on X.

Let $f: X \to Y$ be a morphism of affine algebraic varieties. We know that it arises from a homomorphism of a k-algebra $f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$.

Proposition 4. For any $\varphi \in \mathcal{O}(Y) = \mathrm{Mor}_{Aff/k}(Y, \mathbb{A}^1_k)$,

$$f^*(\varphi) = \varphi \circ f.$$

Proof. This follows immediately from the above definition. ∎

Having got used to the functorial point of view, it will be natural to generalize the notion of an affine algebraic variety.

Definition. A functor $F: \mathrm{Alg}_k \to \mathrm{Sets}$ is said to be an affine algebraic functor over a field k if it is isomorphic to an affine algebraic variety over k. ∎

We define the coordinate algebra of an affine algebraic functor by

$$\mathcal{O}(F) := \mathcal{O}(X)$$

where X is an affine algebraic variety isomorphic to F. It is clear that there is a bijection

$$\mathcal{O}(F) \overset{1:1}{\longleftrightarrow} \mathrm{Mor}(F, \mathbb{A}^1_k)$$

where the morphisms are taken from the category $\mathrm{FCT}(\mathrm{Alg}_k, \mathrm{Sets})$.

Examples. (1) Let A be a finitely generated k-algebra. Choose a surjective homomorphism $k[x_1,\dots,x_n] \to A$ and let I be its kernel. Let I be the affine algebraic variety corresponding to the ideal I (i.e., defined by any set of generators of I). We have already seen that there is an isomorphism of functors $I \cong h_A$. Thus h_A is an affine algebraic functor.

(2) The functor $G_{m,k}: K \longrightarrow K^*$ (= invertible elements in K) is an affine algebraic functor. It is isomorphic to the functor $K \longrightarrow \{(a,b) \in K^2 \mid ab = 1\}$ which is given by a system of algebraic equations $x_1 x_2 - 1 = 0$. The natural inclusion $K^* \subset K$ allows us to identify $G_{m,k}$ with a subvariety of $\mathbb{A}^1_K$. This functor is called the multiplicative group over k. ∎

Now, we specialize the notion of a morphism of affine algebraic varieties to define the notion of a regular map of affine algebraic sets. Recall that an affine algebraic k-set is a subset $V \subset K^n$ of the form $X(K)$ for some affine algebraic variety X over k and K is an algebraically closed extension of k. We can always choose X to be equal to $\underline{I}$ for some radical ideal $I \triangleleft k[x_1, \dots, x_n]$. This ideal is determined uniquely by V and is equal to the ideal $\mathbb{I}(V) = \{f \in k[x_1, \dots, x_n] \mid f|V = 0\}$. Each morphism $f: X \longrightarrow Y$ of algebraic varieties define a map

$$f(K): X(K) = V \longrightarrow W = Y(K)$$

of algebraic sets. Therefore, it is natural to take for the def. of regular maps of algebraic sets the maps arising this way. We know that f is given by a homomorphism of k-algebras

$$f^*: \mathcal{O}(Y) := k[x'_1, \dots, x'_m]/\mathbb{I}(W) \longrightarrow \mathcal{O}(X) := k[x_1, \dots, x_n]/\mathbb{I}(V).$$

If $f^*: x'_i \bmod \mathbb{I}(W) \longmapsto P_i(x_1, \dots, x_n)$, $i = 1, \dots, m$, then for any $V \ni a = (a_1, \dots, a_n)$ viewed as a homomorphism $\mathcal{O}(X) \longrightarrow K$

its image $f(K)(a)$ is a homomorphism $\mathcal{O}(Y) \longrightarrow K$ given by $x'_i \longmapsto P_i(a)$, $i = 1, \dots, m$. Thus the map $f(K)$ is given by

$$f(K)(a) = (P_1(a_1, \dots, a_n), \dots, P_m(a_1, \dots, a_n)).$$

Clearly this map does not depend on the choice of representative P_i of $f^*(x'_i \bmod \mathbb{I}(W))$ since any polynomial from $\mathbb{I}(V)$ vanishes at a. All these motivates the following

Definition. A regular function on V is a map of sets $f: V \longrightarrow K$ s.t. $\exists F(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ with the property

$$F(a_1, \dots, a_n) = f(a_1, \dots, a_n), \quad \forall a = (a_1, \dots, a_n) \in V.$$

A regular map of affine algebraic sets $f: V \longrightarrow W \subset K^m$ is a map of sets s.t. its composition with each projection map $pr_i: K^m \longrightarrow K$, $(a_1, \dots, a_m) \longmapsto a_i$, is a regular function. An invertible regular map is called a biregular map of algebraic sets. ∎

Sometimes a regular map is called a polynomial map. It is easy to see that a regular map is a continuous map of affine algebraic k-sets equipped with the induced Zariski-topology. However, the converse is obviously false.

It follows from the def. that a regular function $f: V \to k$ is given by a polynomial $F(x_1, \dots, x_n)$ which is defined uniquely modulo the ideal $\mathbb{I}(V)$. The set of all regular functions on V is isomorphic to $\mathcal{O}(V) := k[x_1, \dots, x_n]/\mathbb{I}(V)$; it is called the algebra of regular functions on V. Clearly it is isomorphic to the coordinate algebra of the affine algebraic variety $\underline{\mathbb{I}(V)}$. Every regular map $f: V \to W$ defines a homomorphism $f^*: \mathcal{O}(W) \to \mathcal{O}(V)$, $\varphi \mapsto \varphi \circ f$, and conversely, any homomorphism $\alpha: \mathcal{O}(W) \longrightarrow \mathcal{O}(V)$ defines a unique map $f: V \longrightarrow W$ s.t. $f^* = \alpha$. All these follow from the discussion above. ∎

Section D. Local studies:

Tangent spaces, tangent cones and singularities.

Our aim in this section is to study the structure of a variety near a point.

(D.1) The case of a plane curve.

Let $F(x,y) \in k[x,y]$ be a nonconstant polynomial and consider the curve $V = Z(F) \subset \mathbb{A}^2$. Suppose $F(x,y)$ has no multiple factors, so that $(F(x,y))$ is a radical ideal and $\mathbb{I}(V) = (F(x,y))$. Factoring F into the product of irreducible polynomials, $F(x,y) = \prod F_i(x,y)$, then $V = \bigcup_i Z(F_i)$ expresses V as a union of its irreducible components and each component has dimension 1 (see the discussion on dimension) and so V has pure dimension 1. To clarify this suppose for simplicity that $F(x,y)$ itself is irreducible, then

$$k[V] = k[x,y]/(F(x,y)) = k[\bar{x},\bar{y}]$$

is an integral domain. If $F \neq x-a$, then $\bar{x}$ is transcendental over k and y is algebraic over $k(\bar{x})$, and so $\bar{x}$ is a transcendence basis for $k(V)$ over k. Similarly if $F \neq y-c$, then y is a transcendence basis for $k(V)$ over k. $\Rightarrow \dim V = 1$.

From elementary analytic geometry we know that if $P = (a,b) \in V$, the equation of the tangent to V at this point is

$$\frac{\partial F}{\partial x}(a,b)(x-a) + \frac{\partial F}{\partial y}(a,b)(y-b) = 0 \quad \leftarrow (1)$$

This is the equation of a line unless both $\frac{\partial F}{\partial x}(a,b)$ and $\frac{\partial F}{\partial y}(a,b)$ are zero, in which case it is the equation of a plane.

Def. The tangent space $T_P(V)$ to V at $P=(a,b) \in V$ is the space defined by equation (1). ∎

When $\frac{\partial F}{\partial x}(a,b)$ and $\frac{\partial F}{\partial y}(a,b)$ are not both zero, $T_P(V)$ is a line (and hence is 1-dimensional); in this case we say that P is a non-singular or smooth point of V. Otherwise $T_P(V)$ has dimension 2 and we say that P is singular or multiple.

The curve V is said to be nonsingular or smooth if all its points are nonsingular.

We regard $T_P(V)$ as a subspace of the 2-dim. vector space $T_P(\mathbb{A}^2)$ = 2-dim. space of vectors with origin P. ∎

Examples. (a) $x^m + y^m = 1$. All points are non-singular unless the characteristic of the ground field divides m (in which case $x^m + y^m - 1$ has multiple factors:
$(x+y-1)^m = (x+y)^m - 1 = x^m + y^m - 1$).

(b) $y^2 = x^3$. Here only (0,0) is singular.

(c) $y^2 = x^2(x+1)$. Again only (0,0) is singular.

(d) $y^2 = x^3 + ax + b$. In this case

V is singular $\Leftrightarrow$ $y^2 - x^3 - ax - b$, $2y$, $3x^2 + a$ have a common zero. $\Leftrightarrow$ $x^3 + ax + b$ and $3x^2 + a$ have a common zero. Since $3x^2 + a = (x^3 + ax + b)'$ we see that V is singular iff $x^3 + ax + b$ has a multiple root.

(e) $(x^2+y^2)^2 + 3x^2y - y^3 = 0$. The origin is (very) singular.

(f) $(x^2+y^2)^3 - 4x^2y^2 = 0$. The origin is (even more) singular.

(g) $V = V(F,G)$ where FG has no multiple factors and F and G are relatively prime. Then $V = \mathbb{Z}(F) \cup \mathbb{Z}(G)$, and a point $P = (a,b)$ is singular iff it is a singular point of $\mathbb{Z}(F)$, or a singular point of $V(G)$, or a point of $\mathbb{Z}(F) \cap \mathbb{Z}(G)$; the last example follows from product rule:

$$\frac{\partial(FG)}{\partial x} = F\cdot\frac{\partial G}{\partial x} + \frac{\partial F}{\partial x}\cdot G, \quad \frac{\partial(FG)}{\partial y} = F\cdot\frac{\partial G}{\partial y} + \frac{\partial F}{\partial y}\cdot G. \ ∎$$

Proposition 1. Let V be a curve defined by a non-constant polynomial $F(x,y)$ without multiple factors. Then the set of non-singular points of V is an open dense subset of V.

Proof. Without loss of generality we can assume that F is irreducible. We must show that the set of singular points (the singular locus) is a proper closed subset of V. Because it is defined by the equations

$$F=0\,,\quad \frac{\partial F}{\partial x}=0\,,\quad \frac{\partial F}{\partial y}=0$$

it is obviously closed. It will be proper unless $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are identically zero on V (i.e. all points of V are singular in this case) are therefore both multiples of F, but since they have lower degree this is impossible unless they are both zero. Clearly,

$$\frac{\partial F}{\partial x}=0 \iff \begin{cases} F \text{ is a polynomial in } y \text{ (in case the characteristic of } k \text{ is zero), or } F \text{ is a polynomial in} \\ x^p \text{ and } y, \text{ in case the characteristic of } k \text{ is } p. \end{cases}$$

A similar remark applies to $\frac{\partial F}{\partial y}$. Thus if $\frac{\partial F}{\partial x}=0=\frac{\partial F}{\partial y}$, then F is constant (char $k=0$) or a polynomial in x^p, y^p, and hence a p-th power (char $=p$). But these are contrary to our assumptions. ∎

Tangent cones to plane curves.

It is clear that if $P=(0,0)$, then the equation defining the tangent space is the linear term of F:

since $(0,0)\in V$, we have

$F(x,y)=ax+by+$ terms of higher degrees in x,y.

∴ The equation of the tangent space is given by

$$F_\ell(x,y):=ax+by=0 \quad \longleftarrow (2)$$

In general case $F(x,y)$ can be written (uniquely) as a finite sum:

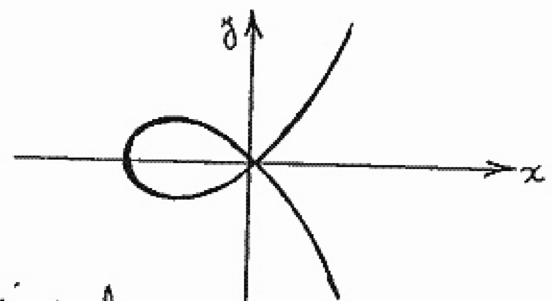
$$F=F_0+F_1+\cdots+F_m+\cdots \quad \text{(finite terms)},$$

where F_m is a homogeneous polynomial of degree m (i.e., a sum of terms $a x^i y^j$ with $i+j=m$). The first non-zero term on the RHS, i.e., the homogeneous summand of least degree, will be written F_* and called the leading form of F.

Def. Suppose $(0,0)\in Z(F)$, F a polynomial without square factors. The geometric tangent cone at $(0,0)$ is the zero set of F_*; the tangent cone is the pair $(Z(F_*), F_*)$. To obtain the tangent cone at any other point; translate to the origin, and then translate back. ∎

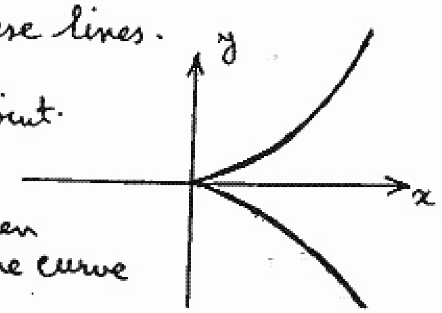
When $P\in V$ is a nonsingular point of the curve V, then V is nicely approximated by its tangent (line) space, $T_P(V)$. This clearly fails when P is singular, for we showed that in this case the tangent space has the wrong dim. (it is too big). To approximate V near a singular point, we need something different; this is the geometric tangent cone as the following examples show.

Examples. (a) $y^2=x^2(x+1)$. The curve is the following



The origin is a singular point. $F(x,y)=y^2-x^2-x^3$. $F_*(x,y)=y^2-x^2$. $\Longrightarrow$ the geometric tangent cone at $(0,0)$ is given by $y^2=x^2$, i.e. $y=\pm x$. The curve at $(0,0)$ is approximated by these lines.

(b) $y^2=x^3$. The origin is a singular point. $F(x,y)=y^2-x^3$. $F_*=y^2$. The geometric tangent cone at $(0,0)$ is given by $y^2=0$, that is the x-axis. Hence the curve at $(0,0)$ is approximated by the x-axis.



(c) $(x^2+y^2)^2+3x^2y-y^3=0$.

$F(x,y) = 3x^2y - y^3 + (x^2+y^2)^2$ and $F_* = 3x^2y - y^3$. The geometric tangent cone at $(0,0)$ is given by the equation $3x^2y-y^3=0$, i.e. the union of the lines $y=0$ and $y=\pm\sqrt{3}\,x$.

(d) $(x^2+y^2)^3-4x^2y^2=0$. The geometric tangent cone at $(0,0)$ is given by $4x^2y^2=0$, i.e., the union of x and y axes (each doubled). ∎

The local ring at a point on a curve.

Proposition 2. Let $P \in V = \mathbb{Z}(F(x,y))$ be a plane curve and let m_p be the maximal ideal in $k[V] := k[x,y]/(F(x,y)) = k[\bar{x},\bar{y}]$ corresponding to P. If P is nonsingular, then $\dim_k m_p/m_p^2 = 1$, and otherwise $\dim_k m_p/m_p^2 = 2$.

Proof. Assume first that $P=(0,0)$. Then $m_p = (\bar{x}-0, \bar{y}-0) = (\bar{x},\bar{y})$ in $k[V] = k[\bar{x},\bar{y}]$. (Notice that

$$(x,y)/(F(x,y)) = (\bar{x},\bar{y}) := m_p \lhd k[\bar{x},\bar{y}].)$$

Now,

$$m_p^2 = (\bar{x}^2, \bar{x}\bar{y}, \bar{y}^2) = \big((x^2,xy,y^2)+(F(x,y))\big)/(F(x,y))$$

$$= (x^2,xy,y^2,F(x,y))/(F(x,y)).$$

Both m_p and m_p^2 are submodules of $k[\bar{x},\bar{y}]$ and we have

$$m_p/m_p^2 = \Big((x,y)/(F(x,y))\Big)\Big/\Big((x^2,xy,y^2,F(x,y))/(F(x,y))\Big)$$

$$\cong (x,y)/(x^2,xy,y^2,F(x,y)).$$

Clearly every element of this quotient is represented by a linear polynomial $cx+dy$ and the only relation is $F_\ell(x,y)=0$. Therefore $\dim_k m_p/m_p^2 = 1$ if $F_\ell \neq 0$ and $\dim_k m_p/m_p^2 = 2$ otherwise.

The same argument works in general except that one must work with the variables $x' = x - a$, $y' = y - b$, i.e. translate the point $P = (a, b)$ to the origin. ∎

Important remark (related to our later discussions)

We explain what the condition $\dim_k(m_P/m_P^2) = 1$ means for the local ring $\mathcal{O}_P := k[V]_{m_P}$ (see later for more details). Let n_P be the maximal ideal of the local ring $k[V]_{m_P}$, i.e.

$$n_P = m_P\, k[V]_{m_P}.$$

Then the map $m_P \longmapsto n_P$ induces an isomorphism $m_P/m_P^2 \to n_P/n_P^2$ and so we have

$$P \text{ nonsingular} \iff \dim_k m_P/m_P^2 = 1 \iff \dim_k n_P/n_P^2 = 1.$$

Nakayama's lemma shows that the last condition is equivalent to n_P being a principal ideal. Since $\mathcal{O}_P$ is of dimension 1, n_P being principal means that $\mathcal{O}_P$ is a regular local ring of dimension 1, and hence it is a discrete valuation ring, i.e., a principal ideal domain with exactly one prime element (up to associates). Thus

$$P \text{ nonsingular} \iff \mathcal{O}_P \text{ regular} \iff \mathcal{O}_P \text{ is a discrete valuation ring.} \;∎$$

(D.2) Tangent spaces of subvarieties of $\mathbb{A}^n$

First a reminder from linear algebra:

Consider the vector space k^m. Let x_i be the i-th coordinate function $a = (a_1, \dots, a_n) \longmapsto a_i$. Thus $x_1, \dots, x_m$ is the dual basis to the standard basis of k^m. A linear form $\sum a_i x_i$ can be regarded as an element of the dual vector space $(k^m)^* := \mathrm{Hom}_k(k^m, k)$.

Let $A = (a_{ij})$ be an $n \times m$ matrix; it defines a linear map

$$\alpha : k^m \to k^n, \text{ by } \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \longmapsto A \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}.$$

Thus, if $\alpha(a) = b$, then $b = \sum_{j=1}^m a_{ij} a_j$. Write $x_1, \dots, x_m$ for the coordinate functions on k^m and $y_1, \dots, y_n$ for coordinate functions on k^n. Then the equation $b_i = \sum_{j=1}^m a_{ij} a_j$ can be written as

$$y_i \circ \alpha = \sum_{j=1}^m a_{ij} x_j \qquad \leftarrow ③$$

This shows that if we apply α to $a \in k^m$, then the i-th coordinate of the result is

$$\sum_{j=1}^m a_{ij}(x_j a) = \sum_{j=1}^m a_{ij} a_j. \qquad ∎$$

Tangent spaces. Consider a variety $V \subset k^m$, and let $I = \mathbb{I}(V)$. $T_a(\mathbb{A}^m)$, the tangent space at $a = (a_1, \dots, a_m)$ to $\mathbb{A}^m$ is an m-dimensional vector space with origin a, and $T_a(V)$, the tangent space to V at a, is a subspace of this vector space defined by the linear equations

$$\sum_{i=1}^m \frac{\partial F}{\partial x_i}\Big|_a (x_i - a_i) = 0, \quad \forall F \in I. \qquad \leftarrow ④$$

Let us write

$$(dx_i)_a := (x_i - a_i),$$

then $(dx_i)_a$ form a basis for the dual vector space $T_a(\mathbb{A}^m)^*$ to $T_a(\mathbb{A}^m)$. In fact they are coordinate functions on $T_a(\mathbb{A}^m)$. As in advanced calculus, for a function $F \in k[x_1, \dots, x_m]$, we define the differential of F at a by the equation

$$(dF)_a = \sum \frac{\partial F}{\partial x_i}\Big|_a (dx_i)_a \qquad \leftarrow ⑤$$

This is again a linear form on $T_a(\mathbb{A}^m)$. We can now write the subspace $T_a(V) \subset T_a(\mathbb{A}^m)$, defined by ④, as

$$(dF)_a = 0 \ , \quad \forall F \in I \ . \qquad \leftarrow ⑥$$

In ④ and ⑥ it suffices to take F in the generating set for I ; this is seen as follows:

the product rule for the differential shows that if $G = \sum_j H_j F_j$, then

$$(dG)_a = \sum_j \{ H_j(a) \cdot (dF_j)_a + F_j(a) (dH_j)_a \} ,$$

so if $I = (F_1, \dots, F_r)$ and $a \in V = Z(I)$, then $F_j(a) = 0$ and we get

$$(dG)_a = \sum_j H_j(a) (dF_j)_a \ .$$

$$\therefore \quad (dG)_a(t) = 0 \Longleftarrow (dF_j)_a(t) = 0 , \quad \forall t \in T_a(V), \forall j.$$

Def. When V is irreducible, a point $a \in V$ is said to be nonsingular (or smooth) if the dimension of the tangent space at a is equal to the dimension of V; otherwise it is singular (or multiple). ∎

Now, let $I = (F_1, \dots, F_r)$, and let

$$J = \mathrm{Jac}(F_1, \dots, F_r) := \left(\frac{\partial F_i}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial F_r}{\partial x_1} & \cdots & \frac{\partial F_r}{\partial x_m} \end{pmatrix}$$

Then the linear equations ⑤ (or ⑥) defining $T_a(V)$ as a subspace of $T_a(\mathbb{A}^m)$ have matrix $J(a)$. It then follows that

$$\dim_k T_a(V) = m - \mathrm{rank}\, J(a) \qquad \leftarrow ⑦$$

because by ⑤ , $T_a(V)^* = \mathrm{Ker}(J(a))$.

$\therefore$ $a \in V$ is nonsingular $\Longleftrightarrow$ $\mathrm{rank}\, J(F_1, \dots, F_r)(a) = m - \dim_a(V)$.

For example, if V is a hypersurface, say $I(V) = (F(x_1, \dots, x_m))$ then $\mathrm{Jac}(F)(a) = \left(\frac{\partial F}{\partial x_1}(a), \dots, \frac{\partial F}{\partial x_m}(a)\right)$, and hence a is non-singular iff not all partial derivatives

$\frac{\partial F}{\partial x_i}$ vanish at a, in which case $\mathrm{rank}\, J(F)(a) = 1$ and $\dim_a(V) = m-1$. ∎

Remark. We can regard J as a matrix of regular functions on V. For each d the set

$$\{ a \in V \mid \mathrm{rank}\ J(a) \leq d \}$$

is a closed subset of V, because it is the set where certain determinants vanish. Therefore, there is an open subset U of V on which rank $J(a)$ attains its maximum value, and the rank drops on closed subsets. We shall see that the maximum value of rank $J(a)$ is $m - \dim(V)$ and so nonsingular points of V form nonempty open subsets. ∎

The differential of a map

Consider a regular map

$$\alpha : \begin{cases} \mathbb{A}^m \longrightarrow \mathbb{A}^n \\ a \longmapsto b = (P_1(a_1, \dots, a_m), \ \dots \ , P_n(a_1, \dots, a_m)) \end{cases}$$

We may assume α as being given by the equations

$$y_i = P_i(x_1, \dots, x_m) , \quad i = 1, \dots, n .$$

It corresponds to to the algebra morphism

$$\alpha^* : \begin{cases} k[y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_m] \\ y_i \longmapsto P_i(x_1, \dots, x_m) \end{cases} , i = 1, \dots, n .$$

We define the map

$$(d\alpha)_a : T_a(\mathbb{A}^m) \longrightarrow T_b(\mathbb{A}^n)$$

to be the map such that

$$(dy_i)_b \circ (d\alpha)_a = \sum_{j=1}^m \frac{\partial P_i}{\partial x_j}\Big|_a (dx_j)_a \qquad \leftarrow ⑧$$

(compare with ③), i.e., relative to the standard bases $(d\alpha)_a$ is the map represented by the matrix

$$J(P_1, \dots, P_n)(a) = \left(\frac{\partial P_i}{\partial x_j}(a)\right)_{i,j} .$$

For example, take $a=(0,\dots,0)$, $b=(0,\dots,0)$, so that

$$T_a(\mathbb{A}^m)=k^m \quad , \quad T_b(\mathbb{A}^n)=k^n$$

and let

$$P_i=\sum_{j=1}^m c_{ij}x_j + (\text{higher terms}) \quad , \quad i=1,\dots,n.$$

Then

$$y_i\circ(d\alpha)_a=\sum_{j=1}^m c_{ij}x_j$$

and the map $d\alpha$ on the tangent space is given by the matrix (c_{ij}), i.e., simply $t\longmapsto(c_{ij})t$, $t\in T_a(\mathbb{A}^m)$.

Now, let $F\in k[x_1,\dots,x_m]$. We can regard F as a regular map $\mathbb{A}^m\longrightarrow\mathbb{A}^1$, whose differential is a map

$$(dF)_a: T_a(\mathbb{A}^m)\longrightarrow T_b(\mathbb{A}^1) \quad , \quad b=F(a).$$

Identifying $T_b(\mathbb{A}^1)$ with k, we obtain an identification of the differential of F (F regarded as a regular map) with the differential of F (F as a regular function).

Lemma 3. Let $\alpha:\mathbb{A}^m\longrightarrow\mathbb{A}^n$ be given as above. If α maps $V=Z(I)\subset k^m$ into $W=Z(J)\subset k^n$, then $(d\alpha)_a$ maps $T_a(V)$ into $T_b(W)$; $b=\alpha(a)$.

Proof. Recall that α maps $Z(I)$ into $Z(J)$ iff $\alpha^*(J)\subset I$. Therefore, we are given that

$$f\in J \Rightarrow \alpha^*(f):=f\circ\alpha\in I$$

and we want to prove that

$$f\in J \Rightarrow (df)_b\circ(d\alpha)_a\Big|_{T_a(V)}=0 \ .$$

The chain rule gives

$$\frac{\partial f}{\partial x_i}\Big|_a=\sum_{j=1}^n\frac{\partial f}{\partial y_j}\Big|_b\cdot\frac{\partial y_j}{\partial x_i}\Big|_a \ ;$$

therefore, if α is the map given by the equations $y_j=P_j(x_1,\dots,x_m)$, $j=1,\dots,n$, the chain rule implies

$$d(f\circ\alpha)_a=(df)_b\circ(d\alpha)_a \quad , \quad b=\alpha(a).$$

Let $t\in T_a(V)$; then

$$(df)_b\circ(d\alpha)_a(t)=d(f\circ\alpha)_a(t)$$
$$=0 \quad , \quad \text{for } f\in J \text{ and hence } f\circ\alpha\in I.$$

$\therefore$ $(d\alpha)_a(t)\in T_b(W)$, because $(df)_b$ kills it. ∎

As a consequence we have a map $(d\alpha)_a: T_a(V)\to T_b(W)$. The usual rules from advanced calculus and differential geometry hold; e.g.

$$(d\beta)_b\circ(d\alpha)_a = d(\beta\circ\alpha)_a \quad , \quad b=\alpha(a).$$

(D.3) Intrinsic definition of the tangent space.

The definition of a tangent space as given above requires the variety to be embedded in an affine (ambient) space. We now give a definition which is independent of such embeddings and hence intrinsic to the variety itself.

Let $x_1,\dots,x_n$ be commuting variables. A linear form in these variables is a linear combination $\sum c_ix_i$, $c_i\in k$. Linear forms constitute a vector space of dim. $=n$, which is naturally dual to k^n.

Lemma 4. Let $L\triangleleft k[x_1,\dots,x_n]$ be an ideal generated by linear forms $l_1,\dots,l_r$, which we may assume to be linearly independent. Let $x_{i_1},\dots,x_{i_{n-r}}$ be such that $\{l_1,\dots,l_r,x_{i_1},\dots,x_{i_{n-r}}\}$ is a basis for linear forms in $x_1,\dots,x_n$. Then

$$k[x_1,\dots,x_n]/L \cong k[x_{i_1},\dots,x_{i_{n-r}}].$$

Proof. This is obvious if the linear forms $l_1,\dots,l_r$ are $x_1,\dots,x_r$. In general case since $\{x_1,\dots,x_n\}$ and

$\{l_1, \dots, l_r, x_{i_1}, \dots, x_{i_{n-r}}\}$ are both bases of the linear forms, each element of one set is a linear combination of the elements of the second set. Therefore,

$$k[x_1, \dots, x_n] = k[l_1, \dots, l_r, x_{i_1}, \dots, x_{i_{n-r}}]$$

and so

$$k[x_1, \dots, x_n]/L = k[l_1, \dots, l_r, x_{i_1}, \dots, x_{i_{n-r}}]/(l_1, \dots, l_r)$$

$$\cong k[x_{i_1}, \dots, x_{i_{n-r}}]. \quad \blacksquare$$

Let $V = \mathbb{Z}(I) \subset k^n$, and suppose $P = (0, 0, \dots, 0) \in V$. Since V contains the origin, every $f \in I$ has a constant term equal zero. Let I_ℓ be the ideal generated by the linear terms f_ℓ, $f \in I$. It follows that

$$T_P(V) = \mathbb{Z}(I_\ell) \quad \leftarrow (9)$$

which gives the tangent space to V at the origin as a subvariety $T_P(\mathbb{A}^n) \cong \mathbb{A}^n$. Let

$$A_\ell := k[x_1, \dots, x_n]/I_\ell = k[T_P(V)] \quad \leftarrow (10)$$

and let $\mathfrak{m}$ be the maximal ideal in $A = k[V] := k[x_1, \dots, x_n]/I$, corresponding to the origin; i.e.

$$\mathfrak{m} = (\bar{x}_1 - 0, \dots, \bar{x}_n - 0) = (\bar{x}_1, \dots, \bar{x}_n).$$

<u>Proposition 5.</u> There are canonical isomorphisms

$$\mathrm{Hom}_{k\text{-lin}}(\mathfrak{m}/\mathfrak{m}^2, k) \cong \mathrm{Hom}_{k\text{-alg}}(A_\ell, k) \cong T_P(V). \quad \leftarrow (11)$$

<u>Proof.</u> <u>1st isomorphism.</u> Let $\mathfrak{J} = (x_1, \dots, x_n)$ be the maximal ideal at the origin in $k[x_1, \dots, x_n]$, so $\mathfrak{m} = J/I$. We have, as in proposition 2, that

$$\mathfrak{m}/\mathfrak{m}^2 = (x_1, \dots, x_n)/(x_i^2, x_i x_j, I)$$

$$= J/(J^2 + I), \quad \text{where } J^2 = (x_i^2, x_i x_j \mid i, j = 1, \dots, n); \; J = (x_1, \dots, x_n).$$

$$= J/(J^2 + I_\ell), \quad \text{because } f - f_\ell \in J^2, \; \forall f \in I.$$

Let $f_{1,\ell}, \dots, f_{r,\ell}$ be a basis for the vector space I_ℓ; there are $n-r$ indeterminates $x_{i_1}, \dots, x_{i_{n-r}}$ (see the lemma 4) such that they form together with $f_{i,\ell}$ a basis for the linear forms on k^n. Then $x_{i_1}, \dots, x_{i_{n-r}}$ forms a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space (note that there is an action of $A/\mathfrak{m} \cong k$ on $\mathfrak{m}/\mathfrak{m}^2$).

Now, lemma 4 shows that $A_\ell = k[x_{i_1}, \dots, x_{i_{n-r}}]$. But then any homomorphism $\alpha : A_\ell \to k$ of k-algebras is determined by its values $\alpha(x_{i_1}), \dots, \alpha(x_{i_{n-r}})$ and these can be arbitrarily chosen. Since a k-linear map $\mathfrak{m}/\mathfrak{m}^2 \to k$ has a similar description, the first isomorphism is now obvious.

<u>2nd isomorphism.</u> To specify a k-algebra homomorphism $A_\ell \to k$ is the same as to give an element $(a_1, \dots, a_n) \in k^n$ s.t. $f(a_1, \dots, a_n) = 0$ for all $f \in I_\ell$. But this is the same as to give an element of $T_P(V)$. $\blacksquare$

<u>Lemma 6.</u> Let $\mathfrak{m}$ be a maximal ideal of a unital ring A, and let $\mathfrak{n} := \mathfrak{m} A_\mathfrak{m}$. For all $q \in \mathbb{N}^+$ the map

$$f_q : \begin{cases} A/\mathfrak{m}^q \longrightarrow A_\mathfrak{m}/\mathfrak{n}^q \\ a + \mathfrak{m}^q \longmapsto a/1 + \mathfrak{n}^q \end{cases}$$

is an isomorphism. Moreover, it induces the following isomorphisms

$$m^r/m^q \xrightarrow{\cong} n^r/n^q$$

for all $r<q$.

Proof (sketchy). The first statement follows by showing that the map is well-defined, it is injective and surjective. The second statement follows from the 3-lemma applied to the 2-row exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & m^r/m^q & \to & A/m^q & \to & A/m^r & \to & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & n^r/n^q & \to & A_m/n^q & \to & A_m/n^r & \to & 0 \end{array}$$

∎

As a consequence of proposition 5 and lemma 6 we have the following canonical isomorphism

$$T_p(V) \cong \mathrm{Hom}_{k\text{-lin}}\left(n_p/n_p^2, k\right) \qquad \leftarrow (12)$$

where n_p is the maximal ideal of the local ring $\mathcal{O}_p := A_{m_p}$ (i.e. $n_p = m_p A_{m_p}$, $A := k[V]$).

Def. (Intrinsic def. of the tangent space)

The tangent space $T_p(V)$ at a point P of a variety is $\mathrm{Hom}_{k\text{-lin}}\left(\frac{n_p}{n_p^2}, k\right)$, where n_p is the maximal ideal in $\mathcal{O}_p$. ∎

This new definition has the advantage of being local, i.e., it depends only on a (small) neighborhood of $P\in V$; in particular it is independent of any affine embedding of V.

Let $\alpha: V \to W$ be a regular map of affine varieties sending $P\in V$ to $Q\in W$. This defines a local

homomorphism $\mathcal{O}_Q \to \mathcal{O}_P$ which induces maps $m_Q \to m_P$, $m_Q/m_Q^2 \to m_P/m_P^2$ and $T_p(V) \to T_Q(W)$. The last map is written $(d\alpha)_P$.

When some open neighborhoods of P and Q are realized as closed subvarieties of affine spaces, then $(d\alpha)_P$ becomes identified with the map $(d\alpha)_P$ defined earlier (see equation (8)).

In particular, if $f\in m_P$, then f is represented by a regular map $U \to \mathbb{A}^1$, $P \mapsto 0$, and defines (as before) a linear map $(df)_P : T_p(V) \to k$. This is just the map sending a tangent vector (i.e. an element of $\mathrm{Hom}_{k\text{-lin}}(m_P/m_P^2, k)$) to its value at $f \bmod (m_P^2)$:

$$\forall f\in m_P,\ \forall h\in T_p(V) := \mathrm{Hom}_{k\text{-lin}}(m_P/m_P^2, k):$$
$$(df)_P : h \longmapsto h(f|_{\mathrm{mod}\,(m_P^2)}).$$

In general, for $f\in\mathcal{O}_P$, the local ring at P, we define

$$(df)_P := f - f(P) \mod (m_P^2). \qquad \leftarrow (13)$$

The vector space generated by all such differentials at P is dual to the tangent space at P.

As an example let $P=(p_1,\dots,p_n)\in V = Z(I) \subset \mathbb{A}^n$, where I is a radical ideal. For every $f\in k[\mathbb{A}^n] = k[x_1,\dots,x_n]$, we have (a trivial Taylor expansion)

$$f = f(P) + \sum_{i=1}^n c_i(x_i - p_i) + \text{terms of degree} \geq 2 \text{ in } (x_i - p_i)$$

$$\therefore\ f - f(P) \equiv \sum_{i=1}^n c_i(x_i - a_i) \mod (m_P^2),$$

for $m_P = (x_1 - p_1, \dots, x_n - p_n)$, $m_P^2 = \left((x_i-p_i)^2, (x_i-p_i)(x_j-p_j) \mid i,j = 1,\dots,n\right)$.

Therefore, $(df)_P$ given by (13) can be identified with

$$(df)_P := f - f(P)|_{\mathrm{mod}\,(m_P^2)} = \sum_i c_i(x_i - p_i) = \sum_i \frac{\partial f}{\partial x_i}\Big|_P (x_i - p_i)$$

which is how we originally defined the differential in ⑥.

(Remark. The same discussion applies to $f \in \mathcal{O}_p$. Such an f is of the form g/h with $h(p) \neq 0$ and has, not a quite trivial, Taylor expansion of the same form, but with an infinite number of terms, i.e., it lies in the power series ring $k[[x_1 - p_1, \dots, x_n - p_n]]$.)

From these considerations and what we had said earlier about the linear equations defining $T_p(V)$, where $V = Z(I)$, we see that $T_p(V)$ is given by the equations

$$(df)_p = 0, \quad \forall f \in I. \qquad \leftarrow (14)$$

If $I = (f_1, \dots, f_m)$ and $p \in V = Z(I)$, then (14) implies

$$\forall i: (df_i)_p := f_i - f_i(p) \mod (m_p^2)$$
$$= f_i \mod (m_p^2), \quad \text{for } f_i(p) = 0, \text{ as } p \in Z(I)$$

$\therefore$ $f_i \mod (m_p^2) = 0$, $i = 1, \dots, m$, give the equation of tangent space at $p \in Z(I) = V$. The set

$$\{ (df)_p |_{T_p(V)} \mid f \in k[x_1, \dots, x_n] \}$$
$$= \{ (df)_p := f - f(p) |_{\text{mod } (m_p^2)} \mid f \in m_p \}$$

is the dual space to $T_p(V)$. ∎

(D.4) Derivations, dual numbers and yet another definition of the tangent space.

When an affine variety is looked upon as a functor, we need the concept of a tangent vector to such functors to be able to define tangent spaces. This is what we shall do now.

Def. Let K be a k-algebra. We define the K-algebra of dual numbers by

$$K[\varepsilon] := K[t]/(t^2), \quad \varepsilon = \bar{t} := t \mod (t^2). ∎$$

As a K-vector space $K[\varepsilon]$ has as a basis the set $\{1, \varepsilon\}$ and every element of $K[\varepsilon]$ has a form $a + b\varepsilon$, $a, b \in K$. In $K[\varepsilon]$ we add elements coordinate-wise and multiply them using $\varepsilon^2 = 0$, so $(a+b\varepsilon)(a'+b'\varepsilon) = aa' + (ab' + a'b)\varepsilon$. This makes $K[\varepsilon]$ into a ring (or an algebra) called the K-algebra of dual number, as defined above.

Consider the natural homomorphism $\pi_1: \begin{cases} K[\varepsilon] \to K \\ a + b\varepsilon \mapsto a. \end{cases}$ Clearly, $\operatorname{Ker}(\pi_1) = (\varepsilon)$ is the ideal $(\varepsilon) = \{ b\varepsilon \mid b \in K \} \cong K$. This is a unique maximal ideal; hence $K[\varepsilon]$ is a local ring.

Def. Let $F: Alg_k \to Sets$ be a functor from the category of algebras to the category of sets. Let $x \in F(K)$ be a K-point of F. A tangent vector t_x of F at x is a $K[\varepsilon]$-point $t_x \in F(K[\varepsilon])$ such that

$$F(\pi_1)(t_x) = x. \qquad \leftarrow (15)$$

The set of tangent vectors of F at x is denoted by $T(F)_x$ and is called the tangent space of F at x. ∎

Example. Let $X: Alg_k \to Sets$ be an affine algebraic variety given by a system of algebraic equations

$$S: \; f_1(y_1, \dots, y_n), \; \dots, \; f_m(y_1, \dots, y_n).$$

A point $x \in X(K)$ is a solution $(a_1, \dots, a_n) \in K^n$ of this system. A tangent vector t_x is a solution

$$(a_1 + b_1\varepsilon, \dots, a_n + b_n\varepsilon) \in X(K[\varepsilon])$$

of the same system. For every $i = 1, \dots, m$ we have (a trivial Taylor expansion)

$$0 = f_i(a_1+b_1\varepsilon, \dots, a_n+b_n\varepsilon) = f_i(a_1,\dots,a_n) + \sum_{j=1} \alpha_j^{(i)} b_i \varepsilon + \sum_{j,k=1}^{n} \alpha_{jk}^{(i)} b_i b_j \varepsilon^2 + (\dots)\varepsilon^3 + \dots =$$

$$= \sum_{j=1}^{n} \alpha_j^{(i)} b_i \varepsilon \quad , \quad \text{for } \varepsilon^2 = 0 \text{ and } f_i(a_1,\dots,a_n) = 0 \text{ since } (a_1,\dots,a_n) \in X(K).$$

But the coefficients $\alpha_j^{(i)}$ are the partial derivatives $\frac{\partial f_i}{\partial y_j}(x)$, $x = (a_1,\dots,a_n)$. We therefore deduce that $(b_1,\dots,b_n)$ satisfies the system of linear homogeneous equations

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}(x)\, b_i = 0 \quad , \quad i = 1,\dots,m. \qquad (*)$$

This shows that the set of tangent vectors $T(X)_x$ is bijective to the linear space of solutions of $(*)$. In particular this introduces the structure of a K-module on $T(X)_x$. ∎

In the above example, however, it is not so clear that the K-module structure of $T(X)_x$ is independent of a choice of the system of equations defining X. To overcome this difficulty we shall now give a more invariant definition of $T(X)_x$.

Let A be a commutative k-algebra and M be an A-module. A map $\delta: A \to M$ is called a M-derivation of A if the following relations hold:

$$\begin{cases} \delta(ab) = a\cdot\delta(b) + b\cdot\delta(a) & \text{(Leibnitz rule)} \quad \forall a,b \in A \\ \delta(a+b) = \delta(a) + \delta(b) \\ \delta(c) = 0, & \forall c \in k \end{cases}$$

These conditions imply that δ is k-linear (but not A-linear). Let us write

$$Der_k(A,M) = \{\delta: A \to M \mid \delta \text{ is a k-derivation}\}.$$

This set has a natural structure of an A-module via

$$(a\delta)(b) := a\,\delta(b) \quad , \quad \forall a,b \in A.$$

Lemma 7. Let $A \to B$ be a homomorphism of k-algebras, and $\delta: B \to M$ be a M-derivation of B. Then the composition $\delta \circ f: A \to B \to M$ is a $M_{[f]}$-derivation of A, where $M_{[f]}$ is the A-module obtained from M by restriction of scalars (i.e., $a.m = f(a).m$, $\forall a \in A$, $\forall m \in M$).

Proof. Easy verification. ∎

We now apply this result to the situation under consideration. Notice that the k-linear map

$$\pi_2: \begin{cases} K[\varepsilon] \to K \\ a+b\varepsilon \mapsto b \end{cases}$$

is a K-derivation of $K[\varepsilon]$ considered as a K-algebra; K can be considered as a $K[\varepsilon]$-module via the homomorphism $\pi_1: K[\varepsilon] \to K$, $a+b\varepsilon \mapsto a$.

Let us consider the functor $F := Hom_{k\text{-alg}}(\mathcal{O}(X), -)$. We can use the following identification

$$X(K) \ni x \xleftrightarrow{1:1} ev_x \in Hom_{k\text{-alg}}(\mathcal{O}(X), K) \qquad \leftarrow (16)$$

which implies, using def. (15) of t_x, that the tangent vector $t_x \in T(X)_x$ can be identified with the k-algebra homomorphism

$$t_x: \mathcal{O}(X) \to K[\varepsilon] \quad \left(\text{i.e. } t_x \in Hom_{k\text{-alg}}(\mathcal{O}(X), K[\varepsilon])\right)$$

such that its composition with π_2 is a K-derivation of $\mathcal{O}(X)$. Here K is considered as a $\mathcal{O}(X)$-module via the homomorphism ev_x. This establishes a map

$$T(X)_x \xrightarrow{f} Der_k(\mathcal{O}(X), K)_x$$

where the subscript x reminds us about the structure of $\mathcal{O}(X)$-module on K. By definition

$$Der_k(\mathcal{O}(X),K)_x = \{\delta \in Hom_k(\mathcal{O}(X),K) \mid \delta(pq) = p(x)\delta(q)+q(x)\delta(p),\ \forall p,q\in\mathcal{O}(X)\}.$$

Lemma 8. The map

$$T(X)_x \xrightarrow{f} Der_k(\mathcal{O}(X),K)_x$$

is a bijection.

Proof. Let $\delta \in Der_k(\mathcal{O}(X),K)_x$. Define a mapping

$$\alpha_\delta : \begin{cases} \mathcal{O}(X) \longrightarrow K[\varepsilon] \\ p \longmapsto p(x) + \varepsilon\,\delta(p) \end{cases}$$

It follows that α_δ is a homomorphism of k-algebras and its composition with $\pi_1 : K[\varepsilon] \longrightarrow K$ is equal to ev_x ($\equiv x$ by (16)). Thus α_δ defines a tangent vector at x and $\delta \longmapsto \alpha_\delta$ is the inverse of f. ∎

Therefore, we have proved

$$T(X)_x \xrightarrow{\cong} Der_k(\mathcal{O}(X),K)_x \qquad \leftarrow (17)$$

As we said earlier $Der_k(\mathcal{O}(X),K)_x$ has a structure of a K-module by $(a\delta)(p) = a\,\delta(p)$, $\forall a \in K$, $\forall p \in \mathcal{O}(X)$. We can transfer this structure to $T(X)_x$ by means of the bijection (17). This K-module structure on $T(X)_x$ is obviously independent of the choice of equations defining X. ∎

Let us now specialize to the case $K=k$. For any point $x \in X(k)$ (a rational point of X) the kernel of $x (\equiv ev_x) : \mathcal{O}(X) \longrightarrow k$ is a maximal ideal $m_x \lhd \mathcal{O}(X)$ and $\mathcal{O}(X)/m_x \cong k$. Let $\delta \in Der_k(\mathcal{O}(X),k)_x$ be a k-derivation of $\mathcal{O}(X)$. Then

$$\forall p,q \in m_x : \quad \delta(p\cdot q) = p(x)\,\delta(q) + q(x)\,\delta(p) = 0$$

therefore, the restriction of δ to m_x^2 is identically zero. The following lemma establishes the link between the present approach to our previous approach of the tangent space.

Lemma 9. The map

$$\begin{cases} Der_k(\mathcal{O}(X),k)_x \longrightarrow Hom_k\left(\frac{m_x}{m_x^2}, k\right) \\ \delta \longmapsto \delta|_{m_x} \end{cases} \qquad \leftarrow (18)$$

defines an isomorphism of k-linear spaces

$$T(X)_x \xrightarrow{\cong} Hom_k\left(\frac{m_x}{m_x^2}, k\right).$$

Proof. $\mathcal{O}(X)/m_x^2$ has a natural structure of a k-algebra, hence there exists a canonical homomorphism $k \hookrightarrow \mathcal{O}(X)/m_x^2$ and its composition with the algebra epimorphism $\mathcal{O}(X)/m_x^2 \longrightarrow \mathcal{O}(X)/m_x \cong k$ is the identity on k. If we identify k with the subring of $\mathcal{O}(X)/m_x^2$ by means of the above embedding, then the restriction of the algebra map $\mathcal{O}(X)/m_x^2 \longrightarrow \mathcal{O}(X)/m_x$ is just the identity map on k.

For every $p \in \mathcal{O}(X)$ let us denote

$$p_x := \bar{p} = p + m_x^2 \in \mathcal{O}(X)/m_x^2.$$

clearly,

$$(p - p(x))(x) = p(x) - p(x) = 0 \ \Longrightarrow\ p - p(x) \in m_x \ ;$$
$$\Longrightarrow\ p_x - p(x) \in m_x/m_x^2.$$

Let us define a map $f \longmapsto \delta_f$ according to

$$\forall f \in Hom_k\left(\frac{m_x}{m_x^2}, k\right),\ \forall p \in \mathcal{O}(X) : \quad \delta_f(p) := f(p_x - p(x)).$$

We show that this is a k-derivation of $\mathcal{O}(X)$. Obviously one has

$$\forall p,q \in \mathcal{O}(X) : \quad (p_x - p(x))(q_x - q(x)) \in m_x^2,$$
$$\therefore\ f[(p_x - p(x))(q_x - q(x))] = 0.$$

We can, therefore, write

$$\begin{aligned}\delta_f(pq) &= f(p_x q_x - p(x)\,q(x)) \\ &= f[(p_x - p(x))(q_x - q(x)) + p(x)(q_x - q(x)) + q(x)(p_x - p(x))] \\ &= f[p(x)(q_x - q(x))] + f[q(x)(p_x - p(x))] \\ &= p(x)\,f(q_x - q(x)) + q(x)\,f(p_x - p(x)) \\ &= p(x)\,\delta_f(q) + q(x)\,\delta_f(p)\end{aligned}$$

$\Rightarrow \delta_f \in \mathrm{Der}_k(\mathcal{O}(X), k)_x$.

It is now easy to show that $f \longmapsto \delta_f$ is the inverse of the map (18). ∎

Finally let $f : F \longrightarrow G$ be a morphism of two functors from Alg_k to Sets. Let $x \in F(K)$ and $y = f(K)(x) \in G(K)$. By functoriality $f(K[\varepsilon]) : F(K[\varepsilon]) \longrightarrow G(K[\varepsilon])$ induces a natural map

$$(df)_x : T(F)_x \longrightarrow T(G)_y.$$

This is called the differential of f at the point x. If $f : X \to Y$ is a morphism of affine k-varieties corresponding to the homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ of k-algebras, $x \in X(k)$, $y = f(k)(x) \in Y(k)$, then it is straight forward to verify that the differential $(df)_x$ coincides with the transpose of the linear map $m_y/m_y^2 \longrightarrow m_x/m_x^2$ induced by the homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. It also follows from the definition of the functors that the following result holds.

Proposition 10. (chain rule)
Let $f : F \to G$, $g : G \longrightarrow H$ be morphisms of functors from Alg_k to Sets. Let $x \in F(K)$ and $y = f(K)(x) \in G(K)$. Then

$$d(g \circ f)_x = (dg)_y \circ (df)_x. \quad ∎$$

(D.5) Dimension of affine varieties and of tangent spaces

First we shall give an algebraic characterization of the dimension of affine varieties. Let V be an irreducible algebraic set. Then $\mathbb{I}(V)$ is a prime ideal and so $k[V]$ is an integral domain. Let $k(V)$ be its

field of fractions, usually called the field of rational functions on V. The dimension of V is defined to be the transcendence degree of $k(V)$ over k.

We shall clarify this statement. Let $\Omega \supset k$ be fields (e.g. $\mathbb{C} \supset \mathbb{Q}$). Elements $\alpha_1, \dots, \alpha_n \in \Omega$ are said to be algebraically dependent over k if they satisfy an algebraic relation with coefficients in k, i.e., if $\exists f \in k[x_1, \dots, x_n]$ s.t. $f(\alpha_1, \dots, \alpha_n) = 0$. Otherwise these elements are said to be algebraically independent. An infinite set $A \subset \Omega$ is algebraically independent if every finite subset of A is algebraically independent.

If A is a subset of Ω, we write $k(A)$ for the field generated by A over k, i.e., the smallest subfield of Ω containing k and A. It consists of all quotients of polynomials in elements of A with coefficients in k.

Def. A transcendence basis for Ω over k is an algebraically independent set $A \subset \Omega$ such that Ω is algebraic over $k(A)$. ∎

It can be shown that if Ω has a finite transcendence basis over k, then all transcendence bases are finite and have the same number of elements. It can also be shown that any (possibly infinite) transcendence bases for Ω over k have the same cardinality. The cardinality of a transcendence basis for Ω over k is called the transcendence degree of Ω over k and it is denoted by $\mathrm{tr\,deg}_k \Omega$.

Let V be an irreducible algebraic set. Then, as said above, $k[V]$ is an integral domain and $k(V)$ the field of fractions of $k[V]$. We have defined

$$\dim(V) := \operatorname{tr\,deg}_k k(V). \qquad \leftarrow (19)$$

Examples. (1) $V = k^n (\equiv \mathbb{A}^n_k)$. Then $k(V) = k(x_1, \ldots, x_n)$ and so $\dim(V) = n$.

(2) An irreducible algebraic set has dimension zero iff it consists of a single point. This is because $k(V)$ has transcendence degree 0 over k iff it equals k, k being algebraically closed. Thus

$$k \subset k[V] \subset k(V) = k \implies k[V] \text{ is a field}$$
$$\implies \mathbb{I}(V) \text{ is a maximal ideal}$$
$$\implies V \text{ is a single point. } \blacksquare$$

By a hypersurface in k^n we mean $\mathbb{Z}(f(x_1, \ldots, x_n))$ where $f(x_1, \ldots, x_n)$ is a non-zero non-constant polynomial.

Proposition 11. An irreducible hypersurface in k^n has dimension $n-1$.

Proof. Let $k[\bar{x}_1, \ldots, \bar{x}_n] = k[x_1, \ldots, x_n]/(f)$ and let $k(\bar{x}_1, \ldots, \bar{x}_n)$ be the field of fractions of $k[\bar{x}_1, \ldots, \bar{x}_n]$. Because $\bar{x}_1, \ldots, \bar{x}_n$ generate $k(\bar{x}_1, \ldots, \bar{x}_n)$ and they are algebraically dependent, the transcendence degree must be $< n$. To see it is not $< n-1$ note that if x_n occurs in f, then it occurs in all non-zero multiples of f and so no non-zero polynomial in $x_1, \ldots, x_{n-1}$ belongs to (f). $\implies \bar{x}_1, \ldots, \bar{x}_{n-1}$ are algebraically independent. $\blacksquare$

For a reducible algebraic set V, we define the dimension of V to be the maximum of the dimensions of its irreducible components. When these all have the same dimension d, we say that V has pure dimension d.

Proposition 12. If V is irreducible and Z is a proper closed subvariety of V, then $\dim(Z) < \dim(V)$.

Proof. Suppose that Z is irreducible. Then Z corresponds to a prime ideal $J \lhd k[V]$ and $k[Z] = k[V]/J$. Suppose $V \subset k^n$, so that $k[V] = k[x_1, \ldots, x_n]/\mathbb{I}(V) \cong k[\bar{x}_1, \ldots, \bar{x}_n]$. If x_i is regarded as a function on k^n then $\bar{x}_i$, its image in $k[V]$, is the restriction of this function to V. If $f \in k[V]$, the image $\bar{f}$ of f in $k[Z]$ is the restriction of f to Z. With this notation, $k[Z] = k[\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_n]$. Suppose $\dim(Z) = d$ and that $\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_d$ are algebraically independent. We show for any non-zero $f \in J$ the $d+1$ elements $\{\bar{x}_1, \ldots, \bar{x}_d, f\}$ are algebraically independent which implies $\dim(V) \geq d+1$. Suppose otherwise. Then there is a non-trivial algebraic relation among the $\bar{x}_i$ and f, which we can write as

$$a_0(\bar{x}_1, \ldots, \bar{x}_d) f^m + a_1(\bar{x}_1, \ldots, \bar{x}_d) f^{m-1} + \cdots + a_m(x_1, \ldots, x_d) = 0$$

with $a_i(\bar{x}_1, \ldots, \bar{x}_d) \in k[\bar{x}_1, \ldots, \bar{x}_d]$. Because this relation is non-trivial, at least one of a_i is non-zero. After cancelling by a power of f if necessary, we can assume $a_m(\bar{x}_1, \ldots, \bar{x}_d) \neq 0$ (we have used the fact that $k[V]$ is an integral domain.) On restricting the functions in the above equation to Z, i.e., applying the homomorphism $k[V] \to k[Z]$, we get $a_m(\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_d) = 0$ which contradicts the algebraic independence of $\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_d$. $\blacksquare$

Example. Let $F(x,y)$ and $G(x,y)$ be non-constant polynomials with no common factor. Then $\mathbb{Z}(F(x,y))$ has dimension 1 (by proposition 11) and so

$Z(F(x,y)) \cap Z(G(x,y))$ must have dimension zero; it is therefore a finite set. ∎

Topological characterization of dimension. We start with the observation that the dimension of a linear space V can be defined as follows:

let $V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_d \neq \{0\}$
be a maximal (i.e. it cannot be refined) strictly decreasing chain of linear subspaces. Then $\dim(V) = d$.

Let V be a nonempty topological space. Its Krull dimension is defined as follows:

let $V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_r \neq \emptyset$
be a maximal strictly decreasing chain of closed irreducible subsets of V. Then the Krull dimension of V is equal to r. By definition $\dim \emptyset = -\infty$.

Clearly if V is a non-empty Hausdorff space, then $\dim(V) = 0$ for a point is the only closed irreducible subset of such a space. We can now state the following

Def. The dimension of an algebraic set V is the Krull dimension of the corresponding topological space. ∎

We shall elaborate a little on this definition. The following theorem suggests that when we impose additional polynomial conditions on an algebraic set, the dimension does not go down more than what linear algebra would suggest.

Theorem 13. (Geometric Krull's Hauptidealsatz) Let V be an irreducible affine variety and let $f \in k[V]$. If f is a non-zero non-unit in $k[V]$ then every

irreducible component of $Z(f)$ has dimension equal to $\dim(V) - 1$; i.e., $Z(f)$ has a pure dimension equal to $\dim(V) - 1$. ∎

There is also the following corollary which still shows a similar analogy with linear algebra situation.

Corollary 14. For any irreducible variety V and functions $f_1, \ldots, f_r \in k[V]$, the irreducible components of $Z(f_1, \ldots, f_r)$ have dimension $\geq n - r$. ∎

It also follows from theorem 13 that if V is an irreducible variety and Z is a maximal proper irreducible subset of V, then $\dim(Z) = \dim(V) - 1$.

Corollary 15. (Topological characterization of dimension) Suppose V is irreducible and that

$$V \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_d \neq \emptyset$$

is a maximal chain of closed irreducible subsets of V. Then $\dim(V) = d$.

Proof. By what we said above we have

$$\dim(V) = \dim(V_1) + 1 = \dim(V_2) + 2 = \cdots = \dim(V_d) + d = d.$$ ∎

Def. For every commutative ring A its Krull dimension is defined by:

let $(0) \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_d \subsetneq A$
be a maximal chain of prime ideals of A. Then $d = \text{Kdim}(A)$. ∎

Proposition 16. Let V be an affine algebraic k-set and $A = k[V]$ its coordinate ring. Then

$$\dim(V) = \text{Kdim}(A).$$

Proof. This immediately follows from the existence of of the 1:1 correspondence between closed irreducible subsets of V and prime ideals in $k[V] = A$. ∎

For example $\dim(\mathbb{A}^n_k) = n$; for by proposition 16 we must show $K\dim(k[x_1,\dots,x_n]) = n$. Clearly

$$0 \subsetneq (x_1) \subsetneq (x_1,x_2) \subsetneq \cdots \subsetneq (x_1,\dots,x_n) \subsetneq k[x_1,\dots,x_n]$$

is a strictly increasing chain of proper prime ideals of $k[x_1,\dots,x_n]$; hence the required result. ∎

Dimension of the tangent space and system of local parameters.

Def. Let V be an algebraic k-set and $T_x(V)$ the tangent space to V at $x \in V$. Since $n_x \triangleleft \mathcal{O}_x$ is finitely generated the linear space n_x/n_x^2 is finite dimensional. We define

$$\dim_k T_x(V) = \dim_k \left(\frac{n_x}{n_x^2}\right)^* = \dim_k \left(\frac{n_x}{n_x^2}\right). \qquad \leftarrow (20)$$

∎

Def. Let V be an algebraic k-set. The elements $f_1,\dots,f_n \in n_x \triangleleft \mathcal{O}_x$ are said to be a set of <u>local parameters</u> at x if their residue modulo n_x^2 form a basis of the linear space n_x/n_x^2 over the field $k(x) = \mathcal{O}_x/n_x$. ∎

We will see that a set of local parameters is a generating set for the maximal ideal n_x and in particular if x is a non-singular point of an algebraic set V then $n_x \triangleleft \mathcal{O}_x$ can be generated by $n = \dim_x(V)$ elements. Clearly, the residues mod (n_x^2) of every set of generators of n_x span n_x/n_x^2. Thus the maximal number of generators of n_x does not exceed n and it is equal to n when x is a non-singular point.

Lemma 17. (Nakayama)

Let A be a local Noetherian ring with maximal ideal m, and let M be a finitely generated A-module. If $M = N + mM$ for some submodule N of M, then $M = N$.

Proof. Replacing M by the factor module M/N, we may assume that $N = 0$. Let $f_1,\dots,f_r$ be a set of generators of M. Since $mM = M$ we may write

$$f_i = \sum_{j=1}^{r} a_{ij} f_j \quad , \; i = 1,\dots,r \; , \; a_{ij} \in m .$$

Therefore, $f_1,\dots,f_r$ can be considered to be the solution of the system of equations in r variables

$$\sum_{j=1}^{r} (\delta_{ij} - a_{ij}) f_j = 0 \quad ;$$

then by Cramer's rule

$$\det(\delta_{ij} - a_{ij}) \cdot f_i = 0 \quad , \; \forall i = 1,\dots,r. \qquad (*)$$

However, $\det(\delta_{ij} - a_{ij}) = 1 + q$, $q \in m$. Hence $\det(\delta_{ij} - a_{ij}) \notin m$ and so it is a unit. Then (*) implies $f_i = 0$ for all i. $\Rightarrow M = 0$. ∎

Corollary 18. Let A be a local Noetherian ring and m be its maximal ideal. Elements $f_1,\dots,f_r$ generate m iff their residue modulo m^2 span m/m^2 as a vector space over $k = A/m$. In particular the number of generators of the maximal ideal m is equal to the dimension of the vector space m/m^2.

Proof. Let $M = m$, $N = (f_1,\dots,f_r)$. Since A is Noetherian, M is a finitely generated A-module and N is its submodule. By assumption, $M = mM + N$. By Nakayama lemma $M = N$. ∎

Proposition 19. Let V be an algebraic k-set, $x \in V$, and $f_1,\dots,f_r \in n_x \triangleleft \mathcal{O}_x$. The following statements are equivalent:

(i) $f_1, \dots, f_r$ is a set of local parameters;
(ii) $f_1, \dots, f_r$ generate n_x and $r = \dim T_x(V)$;
(iii) $f_1, \dots, f_r$ is a minimal set of generators of n_x.

Proof. (i) $\Rightarrow$ (ii): By def. $r = \dim_{k(x)} \frac{n_x}{n_x^2}$. It follows from Corollary 18 that $f_1, \dots, f_r$ generate n_x.

(ii) $\Rightarrow$ (iii): If $\{f_1, \dots, f_r\}$ contains a proper subset of generators of n_x, then the cosets of the elements of this subset span n_x/n_x^2. Thus the dimension of n_x/n_x^2 is less than $r = \dim_{k(x)} n_x/n_x^2$ which is absurd.

(iii) $\Rightarrow$ (i): The cosets $(f_i)_x := f_i \bmod (n_x^2)$ span n_x/n_x^2. It suffices to show they are linearly independent. Assume that this is not true. Then a proper subset of $\{(f_1)_x, \dots, (f_r)_x\}$ span n_x/n_x^2. By Corollary 19 this proper subset of $\{(f_1)_x, \dots, (f_r)_x\}$ generate n_x. This contradicts the minimality of $\{f_1, \dots, f_r\}$. ∎

Def. A Noetherian local ring A is called regular if its maximal ideal m is generated by $n = \dim(A)$ elements. ∎

Corollary 20. Let V be an algebraic k-set and $x \in V$ such that $r = \dim_x(V)$. The following properties are equivalent:

(i) x is a non-singular point;
(ii) any set of local parameters at x consists of r elements;
(iii) the minimal number of generators of n_x is equal to r.
(iv) the local ring $\mathcal{O}_x$ is regular. ∎

Finally we quote the following result

Theorem 21. Let V be irreducible; then

$$\dim T_x(V) \geq \dim(V)$$

and equality holds iff $\mathcal{O}_x$ is regular. ∎

This implies at any singular point $x \in V$ $\dim T_x(V) > \dim_x(V)$ and at any non-singular point x $\dim T_x(V) = \dim_x(V)$.

Examples. (1) Let $V = \mathbb{Z}(x^3+y^2)$; so $a = (0,0) \in V$. The maximal ideal $m_a = (x,y)/(x^3+y^2) = (\bar{x}, \bar{y})$ is generated by two elements over $k[V]$. Therefore, n_a is also generated by two elements over $\mathcal{O}_a = k[V]_{m_a}$ (so it is not a principal ideal). This result was to be expected since $a = (0,0)$ is a singular point of $x^3+y^2=0$ and hence

$$\dim T_a(V) = 2 > \dim_a(V) = 1.$$

(2) Let $x = (a_1, \dots, a_n) \in V = \mathbb{A}^n_k$. The germ of polynomials $z_i - a_i$, $i = 1, \dots, n$, form a set of local parameters at the point x. For any polynomial $F(z_1, \dots, z_n)$ we can write

$$F(z_1, \dots, z_n) = F(x) + \sum \frac{\partial F}{\partial z_i}(x)(z_i - a_i) + G(z_1, \dots, z_n) \quad (*)$$

where $G(z_1, \dots, z_n) \in m_x^2$. Thus the cosets

$$dz_i := (z_i - a_i) \bmod (n_x^2)$$

form a basis of the linear space n_x/n_x^2 and (it follows from (*) that) the germ $F_x - F(x)$ is a linear combination of dz_i:

$$F_x - F(x) := (F(z_1, \dots, z_n) - F(x)) \bmod (n_x^2)$$
$$= \sum_{i=1}^{n} \left(\frac{\partial F}{\partial z_i}\right)(x)\, dz_i .$$

Let $\frac{\partial}{\partial z_i}$, $i=1,\dots,n$ denote a basis of $T_x(V)$, dual to the basis $dz_1,\dots,dz_n$. Then the value of the tangent vector $\sum_i \alpha_i \frac{\partial}{\partial z_i}$ at $F_x - F(x)$ is equal to (using $\frac{\partial}{\partial z_i}(dz_j)=\delta_{ij}$) $\sum_{i=1}^n \alpha_i \frac{\partial F}{\partial z_i}(x)$. This is also the value at F of the derivation of $k[z_1,\dots,z_n]$ defined by the tangent vector $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial z_i}$.

(3) Let $f: V=\mathbb{A}^n_k \longrightarrow \mathbb{A}^m_k = W$ be a regular map given by the homomorphism

$$f^*: \begin{cases} k[t_1,\dots,t_m] \longrightarrow k[z_1,\dots,z_n] \\ t_i \longmapsto P_i(z_1,\dots,z_n). \end{cases}$$

Let $\partial_x = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial z_i} \in T_x(V)$. Then

$$(df)_x(\partial_x)(t_i) = \partial_x(f^*(t_i)) = \partial_x(P_i(z_1,\dots,z_n)) =$$
$$= \sum_{j=1}^n \alpha_j \frac{\partial P_i}{\partial z_j}(x) = \sum_{k=1}^m \sum_{j=1}^n \alpha_j \frac{\partial P_i}{\partial z_j}(x) \frac{\partial}{\partial t_k}(t_i).$$

From this we conclude that the matrix of the differential $(df)_x$ with respect to bases $\{\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\}$ and $\{\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_m}\}$ of $T_x(V)$ and $T_{f(x)}(W)$ respectively, is equal to

$$\begin{pmatrix} \frac{\partial P_1}{\partial z_1} & \cdots & \frac{\partial P_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial P_m}{\partial z_1} & \cdots & \frac{\partial P_m}{\partial z_n} \end{pmatrix}. \quad \blacksquare$$

(D.6) Tangent cone and associated graded ring

As we have seen, at a non-singular point x of a variety V, $\dim_x(V) = \dim T_x(V)$. Therefore, the tangent space is the best linear approximation of a variety at a non-singular point (much the same as for differentiable manifolds).

However, at a singular point x, $\dim T_x(V) > \dim_x(V)$ and hence $T_x(V)$ is no longer the best linear approximation of V at x. In such a case the tangent cone $TC_x(V)$ is the appropriate linear approximation. We shall study this idea in some details.

Let $V = \mathbb{Z}(I) \subset k^m$, $I = \text{rad}(I)$, $p=(0,\dots,0) \in V$, and $k[V] = k[x_1,\dots,x_n]/I$. Let $F \in I$; we can write F as

$$F(x_1,\dots,x_n) = F_0 + F_1 + F_2 + \cdots$$

where $F_i(x_1,\dots,x_n)$ is homogeneous of degree i. Let $F_*(x_1,\dots,x_n)$ be the non-zero homogeneous part of F of least degree; this is called the <u>leading term</u> of F. Let

$$I_* = (F_*(x_1,\dots,x_n) \mid F \in I)$$

be the ideal of $k[x_1,\dots,x_n]$ generated by the leading terms of polynomials in I.

<u>Def.</u> <u>The geometric tangent cone</u> at $p=(0,\dots,0) \in V$, denoted by $TC_p(V)$, is

$$TC_p(V) := \mathbb{Z}(I_*)$$

and the tangent cone is the pair $(\mathbb{Z}(I_*), k[x_1,\dots,x_n]/I_*)$, where clearly $k[x_1,\dots,x_n]/I_* = k[TC_p(V)]$. $\blacksquare$

It is clear that $TC_p(V) \subset T_p(V)$, for $I_\ell \subset I_*$; see ⑨.

<u>Computing the tangent cone.</u> If I is principal, $I = (F)$ say, then clearly $I_* = (F_*)$. But in general when $I = (F_1,\dots,F_r)$ say, it is not generally true that

$$I_* = (F_{1*},\dots,F_{r*})$$

the reason being that

$$\operatorname{ord}(f+g) \geq \min(\operatorname{ord}(f), \operatorname{ord}(g))$$

and the equality does not necessarily hold. For example consider $I = (xy, xz + z(y^2 - z^2))$. It can be shown that this is a radical ideal, for example by showing that it is an intersection of prime ideals (see the reference Cox et al, page 474). Now since

$$yz(y^2-z^2) = y(xz + z(y^2-z^2)) - z(xy) \in I$$

and it is also homogeneous, it belongs to I_*. However, it is easily seen that $yz(y^2-z^2)$ does not belong to the ideal generated by the leading terms

$$(xy)_* = xy \quad , \quad (xz + z(y^2-z^2))_* = xz .$$

Hence, in general $I_* \supset (F_{1*}, \dots, F_{r*})$. This raises the question: given a set of generators for an ideal $I \trianglelefteq k[x_1, \dots, x_n]$, how do we find a set of generators for I_*. There is an algorithm for doing this; see Cox et al, page 485, proposition 4.

Intrinsic definition of tangent cone.

An algebra A is said to be graded if there exists subspaces $(A_i)_{i \in \mathbb{N}}$ such that

$$A = \bigoplus_{i \in \mathbb{N}} A_i \quad \text{and} \quad A_i \cdot A_j \subset A_{i+j}$$

For example free algebras are graded by the length of the words, i.e., the subspace A_i of $A = k\langle X \rangle$, $X = \{x_1, \dots, x_n\}$, is defined as the subspace linearly generated by all monomials of degree i. The elements of X are of degree one. Similarly the polynomial algebra $A = k[X] := k\langle X \rangle / (x_i x_j - x_j x_i \mid i, j)$ is a graded

algebra. This follows from the following general result.

Proposition 22. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded algebra and I be a 2-sided ideal generated by homogeneous elements. Then

$$I = \oplus_{i \in \mathbb{N}} I \cap A_i$$

and the quotient algebra A/I is graded, i.e.,

$$A/I = \bigoplus_{i \in \mathbb{N}} (A/I)_i$$

with $(A/I)_i = A_i / I \cap A_i$, for all i.

Proof. It suffices to show that $I = \bigoplus_{i \in \mathbb{N}} I \cap A_i$. First observe that the sum has to be direct since the subspaces A_i form a direct sum. Therefore, it remains to be checked that $I = \sum_{i \geq 0} I \cap A_i$. The ideal I is generated by homogeneous elements x_i of degree d_i. Consequently, if $x \in I$ then

$$x = \sum_i a_i x_i b_i \quad , \quad a_i, b_i \in A .$$

But $a_i = \sum_j a_i^j$ and $b_i = \sum_j b_i^j$, where a_i^j, b_i^j are homogeneous elements of degree j. It follows that

$$x = \sum_{i,j,k} a_i^j x_i b_i^k$$

is a sum of homogeneous elements of degree $d_i + j + k$ in I. This implies that I is a subspace of $\sum_{i \geq 0} I \cap A_i$. The converse inclusion is clear. ∎

In algebraic geometry a usual context in which we encounter graded algebras is the following. Let I be a 2-sided ideal of an algebra A. Define the subset

$$I^k / I^{k+1} := \{ f + I^{k+1} \mid f \in I^k \} \subset A / I^{k+1} .$$

We add elements in I^k/I^{k+1} as in A/I^{k+1}. Consider the set

$$Gr_I(A) := \bigoplus_{k\in\mathbb{N}} I^k/I^{k+1}$$

i.e., the set of infinite sequences $(a_0, a_1, a_2, \dots)$ with $a_k \in I^k/I^{k+1}$ and with only finitely many $a_i \neq 0$. The elements of $Gr_I(A)$ are added componentwise and we now define a multiplication making $Gr_I(A)$ into an algebra. Let

$$f + I^{k+1} \in I^k/I^{k+1}, \quad g + I^{l+1} \in I^l/I^{l+1}$$

and define

$$(f + I^{k+1})(g + I^{l+1}) := fg + I^{k+l+1}$$

(i.e., if $A_i := I^i/I^{i+1}$, then $A_i \cdot A_j \subset A_{i+j}$), and extend this multiplication to all of $Gr_I(A)$ via distributive law. This makes $Gr_I(A)$ into an algebra called the <u>associated graded algebra of A with respect to I</u>.

Let A be a <u>local algebra</u> with maximal ideal n. The associated graded algebra of A is

$$Gr(A) = \bigoplus_{i\in\mathbb{N}} n^i/n^{i+1}$$

Let B be an affine commutative algebra (so $B \cong k[x_1,\dots,x_n]/I$ for some $I \lhd k[x_1,\dots,x_n]$) and let $m \lhd B$ be a maximal ideal. Consider the local algebra $A := B_m$ with the unique maximal ideal $n = mB_m$. Thus we have

$$Gr(A) = \bigoplus_{i\in\mathbb{N}} n^i/n^{i+1}$$

$$\cong \bigoplus_{i\in\mathbb{N}} m^i/m^{i+1} \quad \text{(where lemma } \underline{6} \text{ is used).}$$

An intrinsic def. of tangent cone follows from the following

<u>Proposition 23.</u> The map

$$k[x_1,\dots,x_n]/I_* \longrightarrow Gr(\mathcal{O}_p)$$

$$\bar{x}_i \longmapsto \bar{\bar{x}}_i$$

where $\bar{\bar{x}}_i$ is the class of x_i in $Gr(\mathcal{O}_p)$ is an isomorphism of graded algebras.

<u>Proof.</u> By proposition <u>22</u> $k[x_1,\dots,x_n]/I_*$ is a graded algebra. Let $m = (x_1,\dots,x_n)$ be the maximal ideal in $k[x_1,\dots,x_n]$ corresponding to $(0,\dots,0) \in k^n$. and $k[V] = k[x_1,\dots,x_n]/I$; so $m_p = m/I$ is the maximal ideal of $k[V]$ corresponding to $p = (0,\dots,0) \in V$. Therefore, we have

$$Gr(\mathcal{O}_p) := \bigoplus_{i\geq 0} \frac{n_p^i}{n_p^{i+1}} \cong \bigoplus_{i\geq 0} m_p^i/m_p^{i+1}$$

$$\cong \bigoplus_{i\geq 0} \frac{(x_1,x_2,\dots,x_n)^i}{(x_1,\dots,x_n)^{i+1} + (I \cap (x_1,\dots,x_n)^i)},$$

where the last step follows from the following computation:

$$m_p^i = \frac{(x_1,\dots,x_n)^i + I}{I}, \quad m_p^{i+1} = \frac{(x_1,\dots,x_n)^{i+1} + I}{I}$$

$$\therefore \; m_p^i/m_p^{i+1} \cong \frac{(x_1,\dots,x_n)^i + I}{(x_1,\dots,x_n)^{i+1} + I}$$

$$= \frac{(x_1,\dots,x_n)^i + (x_1,\dots,x_n)^{i+1} + I}{(x_1,\dots,x_n)^{i+1} + I}, \text{ for } (x_1,\dots,x_n)^{i+1} \subset (x_1,\dots,x_n)^i,$$

$$\cong \frac{(x_1,\dots,x_n)^i}{(x_1,\dots,x_n)^i \cap ((x_1,\dots,x_n)^{i+1} + I)}$$

$$= \frac{(x_1,\dots,x_n)^i}{(x_1,\dots,x_n)^i \cap (x_1,\dots,x_n)^{i+1} + (x_1,\dots,x_n)^i \cap I}$$

$$= \frac{(x_1,\dots,x_n)^i}{(x_1,\dots,x_n)^{i+1} + ((x_1,\dots,x_n)^i \cap I)} .$$

But this is just the i-th homogeneous part of $k[x_1,\dots,x_n]/I_*$. (For, by proposition 22 we have

$$\left(k[x_1,\dots,x_n]/I_*\right)_i = k[x_1,\dots,x_n]_i \big/ I_* \cap k[x_1,\dots,x_i]_i \qquad (*)$$

and

$$k[x_1,\dots,x_n]_i = \frac{(x_1,\dots,x_n)^i}{(x_1,\dots,x_n)^{i+1}} ,$$

$$I_* \cap k[x_1,\dots,x_i] = \frac{(x_1,\dots,x_n)^i \cap I + (x_1,\dots,x_n)^{i+1}}{(x_1,\dots,x_n)^{i+1}}$$

putting in (*) the result follows).

References of chapter 3

(1) Atiyah & MacDonald : Introduction to Commutative algebra.

(2) D. Mumford : The Red Book of varieties and schemes.

(3) Shafarevich : Basic algebraic geometry I.

(4) Cox, Little, O'Shea : Ideals, varieties and algorithms.

(5) Griffiths & Harris : Principle of Algebraic Geometry.

(6) Kunz : Introduction to commutative algebra and algebraic geometry.

(7) Dolgachev : Algebraic geometry.

(8) Milne : Algebraic geometry (Lecture notes).

CHAPTER IV

Representation varieties and schemes

(4.1) Representation varieties $\mathrm{Rep}_n A$

An affine k-algebra A is a finitely generated algebra over k, say by elements $\{a_1, a_2, \cdots, a_m\}$. In case A is commutative, using the freeness of polynomial algebra in this category, we conclude that there exists an algebra epimorphism

$$\phi : \begin{cases} k[x_1, \cdots, x_m] \twoheadrightarrow A \\ x_i \longmapsto a_i \end{cases}$$

Let $I_A := \mathrm{Ker}(\phi)$, which is the ideal of relations holding among the generators of A; then obviously

$$A \cong k[x_1, \cdots, x_m]/I_A$$

Since I_A is a 2-sided ideal in $k[x_1, \cdots, x_m]$, its zero set is an algebraic subset of k^m:

$$\mathbb{Z}(I_A) = \{a \in k^m \mid f(a) = 0, \ \forall f \in I_A\} \hookrightarrow k^m.$$

As we have seen, there is a Zariski topology on $\mathbb{Z}(I_A)$ induced by that of k^m. It is defined by declaring that every closed subset of $\mathbb{Z}(I_A)$ is of the form

$$\mathbb{Z}(J) := \mathbb{Z}(\phi^{-1}(J)) \cdot , \quad \forall J \triangleleft A.$$

We immediately face two problems

(1) One major question that we will consider in this chapter is the following: given the affine algebraic subset $\mathbb{Z}(I_A) \hookrightarrow k^m$, can one reverse the above process and reconstruct the algebra A? The answer is negative

in general. The reason is that any zero set $\mathbb{Z}(I)$ only determines the radical of I; $rad(I) \supset I$. For, suppose $f \in k[x_1, \dots, x_n]$ s.t. $f^n \in I$ for some $n \in \mathbb{N}^+$;

$$\forall a \in \mathbb{Z}(I): \; f^n(a) = (f(a))^n = 0$$
$$\Rightarrow f(a) = 0, \quad \forall a \in \mathbb{Z}(I)$$
$$\Rightarrow f \in \mathbb{I}(\mathbb{Z}(I))$$
$$= rad(I), \text{ by Hilbert's Nullstellensatz.}$$

i.e., the set of all functions vanishing on $\mathbb{Z}(I)$ is just $rad(I)$. This implies that we can only recover A back from the zero set $\mathbb{Z}(I_A)$ iff A is reduced, i.e., when A has no nilpotent elements; and this is exactly when $I_A = rad(I_A)$, for in this case

$$\forall \bar{a} \in A: \quad \bar{a}^n = 0 \Rightarrow a^n \in I_A \; (= rad(I_A))$$
$$\Rightarrow a \in I_A \Rightarrow \bar{a} = 0.$$

Therefore, the topological space $\mathbb{Z}(I_A)$, in general, contains only information about the reduced quotient algebra $\bar{A} := A/rad(0)$, where $rad(0)$ is the nilradical of A. We shall see how to overcome this problem by considering the scheme associated to A rather than the variety $\mathbb{Z}(I_A)$.

(2) The second problem is that the algebra A does not determine $\mathbb{Z}(I_A)$ uniquely, for we could have chosen another set of generators, say $\{a'_1, \dots, a'_\ell\}$, for A giving rise to another closed subset $\mathbb{Z}(I'_A) \hookrightarrow k^\ell$. Therefore, one likes to have an interpretation of $\mathbb{Z}(I_A)$

which is independent of particular embeddings (i.e., particular choices of generators for A) which will then imply that $\mathbb{Z}(I_A)$ and $\mathbb{Z}(I'_A)$ are homeomorphic as topological spaces. This is achieved by the functor point of view considered in chapter 3. This approach leans itself on the following observation:

let $p \in \mathbb{Z}(I_A)$; evaluating polynomials at p gives an algebra epimorphism ev_p which factors through A:

$$\begin{array}{ccc} k[x_1, \dots, x_n] & \xrightarrow{ev_p} & k \\ \phi \downarrow & \nearrow_{e_p} & \\ A = k[x_1,\dots,x_n]/I_A & & \end{array} \qquad \leftarrow (1)$$

clearly e_p is unique if it exists and its existence is guaranteed by the fact that

$$Ker(ev_p) = \mathfrak{m}_p \lhd k[x_1, \dots, x_n]$$

is a maximal ideal which contains $I_A = Ker(\phi)$.

It is clear that every homomorphism of A into k is of the form e_p for some $p \in \mathbb{Z}(I_A)$. Moreover,

$$\forall p, q \in \mathbb{Z}(I_A): \quad e_p = e_q \Rightarrow p = q \qquad \leftarrow (2)$$

because polynomials separate points in k^m.

Therefore, the points of $\mathbb{Z}(I_A)$ parametrize the one-dimensional representations of A; and by (2) this correspondence is 1:1. This gives the following bijection

$$\begin{array}{ccc} \mathbb{Z}(I_A) & \xleftrightarrow{1:1} & Hom_{k\text{-alg}}(A, k) \\ p & \longleftrightarrow & e_p \end{array} \qquad \leftarrow (3)$$

Therefore, instead of $Z(I_A)$, which depends on embeddings, we may take $\mathrm{Hom}_{k\text{-alg}}(A,k)$ as the corresponding algebraic set. This gives the functor interpretation of $Z(I_A)$:

$$\mathrm{Hom}_{k\text{-alg}}(A,-) : \mathrm{Alg}_k \longrightarrow \mathrm{Sets} \qquad \leftarrow (4)$$

which is a covariant functor. In this approach morphisms of algebraic sets are given by the contravariant functor

$$\mathrm{Hom}_{k\text{-alg}}(-,k) : \mathrm{Alg}_k \longrightarrow \mathrm{Sets}$$

which assigns to any k-algebra morphism $A \xrightarrow{\varphi} B$ a morphism of the corresponding varieties

$$\mathrm{Hom}_{k\text{-alg}}(\varphi,k) : \begin{cases} \mathrm{Hom}_{k\text{-alg}}(B,k) \longrightarrow \mathrm{Hom}_{k\text{-alg}}(A,k) \\ e_q \longmapsto e_p \end{cases} \qquad \leftarrow (5)$$

where $\mathrm{Hom}_{k\text{-alg}}(\varphi,k)(e_q) = e_q \circ \varphi$, $\quad A \xrightarrow{\varphi} B$, $e_q\circ\varphi = e_p : A \to k$, $e_q : B \to k$.

We will now generalize the above given argument to higher dimensional representations and to non-commutative algebras.

Let A be a non-commutative affine k-algebra generated by m elements, say $\{a_1, \dots, a_m\}$; then there is an algebra epimorphism

$$\phi : \begin{cases} k\langle x_1, \dots, x_m\rangle \longrightarrow\!\!\!\!\rightarrow A \\ x_i \longmapsto a_i \qquad i = 1, \dots, m. \end{cases}$$

Let $\mathrm{Ker}(\phi) = I_A$, which is a 2-sided ideal in $k\langle x_1, \dots, x_m\rangle$; it is called the ideal of relations in A.

one has the following presentation of A

$$A \cong k\langle x_1, \dots, x_m\rangle / I_A \qquad \leftarrow (6)$$

However, because there is no analogue of Hilbert's basis theorem in this case, I_A is not necessarily finitely generated. In case I_A is finitely generated, we say that A is finitely presented.

An n-dimensional representation of A is an algebra morphism

$$\chi : A \longrightarrow M_n(k)$$

where $M_n(k)$ is the algebra of $n\times n$ matrices with entries in the field k. If A is generated by m elements $\{a_1, \dots, a_m\}$, then χ is fully determined by the point

$$(\chi(a_1), \chi(a_2), \dots, \chi(a_m)) \in M_n^m(k) := \underbrace{M_n(k) \oplus \cdots \oplus M_n(k)}_{m\text{-fold}};$$

notice that $M_n^m(k)$ is isomorphic as an affine variety with $\mathbb{A}_k^{mn^2} \equiv k^{mn^2}$. Let us consider the set of all n-dimensional representations of A, i.e.

$$\mathrm{Rep}_n A := \mathrm{Hom}_{k\text{-alg}}(A, M_n(k)) \qquad \leftarrow (7)$$

We will now show that there is a bijection between this set and a Zariski closed subset of $M_n^m(k)$, and hence they can be identified.

First of all notice that any given m-tuple of $n\times n$ matrices $(A_1, \dots, A_m) \in M_n^m(k)$ determines an algebra morphism

$$\chi : \begin{cases} k\langle x_1, \dots, x_m\rangle \longrightarrow M_n(k) \\ x_i \longmapsto A_i \end{cases}$$

$$\therefore\ \mathrm{Rep}_n(k\langle x_1, \dots, x_m\rangle) = M_n^m \ (\cong k^{mn^2}). \qquad \leftarrow (8)$$

Next, an m-tuple of matrices $(A_1,\dots,A_m) \in M_n^m(k)$ determines a representation of $A := k\langle x_1,\dots,x_m\rangle / I_A$ iff

$$\forall\ r(x_1,\dots,x_m) \in I_A \triangleleft k\langle x_1,\dots,x_n\rangle \ :\ r(A_1,\dots,A_m) = 0, \qquad \Leftarrow (9)$$

where 0 is the zero $n \times n$ matrix.

Let $k[M_n^m(k)] := k[x_{ij}(k) \mid 1 \le i,j \le n,\ 1 \le k \le m]$ be the polynomial k-algebra on the entries of generic $n \times n$ matrices $X_k = (x_{ij}(k))_{i,j}$; it is a commutative algebra in mn^2 variables. Let

$$I_A(n) \triangleleft k[M_n^m(k)] \qquad \Leftarrow (10)$$

be the ideal generated by the entries of all matrices in $M_n(k[M_n^m])$ which are of the form

$$r(X_1,\dots,X_m), \quad \forall r(x_1,\dots,x_m) \in I_A$$

Then (9) implies that

$$\operatorname{Rep}_n A := \operatorname{Hom}_{k\text{-alg}}(A, M_n(k)) \cong \mathbb{Z}(I_A(n)) \hookrightarrow M_n^m \ (\cong k^{mn^2}) \qquad \Leftarrow (11)$$

This shows that $\operatorname{Rep}_n A$ is an affine variety, called the representation variety of A at level n.

Observe that even if A is not finitely presented (i.e., I_A is not finitely generated) $I_A(n)$, as an ideal of the commutative polynomial algebra $k[M_n^m]$, is finitely generated. As has been said before the ideal $I_A(n)$ contains more information than the closed subset $\mathbb{Z}(I_A(n))$ which only determines $\operatorname{rad}(I_A(n)) \supset I_A(n)$. This forces us to consider the representation scheme $\underline{\operatorname{Rep}_n A}$

in the next section. Notice that $k[\operatorname{Rep}_n A] = k[M_n^m] / \operatorname{rad}(I_A(n))$.

It may well happen that for a given algebra A, $\operatorname{Rep}_n A = \phi$. Of course this is the case when A has no n-dimensional representation (notice that by assumption A is unital).

Example. Consider the Weyl algebra

$$A(k) := k\langle x_1, x_2\rangle / (x_1x_2 - x_2x_1 - 1) \qquad \Leftarrow (12)$$

Suppose $(A_1, A_2) \in \operatorname{Rep}_n A$, so these matrices must satisfy $A_1A_2 - A_2A_1 = \mathbb{1}_n$. However taking trace of both sides one gets $0 = n$ which is impossible. Hence $\operatorname{Rep}_n A = \phi$. ∎

$\operatorname{Rep}_n A$ is a GL_n-variety. This means that there is a GL_n-action on $\operatorname{Rep}_n A$. To see this remember that GL_n acts on M_n^m by simultaneous conjugation (see chapter 2). We must show that $\operatorname{Rep}_n A \hookrightarrow M_n^m$ is stable under this GL_n-action on M_n^m. But this is clear for if $\chi : A \longrightarrow M_n(k)$ is a representation of A then

$$\begin{array}{ccc} A & \xrightarrow{\ \chi\ } & M_n(k) \\ & \searrow^{\alpha_g \circ \chi} & \swarrow_{\alpha_g = g \cdot g^{-1}} \\ & M_n(k) & \end{array}$$

$\alpha_g = g \cdot g^{-1}$ = action of GL_n on $M_n(k)$ by conjugation

and $\alpha_g \circ \chi \in \operatorname{Rep}_n A$. Hence if $(A_1,\dots,A_m) \in \operatorname{Rep}_n A$, then $(gA_1g^{-1},\dots,gA_mg^{-1}) \in \operatorname{Rep}_n A$, proving that $\operatorname{Rep}_n A$ is a GL_n-variety.

We will now give an interpretation of the orbits of this action. Let M be an A-module of k-dimension

n ; fixing a basis $\{e_1, \dots, e_n\}$ allows one to identify M with k^n whose elements are represented by column n-vectors. The action of $a \in A$ on M is then represented by the action of $\chi(a) \in M_n(k)$ on the column vectors. The module condition

$$a.(b.m) = (ab).m \quad , \quad \forall a,b \in A, \ \forall m \in M$$

translates into the equivalent condition

$$\chi(ab) = \chi(a)\chi(b) \quad , \quad \forall a,b \in A ,$$

so $A \xrightarrow{\chi} M_n(k)$ is an n-dim. representation of A. Conversely, any n-dimensional representation $\chi : A \longrightarrow M_n(k)$ determines an A-module structure on k^n by the rule

$$a.v := \chi(a)v \quad , \quad \forall v \in k^n .$$

Hence there is a 1:1 correspondence between the n-dim. representations of A and A-module structure on k^n. Denoting the set of all A-module structures on k^n by $\text{mod}_n A$, we have the following bijection

$$\text{mod}_n A \cong \text{Rep}_n A . \qquad \leftarrow (13)$$

Suppose M and N are two isomorphic A-module structures on k^n (where this isomorphism is given by a linear map $g \in Gl_n$). Then for all $a \in A$ we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ {\scriptstyle a.}\downarrow & & \downarrow{\scriptstyle a.} \\ M & \xrightarrow{g} & N \end{array}$$

i.e., if the action of a on M and N are represented by

$n \times n$ matrices $\chi(a)$ and $\gamma(a)$ respectively, then $\gamma(a) = g\,\chi(a)\,g^{-1}$. Therefore, the two A-module structure on k^n are isomorphic iff the points of $\text{mod}_n A$ corresponding to them lie in the same Gl_n-orbit. Hence studying n-dimensional representations of A up to isomorphism is the same as studying Gl_n-orbits in the (reduced) representation variety $\text{Rep}_n A$. ($\text{Rep}_n A$ is reduced if $I_A(n)$ is a radical ideal).

(4.2) The representation scheme $\text{Rep}_n A$.

If the defining ideal $I_A(n)$ is a radical ideal, and we will see this is the case when A is Quillen-smooth, the above considerations suffice. In general the scheme structure of the representation variety $\text{Rep}_n A$ must be considered.

Usually an affine variety is defined to be the algebraic set $V = Z(I) \hookrightarrow k^n$ for some ideal $I \triangleleft k[x_1, \dots, x_n]$. Many ideals have the same set of zeroes and by Hilbert's Nullstellensatz they are all contained in $\mathbb{I}(V) = \text{rad}(I)$. The coordinate ring of V is $k[V] = k[x_1, \dots, x_n]/\mathbb{I}(V)$ which is therefore a reduced affine algebra. Moreover, we defined morphisms between affine varieties such that there is a natural 1:1 correspondence

$$\text{Mor}(V,W) \longleftrightarrow \text{Hom}_{k\text{-alg}}(k[W], k[V]) .$$

Alternatively, we have introduced the category of affine varieties as the category dual to the category

of reduced affine k-algebras.
(Dual because we have reversed arrows;

$$\mathrm{Aff}\ V/k = \text{Reduced } \mathrm{Aff\ Alg}_k^{op}$$

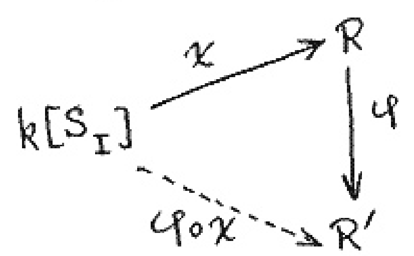
where "op" denotes dual (or opposite).)

This interpretation gives us a trivial (but useful) idea to construct more general geometric objects, fine enough to separate all affine k-algebras. These objects are called affine schemes.

<u>Def.</u> Let $I \triangleleft k[x_1,\dots,x_n]$ be an ideal. The <u>affine scheme</u> S_I is the functor

$$S_I : \text{Affine } k\text{-algebras} \longrightarrow \text{Sets}$$
$$R \longmapsto S_I(R) = \mathrm{Hom}_{k\text{-alg}}(k[S_I], R)$$

where $k[S_I] = k[x_1,\dots,x_n]/I$; and to an algebra morphism $R \xrightarrow{\varphi} R'$ there corresponds a morphism $S_I(R) \xrightarrow{S_I(\varphi)} S_I(R')$ defined by composition



i.e., $S_I(\varphi) = \varphi_* : \chi \longmapsto \varphi \circ \chi$.

We have, as before, the following equivalence of categories

$$\text{Affine schemes} = \mathrm{Aff\ Alg}_k^{op}$$

for,

$$\mathrm{Mor}(S_I, S_J) \xleftrightarrow{1:1} \mathrm{Hom}_{k\text{-alg}}(k[S_J], k[S_I]). \blacksquare$$

Given a scheme S_I with coordinate ring $k[S_I] = k[x_1,\dots,x_n]/I$, we can still consider its

underlying variety $\mathbb{V}(I)$, called <u>the set of geometric points of S_I</u>, and whose coordinate ring is

$$k[\mathbb{V}(I)] = k[x_1,\dots,x_n]/\mathrm{rad}(I) \quad , \quad \mathrm{rad}(I) = \mathbb{I}(\mathbb{V}(I)).$$

Now, since $S_I := \mathrm{Hom}_{k\text{-alg}}(k[x_1,\dots,x_n]/I, -)$, we see that the geometric points, $\mathbb{V}(I)$, are just

$$\mathbb{V}(I) = S_I(k) = \mathrm{Hom}_{k\text{-alg}}(k[x_1,\dots,x_n]/I, k).$$

The sets $S_I(R)$ for other k-algebras R give the additional scheme structure on S_I.

<u>Example.</u> Consider the linear algebraic group GL_n which is an affine variety

$$GL_n = \mathbb{Z}(t\cdot\det X - 1) \hookrightarrow k \times M_n(k)$$

where X is a generic $n\times n$ matrix. Its coordinate ring is

$$k[GL_n] = k[x_{11}, x_{12}, \dots, x_{nn}; t]/(t\cdot\det X - 1)$$

and its associated functor is

$$\underline{GL_n} := \mathrm{Hom}_{k\text{-alg}}(k[GL_n], -).$$

It assigns to any commutative k-algebra R the set $\underline{GL_n}(R) = \mathrm{Hom}_{k\text{-alg}}(k[GL_n], R)$; this can be identified with the set of $n\times n$ invertible matrices with entries in R as follows:

any $\chi \in \underline{GL_n}(R)$ is determined by the images $\chi(x_{ij}) = r_{ij} \in R$, $\chi(t) = d \in R$; so χ determines an $n\times n$ matrix with entries in R

$$m_\chi = (r_{ij})_{i,j} \in M_n(R)$$

which satisfies $d\cdot\det(m_\chi) = 1$. Each of the sets $\underline{GL_n}(R)$ has a group structure ; for this reason $\underline{GL_n}$

is called an affine group scheme. ∎

Def. The representation scheme $Rep_n A$.

$$Rep_n A := Hom_{k\text{-alg}}\left(k[M_n^m]/I_A(n) , - \right) \quad \leftarrow (14)$$

is a functor from the category of k-algebras (which always means commutative k-algebras) to the category of sets. ∎

Any $\psi \in Rep_n A(K)$ is an algebra morphism $k[M_n^m]/I_A(n) \longrightarrow K$, and such a morphism is determined by the images $\psi(x_{ij}(k)) = r_{ij}(k) \in K$. Hence ψ determines an m-tuple of $n\times n$ matrices with coefficients in K

$$(R_1, \dots, R_m) \in \underbrace{M_n(K) \oplus \cdots \oplus M_n(K)}_{m\text{-fold}}$$

where $R_k = \begin{pmatrix} r_{11}(k) & \cdots & r_{1n}(k) \\ \vdots & & \\ r_{n1}(k) & \cdots & r_{nn}(k) \end{pmatrix}$.

clearly, for every $r(x_1,\dots,x_m) \in I_A$ one has $r(R_1,\dots,R_m) = 0 \in M_n(K)$. Therefore, ψ determines uniquely a K-algebra morphism

$$\psi: \begin{cases} K\otimes_k A \longrightarrow M_n(K) \\ x_i \longmapsto R_i \end{cases}$$

This gives an alternative interpretation of $Rep_n A(K)$: this is the set of all left $K\otimes_k A$-module structures on the free K-module $K^{\oplus n}$ of rank n.

Summarizing we have

$$Rep_n A(-) = Hom_{(-)alg.}(-\otimes_k A, M_n(-)) \cong Hom_{k\text{-alg}}\left(k[M_n^m]/I_A^{(n)}, -\right)$$

where $\cong$ is the equivalence of functors. The geometric points of this scheme are

$$Rep_n A(k) = Hom_{k\text{-alg}}(k\otimes_k A, M_n(k)) = Hom_{k\text{-alg}}(A, M_n(k)) :=$$
$$:= Rep_n A,$$

which is the corresponding representation variety. ∎

(4.3) The ring of GL_n-invariants and GL_n-quotient varieties.

An important fact about the group GL_n is that it is a reductive group, i.e., all its representations are completely reducible. Let Γ be the set of isomorphism classes of irreducible GL_n-representations. If V is an IRR of GL_n belonging to the isomorphism class $\gamma \in \Gamma$, we say V is of type γ and write $V \in \gamma$.

Let L be a complex linear space, not necessarily finite dimensional, with a GL_n-action $GL_n \times L \longrightarrow L$. We say that this action is locally finite on L if for any finite dimensional subspace $M \subset L$ there exists a finite dimensional invariant subspace $M' \subset L$ s.t. $M \subset M' \subset L$, i.e., the GL_n-action on L is locally finite if every finite dim. subspace of L is contained in a finite dim. GL_n-stable subspace.

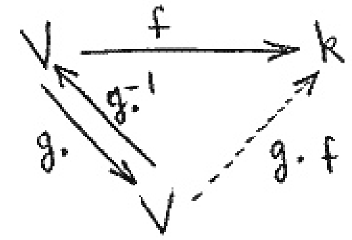
The isotypical component of L of type $\gamma \in \Gamma$ is defined to be the subspace $L_\gamma := \sum \{V \mid V \subset L, V \in \gamma\}$. Alternatively one can write $L_\gamma = \{x \in L \mid \exists \text{ homom. } \varphi: V_\gamma \longrightarrow L \text{ s.t. } x \in Im(\varphi)\}$, where V_γ is irreducible of type γ.

If W is a representation of GL_n, because it is completely reducible, we have

$$W = \bigoplus_{\gamma\in\Gamma} W_\gamma \qquad (*)$$

and every isotypical component $W_\gamma \cong V^{\oplus e_\gamma}$, where $V\in\gamma$ and $e_\gamma \in \mathbb{N}$. obviously $e_\gamma \neq 0$ for finitely many irreducible classes $\gamma\in\Gamma$. (*) is called the isotypical decomposition of W, and we say that the irreducible representation V occurs with multiplicity e_γ in W. It is also clear that when the action $GL_n \times L \to L$ is locally finite, we can reduce to finite dim. representations and obtain the isotypical decomposition of L.

Now, let V be an m-dimensional representation of GL_n. We can view V as an affine space k^m and consider the induced action of GL_n on the polynomial functions $f\in k[V]=k[x_1,\dots,x_m]$, as follows:



where

$$(g\cdot f)(v) = f(g^{-1}\cdot v), \quad \forall v\in V. \quad \leftarrow (15)$$

considering $k[V]=k[x_1,\dots,x_m]$ as a graded algebra with $\deg x_i = +1$, $\forall i=1,\dots,m$, then each homogeneous component of $k[V]$ is a finite dimensional representation of GL_n. Hence the GL_n-action on $k[V]$, as given by (15), is locally finite:

let $\{y_1,\dots,y_\ell\}$ be a basis for a finite dim. subspace $Y\subset k[V]$ and suppose $d = \max.\deg(y_i)$. Then

$$Y' = \bigoplus_{i=0}^{d} k[V]_i$$

is a GL_n-representation containing Y. Therefore, we have an isotypical decomposition

$$k[V] = \bigoplus_{\gamma\in\Gamma} k[V]_\gamma \quad \leftarrow (16)$$

Let $0\in\Gamma$ denote the isomorphism class of the trivial representation of GL_n (i.e., the representation given by $g\cdot x_i = x_i$, $\forall g\in GL_n$, $\forall i=1,\dots,m$). Then we have

$$k[V]_0 = \{f\in k[V] \mid g\cdot f = f, \ \forall g\in GL_n\} := k[V]^{GL_n} \quad \leftarrow (17)$$

These are just the polynomial functions which are constant along the GL_n-orbits in $V\ (\cong k^m)$. We have the following fundamental lemma.

Lemma 1. Let V be a GL_n-representation.

(1) Let $I \lhd k[V]$ be a GL_n-stable ideal, i.e., $g\cdot I\subset I$, $\forall g\in GL_n$. Then

$$\left(k[V]/I\right)^{GL_n} \cong k[V]^{GL_n} \big/ I\cap k[V]^{GL_n}. \quad \leftarrow (18)$$

(2) Every ideal $J \lhd k[V]^{GL_n}$ has the following (lying over) property

$$J = Jk[V] \cap k[V]^{GL_n} \quad \leftarrow (19)$$

where $Jk[V]$ is the ideal generated by J in $k[V]$. Hence $k[V]^{GL_n}$ is Noetherian:

(3) If $(I_j)_j$ is a family of GL_n-stable ideals of $k[V]$ then

$$\Big(\sum_j I_j\Big)\cap k[V]^{GL_n} = \sum_j \left(I_j \cap k[V]^{GL_n}\right). \quad \leftarrow (20)$$

Proof. (1) First of all since I is GL_n-stable, the quotient $k[V]/I$ carries a GL_n-action and hence it makes sense to talk about the ring of invariants $(k[V]/I)^{GL_n}$. Because the GL_n-action on V is locally finite, the induced action on $k[V]$ is locally finite and hence the GL_n-action on I is locally finite; this means that we have the isotypical decomposition $I = \oplus I_\gamma$, where clearly $I_\gamma = k[V]_\gamma \cap I$. One then has

$$k[V]_\gamma/I_\gamma = k[V]_\gamma/I \cap k[V]_\gamma \cong \frac{k[V]_\gamma + I}{I} ;$$

$$\therefore \bigoplus_\gamma k[V]_\gamma/I_\gamma \cong \bigoplus_\gamma \frac{k[V]_\gamma + I}{I} = k[V]/I .$$

On the other hand $k[V]/I = \bigoplus_\gamma (k[V]/I)_\gamma$; so

$$(k[V]/I)_\gamma = k[V]_\gamma/I_\gamma$$

and taking the particular case $\gamma = 0$ gives the desired result.

(2) It is clear that for every $f \in k[V]^{GL_n}$, left multiplication by f in $k[V]$ commutes with GL_n-action on $k[V]$, whence $f \cdot k[V]_\gamma \subset k[V]_\gamma$. This implies that $k[V]_\gamma$ is a $k[V]^{GL_n}$ module. But then, as $J \subset k[V]^{GL_n}$, we have

$$J\,k[V] = \oplus_\gamma (J\,k[V])_\gamma = \oplus_\gamma J\,k[V]_\gamma ,$$

so, $(J\,k[V])_\gamma = J\,k[V]_\gamma$ .. (*)

Taking the special case $\gamma = 0$, we obtain

$$\begin{aligned}(J\,k[V])_0 &= J\,k[V] \cap k[V]^{GL_n}, && \text{clear by def.,}\\ &= J\,k[V]^{GL_n}, && \text{follows from (*)},\\ &= J, && \text{for } J \lhd k[V]^{GL_n}.\end{aligned}$$

The Noetherian statement follows from the fact that $k[V]$ is Noetherian (Hilbert's basis theorem).

(3) Since I_j is GL_n-stable, for every j we have the decomposition $I_j = \oplus_\gamma (I_j)_\gamma$. Therefore,

$$\begin{aligned}\bigoplus_\gamma \left(\sum I_j\right)_\gamma &= \sum_j I_j, && \text{by the fact that } \textstyle\sum_j I_j \text{ is } GL_n\text{-stable}\\ &= \sum_j \bigoplus_\gamma (I_j)_\gamma\\ &= \oplus_\gamma \sum_j (I_j)_\gamma\end{aligned}$$

$\therefore \left(\sum_j I_j\right)_\gamma = \sum_j (I_j)_\gamma$. Taking $\gamma = 0$ immediately gives the required result. ∎

This lemma can be used to prove the finite generation of the ring of polynomial invariants, a result due to D. Hilbert.

Theorem 2. Let V be a GL_n-representation. Then the ring of polynomial invariants $k[V]^{GL_n}$ is an affine k-algebra (i.e., a finitely generated commutative k-algebra).

Proof. Since the action of GL_n on $k[V]$ preserves graduation (because it is a linear action), the ring of invariants, as a subring of $k[V]$, is also graded:

$$k[V]^{GL_n} := R = \mathbb{C} \oplus R_1 \oplus R_2 \oplus \cdots$$

From lemma 1 (2) we know that $k[V]^{GL_n}$ is Noetherian. This implies the ideal $R_+ := R_1 \oplus R_2 \oplus \cdots \lhd k[V]^{GL_n}$ is finitely generated (over R). Hence

$$R_+ = Rf_1 + \cdots + Rf_\ell , \quad \text{for some elements } f_1, \ldots, f_\ell .$$

We claim that, as a k-algebra $k[V]^{GL_n}$ is generated by $f_1, \ldots, f_\ell$; i.e.

$$k[V]^{GL_n} = k[f_1, \ldots, f_\ell] .$$

To prove this, notice that $R_+ = \sum_{i=1}^{\ell} k f_i + R_+$ and also $R_+^2 = \sum_{i,j=1}^{\ell} k f_i f_j + R_+^3$; iterating this procedure one gets, for all powers m, that

$$R_+^m = \sum_{\sum_{i=1}^{\ell} m_i = m} k f_1^{m_1} \cdots f_\ell^{m_\ell} + R_+^{m+1}. \qquad (*)$$

Consider the subalgebra $k[f_1, \dots, f_\ell] \subset R := k[V]^{GL_n}$. Using $(*)$ we see that for any integer $d > 0$ we have

$$R := k[V]^{GL_n} = k[f_1, \dots, f_\ell] + R_+^d .$$

Therefore, for any $i \geq 0$ the homogeneous component of degree i of R is

$$R_i = k[f_1, \dots, f_\ell]_i + (R_+^d)_i$$

which shows that if $d > i$, then $(R_+^d)_i = 0$ and $R_i = k[f_1, \dots, f_\ell]_i$. But this holds for every $i \geq 0$, and this proves the claim. ∎

Example. Consider the action of GL_n on

$$M_n^m = M_n(k) \oplus \cdots \oplus M_n(k)$$

by simultaneous conjugation. Then the above theorem gives another proof of the finite generation of the necklace algebra

$$\mathbb{N}_n^m := k[M_n^m]^{GL_n} .$$ ∎

Algebraic quotients. Let the invariant polynomials $\{f_1, \dots, f_\ell\}$ be a generating set of $k[V]^{GL_n}$. Consider the morphism of affine varieties

$$\phi : \begin{cases} k^m \cong V \longrightarrow k^\ell \\ v \longmapsto (f_1(v), \dots, f_\ell(v)) \end{cases} \qquad \leftarrow (21)$$

and let W be the Zariski closure $\overline{\phi(V)}$ in k^ℓ. There

exists a diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & k^\ell \\ & \searrow^{\pi} & \uparrow \\ & & W \end{array}$$

and the corresponding isomorphism $\pi^* : k[W] \xrightarrow{\cong} k[V]^{GL_n}$ of affine k-algebras.

More generally, let X be a closed GL_n-stable subvariety of V, then $X = \mathbb{Z}_V(I)$ for some GL_n-stable ideal $I \triangleleft k[V]$. From lemma 1 (1) we obtain

$$k[X]^{GL_n} = \left(k[V]/I\right)^{GL_n} = k[V]^{GL_n} \big/ I \cap k[V]^{GL_n} \qquad \leftarrow (22)$$

Therefore, $k[X]^{GL_n}$ is also an affine k-algebra (i.e., finitely generated) and it is generated by the images

$$\bar{f}_i := f_i \bmod (I) , \quad i = 1, \dots, \ell .$$

Let Y be the Zariski closure of $\phi(X)$ in k^ℓ, where ϕ is given by (21). We have the following diagram

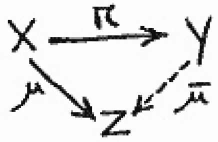
$$\begin{array}{ccc} X & \longrightarrow & k^\ell \\ & \searrow^{\pi} & \uparrow \\ & & Y \end{array} \qquad \pi : \begin{cases} X \longrightarrow Y \\ X \ni v \longmapsto (\bar{f}_1(v), \dots, \bar{f}_\ell(v)) \end{cases}$$

and an isomorphism $\pi^* : k[Y] \xrightarrow{\cong} k[X]^{GL_n}$. The morphism $\pi : X \longrightarrow Y$ defined this way is called the algebraic quotient of X under GL_n. (π maps an orbit of GL_n action in X to a point in Y). Y is usually denoted by $X /\!/ GL_n$.

The important properties of algebraic quotients are given in the next three propositions.

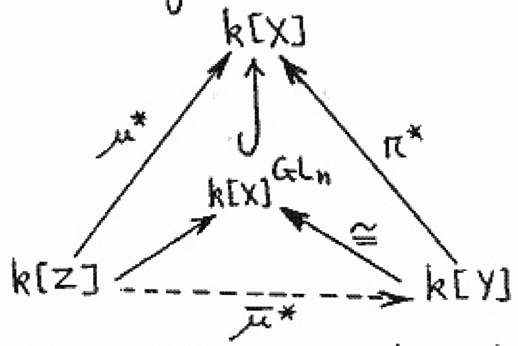
Proposition 3. (Universal property)

Given any morphism of varieties $X \xrightarrow{\mu} Z$ which is constant on GL_n-orbits in X, there exists a unique (factorizing) morphism $\bar{\mu}$ s.t. the diagram



is commutative.

Proof. Because μ is constant on GL_n-orbits in X we have an inclusion map $\mu^*(k[Z]) \subset k[X]^{GL_n}$. This gives the following commutative diagram



from which the existence and uniqueness of $\bar{\pi}$ follow. ∎

As a consequence of the universal property the algebraic quotient is uniquely determined up to isomorphism; this implies that we might have started from a different generating set of invariants and still obtain the same quotient variety up to isomorphism.

Proposition 4. (onto property)
The algebraic quotient $X \xrightarrow{\pi} Y$ is surjective. Moreover, if $Z \hookrightarrow X$ is a closed GL_n-stable subset, then $\pi(Z)$ is closed in Y and the morphism

$$\pi_X|Z : Z \longrightarrow \pi(Z)$$

is an algebraic quotient, i.e., $k[\pi(Z)] \cong k[Z]^{GL_n}$.

Proof. Let $y \in Y$ and the corresponding maximal ideal $m_y \lhd k[Y] := k[X]^{GL_n}$. By lemma 1 (2) $m_y k[X] \neq k[X]$, therefore, there exists a maximal ideal $m_x \lhd k[X]$ which contains $m_y k[X]$; this implies $\pi(x) = y$.

Now, let $I \lhd k[X]$ be a GL_n-stable ideal and $Z = Z_X(I)$. Then $\overline{\pi(Z)} = Z_Y(I \cap k[Y])$; this implies that

$$k[\overline{\pi(Z)}] = k[Y]/I \cap k[Y]$$
$$\cong (k[X]/I)^{GL_n} \quad , \text{ by lemma 1 (1)}$$
$$= k[Z]^{GL_n} .$$

Finally the surjectivity of $\pi|Z$ is established in the first part of this proof. ∎

As a consequence of this proposition the Zariski topology on Y is the quotient of the topology on X. For, take $U \subset Y$ s.t. $\pi^{-1}(Y)$ is Zariski open in X. Then $X - \pi^{-1}(U)$ is a GL_n-stable closed subset of X. Thus $\pi(X - \pi^{-1}(U)) = Y - U$ is Zariski closed in Y and hence U is Zariski open in Y.

Proposition 5. (separation property)
The quotient $X \xrightarrow{\pi} Y$ separates disjoint closed GL_n-stable subvarieties of X.

Proof. Let $(Z_j)_j$ be closed GL_n-stable subvarieties of X with defining ideals I_j, i.e., $Z_j = Z_X(I_j)$. Then $\bigcap_j Z_j = Z_X(\sum_j I_j)$. Then

$$\overline{\pi(\bigcap_j Z_j)} = Z_Y((\sum_j I_j) \cap k[Y])$$
$$= Z_Y(\sum_j (I_j \cap k[Y])) \quad , \text{ where lemma 1 (3) is used ,}$$
$$= \bigcap_j Z_Y(I_j \cap k[Y])$$
$$= \bigcap_j \overline{\pi(Z_j)} . \qquad (*)$$

Now, the onto property implies $\pi(Z_j) = \pi(Z_j)$. Thus (*) implies $\pi(\bigcap_j Z_j) = \bigcap_j \pi(Z_j)$, from which the separation property follows. ∎

It follows from the universal property that the quotient variety $Y = X/\!\!/GL_n$ determined by the ring of polynomials invariants $k[X]^{GL_n}$ is the best continuous approximation to the orbit space problem. In fact from the separation property a stronger fact follows.

Proposition 6.

The algebraic quotient is the best continuous approximation to the orbit space. That is, points of Y parametrize the closed GL_n-orbits in X. Moreover, every fiber $\pi^{-1}(y)$ contains exactly one closed orbit C and we have

$$\pi^{-1}(y) = \{x \in X \mid C \subset \overline{GL_n \cdot x}\}.$$

Proof. Clearly the fiber $F = \pi^{-1}(y)$ is a GL_n-stable closed subvariety of X. Take any orbit $GL_n \cdot x \subset F$; then this is either closed or contains in its closure an orbit of strictly smaller dimension. Induction on the dimension then shows that $\overline{GL_n \cdot x}$ contains a closed orbit C. On the other hand, assume that F contains two closed orbits, then they have to be disjoint contradicting the separation property. ∎

Example. Let A be an affine k-algebra generated by m-elements $a_1, \ldots, a_m$. Then $Rep_n A$ is a closed GL_n-stable subvariety of M_n^m.

Let $f_1, \ldots, f_\ell$ be a set of generating necklaces for the necklace algebra $\mathbb{N}_n^m := k[M_n^m]^{GL_n}$ and consider the

induced morphism

$$M_n^m \xrightarrow{\pi} \pi(M_n^m) \hookrightarrow k^\ell.$$

Then $\pi(M_n^m)$ is a closed subset of k^ℓ and there is a 1:1 correspondence between the points of $\pi(M_n^m)$ and closed GL_n-orbits in M_n^m; the coordinate ring of $\pi(M_n^m)$ is isomorphic with $\mathbb{N}_n^m$.

Moreover, for $Rep_n A$ we have the diagram

$$\begin{array}{ccc} M_n^m & \xrightarrow{\pi} & \pi(M_n^m) \hookrightarrow k^\ell \\ \uparrow & & \uparrow \\ Rep_n A & \xrightarrow{\pi \mid Rep_n A} & \pi(Rep_n A) \end{array}$$

and $\pi(Rep_n A)$ is again a closed subset of k^ℓ and its points parametrize the closed orbits of $Rep_n A$. The quotient $\pi(Rep_n A)$ is denoted by $iss_n A := Rep_n A/\!\!/GL_n$ and the coordinate ring of this affine variety is

$$k[Rep_n A/\!\!/GL_n] = k[Rep_n A]^{GL_n} = \left(k[M_n^m]/I_A(n) \right)^{GL_n}. \quad ∎$$

(4.4) Invariant theoretic reconstruction of Cayley-Hamilton algebras

We will show that a large class of finitely generated k-algebras can be reconstructed from geometric data associated with them.

Def. A trace map on a not necessarily finitely generated k-algebra A is a k-linear map $A \xrightarrow{tr} A$ such that

$$\begin{cases} \text{(i)} & tr(a)\, b = b\, tr(a) \\ \text{(ii)} & tr(ab) = tr(ba) \\ \text{(iii)} & tr(tr(a)\, b) = tr(a)\, tr(b) \end{cases} \qquad \forall a, b \in A.$$

∎

Notice that (i) implies $\mathrm{Im}(tr) \subseteq Z(A)$, the center of A.

<u>Def.</u> Given two k-algebras A, B equipped with trace maps tr_A, tr_B, a <u>trace morphism</u> $A \xrightarrow{\phi} B$ is a k-algebra morphism which is compatible with the trace maps, i.e., the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr_B} \\ A & \xrightarrow{\phi} & B \end{array}$$

This makes algebras with a trace into a category, denoted by Alg_k^{tr}. ∎

<u>Def.</u> A k-algebra is said to be <u>trace generated</u> by a subset $S \subset A$ if $A = \langle B, tr(B) \rangle_k$, where $B = \langle S \rangle_k$ is the subalgebra of A generated by S. ∎

Observe that this definition does not imply that A is generated as a k-algebra by S. For example the formal trace algebra $\mathbb{T}^\infty$ (see chapter 2) is trace generated by $\{x_1, x_2, \ldots, x_n, \ldots\}$ but not generated as a k-algebra by these elements. <u>In fact $\mathbb{T}^\infty$ is the free algebra in generators $\{x_1, x_2, \ldots, x_n, \ldots\}$ in the category Alg_k^{tr}.</u> It follows that if A is an algebra with a trace map, trace generated by $\{a_1, a_2, \ldots\}$ then there exists a trace preserving algebra epimorphism $\mathbb{T}^\infty \overset{\pi}{\twoheadrightarrow} A$; i.e. the diagram

$$\begin{array}{ccc} \mathbb{T}^\infty & \overset{\pi}{\twoheadrightarrow} & A \\ {\scriptstyle Tr}\downarrow & & \downarrow{\scriptstyle tr_A} \\ \mathbb{T}^\infty & \overset{\pi}{\twoheadrightarrow} & A \end{array}$$

is commutative; Tr is the formal trace map on $\mathbb{T}^\infty$. π is explicitly given by

$$\pi(x_i) = a_i$$
$$\pi(Tr(x_{i_1} \cdots x_{i_l})) = tr(\pi(x_{i_1}) \cdots \pi(x_{i_l})).$$

Similarly the trace algebra $\mathbb{T}^m$ is a subalgebra of $\mathbb{T}^\infty$, trace generated by $\{x_1, \ldots, x_m\}$. <u>It is the free algebra in the category of algebras with trace which are trace generated by m elements.</u>

In chapter 2 we have seen that given a trace map on A, for any $a \in A$ one can define a <u>formal Cayley-Hamilton polynomial of degree n</u> as follows:

express $f(t) = \prod_{i=1}^{n} (t - \lambda_i)$ as a polynomial in t with coefficients as polynomial functions in Newton functions $\sum_{i=1}^{n} \lambda_i^k$; replace the Newton function $\sum_{i=1}^{n} \lambda_i^k$ by $tr(a^k)$ and we obtain a Cayley-Hamilton polynomial of degree n, $\chi_a^{(n)} \in A[t]$. This allows for the following important definition.

<u>Def.</u> (Cayley-Hamilton algebra of degree n)
A unital, not necessarily finitely generated k-algebra A with a trace map $A \xrightarrow{tr} A$ is said to be a Cayley-Hamilton algebra of degree n if the following properties are satisfied

(1) $tr(1) = n$
(2) $\forall a \in A$: $\chi_a^{(n)}(a) = 0$ in A. ∎

<u>Examples.</u> (1) For any commutative k-algebra K, $M_n(K)$ with the usual trace map

$$M_n(K) \xrightarrow{tr} K \xhookrightarrow{\cdot 1_n} M_n(K)$$

(inclusion via scalar matrices) is a Cayley-Hamilton algebra of degree n.

(2) The infinite trace algebra $\mathbb{T}_n^\infty$ has a trace map induced by natural inclusions:

$$\begin{array}{ccc} \mathbb{T}_n^\infty & \subset\!\longrightarrow & M_n(k[M_n^\infty]) \\ {\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle tr} \\ \mathbb{N}_n^\infty & \subset\!\longrightarrow & k[M_n^\infty] \end{array}$$

whose image $tr(\mathbb{T}_n^\infty) = \mathbb{N}_n^\infty$ is the infinite necklace algebra. Because the inclusions are trace preserving, it follows that $\mathbb{T}_n^\infty$ is a Cayley-Hamilton algebra of degree n. ∎

Let Alg_k, Alg_k^{tr}, Alg_n^{CH} denote the categories of algebras over k, trace algebras over k and the Cayley-Hamilton algebras of degree n over k. Given any $A \in Alg_k$ we can construct an algebra $A^\tau \in Alg_k^{tr}$ by adjoining the traces, $t_b, b \in A$; i.e.

$$A^\tau = \langle a, t_b \mid a, b \in A \rangle / (at_b = t_b a, t_a t_b = t_b t_a) \cong A \otimes A/[A,A]$$

and the trace map in A^τ is given by

$$tr: a \otimes [b] \longmapsto 1 \otimes [ab].$$

Once A^τ is constructed one may impose the Cayley-Hamilton relations of degree n to obtain a Cayley-Hamilton algebra $A_n \in Alg_n^{CH_n}$,

$$A_n = A^\tau / (tr(1) - n, \chi_a^{(n)}(a) = 0, \forall a \in A).$$

If we start off with a free algebra $A := k\langle x_1, \dots, x_m \rangle$ and apply the above procedure we shall obtain the free algebra generated by m elements in the categories Alg_k^{tr} and $Alg_k^{CH_n}$ respectively. Similarly if we start with $A = k\langle x_1, \dots, x_m, \dots \rangle$, the free algebra on infinite number of variables, we obtain the free trace algebra and the free Cayley-Hamilton algebra in their respective categories.

We can now give the following categorical description of the infinite trace algebra $\mathbb{T}_n^\infty$.

<u>Proposition 7.</u> The trace algebra $\mathbb{T}_n^\infty$ is the free algebra on the generic matrix generators $\{X_1, \dots X_i, \dots\}$ in the category of Cayley-Hamilton algebras of degree n. Moreover, for any m, the trace algebra $\mathbb{T}_n$ is the free algebra on the generic matrix generators $\{X_1, \dots, X_m\}$ in the category of C.H. algebras of degree n which are trace generated by at most m elements.

<u>Proof.</u> Let F_n be the free algebra on generators $\{y_1, y_2, \dots\}$ in the category of Cayley-Hamilton algebras of degree n. By freeness of $\mathbb{T}^\infty$ (see chapter 2) <u>there is a trace preserving algebra epimorphism</u> $\pi: \mathbb{T}^\infty \twoheadrightarrow F_n$, $x_i \mapsto y_i$, and by the universal property of the free object F_n, the ideal $I := Ker(\pi) \lhd \mathbb{T}^\infty$ must be the minimal ideal of $\mathbb{T}^\infty$ s.t. $\mathbb{T}^\infty/I$ is C.H. of degree n. We first show that $Ker(\pi)$ is substitution invariant. Let $\phi: \mathbb{T}^\infty \to \mathbb{T}^\infty$ be a substitution endomorphism of $\mathbb{T}^\infty$ and consider the following diagram

$$\begin{array}{ccc} \mathbb{T}^\infty & \xrightarrow{\ \pi\ } & F_n \\ {\scriptstyle\phi}\uparrow & \nearrow{\scriptstyle \chi = \pi\circ\phi} & \uparrow \\ \mathbb{T}^\infty & \dashrightarrow & \mathbb{T}^\infty/Ker(\chi) \end{array}$$

clearly $Ker(\chi)$ is an ideal stable under traces s.t. $\mathbb{T}^\infty/Ker(\chi)$ is a C.H. algebra of degree n (being a subalgebra of F_n). By minimality of $Ker(\pi)$ we have $Ker(\pi) \subset Ker(\chi)$ and hence χ factors over F_n, i.e., the substitution endomorphism descends to an endomorphism $\overline{\phi}: F_n \to F_n$, implying that $Ker(\pi)$ is invariant under ϕ, proving the claim. Now, any formal C.H. polynomial $\chi_a^{(n)}(a)$ of degree n for $a \in \mathbb{T}^\infty$ maps to zero under π. By substitution invariance it follows that the ideal of trace relations $Ker(\tau) \subset Ker(\pi)$ (see def. of π given above). By what we have seen in chapter 2, $\mathbb{T}^\infty/Ker(\tau) = \mathbb{T}_n^\infty$ is the infinite trace algebra; it is a Cayley-Hamilton algebra of degree n (by the previous example). Thus by the minimality of $Ker(\pi)$ we must have $Ker(\tau) = Ker(\pi)$. $\Rightarrow F_n \cong \mathbb{T}_n^\infty$.

The second statement in the proposition follows immediately from the general case just proved. ∎

Let A be a C.H. algebra of degree n, trace generated by m elements $\{a_1, \dots, a_m\}$ with respect to the trace map tr_A. By freeness of $\mathbb{T}_n^m$ we have a trace preserving algebra epimorphism φ_A defined by $\varphi_A(X_i) = a_i$, $i = 1, \dots, m$; that is the commutative diagram

$$\begin{array}{ccc} \mathbb{T}_n^m & \xrightarrow{\varphi_A} & A \\ {\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle tr_A} \\ \mathbb{T}_n^m & \xrightarrow{\varphi_A} & A \end{array}$$

and we have a presentation $A \cong \mathbb{T}_n^m / T_A$; ← (23)

where $T_A := \operatorname{Ker}(\varphi_A)$ is the ideal of trace relations holding among the generators of A.

The fundamental result of this section is proved in the following several steps:

(1) We recall the invariant theoretic description of the trace algebra $\mathbb{T}_n^m$:

the action of GL_n on the coordinate ring $k[M_n^m]$ is given by the rule

$$g \cdot x_{ij}(k) := (g^{-1} X_k g)_{ij} \quad , \quad \forall g \in GL_n. \qquad \leftarrow (24)$$

This action is locally finite and the ring of invariants

$$k[M_n^m]^{GL_n} = \mathbb{N}_n^m$$

is the necklace algebra. Consider the natural action of GL_n on the tensor product

$$M_n(k[M_n^m]) \cong M_n(k) \otimes k[M_n^m]$$

i.e., by conjugation on $M_n(k)$ and by (24) on $k[M_n^m]$, so

$$\begin{array}{l} \forall f_{ij} \in k[M_n^m] \\ \quad i,j = 1, \dots, n \\ \forall g \in GL_n \end{array} : \quad g \cdot \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix} := g \begin{bmatrix} g \cdot f_{11} & \cdots & g \cdot f_{1n} \\ \vdots & & \vdots \\ g \cdot f_{n1} & \cdots & g \cdot f_{nn} \end{bmatrix} g^{-1} \qquad \leftarrow (25)$$

This action of GL_n on $M_n(k[M_n^m])$ is locally finite and the (non-commutative) algebra of matrices f which are invariant under this action is precisely the algebra of GL_n-equivariant polynomial maps $M_n^m \xrightarrow{f} M_n$, i.e.,

$$M_n(k[M_n^m])^{GL_n} = \mathbb{T}_n^m. \qquad \leftarrow (26)$$

Notice that $M_n(k[M_n^m])^{GL_n} \neq M_n(k[M_n^m]^{GL_n})$, or, $\mathbb{T}_n^m \neq M_n(\mathbb{N}_n^m)$. We also have the diagonal embedding $\mathbb{N}_n^m \xrightarrow{\cdot 1_n} \mathbb{T}_n^m$ via scalar matrices:

$$z \cdot 1_n \in \mathbb{T}_n^m \quad , \quad \forall z \in \mathbb{N}_n^m.$$

We know that for any commutative k-algebra R any 2-sided ideal $J \triangleleft M_n(R)$ is of the form $M_n(I)$ for an ideal $I \triangleleft R$. The ideal

$$T_A \triangleleft \mathbb{T}_n^m \hookrightarrow M_n(k[M_n^m])$$

extends to an ideal in $M_n(k[M_n^m])$ and we have thus the extended ideal

$$J = M_n(k[M_n^m])\, T_A\, M_n(k[M_n^m]) \triangleleft M_n(k[M_n^m])$$

and hence, by what we said above, this ideal is of the form

$$J = M_n(N_A) \qquad \leftarrow (27)$$

for some ideal $N_A \triangleleft k[M_n^m]$. Observe that both the extended ideal J and N_A are stable under the respective GL_n-actions. Therefore

$$k[M_n^m]/N_A \quad \text{carries a } GL_n\text{-action.}$$

(2) Let V and W be (not necessarily finite dim.) k-vector spaces with a locally finite GL_n-action. Let $f : V \longrightarrow W$ be a linear map which commutes with the

GL_n-actions. Decomposing V and W into their isotypical components and recalling that $V_0 = V^{GL_n}$ and $W_0 = W^{GL_n}$, we obtain a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ V^{GL_n} & \xrightarrow{\ f_0\ } & W^{GL_n} \end{array}$$

where R is the canonical projection to the isotypical component of the trivial representation. It is called the Reynolds operator. Clearly this operator commutes with the GL_n-action. Moreover, using the complete decomposability of V and W into their isotypical components, it follows that f_0 is surjective (respectively injective) if f is surjective (resp. injective).

(3) Because both $M_n(k[M_n^m])$ and $M_n(k[M_n^m]/N_A)$ carries GL_n-actions which are locally finite, applying the above argument we obtain the following commutative diagram

$$\begin{array}{ccc} M_n(k[M_n^m]) & \xrightarrow{\ \pi\ } & M_n\left(k[M_n^m]/N_A\right) = \dfrac{M_n(k[M_n^m])}{M_n(N_A)} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ \mathbb{T}_n^m & \xrightarrow{\ \pi_0\ } & M_n\left(k[M_n^m]/N_A\right)^{GL_n} \end{array}$$

and the map π_0 factorizes through $A = \mathbb{T}_n^m/T_A$ giving an epimorphism

$$A \twoheadrightarrow M_n\left(k[M_n^m]/N_A\right)^{GL_n}. \qquad \leftarrow (28)$$

(4) We will show that this map is injective and hence it is an isomorphism. To prove this observe that

$$M_n\left(k[M_n^m]/N_A\right)^{GL_n} \cong M_n(k[M_n^m]) \big/ M_n(N_A)^{GL_n}$$

$$= \mathbb{T}_n^m \big/ M_n(N_A)^{GL_n} ;$$

and since $A = \mathbb{T}_n^m/T_A$, using the fact that

$$M_n(N_A)^{GL_n} = \left(M_n(k[M_n^m])\, T_A\, M_n(k[M_n^m])\right)^{GL_n}$$
$$= M_n(k[M_n^m])\, T_A\, M_n(k[M_n^m]) \cap \mathbb{T}_n^m$$

it suffices to prove

$$T_A = M_n(k[M_n^m])\, T_A\, M_n(k[M_n^m]) \cap \mathbb{T}_n^m \qquad \leftarrow (29)$$

to establish the required isomorphism.

<u>Proof of (29)</u> The Reynolds operator commutes with multiplication in $M_n(k[M_n^\infty])$ with an element $x \in \mathbb{T}_n^m$ and also with respect to the trace map (since both of these operations commute with GL_n-action). Therefore, we have that

(i) $\forall x \in \mathbb{T}_n^m,\ \forall z \in M_n(k[M_n^\infty])$: $\begin{array}{l} R(xz) = x R(z) \\ R(zx) = R(z)x \end{array}$

(ii) $\forall z \in M_n(k[M_n^\infty])$: $R(tr(z)) = tr(R(z))$.

Now, assume that

$$z = \sum_i m_i t_i n_i \in M_n(k[M_n^m])\, T_A\, M_n(k[M_n^m]) \cap \mathbb{T}_n^m ,$$

with $m_i, n_i \in M_n(k[M_n^m])$, $t_i \in T_A$. Let $X_{m+1} \in \mathbb{T}_n^\infty$; using the cyclic property of traces we have

$$tr(z X_{m+1}) = \sum_i tr(m_i t_i n_i X_{m+1}) = \sum_i tr(n_i X_{m+1} m_i t_i)$$

$$\therefore tr(R(z X_{m+1})) = \sum_i tr(R(n_i X_{m+1} m_i t_i)) \quad ; \text{ so we have}$$

$$tr(z X_{m+1}) = tr\left(\sum_i R(n_i X_{m+1} m_i) t_i\right) , \text{ for } t_i \in T_A \subset \mathbb{T}_n^m . \quad (*)$$

Now, for any i, the term $R(n_i X_{m+1} m_i)$ is invariant and so it belongs to $\mathbb{T}_n^{m+1}$ and it is <u>linear in X_{m+1}</u>. Knowing the generating elements of $\mathbb{T}_n^{m+1}$ we can write

$$R(n_i X_{m+1} m_i) = \sum_j s_{ij} X_{m+1} t_{ij} + \sum_k tr(u_{ik} X_{m+1} v_{ik})$$

where all elements $s_{ij}, t_{ij}, u_{ik}, v_{ik} \in \mathbb{T}_n^m$.
Substituting this in (*) and using the cyclic property of traces we obtain

$$tr(z X_{m+1}) = tr\Big[\Big(\sum_{i,j,k} t_{ij} t_i s_{ij} + tr(v_{ik} t_i u_{ik})\Big) X_{m+1}\Big]$$

and by non-degeneracy of the trace it follows that

$$z = \sum_{i,j,k} [\, t_{ij} t_i s_{ij} + tr(v_{ik} t_i u_{ik}) \,] .$$

Now, since $t_i \in T_A$ and T_A is stable under the operation of taking trace, both $t_{ij} t_i s_{ij}$ and $tr(u_{ik} v_{ik} t_i)$ are in T_A. $\Rightarrow$ $z \in T_A$. Thus we have established (29).
We therefore have $A = M_n(k[M_n^m]/N_A)^{GL_n}$. We can now apply the fact that the Reynolds operator commutes with trace and get the following commutative diagram

$$\begin{array}{ccc} M_n\left(k[M_n^m]/N_A\right) & \underset{\cdot 1_n}{\overset{tr}{\rightleftarrows}} & k[M_n^m]/N_A \\ \downarrow R & & \downarrow R \\ A & \underset{\cdot 1_n}{\overset{tr_A}{\rightleftarrows}} & \left(k[M_n^m]/N_A\right)^{GL_n} \end{array}$$

from which we conclude that $tr_A(A) = \left(k[M_n^m]/N_A\right)^{GL_n}$.
Summarizing the above discussion we have the following important theorem.

<u>Theorem 8.</u> (Invariant theoretic reconstruction of C.H. algebras)
Let A be a C.H. algebra of degree n, with a trace map tr_A,

which is trace generated by at most m elements. Then there is a canonical ideal $N_A \triangleleft k[M_n^m]$ from which we can reconstruct the algebra A and $tr_A A$ as invariants algebras

$$A = M_n\left(k[M_n^m]/N_A\right)^{GL_n}, \qquad \leftarrow (30)$$

$$tr_A(A) = \left(k[M_n^m]/N_A\right)^{GL_n}. \qquad \leftarrow (31)$$

■

A direct consequence of this result is the following <u>universal property of the embedding $A \xrightarrow{i_A} M_n(k[M_n^m]/N_A)$</u>:

If R is any commutative k-algebra, then $M_n(R)$ with its usual trace map is a C.H. algebra of degree n. If $A \xrightarrow{f} M_n(R)$ is a trace preserving algebra map we claim that there exists a natural morphism of algebras $\bar{f} : k[M_n^m]/N_A \longrightarrow R$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & M_n(R) \\ i_A \downarrow & \nearrow{\scriptstyle M_n(\bar f)} & \\ M_n\left(k[M_n^m]/N_A\right) & & \end{array} \qquad \leftarrow (32)$$

is commutative, where $M_n(\bar{f})$ is the algebra morphism defined entrywise. To see this consider the composed trace preserving morphism $\phi : \mathbb{T}_n^m \longrightarrow A \xrightarrow{f} M_n(R)$. Its image is fully determined by the images of the trace generators X_k of $\mathbb{T}_n^m$ which are, say, $m_k = (m_{ij}(k))_{ij}$. But this gives an algebra morphism $k[M_n^m] \xrightarrow{g} R$, $x_{ij}(k) \longmapsto m_{ij}(k)$. We have $T_A \subset Ker(\phi)$ (for the kernel becomes larger under the composition of mappings), and after inducing to $M_n(k[M_n^m])$ it follows that $N_A \subset Ker(g)$ and this proves that g factors through $k[M_n^m]/N_A \longrightarrow R$.

This morphism is the required morphism f. ∎

(4.5) Geometric reconstruction of Cayley-Hamilton algebras

We shall now give a geometric interpretation of the reconstruction result discussed above.

Again let A be a Cayley-Hamilton algebra of degree n, with trace map tr_A, generated by m elements $a_1, \dots, a_m$. We shall describe the scheme determined by the ideal $N_A \triangleleft k[M_n^m]$ by its associated functor.

The ideal N_A defines an affine variety

$$Z(N_A) \hookrightarrow M_n^m \cong k^{mn^2},$$

with coordinate ring $k[Z(N_A)] = k[M_n^m]/rad(N_A)$. A point $p = (p_1, \dots, p_m) \in Z(N_A)$ determines a maximal ideal of $k[Z(N_A)]$ and hence it determines an algebra map

$$f_p : k[M_n^m]/N_A \longrightarrow k$$

(the kernel of f_m is m_p/N_A where $m_p \triangleleft k[M_n^m]$ is a maximal ideal and it is determined by the fact that $m_p/rad(N_A)$ is the maximal ideal corresponding to $p \in Z(N_A)$)

and this extends to an algebra morphism

$$M_n(f_p) : M_n\left(k[M_n^m]/N_A\right) \longrightarrow M_n(k)$$

and hence we have a trace preserving morphism ϕ_p (i.e., a trace preserving representation of A):

$$\begin{array}{ccc} A & \overset{\phi_p}{\dashrightarrow} & M_n(k) \\ \downarrow & \nearrow_{M_n(f_p)} & \\ M_n\left(k[M_n^m]/NA\right) & & \end{array}$$

Conversely, by the universal property of the embedding $A \overset{i_A}{\hookrightarrow} M_n\left(k[M_n^m]/NA\right)$, every trace preserving algebra morphism $A \longrightarrow M_n(k)$ is of this form by considering the images of the trace generators $a_1, \dots, a_m$ of A.

∴ <u>The points of $Z(N_A)$ parametrize n-dim. trace preserving representation of A. For this reason we denote $Z(N_A)$ by $Rep_n^{tr} A$ and call it the trace preserving representation variety of A.</u>

Recall that for any affine k-algebra we have defined an ideal $I_A(n) \triangleleft k[M_n^m]$ and the corresponding variety $Z(I_A(n)) \hookrightarrow M_n^m$ whose points parametrize n-dim. representation of A; we denoted $Z(I_A(n))$ by $Rep_n A$.

<u>Lemma 9.</u> Let A be a Cayley-Hamilton algebra of degree n generated by $\{a_1, \dots, a_m\}$. Then the reduced trace preserving representation variety

$$Rep_n^{tr} A \hookrightarrow Rep_n A$$

is a closed subvariety of the reduced representation variety.

<u>Proof.</u> Assume that A is generated as a k-algebra by $\{a_1, \dots, a_m\}$; this is no restriction for trace affine algebras are affine. Then clearly $I_A(n) \subset N_A$.

$\Rightarrow Rep_n^{tr} A \hookrightarrow Rep_n A$. We can determine the additional defining equations of $Rep_n^{tr} A$ as follows:

write any trace monomial in generators as

$$tr_A(a_{i_1} \cdots a_{i_k}) = \sum \alpha_{j_1 \dots j_\ell}\, a_{j_1} \cdots a_{j_\ell}.$$

Then a point $p = (p_1, \dots, p_m) \in \mathrm{Rep}_n A$ belongs to $\mathrm{Rep}_n^{tr} A$ if it satisfies the relations of the form

$$tr(p_{i_1} \cdots p_{i_k}) = \sum \alpha_{j_1 \dots j_\ell}\, p_{j_1} \cdots p_{j_\ell}$$

where here the trace is the usual trace on $M_n(k)$. These relations define a closed (usually proper) subvariety $\mathrm{Rep}_n^{tr} A$ of $\mathrm{Rep}_n A$. ∎

Let us now consider the scheme structure of $\mathbb{Z}(N_A)$. The corresponding functor $\mathrm{Alg}_k \longrightarrow \mathrm{Sets}$ is given by

$$\underline{\mathrm{Rep}_n^{tr}}(-) := \mathrm{Hom}_{k\text{-alg}}\left({}^{k[M_n^m]}/_{N_A}, - \right) \qquad \leftarrow (33)$$

which assigns to any commutative k-algebra the set

$$\underline{\mathrm{Rep}_n^{tr}}(R) = \mathrm{Hom}_{k\text{-alg}}\left({}^{k[M_n^m]}/_{N_A}, R \right).$$

Let A be a Cayley-Hamilton algebra of degree n, generated by $\{a_1, \dots, a_m\}$. The functor $\mathrm{Alg}_k^{tr} \longrightarrow \mathrm{Sets}$, given by

$$\underline{\mathrm{Rep}_n^{tr}} A(-) := \mathrm{Hom}^{tr}_{k\text{-alg}}(A, M_n(-)) \qquad \leftarrow (34)$$

is called <u>the trace preserving representation scheme of A.</u>
(We can also write this as $\mathrm{Hom}^{tr}_{(-)\text{alg}}(A \otimes_k -, M_n(-))$.)
It assigns to every commutative k-algebra R the set $\mathrm{Hom}^{tr}_{k\text{-alg}}(A, M_n(R)) = \{A \xrightarrow{\phi} M_n(R) \mid \phi \text{ is tr. preserving}\}$

<u>Lemma 10.</u> The functor $\underline{\mathrm{Rep}_n^{tr}} A(-)$ is representable and it is represented by $\underline{\mathrm{Rep}_n^{tr}}(-)$; i.e.,

$$\mathrm{Hom}^{tr}_{k\text{-alg}}(A, M_n(-)) \cong \mathrm{Hom}_{k\text{-alg}}\left({}^{k[M_n^m]}/_{N_A}, - \right).$$

<u>Proof.</u> An algebra morphism $\psi : {}^{k[M_n^m]}/_{N_A} \longrightarrow R$ specifies uniquely an m-tuple of $n \times n$ matrices with coefficients in R according to

$$r_k = \begin{bmatrix} \psi(\overline{x_{11}(k)}) & \cdots & \psi(\overline{x_{1n}(k)}) \\ \vdots & & \\ \psi(\overline{x_{n1}(k)}) & \cdots & \psi(\overline{x_{nn}(k)}) \end{bmatrix}$$

Composing with the canonical embedding i_A,

$$\begin{array}{ccc} A & \overset{\phi}{\dashrightarrow} & M_n(R) \\ {\scriptstyle i_A}\downarrow & \nearrow {\scriptstyle M_n(\psi)} & \\ M_n\left({}^{k[M_n^m]}/_{N_A} \right) & & \end{array}$$

one obtains a trace preserving algebra morphism ϕ where the trace map on $M_n(R)$ is the usual trace. By the universal property any trace preserving morphism $A \to M_n(R)$ is of this form. ∎

It is also clear that $\underline{\mathrm{Rep}_n^{tr}} A$ is a closed subscheme of $\underline{\mathrm{Rep}_n} A$ (in the same way for the corresponding varieties $\mathrm{Rep}_n^{tr} A \hookrightarrow \mathrm{Rep}_n A$.)

<u>The quotient representation schemes.</u>

Going back to the action of GL_n on $k[M_n^m]$ and using the fact that the ideals $I_A(n)$ and N_A, by their very definitions, are GL_n-stable, we conclude that there is an action of GL_n by automorphisms on the quotient algebras

$${}^{k[M_n^m]}/_{I_A(n)} \quad \text{and} \quad {}^{k[M_n^m]}/_{N_A}.$$

The corresponding algebra of invariants are

$$k[\underline{\mathrm{Rep}_n} A]^{GL_n} = \left(k[M_n^m]/_{I_A(n)} \right)^{GL_n} = \frac{\mathbb{N}_n^m}{I_A(n) \cap \mathbb{N}_n^m}, \quad \leftarrow (35)$$

$$k[\underline{\mathrm{Rep}_n^{tr}} A]^{GL_n} = \left(k[M_n^m]/_{N_A} \right)^{GL_n} = \frac{\mathbb{N}_n^m}{N_A \cap \mathbb{N}_n^m}; \quad \leftarrow (36)$$

where the last equalities are the consequence of the following consideration: when $I \triangleleft k[V]$ is a GL_n-stable ideal, there is an action of GL_n on $k[V]/_I$ and for the ring of invariants we have

$$\left(k[V]/_I\right)^{GL_n} \cong \frac{k[V]^{GL_n}}{I \cap k[V]^{GL_n}} \qquad (\text{lemma } \underline{1}\ (1))$$

$$\therefore \left(k[M_n^m]/_{I_A(n)} \right)^{GL_n} \cong \frac{k[M_n^m]^{GL_n}}{I_A(n) \cap k[M_n^m]^{GL_n}} = \frac{\mathbb{N}_n^m}{I_A(n) \cap \mathbb{N}_n^m},$$

for $\mathbb{N}_n^m := k[M_n^m]^{GL_n}$.

These rings of invariants define closed subschemes of the affine (reduced) varieties associated to the n cklace algebra $\mathbb{N}_n^m$. We call these schemes the <u>quotient schemes for GL_n-action</u> and denote them by

$$\underline{ISS_n} A := \underline{\mathrm{Rep}_n} A /\!/ GL_n,$$
$$\underline{ISS_n^{tr}} A := \underline{\mathrm{Rep}_n^{tr}} A /\!/ GL_n. \quad \leftarrow (37)$$

<u>The geometric points of the affine quotient scheme $\underline{ISS_n A}$, corresponding to the reduced variety $ISS_n A$, parametrize the isomorphism classes of n-dim. semisimple representations of A.</u> Similarly the geometric points of the

quotient scheme $\underline{ISS_n^{tr}} A$, corresponding to the reduced affine variety $ISS_n^{tr} A$ parametrize the isomorphism classes of the trace preserving n-dim. representations of A.

We therefore have the following

<u>Proposition 11.</u> Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A. Then

$$tr_A(A) = k[\underline{ISS_n^{tr}} A] \quad \leftarrow (38)$$

the coordinate ring of the quotient scheme $\underline{ISS_n^{tr}} A$. clearly the maximal ideals of $tr_A(A)$ parametrize the isomorphism classes of trace preserving n-dim. semi-simple representation of A. ∎

Consider the GL_n-equivariant maps between the GL_n-schemes

$$\underline{\mathrm{Rep}_n^{tr}} A \xrightarrow{f} \underline{M_n} = M_n$$

i.e., for any commutative k-algebra R the corresponding map

$$\underline{\mathrm{Rep}_n^{tr}} A(R) \xrightarrow{f(R)} M_n(R)$$

commutes with the actions of GL_n. Alternatively, one can consider the ring of all morphisms $\underline{\mathrm{Rep}_n^{tr}} A \to M_n$ which is the matrix algebra $M_n\left(k[M_n^m]/_{N_A}\right)$ and those which commute with the GL_n-actions are exactly the invariants; and since by theorem $\underline{8}$ we have

$$A = M_n\left(k[M_n^m]/_{N_A}\right)^{GL_n}$$

we have the following description of A.

<u>Proposition 12.</u> Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A. Then A can be

recovered as the ring of GL_n-equivariant maps

$$A = \{ f : \operatorname{Rep}_n^{tr} A \longrightarrow M_n \ , \text{ equivariant} \}$$

of affine GL_n-schemes. ∎

We can now summarize the geometric description of A in the following

Theorem 13. The functor which assigns to a C.H. algebra A of degree n the GL_n-affine scheme $\operatorname{Rep}_n^{tr} A$ of trace preserving n-dimensional representations, has a left inverse.

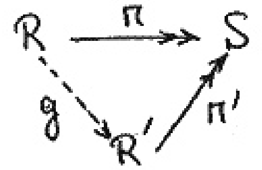
This left inverse functor assigns to a GL_n-affine scheme $\underline{X}$ its witness algebra $M_n(k[\underline{X}])^{GL_n}$, which is a C.H. algebra of degree n. ∎

It is important to notice that this functor is not an equivalence of categories, for in general there are many GL_n-schemes having the same witness algebras.

(4.6) Cayley smoothness

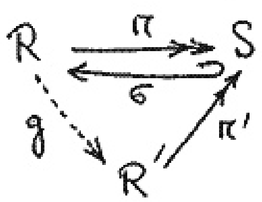
Recall that an epimorphism of k-algebras $R \overset{\pi}{\twoheadrightarrow} S$ with square zero kernel is called an infinitesimal extension of S. This extension is trivial (or cleft, or split) if there is an algebra section $S \overset{\sigma}{\hookrightarrow} R$ (i.e., a homomorphism $\sigma : S \to R$ s.t. $\pi \circ \sigma = id_S$). We have seen that every infinitesimal extension of a formally smooth algebra splits (i.e., it is trivial). In particular this result holds for Grothendieck smooth k-algebras when we confine to the category of commutative k-algebras.

Now, let $R \xrightarrow{\pi} S$ be an infinitesimal extension of k-algebras. This extension is called a versal infinitesimal extension of S if for any other infinitesimal extension of S, $R' \xrightarrow{\pi'} S$, there exists a k-algebra morphism g such that the diagram



(notice that g is not required to be unique.)

is commutative. It is clear that if a versal infinitesimal extension of S is cleft (or trivial) then every infinitesimal extension of S is trivial for, suppose there exists an algebra section $\sigma : S \hookrightarrow R$; so $\pi \circ \sigma = id_S$. This gives a morphism $\sigma' : S \to R'$



: $\sigma' = g \circ \sigma$

and we have

$$\pi' \circ \sigma' = \pi' \circ (g \circ \sigma) = (\pi' \circ g) \circ \sigma = \pi \circ \sigma = id_S ,$$

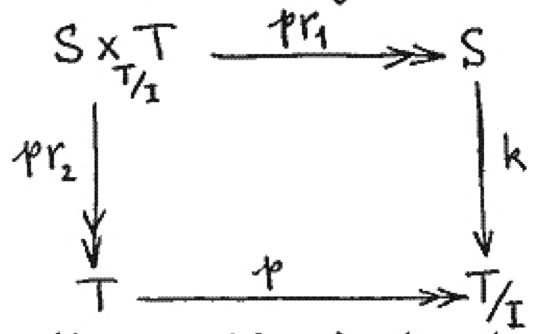
so $R' \xrightarrow{\pi'} S$ is a trivial extension.

Proposition 14. S is Grothendieck smooth iff a versal infinitesimal extension of S is trivial.

Proof. Clearly when S is G-smooth every infinitesimal extension of S (including any versal infinitesimal extension of S) is trivial.

Conversely, suppose a versal infinitesimal extension of S splits (so every infinitesimal extension of S splits) and let $T \xrightarrow{p} T/_I$ be an infinitesimal extension of k-algebras (so, $I^2 = 0$), and let an

algebra morphism $S \xrightarrow{k} T/_I$ be given. Then we have the following commutative diagram



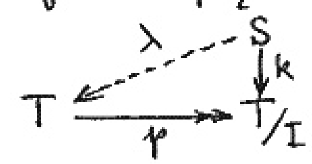
where the pull-back algebra $S \times_{T/I} T$ is defined by

$$S \times_{T/I} T = \{(s,t) \in S \times T \mid k(s) = p(t)\} .$$

Clearly pr_1 is a k-algebra morphism and $Ker(pr_1) = 0 \times_{T/I} I$; so $(Ker(pr_1))^2 = 0$, since $I^2 = 0$. Therefore, $S \times_{T/I} T$ is an infinitesimal extension of S and by assumption it splits:

$$\exists\, \sigma : S \hookrightarrow S \times_{T/I} T \quad s.t. \quad pr_1 \circ \sigma = id_S .$$

This gives the lift $\lambda = pr_2 \circ \sigma$ s.t. the diagram



is commutative. Hence S is G-smooth. ∎

By some rather standard arguments in algebraic geometry one can establish the following results. (For the proofs see for example Le Bruyn, as given in references).

Proposition 15. The affine scheme $\underline{X}$ is non-singular at the geometric point x iff the local algebra $\mathcal{O}_x(\underline{X})$ is Grothendieck smooth. ∎

Proposition 16. Let $\underline{X}$ be an affine scheme. Then, $k[\underline{X}]$ is Grothendieck smooth iff $\underline{X}$ is non-singular in all of its geometric points. In this case X is a reduced affine variety. ∎

Next we look for the concept of smoothness in the category of Cayley-Hamilton algebras. One first notices that the commutative k-algebras are precisely the C.H. algebras of degree 1. Therefore, the notion of Grothendieck smoothness is a special case of the following more general definition.

Def. A Cayley-Hamilton algebra A of degree n with trace map tr_A is said to be Cayley smooth if it satisfies the following lifting property:

Let T be a Cayley-Hamilton algebra of degree n with trace map tr_T and $I \triangleleft T$ be a 2-sided nilpotent ideal which is trace stable (i.e., $tr_T(I) \subset I$). Given any trace preserving k-algebra morphism $k : A \longrightarrow T/I$, there exists a trace preserving k-algebra lift $\lambda : A \longrightarrow T$ such that the diagram

$$\begin{array}{ccc} & & A \\ & \overset{\exists\lambda}{\swarrow} & \downarrow k \\ T & \longrightarrow & T/I \end{array}$$

is commutative. ∎

Using the fact that the trace algebra $\mathbb{T}_n^m$ is the free object in the category of C.H. algebras of degree n, generated by m elements, given any C.H. algebra B of degree n generated by elements, say $\{b_1, \dots, b_m\}$, we have

$$B = \mathbb{T}_n^m / T_B \quad , \quad \mathbb{T}_n^m := M_n\left(k[M_n^m]\right)^{GL_n} ;$$

where the ideal T_B is trace stable. Extending this ideal with respect to the embedding

$$\mathbb{T}_n^m \hookrightarrow M_n(k[M_n^m])$$

we obtain a 2-sided ideal E_B :

$$M_n(k[M_n^m]) \trianglerighteq E_B = M_n(k[M_n^m]) \cdot T_B \cdot M_n(k[M_n^m]) = M_n(N_B) \quad \leftarrow (39)$$

for some ideal $N_B \triangleleft k[M_n^m]$.

We have seen (e.g. (33)) that

$$k[\underline{Rep}_n^{tr} B] = k[M_n^m]/N_B \quad . \quad \leftarrow (40)$$

To be able to prove the fundamental result of this section, we need, at this point, a technical result.

Lemma 17. In the above stated setting, for all positive integers k we have

$$E_B^{kn^2} \cap \mathbb{T}_n^m \subset T_B \quad \leftarrow (41)$$

Proof. (see Le Bruyn as given in the references of chapter 1.) ∎

As a consequence of this lemma one has the following result:

Let B be a C.H. algebra of degree n with trace map tr_B. Let $I \triangleleft B$ be a 2-sided ideal which is trace-stable. Let E(I) be the extended ideal with respect to the canonical embedding

$$B \overset{i_B}{\hookrightarrow} M_n\left(k[M_n^m]/N_B\right)$$

i.e.,

$$E(I) = M_n(k[\underline{Rep}_n^{tr} B]) \cdot I \cdot M_n(k[\underline{Rep}_n^{tr} B]) \quad \leftarrow (42)$$

where $k[\underline{Rep}_n^{tr} B]$ is given by (40). Then for all integers $k \in \mathbb{N}^+$ we have

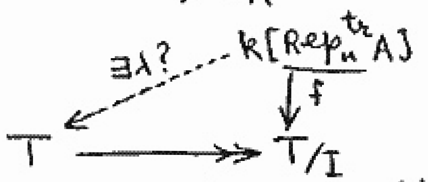
$$E(I)^{kn^2} \cap B \subset I^k . \quad \leftarrow (43)$$

We can now state and prove the following fundamental result.

Theorem 18. Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A. Then A is Cayley-smooth iff the trace preserving representation scheme $\underline{Rep}_n^{tr} A$ is non-singular in all of its points. In particular $\mathrm{Rep}_n^{tr} A$ is reduced, so it is a smooth affine variety.

Proof. Suppose A is cayley-smooth; we will show that $k[\mathrm{Rep}_n^{tr} A]$ is Grothendieck smooth which implies that $\mathrm{Rep}_n^{tr} A$ is a smooth affine variety.

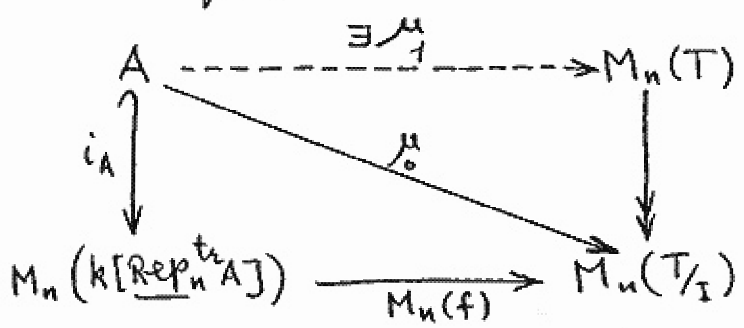
Take a Grothendieck smoothness test object (T, I), $I \triangleleft T$ is a nilpotent ideal, and let an algebra map $f: k[\mathrm{Rep}_n^{tr} A] = k[M_n^m]/N_A \longrightarrow T/I$ be given:



Composing with the (canonical) embedding

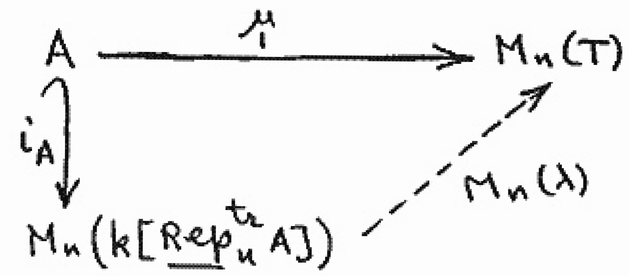
$$A \overset{i_A}{\hookrightarrow} M_n\left(k[M_n^m]/N_A\right)$$

we obtain a trace preserving morphism μ_0 and a commutative diagram



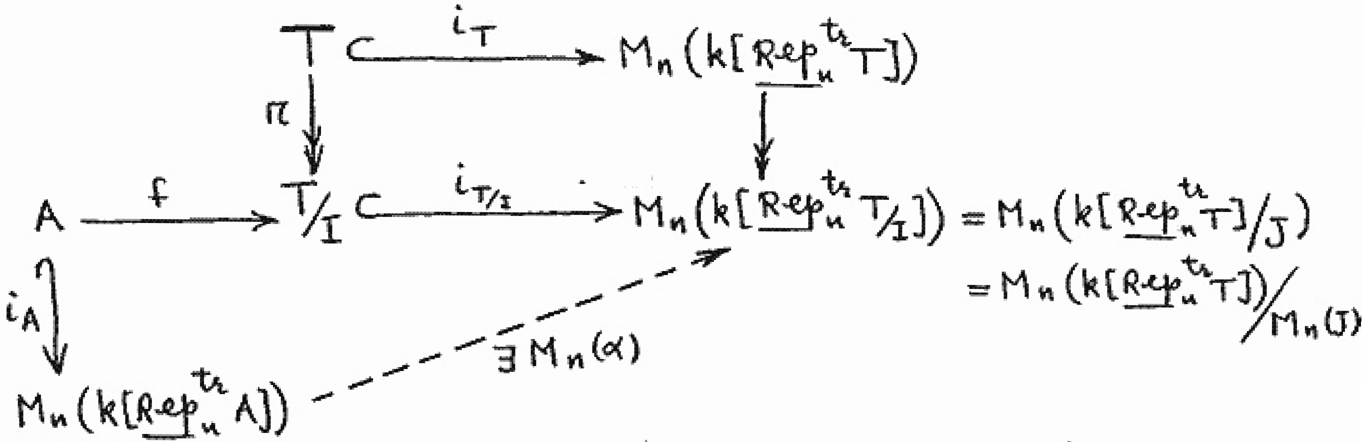
where μ_1 is obtained as follows: $M_n(T)$ with its usual trace is a cayley-Hamilton algebra of degree n and $M_n(I)$ is a trace stable ideal; by cayley-smoothness of A there exists a trace preserving algebra map μ_1, the lift of μ_0.

But then by the universal property of the embedding i_A, there exists a k-algebra morphism $M_n(\lambda)$ s.t. the following diagram is commutative



for some algebra morphism $\lambda: k[\mathrm{Rep}_n^{tr} A] \longrightarrow T$. This is the required lift of f we were looking for. Therefore, $k[\mathrm{Rep}_n^{tr} A]$ is G. smooth, and this implies that $\mathrm{Rep}_n^{tr} A$ is a smooth variety.

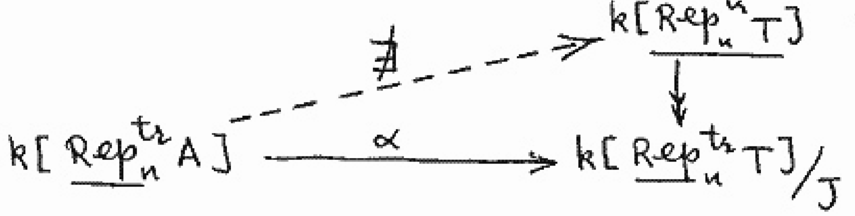
Conversely, assume that $\mathrm{Rep}_n^{tr} A$ is a smooth variety. This implies that $k[\mathrm{Rep}_n^{tr} A]$ is G. smooth. We will show that A is cayley-smooth. Choose a cayley-smoothness test object (T, I), i.e., T is a Cayley-Hamilton algebra of degree n with a trace map tr_T and $I \triangleleft T$ is a trace stable 2-sided nilpotent ideal of T; and suppose $f: A \longrightarrow T/I$ is a given trace preserving algebra map. Combining this test data with the respective universal embeddings, we obtain the following commutative diagram



where $M_n(J) = M_n(k[\mathrm{Rep}_n^{tr} T]) \cdot I \cdot M_n(k[\mathrm{Rep}_n^{tr} T])$, and J is again the extended ideal with respect to the universal embedding $T \overset{i_T}{\hookrightarrow} M_n\left({}^{k[M_n^m]}/_{N_T}\right) := M_n(k[\underline{Rep}_n^{tr} T])$, and we know that $J \cap T = I$. Therefore, the universal property of i_A yields a k-algebra morphism $M(\alpha)$ (in the above diagram) and thus we get an algebra map

$$k[\underline{Rep}_n^{tr} A] \xrightarrow{\alpha} k[\underline{Rep}_n^{tr} T]/_J$$

which we would like to lift to $k[\underline{Rep}_n^{tr} T]$. However, this does not follow from G. smoothness of $k[\underline{Rep}_n^{tr} A]$ (our assumption), since the ideal J is in general not a nilpotent ideal; so



this extension is not generally an infinitesimal extension.

But since I is nilpotent, $I^m = 0$ for some m, and as I is closed under trace, by (43) we have

$$E(I)^{mn^2} \cap T \subset I^m = 0 \qquad (*)$$

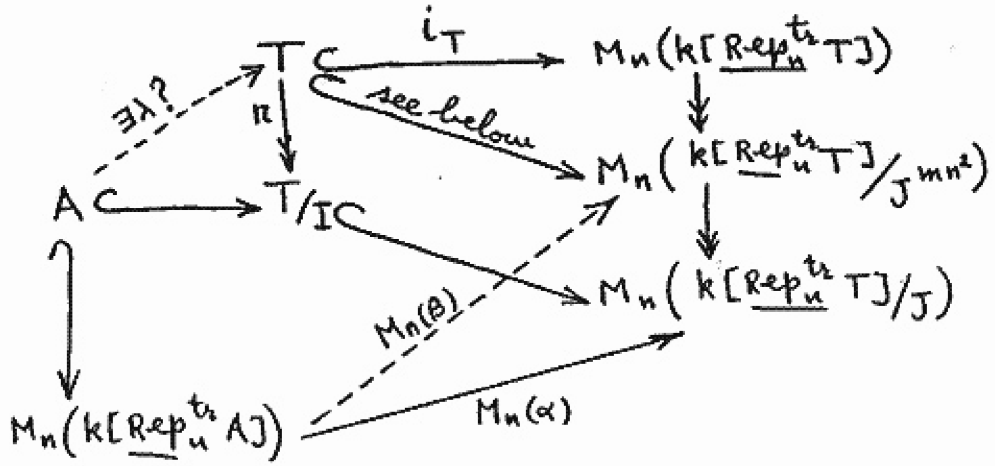
where

$$E(I) := M_n(k[\underline{Rep}_n^{tr} T]) \cdot I \cdot M_n(k[\underline{Rep}_n^{tr} T]) := M_n(J);$$

so $(*)$ implies $M_n(J)^{mn^2} \cap T = 0$. (We now use the fact that if T is a subring of R and $I \lhd R$ s.t. $T \cap I = 0$ then there is an embedding $T \hookrightarrow R/_I$, $a \mapsto a + I$, $\forall a \in T$.) It follows that there is a trace preserving embedding

$$T \hookrightarrow M_n\left({}^{k[\underline{Rep}_n^{tr} T]}/_{J^{mn^2}}\right)$$

We, therefore, have the following commutative diagram



where,

(i) $T \overset{i_T}{\hookrightarrow} M_n(k[\underline{Rep}_n^{tr} T]) \twoheadrightarrow M_n\left({}^{k[\underline{Rep}_n^{tr} T]}/_{J^{mn^2}}\right)$ is injective, for the first map is injective; The second map is injective on T since $T \cap J^{mn^2} = 0$.

(ii) $\beta : k[\underline{Rep}_n^{tr} A] \longrightarrow k[\underline{Rep}_n^{tr} T]/J^{mn^2}$ exists by G. smoothness of $k[\underline{Rep}_n^{tr} A]$ and the fact that $k[\underline{Rep}_n^{tr} T]/J^{mn^2} \overset{p}{\twoheadrightarrow} k[\underline{Rep}_n^{tr} T]/J$ is an infinitesimal extension because

$$(\mathrm{Ker}(p))^{mn^2} = \left(J/_{J^{mn^2}}\right)^{mn^2} = 0 \;;$$

then β extends to an algebra morphism $M_n(\beta)$.

But then this gives a trace preserving morphism $A \xrightarrow{\lambda} M_n(k[\underline{Rep}_n^{tr} T]/J^{mn^2})$ the image of which is contained in the algebra of GL_n invariants: this is because A is by assumption a C.H. algebra of degree n and hence

$$A = M_n\left({}^{k[M_n^m]}/_{N_A}\right)^{GL_n} \equiv M_n(k[\underline{Rep}_n^{tr} A])^{GL_n}$$

and the image of A is contained in

$$M_n\left({}^{k[\underline{Rep}_n^{tr} T]}/_{J^{mn^2}}\right)^{GL_n}.$$

Since $T \hookrightarrow M_n(k[\underline{\mathrm{Rep}}_n^{tr} T]/_J m n^2)$ and because T is also a C.H. algebra of degree n, it is also contained in the subalgebra of invariants, and because the invariants map surjectively under epimorphisms, the commutative triangle in the lower right hand side of the diagram ensures the existence of the lift $A \xrightarrow{\lambda} T$. This implies A is Cayley-smooth. ∎

We now arrive at a crucial point of our studies. Given a Quillen smooth algebra A, for any positive integer n one can construct a Cayley-smooth algebra A_n (read A at level n) in the following manner:

let Alg_k^{tr} be the category of all k-algebras with a trace map and with trace preserving morphisms. The forgetful functor

$$\mathrm{Alg}_k^{tr} \xrightarrow{\mu} \mathrm{Alg}_k$$

has a left adjoint

$$\mathrm{Alg}_k \xrightarrow{\tau} \mathrm{Alg}_k^{tr}$$

(i.e., there exists a natural equivalence of the following hom. functors

$$\mathrm{Hom}_{\mathrm{Alg}_k}(\mu(-), -) \cong \mathrm{Hom}_{\mathrm{Alg}_k^{tr}}(-, \tau(-))$$

which are bifunctors $(\mathrm{Alg}_k^{tr})^{op} \times \mathrm{Alg}_k \longrightarrow \mathrm{Sets}$.)

Given a k-algebra A we construct an algebra A^τ by formally adjoining traces (as we have explained in section 4 of this chapter). If $tr : A^\tau \longrightarrow A^\tau$ is the trace map

on A^τ we define for each $n \in \mathbb{N}^+$ the Cayley-Hamilton algebra

$$A_n := A^\tau / (tr(1) - n,\ \chi_a^{(n)}(a),\ \forall a \in A) \quad \longleftarrow (44)$$

Clearly, $A_n = 0$ if A has no n-dim. representation. Also notice that $A_1 = A/[A,A]$ since $\chi_a^{(1)}(a) = a - tr(a)$, $\forall a \in A$; and if A is commutative then $A_1 = A$, because the center of A is A in this case.

By its very definition, (44), it is clear that A_n has the following universal property:

given any k-algebra map $A \xrightarrow{\phi} B$ s.t. B is a Cayley-Hamilton algebra of degree n, ϕ factors uniquely through $A \xrightarrow{\mathrm{can.}} A_n$, i.e., there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{can.}} & A_n \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle \exists \phi_n} & \\ & B & \end{array}$$

$\exists \phi_n$: a unique trace preserving algebra morphism.

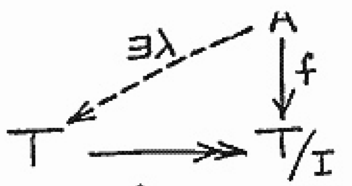
From this universal property, one immediately concludes the following result:

Proposition 19. If A is Quillen smooth, then for any $n \in \mathbb{N}^+$, the Cayley-Hamilton algebra of degree n, A_n, is Cayley-smooth. Moreover

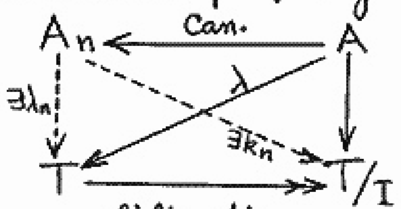
$$\mathrm{Rep}_n A \cong \mathrm{Rep}_n^{tr} A_n.$$

is a smooth affine variety.

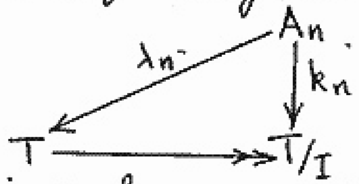
Proof. Suppose (T, I) is a Cayley-smoothness test object and A is a k-algebra which is Quillen smooth. Treating (T, I) as a Q-smoothness test object we have the following lift diagram



Using the universal property of A_n, this gives



and this gives a lift diagram for Cayley-smoothness:



and hence A_n is Cayley-smooth. For the second statement notice that

$$R \xrightarrow{\mathrm{Rep}_n A} \mathrm{Hom}_{Alg_k}(A, M_n(R)) = \mathrm{Hom}^{tr}_{Alg_k}(A^{\tau}, M_n(R))$$
$$= \mathrm{Hom}^{tr}_{Alg_k}(A_n, M_n(R))$$
$$:= \mathrm{Rep}^{tr}_n A_n(R). \quad \blacksquare$$

This result can, in principle, be used to study the étale local structure of Quillen-smooth algebras. We know that the algebra A_n is given by the GL_n-equivariant maps from $\mathrm{Rep}^{tr}_n A_n$ to $M_n(k)$. Because $\mathrm{Rep}^{tr}_n A_n$ is a smooth affine variety, we can apply Luna's étale slices result to determine the étale local structure of this variety and as a consequence that of A_n. In the next sections it will be shown (although not in full details) that this local structure is fully determined by particular types of quivers. Therefore, the local study of arbitrary Quillen smooth algebras can be reduced to that of the better understood subclass of path algebra of quivers. $\blacksquare$

(4.7) The tangent and the Normal spaces of $\mathrm{Rep}_n A$.

The following facts follow from our considerations on tangent spaces as given in chapter 3.

(1) Let $\underline{X}$ be an affine scheme defined by the ideal $I \triangleleft k[x_1, \ldots, x_n]$, $I = (f_1, \ldots, f_r)$; $k[\underline{X}] = k[x_1, \ldots, x_n]/_I$. Consider the affine (not generally reduced) variety $Z(I)$ and suppose $k^n \ni 0 = (0, \ldots, 0) \in Z(I)$. This implies that the elements of I have zero constant terms. The tangent space to $\underline{X}$ at $0 \in k^n$, denoted by $T_0(\underline{X})$, is given by

$$T_0(\underline{X}) = Z(I_\ell) \quad \leftarrow (45)$$

where I_ℓ is the ideal generated by the linear terms of the generators f_i. This is a subvariety of k^n with coordinate ring

$$k[T_0(\underline{X})] = k[x_1, \ldots, x_n]/_{I_\ell} . \quad \leftarrow (46)$$

If $I = (f_1, \ldots, f_r)$, then

$$I_\ell = (a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{r1}x_1 + \cdots + a_{rn}x_n)$$

for some $r \times n$ matrix $A = (a_{ij})_{i,j}$ of rank r. Thus (46) gives

$$K[T_0(\underline{X})] \cong k[x_{i_1}, \ldots, x_{i_{n-r}}] . \quad \leftarrow (47)$$

Because this is an integral domain, $T_0(\underline{X})$ is a reduced subvariety of k^n; it is a linear subspace of k^n of dim. $n-r$ through $0 \in k^n$.

(2) Now, consider an arbitrary geometric point $x \in \underline{X}$ with coordinates $(a_1, \ldots, a_n) \in k^n$. Translating x to

origin $0 \in k^n$, we get the translate of $\underline{X}$ to be the scheme (resp. variety) defined by the ideal

$$I = (f_1(x_1+a_1, \dots, x_n+a_n), \dots, f_r(x_1+a_1, \dots, x_n+a_n)).$$

The linear term of the translated polynomial

$$f_i(x_1+a_1, \dots, x_n+a_n)$$

is equal to

$$\frac{\partial f_i}{\partial x_1}(a_1, \dots, a_n)x_1 + \dots + \frac{\partial f_i}{\partial x_n}(a_1, \dots, a_n)x_n ;$$

therefore, the tangent space to $\underline{X}$ at x, $T_x(\underline{X})$, is the linear subspace of k^n defined by the set of zeroes of these linear terms, i.e.,

$$T_x(\underline{X}) = \mathbb{Z}\left(\sum_{j=1}^n \frac{\partial f_1}{\partial x_j}(x)x_j, \dots, \sum_{j=1}^n \frac{\partial f_r}{\partial x_j}(x)x_j\right) \hookrightarrow k^n \quad \leftarrow (48)$$

and the dimension of this linear subspace is given by

$$\dim T_x(\underline{X}) = n - \operatorname{rank}\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial f_r}{\partial x_1}(x) & \cdots & \frac{\partial f_r}{\partial x_n}(x) \end{bmatrix}.$$

At a non-singular point x the rank of Jacobian matrix is maximal, hence at such a point on has $\dim T_x(\underline{X}) = n - r$.

(3) We have also seen a functorial approach to the tangent space. This approach is based on the following observation.

Let $k[\varepsilon] \cong k[y]/(y^2)$ be the algebra of dual numbers and consider a k-algebra morphism

$$\phi : \begin{cases} k[x_1, \dots, x_n] \longrightarrow k[\varepsilon] \\ x_i \longmapsto a_i + c_i\varepsilon \end{cases} \quad \leftarrow (49)$$

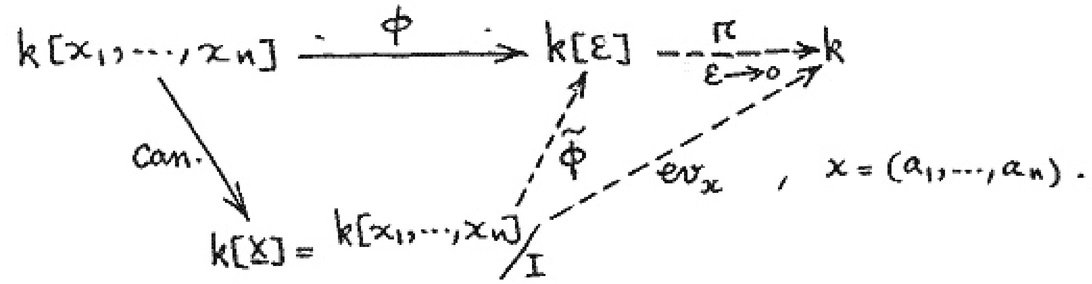
Because $\varepsilon^2 = 0$, one easily verifies that the image of a polynomial $f \in k[x_1, \dots, x_n]$ is of the form

$$\phi(f(x_1, \dots, x_n)) = f(a_1, \dots, a_n) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a_1, \dots, a_n)c_j \cdot \varepsilon \quad \leftarrow (50)$$

Suppose that in (49) we set $x = (a_1, \dots, a_n) \in \underline{X}$ and $(c_1, \dots, c_n) \in T_x(\underline{X})$. Then by (48)

$$f \in I \Rightarrow \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a_1, \dots, a_n)c_j = 0$$

and by (45), $f(a_1, \dots, a_n) = 0$; hence $\phi(f(x_1, \dots, x_n)) = 0$.
$\Rightarrow \operatorname{Ker}(\phi) \supset I$. Therefore, ϕ factorizes through the quotient by I, i.e.,



Hence, we can also identify the tangent space to $\underline{X}$ at $x = (a_1, \dots, a_n) \in \underline{X}$ with the algebra morphisms $k[\underline{X}] \xrightarrow{\tilde{\Phi}} k[\varepsilon]$ whose composition with the projection $\pi : k[\varepsilon] \longrightarrow k$ (sending $\varepsilon \to 0$) is evaluation at x.

Thus if $\underline{X}$ is given by its associated functor $\underline{X} := \operatorname{Hom}_{k\text{-alg}}(k[\underline{X}], -)$, $\underline{X}(k[\varepsilon]) = \{\tilde{\Phi} \mid k[X] \xrightarrow{\tilde{\Phi}} k[\varepsilon],\ \text{alg. morphism}\}$ and if we let $ev_x \in \underline{X}(k)$ be the point corresponding to evaluation at x, then

$$T_x(\underline{X}) = \{\tilde{\Phi} \in \underline{X}(k[\varepsilon]) \mid \underline{X}(\pi \circ \tilde{\Phi}) = ev_x\} \quad \leftarrow (51)$$

Example 1. GL_n as an affine variety is

$$GL_n = \mathbb{Z}(t \det X - 1) \hookrightarrow M_n(k) \times k,$$

with a coordinate ring

$$k[GL_n] = k[x_{11}, x_{12}, \dots, x_{nn}; t]/(t \det X - 1).$$

The corresponding scheme $\underline{GL}_n$ (is) given by its associated functor $\underline{GL}_n(-) = \operatorname{Hom}_{k\text{-alg}}(k[GL_n], -)$; a functor from the category of commutative (affine) k-algebras to Sets. $GL_n(k[\varepsilon])$ is the group of invertible $n \times n$ matrices with coefficients in $k[\varepsilon]$. By (51) we have that for any $g \in GL_n$

$$T_g(\underline{GL}_n) = \{m \in M_n(k) \mid g + m\varepsilon \text{ is invertible in } M_n(k[\varepsilon])\} = M_n(k);$$

for $\varepsilon \to 0$ gives g, which is evaluation at g, and
$(g+m\varepsilon)^{-1} = g^{-1} - g^{-1} m \cdot g^{-1}\varepsilon$, $\forall m \in M_n(k)$.

For any affine algebraic group scheme $\underline{G}$ one defines the Lie algebra $\mathfrak{g}$ of $\underline{G}$ to be the tangent vectorspace $T_e(\underline{G})$ to $\underline{G}$ at the neutral element e. In particular the Lie algebra $\mathfrak{gl}_n$ of GL_n is the vector space $M_n(k)$. ∎

Example 2. Tangent space to the representation variety $\mathrm{Rep}_n A$

Let A be an affine k-algebra generated by m elements $\{a_1, \dots, a_m\}$. Let $\rho: A \to M_n(k)$ be an algebra morphism, i.e., $\rho \in \mathrm{Rep}_n A$.

A linear map $A \xrightarrow{D} M_n(k)$ is said to be a ρ-derivation of A if

$$\forall a, a' \in A: \quad D(aa') = D(a)\rho(a') + \rho(a)D(a'). \quad \leftarrow (52)$$

The vector space of all ρ-derivations of A is denoted by $\mathrm{Der}_\rho(A)$. Clearly any ρ-derivation of A is determined by its image on the generators a_i of A. Thus, $\mathrm{Der}_\rho(A) \subset M_n^m$. We will demonstrate that

$$T_\rho(\underline{\mathrm{Rep}}_n A) = \mathrm{Der}_\rho(A). \quad \leftarrow (53)$$

Since $\mathrm{Rep}_n A(-) := \mathrm{Hom}_{k\text{-alg}}(A, M_n(-))$, then $\underline{\mathrm{Rep}}_n A(k[\varepsilon])$

is the set of k-algebra morphisms $A \to M_n(k[\varepsilon])$. Therefore, by (51) we have

$$T_\rho(\underline{\mathrm{Rep}}_n A) = \{D: A \xrightarrow{\text{lin.}} M_n(k) \mid \rho + \varepsilon D: A \to M_n(k[\varepsilon]) \text{ is an algebra morphism}\} \quad \leftarrow (54)$$

However,

$\rho + \varepsilon D$ is an algebra map $\Leftrightarrow$ D is a ρ-derivation.

For, suppose $\rho + \varepsilon D$ is an algebra morphism; then

$$(\rho+\varepsilon D)(aa') = (\rho+\varepsilon D)(a)\cdot(\rho+\varepsilon D)(a') = (\rho(a)+\varepsilon D(a))\cdot(\rho(a')+\varepsilon D(a'));$$

$\therefore$ $\rho(aa') + \varepsilon D(aa') = \rho(a)\rho(a') + \varepsilon(D(a)\rho(a') + \rho(a)D(a'))$

and since ρ is an algebra morphism, $\rho(aa') = \rho(a)\rho(a')$, we have

$$D(aa') = D(a)\rho(a') + \rho(a)D(a').$$

Similarly one shows that if D is a ρ-derivation, $\rho + \varepsilon D$ is an algebra morphism. Therefore (54) $\Rightarrow$ (53). ∎

Example 3. Tangent space to the trace preserving representation variety $\mathrm{Rep}_n^{tr} A$.

Let A be a Cayley-Hamilton algebra of degree n with trace map tr_A and trace generated by $\{a_1, \dots, a_m\}$. Let $\rho \in \mathrm{Rep}_n^{tr} A$, i.e., $\rho: A \to M_n(k)$ is a trace preserving morphism. Because $\underline{\mathrm{Rep}}_n^{tr} A(-) := \mathrm{Hom}_{k\text{-alg}}^{tr}(A, M_n(-))$, $\underline{\mathrm{Rep}}_n^{tr} A(k[\varepsilon])$ is the set of all trace preserving algebra morphisms $A \to M_n(k[\varepsilon])$ (with the usual trace on $M_n(k[\varepsilon])$). By previous example

$$T_\rho(\underline{\mathrm{Rep}}_n^{tr} A) = \mathrm{Der}_\rho^{tr}(A) \subset \mathrm{Der}_\rho(A) \quad \leftarrow (54)$$

where $\mathrm{Der}_\rho^{tr}(A)$ is the set of trace preserving ρ-derivations of A, i.e., the ρ-derivation of A such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{D} & M_n(k) \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr} \\ A & \longrightarrow & M_n(k) \end{array}$$

is commutative; $D\circ tr_A = tr\circ D$. Again using this property and the fact that A is generated by m elements $\{a_1,\dots,a_m\}$ and a trace preserving ρ- derivation of A is determined by its image on the a_i's, we have that $Der_\rho^{tr}(A)\subset M_n^m$, as expected from (54). ∎

(4) Let us again consider the following data

$$o\in Z(I)=\underline{X}\ ,\quad k[\underline{X}]=k[x_1,\dots,x_n]/I\ ,\quad I=(f_1,\dots,f_r).$$

In chapter 3 we have seen that the tangent cone to $\underline{X}$ at o, denoted by $TC_o(\underline{X})$, is the subscheme of k^n determined by the ideal I_m, i.e.,

$$k[TC_o(\underline{X})]=k[x_1,\dots,x_n]/I_m$$

where I_m is generated by the homogeneous part of lowest degree of every element of I. Notice that here it is no more true that I_m is generated by the homogeneous parts of lowest degree of each generator f_i of I. Moreover, since $I_\ell\subset I_m$, the tangent cone is a closed subscheme of the tangent space $T_o(\underline{X})$. Again if x is an arbitrary geometric point of $\underline{X}$, we define the tangent cone to $\underline{X}$ at x, $TC_x(\underline{X})$ as the tangent cone $TC_o(\underline{X}')$ where X' is the translated scheme of $\underline{X}$ under the translation taking x to o.

Both tangent space and tangent cone contain local informations of the scheme $\underline{X}$ in a neighborhood of $x\in\underline{X}$. We have also seen a description of both tangent space

and tangent cone in terms of the local algebra $\mathcal{O}_x(\underline{X})$. The advantage of this point of view is that it provides a description of the tangent space and the tangent cone which is intrinsic, i.e., it is independent of particular embedding of X (which is the same thing as a particular choice of generators for I). This method is as follows:

Let $m_x\triangleleft k[\underline{X}]$ be the maximal ideal corresponding to $x\in\underline{X}$. Let $\mathcal{O}_x(\underline{X}) := k[\underline{X}]_{m_x}$. This is a local k- algebra with the maximal ideal $n_x=m_x\mathcal{O}_x(\underline{X})$, and $\mathcal{O}_x(\underline{X})/n_x\cong k$. We equip $\mathcal{O}_x=\mathcal{O}_x(\underline{X})$ with the <u>n_x- adic filteration</u> which is the following $\mathbb{Z}$- filteration

$$\cdots\subset n_x^i\subset n_x^{i-1}\subset\cdots\subset n_x\subset\mathcal{O}_x=\mathcal{O}_x=\cdots=\mathcal{O}_x=\cdots$$

The <u>associated graded algebra</u> of this filteration is

$$gr(\mathcal{O}_x)=\cdots\oplus\frac{n_x^i}{n_x^{i+1}}\oplus\frac{n_x^{i-1}}{n_x^i}\oplus\cdots\oplus\frac{n_x}{n_x^2}\oplus k\oplus 0\oplus 0\oplus\cdots$$

and we have seen that

(i) $k[T_x(\underline{X})]\cong k[\frac{n_x}{n_x^2}]$,

(ii) $k[TC_x(\underline{X})]\cong gr(\mathcal{O}_x(\underline{X}))$.

It follows from (i) that

$$T_x(\underline{X})=Hom_{k\text{-}lin}\left(\frac{n_x}{n_x^2},k\right),\quad\text{by dualizing,}$$

$$\cong Hom_{k\text{-}lin}\left(\frac{m_x}{m_x^2},k\right),\quad\text{as shown before.}$$

We have also seen that this interpretation of the tangent space is equivalent to the view in which the tangent space $T_x(\underline{X})$ is the space of point derivations $Der_x(\mathcal{O}_x(\underline{X}))$ (or equivalently of point derivations $Der_x(k[\underline{X}])$) as follows: a linear map $D:\mathcal{O}_x\longrightarrow k$ (resp. $D:k[\underline{X}]\longrightarrow k$)

is a point derivation at x, if

$$D(fg) = D(f)\,g(x) + f(x)\,D(g)$$

for all $f, g \in \mathcal{O}_x$ (resp. $f, g \in k[X]$); we have seen that there is an isomorphism

$$\mathrm{Hom}_k\left(\frac{n_x}{n_x^2}, k\right) \cong \mathrm{Der}_x(\mathcal{O}_x(X)) \quad \leftarrow (55)$$

because any linear map $D : \mathcal{O}_x \longrightarrow k = \mathcal{O}_x/n_x$ (resp. $D : k[X] \longrightarrow k = k[X]/m_x$) is zero on n_x^2 (resp. on m_x^2), hence it induces a linear mapping $\delta : \frac{n_x}{n_x^2} \longrightarrow k$, i.e., an element $\delta \in \mathrm{Hom}_k(\frac{n_x}{n_x^2}, k)$. Conversely, for any given $\delta \in \mathrm{Hom}_k(\frac{n_x}{n_x^2}, k)$ we may define a mapping

$$D_\delta : f \longmapsto \delta(f_x - f(x)) \quad , \quad \forall f \in \mathcal{O}_x(X)$$

where $f_x = f \bmod n_x^2$. Since $(f - f(x))(x) = f(x) - f(x) = 0$, $f - f(x) \in n_x$. $\Rightarrow f_x - f(x) \in n_x/n_x^2$. Then one easily verifies that D_δ is a point derivation, i.e.,

$$D_\delta(fg) = f(x)\,D_\delta(g) + g(x)\,D_\delta f(x) ,$$

proving the isomorphism (55). ∎

We continue our discussion with the following observation. Recall that a morphism $X \xrightarrow{\phi} Y$ of irreducible affine varieties is said to be dominant if the image $\phi(X)$ is Zariski dense in Y.

<u>The morphism $X \xrightarrow{\phi} Y$ is dominant iff the induced algebra map $\phi^* : k[Y] \longrightarrow k[X]$ is injective.</u>

To see this let $f \in k[Y]$; then $\phi^*(f)$ is by def. the composition $X \xrightarrow{\phi} Y \xrightarrow{f} k$.

$$\therefore \phi^*(f) = 0 \iff f(\phi(X)) = 0 \iff f(\overline{\phi(X)}) = 0$$

$$\iff f = 0 , \quad \text{for } \phi(X) \text{ is dense in } Y, \text{ so } \overline{\phi(X)} = Y .$$

Therefore, a dominant morphism

$$\phi : X \longrightarrow Y$$

induces a field extension $\phi^* : k(Y) \hookrightarrow k(X)$ of the function fields.

By some standard argument in algebraic geometry one can establish the following result.

<u>Proposition 20.</u> Let $X \xrightarrow{\phi} Y$ be a dominant morphism of irreducible affine varieties. Then there exists a Zariski dense subset $\mathcal{U} \hookrightarrow X$ such that $d\phi_x$ is surjective for all $x \in \mathcal{U}$.

<u>Proof</u>. See the refrences on algebraic geometry, or see LeBruyn. ∎

We will use this result to compute the tangent space to the orbit closures.

<u>Example</u>. Consider the algebraic group scheme $\underline{GL_n}$, given by its associated functor

$$\underline{GL_n}(-) = \mathrm{Hom}_{k\text{-alg}}(k[\underline{GL_n}], -) \quad , \quad k[\underline{GL_n}] = \frac{k[x_{11}, x_{12}, \ldots, x_{nn}; t]}{(t \cdot \det X - 1)} .$$

This associates to any commutative k-algebra R the set $\underline{GL_n}(R) = \mathrm{Hom}_{k\text{-alg}}(k[\underline{GL_n}], R)$ and every element $\chi \in \underline{GL_n}(R)$ is fully determined by the images $\chi(x_{ij}) = r_{ij} \in R$ and $\chi(t) = d \in R$. Therefore, χ specifies an $n \times n$ matrix with coefficients in R

$$m_\chi (r_{ij})_{i,j} \in M_n(R) \quad , \quad \text{s.t.,} \quad d \cdot \det(m_\chi) = 1 .$$

Thus, $\underline{GL_n}(R)$ is the set of $n \times n$ invertible matrices with entries in R.

Let $\underline{X}$ be a <u>closed</u> GL_n-stable subscheme of $\underline{GL_n}$ (this corresponds to a normal subgroup) and let $x \in \underline{X}$ be a geometric point of $\underline{X}$, and consider the

the orbit closure $\overline{O(x)} = \overline{GL_n \cdot x}$ of x. Because the orbit map $GL_n \twoheadrightarrow GL_n \cdot x \hookrightarrow \overline{O(x)}$ is dominant, we have that $k[\overline{O(x)}] \hookrightarrow k[GL_n]$ and because this is an inclusion into a connected component of GL_n (which is, therefore, irreducible) $k[\overline{O(x)}]$ is an integral domain, since every subring of an integral domain is integral.
$\Rightarrow$ $\overline{O(x)}$ is an irreducible affine variety.

Consider the orbit map

$$GL_n \xrightarrow{\mu} \overline{O(x)} \hookrightarrow \underline{X}$$
$$\overline{O(x)} \cong GL_n/Stab(x)$$

where the stabilizer subgroup $Stab(x)$ is the fiber $\mu^{-1}(x)$; it is a closed subgroup of GL_n. The Lie algebra $\mathfrak{gl}_n = T_e(GL_n) = M_n$. We will show that the differential of the orbit map μ at $e = 1_n$, i.e.

$$d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\underline{X})$$

satisfies the following properties

$$\begin{cases} Ker(d\mu_e) = stab(x) & \text{, the Lie algebra of } Stab(x), \\ Im(d\mu_e) = T_x(\overline{O(x)}) & \end{cases} \quad \leftarrow (56)$$

By proposition 20 there exists a dense open subset $\mathcal{U} \subset GL_n$ s.t. $d\mu_g$ is surjective for all $g \in \mathcal{U}$. By GL_n-equivariance of μ, i.e., by the commutativity of

$$\begin{array}{ccc} GL_n & \xrightarrow{\mu} & \overline{O(x)} \\ g\cdot \downarrow & & \downarrow g\cdot \\ GL_n & \xrightarrow{\mu} & \overline{O(x)} \end{array}$$

it follows that $d\mu_g$ is surjective for all GL_n, and in particular

$$d\mu_e : \mathfrak{gl}_n \longrightarrow T_x(\overline{O(x)}) \quad \leftarrow (57)$$

is surjective. $\Rightarrow Im(d\mu_e) = T_x(\overline{O(x)})$.

Next, notice that all fibers of μ over $O(x)$ have the same dimension for

suppose $x' = g\cdot x$. $\Rightarrow$
$$Stab(x') := \mu^{-1}(x') = \mu^{-1}(g\cdot x) = g\cdot \mu^{-1}(x) \text{, by equivariance} = g\cdot Stab(x)$$

where $g\cdot$ denotes the action of GL_n by conjugation. Therefore,

$$\dim \mu^{-1}(x') = \dim Stab(x) = \dim \mu^{-1}(x).$$

We now use the following result from algebraic geometry.

<u>Proposition 21.</u> Let $X \xrightarrow{\phi} Y$ be a dominant morphism of irreducible affine varieties. Then for every $x \in X$ and every irreducible component C of the fiber $\phi^{-1}(\phi(x))$ we have

$$\dim C \geq \dim X - \dim Y.$$

Moreover, there is a nonempty open subset U of Y contained in the image $\phi(X)$ s.t. for all $u \in U$ we have

$$\dim \phi^{-1}(u) = \dim X - \dim Y. \quad \blacksquare$$

We can, therefore, write (using $GL_n \twoheadrightarrow \overline{O(x)} = \overline{GL_n/Stab(x)}$)

$$\dim \mu^{-1}(x) = \dim GL_n - \dim Stab(x), \quad \text{or}$$
$$\dim GL_n = \dim \overline{O(x)} + \dim Stab(x)$$

which when combined with the surjectivity (57), a dimension count, gives $Ker(d\mu_e) = stab(x)$. $\blacksquare$

<u>The Normal spaces.</u> We now want to give an algebraic interpretation of the normal spaces to the orbits of GL_n-action on $\underline{Rep_n} A$ and $\underline{Rep_n^{tr}} A$, for an affine Cayley-Hamilton algebra A.

Let K and L be two representations of A of dimensions k and l respectively. A representation V of A of dim. k+l is said to be an extension of L by K if there exists a short exact sequence of A-modules

$$e: \; 0 \longrightarrow K \longrightarrow V \longrightarrow L \longrightarrow 0 \; .$$

On the pairs (V, e) specifying extensions of L by K one defines an equivalence relation $(V,e) \sim (V', e')$ iff there exists an isomorphism $\phi : V \longrightarrow V'$ of left A-modules s.t. the diagram below is commutative

$$\begin{array}{ccccccccc} e: & 0 \longrightarrow & K & \longrightarrow & V & \longrightarrow & L & \longrightarrow 0 \\ & & \downarrow{\scriptstyle id_K} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_L} & \\ e': & 0 \longrightarrow & K & \longrightarrow & V' & \longrightarrow & L & \longrightarrow 0 \end{array}$$

The set of equivalence classes of extensions of L by K is denoted by $Ext^1_A(L, K)$.

There is an alternative, and particularly useful, interpretation of $Ext^1_A(L,K)$ as follows:

let $\rho : A \longrightarrow M_k(k)$ and $\sigma : A \longrightarrow M_l(k)$ be representations of A defining the left A-modules K and L respectively. For the extension module (V, e) the underlying k-vector space is $K \oplus L$ and the left A-module structure on V gives an algebra map (i.e., a representation of A)

$$\mu : A \longrightarrow M_{k+l}(k).$$

We can represent the action of $a \in A$ on V by the left multiplication of the block matrix

$$\mu(a) = \begin{pmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{pmatrix}$$

on $K \oplus L$, where $\lambda(a) \in Mat(k \times l, k)$; so λ is a linear mapping $\lambda : A \longrightarrow Hom_k(L, K)$.

Applying the fact that $\mu(a)$ is an algebra map one gets:

$$\mu(a)\mu(a') = \mu(aa') \Rightarrow \underbrace{\begin{pmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{pmatrix}\begin{pmatrix} \rho(a') & \lambda(a') \\ 0 & \sigma(a') \end{pmatrix}}_{\begin{pmatrix} \rho(a)\rho(a') & \rho(a)\lambda(a') + \lambda(a)\sigma(a') \\ 0 & \sigma(a)\sigma(a') \end{pmatrix}} = \begin{pmatrix} \rho(aa') & \lambda(aa') \\ 0 & \sigma(aa') \end{pmatrix}$$

$$\therefore \quad \lambda(aa') = \rho(a)\lambda(a') + \lambda(a)\sigma(a') \qquad \leftarrow (58)$$

Let us denote the set of all linear maps
$$\lambda : A \longrightarrow Hom_k(L, K)$$
satisfying (58) by $Z(L,K)$ and call it the space of cycles.

(Notice that if $K = L$, then $Z(K,K)$ is the vector space of ρ-derivations $Der_\rho(A)$ from A to $M_k(k)$.) Clearly, two extensions of L by K corresponding to two cycles $\lambda, \lambda' \in Z(L,K)$ are equivalent if there is an A-module isomorphism ϕ, with the block form

$$\phi = \begin{pmatrix} id_K & \beta \\ 0 & id_L \end{pmatrix}, \quad \beta \in Hom_k(L,K),$$

between them. The A-linearity of this map (i.e., $\phi(av) = a\phi(v)$, $\forall a \in A$, $\forall v \in V$) translates into the matrix relation

$$\forall a \in A: \begin{pmatrix} id_K & \beta \\ 0 & id_L \end{pmatrix}\begin{pmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{pmatrix} = \begin{pmatrix} \rho(a) & \lambda'(a) \\ 0 & \sigma(a) \end{pmatrix}\begin{pmatrix} id_K & \beta \\ 0 & id_L \end{pmatrix}$$

$$\Rightarrow \quad \forall a \in A: \quad \lambda(a) - \lambda'(a) = \rho(a)\beta - \beta\sigma(a) \; . \qquad \leftarrow (59)$$

Define the subspace of boundaries $B(L,K)$ by

$$Z(L,K) \supset B(L,K) = \{\delta \in Hom_k(L,K) \mid \exists \beta \in Hom_k(L,K) : \forall a \in A, \; \delta(a) = \rho(a)\beta - \beta\sigma(a)\} \quad (60)$$

Then, according to the above given def. of $Ext^1_A(L,K)$, we have

$$Ext^1_A(L,K) = \frac{Z(L,K)}{B(L,K)} \qquad \leftarrow (61)$$

This definition is useful for computational purposes as the following examples show.

Example 1. Let A be an affine k-algebra generated by m elements $\{a_1, \dots, a_m\}$ and $\rho : A \to M_n(k)$ be an algebra morphism, i.e., $\rho \in Rep_n A$ is an n-dim. representation of A which defines a left A-module M.

We will show that the normal space to the orbit-closure $C_\rho = \overline{O(\rho)}$ of ρ has the following description:

$$N_\rho(C_\rho) := \frac{T_\rho(Rep_n A)}{T_\rho(C_\rho)} = Ext^1_A(M,M). \quad \leftarrow (62)$$

We have already noticed that the vector space of cycles $Z(M,M)$ = the space of ρ-derivations of A in $M_n(k)$, $Der_\rho(A)$. However, we have seen that $Der_\rho(A) = T_\rho(Rep_n A)$; moreover, for the orbit map $GL_n \xrightarrow{\mu} C_\rho \hookrightarrow M_n^m$ the differential $d\mu_e : gl_n = M_n(k) \to T_\rho(C_\rho)$ is surjective. Now, ρ is determined by its values on the generators of A: $\rho = (\rho(a_1), \dots, \rho(a_m)) \in M_n^m$, and the action of GL_n on ρ is given by simultaneous conjugation. It follows that for every $R \in gl_n = M_n(k)$

$$d\mu_e(R)(\rho(a_i)) = (1_n + \varepsilon R)\cdot\rho(a_i)\cdot(1_n - \varepsilon R) = \rho(a_i) + \varepsilon(R\rho(a_i) - \rho(a_i)R),$$

and hence by definition of the differential, we get

$$d\mu_e(R)(\rho(a)) = R\rho(a) - \rho(a)R, \quad \forall a \in A$$

$$\therefore \; d\mu_e(R) \in B(M,M).$$

As the differential map is surjective, i.e., its image at e is $T_\rho(C_\rho)$, we have $T_\rho(C_\rho) = B(M,M)$. Hence

$$N_\rho(C_\rho) = \frac{Der_\rho(A) = Z(M,M)}{B(M,M)} = Ext^1_A(M,M). \quad \blacksquare$$

Example 2. Let A be a Cayley-Hamilton algebra with a trace map tr_A and trace generated by $\{a_1, \dots, a_m\}$. Let $\rho \in Rep_n^{tr} A$, i.e., $\rho : A \to M_n(k)$ is a trace preserving algebra morphism; so the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & M_n(k) \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr} \\ A & \xrightarrow{\rho} & M_n(k) \end{array}$$

is commutative. This determines a left A-module M.

Any cycle $A \xrightarrow{\lambda} M_n(k)$, $\lambda \in Z(M,M) = Der_\rho(A)$ determines an algebra morphism.

$$\rho + \varepsilon\lambda : A \to M_n(k[\varepsilon])$$

(because λ is a ρ-derivation, $\rho + \lambda\varepsilon$ is an algebra morphism).

We have already seen that the tangent space $T_\rho(Rep_n^{tr} A)$ is the subspace $Der_\rho^{tr}(A) \subset Der_\rho(A)$ of all trace preserving ρ-derivations; i.e., those satisfying

$$\lambda(tr_A(a)) = tr(\lambda(a)), \quad \forall a \in A.$$

Also notice that all boundaries

$$\delta \in B(M,M) = \{\delta \in Hom_k(M,M) \mid \exists m \in M_n(k) = Hom_k(M,M) : \delta(a) = \rho(a)\cdot m - m\cdot\rho(a)\}$$

are trace preserving, because

$$\begin{aligned} \delta(tr_A(a)) &= \rho(tr_A(a))\cdot m - m\cdot\rho(tr_A(a)) \\ &= tr(\rho(a))\cdot m - m\cdot tr(\rho(a)), \quad \text{since } \rho \text{ is } tr\text{-preserving} \\ &= 0, \quad \text{because } tr(\rho(a)) \in \text{center of } M_n(k) \\ &= tr(m\cdot\rho(a) - \rho(a)\cdot m), \quad \text{by the property of } tr \\ &= tr(\delta(a)); \end{aligned}$$

$\therefore \; B^{tr}(M,M) = B(M,M)$. We can, therefore, define the space of trace preserving self-extensions

$$Ext^{1\,tr}_A(M,M) = Der_\rho^{tr}(A)/B(M,M)$$

and as before, the normal space to the orbit closure $C_\rho = \overline{O(\rho)}$ is equal to

$$N_\rho(C_\rho) := \frac{T_\rho(Rep_n^{tr} A)}{T_\rho(C_\rho)} = Ext^{1\,tr}_A(M,M). \quad \blacksquare$$

(4.8) Luna's étale slices

Let A be an affine k-algebra and $\xi \in \underline{\mathrm{ISS}_n A}$ be a point of the scheme under consideration corresponding to an n-dim. semi-simple representation of A.

Our purpose in this and the next section is to study the étale local structure of $\underline{\mathrm{ISS}_n A}$ near ξ and the étale local GL_n-structure of the representation variety $\underline{\mathrm{Rep}_n A}$ near the closed orbit $\mathcal{O}(M_\xi) = GL_n \cdot M_\xi$.

We shall first outline the basic idea in the setting of differential geometry.

(A) C^∞-slices

Let M be a C^∞-manifold on which a compact Lie group G acts differentiably. By the usual averaging process one can define a G-invariant Riemannian metric on M as follows:

let $\gamma_{ab}(x)$ be any Riemannian metric on M with the corresponding scalar product $\langle\ ,\ \rangle_x$ on the tangent space T_x to M at x. Define a new scalar product $(\ ,\ )_x$ on T_x (and thereby a new metric on M) by setting

$$(\xi, \eta)_x = \frac{1}{\mu(G)} \int_G \langle g_*(\xi), g_*(\eta) \rangle_{g(x)}\, d\mu(g)$$

where $x \in M$, $\xi, \eta \in T_x$, $g \in G$ and g_* is the map induced on the tangent spaces by the smooth g action on the manifold and $\mu(G)$ is the volume of G, and $d\mu(g)$ is a right invariant measure on G. We have

$$(g'_*(\xi), g'_*(\eta))_{g'(x)} = \frac{1}{\mu(G)} \int_G \langle (gg')_*(\xi), (gg')_*(\eta) \rangle_{gg'(x)}\, d\mu(gg')$$
$$= 1/\mu(G) \int_G \langle g''_*(\xi), g''_*(\eta) \rangle_{g''(x)}\, d\mu(g'')$$

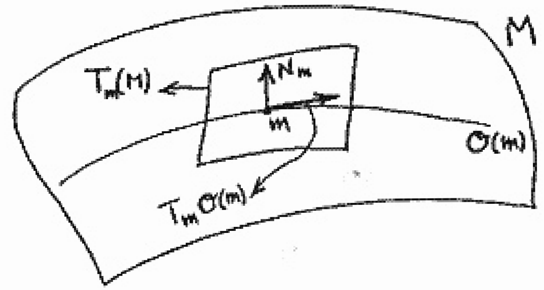
$= (\xi, \eta)_x$,

which shows the G-invariance of the new metric under the action of G.

We shall denote the G-orbit of $m \in M$ by $\mathcal{O}(m) := G \cdot m$. We also define the stabilizer subgroup $H \subset G$ by

$$H = \mathrm{Stab}_G(m) = \{ g \in G \mid g \cdot m = m \}.$$

The $\underline{\text{normal space } N_m}$ is defined to be the orthogonal complement of the tangent space to $\mathcal{O}(m)$ at m in the tangent space $T_m(M)$.



This gives a decomposition of $\underline{H\text{-vector spaces}}$

$$T_m(M) = T_m\mathcal{O}(m) \oplus N_m.$$

Remark. When G acts smoothly on M, if $x, y \in M$ s.t. $y = g \cdot x$ then there is an induced map $g_* : T_x(M) \to T_y(M)$. Therefore, if $H = \mathrm{Stab}_G(m)$, then $h_* : T_m(M) \to T_m(M)$ is a mapping of tangent space $T_m(M)$ and moreover $h_*(N_m) \subset N_m$; hence the term H-vector space. ∎

The normal spaces N_x as x moves over $\mathcal{O}(m)$ defines a vector bundle (with the base $\mathcal{O}(m) \cong G/H$) called the $\underline{\text{normal bundle}}$, denoted by $\mathcal{N}$; one then establishes that $\mathcal{N} \cong G \times_H N_m := G \times N_m / H$,

where the equivalence relation in this definition is given by the following action of H on $G \times N_m$:

$$h \cdot (g, n) = (gh, h^{-1} n).$$

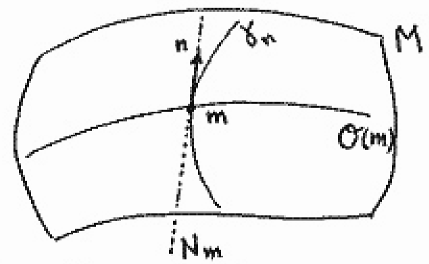
Notice that $\mathcal{N}$ can also be considered as the associated fiber bundle, with typical fiber N_m, of the principal G- bundle $G(G/H, H)$.

Any point $n \in \mathcal{N}$ determines a geodesic in M:

$$\gamma_n : \mathbb{R} \longrightarrow M \quad \text{defined by} \quad \begin{cases} \gamma_n(0) = p(n) \\ \frac{d\gamma_n}{dt}(0) = n \end{cases}$$

where p is the projection in $\mathcal{N}$. Using this geodesic we can define a G- equivariant exponential map from the normal bundle $\mathcal{N}$ to the manifold M via

$$\mathcal{N} \xrightarrow{exp} M$$
$$exp(n) := \gamma_n(1)$$



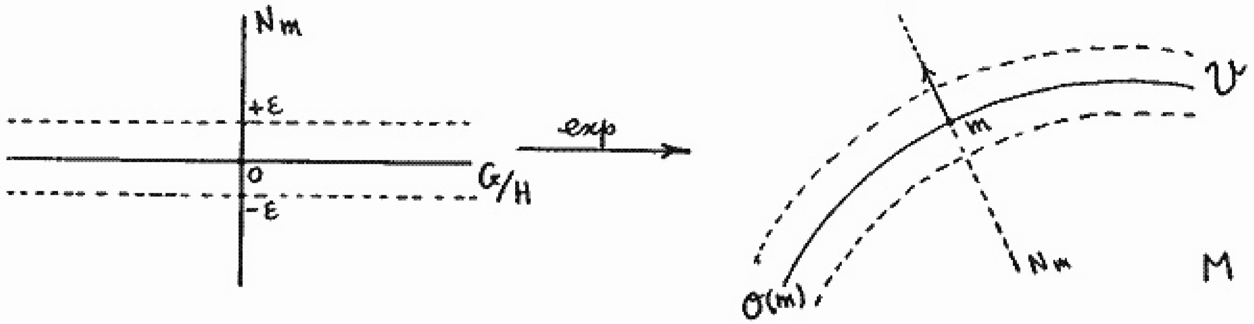
For $\varepsilon > 0$ we define the C^∞- slice S_ε to be

$$S_\varepsilon = \{ n \in N_m \mid \|n\| < \varepsilon \}$$

where the norm is defined by the G- invariant scalar product on T_m. Then $G \times_H S_\varepsilon$ is the neighborhood of the zero section in the normal bundle $G \times_H N_m$. It then follows that there exists a G- equivariant exponential map

$$G \times_H S_\varepsilon \xrightarrow{exp} M$$

which for small enough ε gives a diffeomorphism with a G- stable tubular neighborhood $\mathcal{U}$ of the orbit $O(m)$ in M:



If we assume moreover that the actions of G on M and of H on N_m are such that M/G and N_m/H are manifolds, we then have the following situation

$$\begin{array}{ccccc} G \times_H S_\varepsilon & \xrightarrow[\cong]{exp} & \mathcal{U} & \hookrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ S_\varepsilon/H & \xrightarrow{\cong} & \mathcal{U}/G & \hookrightarrow & M/G \end{array}$$

where since $G \times_H S_\varepsilon := G \times S_\varepsilon / H$, then $G \times_H S_\varepsilon / G \cong S_\varepsilon / H$.

The lower line in the above diagram gives a local diffeomorphism between a neighborhood of $\bar{0}$ in N_m/H and a neighborhood of the point $\bar{m}$ in M/G corresponding to the orbit $O(m)$.

Returning to the setting of the orbit $O(M_\xi)$ in $\underline{Rep_n A}$ we would like to define a GL_n-equivariant morphism from an associated fiber bundle to $\underline{Rep_n} A$

$$GL_n \times_{GL(\xi)} N_\xi \xrightarrow{e} \underline{Rep_n} A$$

where $GL(\xi)$ is the stabilizer subgroup of M_ξ and N_ξ is the normal space to the orbit $O(M_\xi)$.

However, we do not have an exponential map in the setting of algebraic geometry; the map e has to be something which is called an etale map.

(B) Étale slices & Luna's theorem.

Def. Let V and W be smooth varieties. A regular map $\alpha : V \longrightarrow W$ is said to be étale at $a \in V$, if $(d\alpha)_a : T_a(V) \longrightarrow T_b(W)$, $b = \alpha(a)$, is an isomorphism. α is étale if it is étale at all points of V. ∎

Example. A regular map $\alpha = (P_1, \dots, P_n) : \mathbb{A}^n \to \mathbb{A}^n$ is étale at a iff

$$\text{rank Jac}(P_1, \dots P_n)(a) = n$$

i.e.,

$$\det\left(\frac{\partial P_i}{\partial x_j}\right)(a) \neq 0 .$$

This is because the induced map on the tangent spaces (i.e. $(d\alpha)_a$) has matrix $\text{Jac}(P_1, \dots, P_n)(a)$. ∎

In differential geometry the inverse function theorem says that a map α that is étale at a point a is a local isomorphism there, i.e., $\exists$ open neighborhoods U, U' of a and $\alpha(a)$ such that $\alpha | U : U \to U'$ is an isomorphism. This is not true in algebraic geometry at least for the Zariski topology; a map can be etale at a point without being a local isomorphism.

Lemma 22. Let W and V be non-singular varieties. If $\alpha : W \to V$ is etale at P then it is étale at all points in an open neighborhood of P. ∎

The results presented in the rest of this section hold for any reductive group G; in particular we will be mainly interested in the case $G = GL_n$ or subgroups of GL_n of the form $GL_{e_1} \times GL_{e_2} \times \cdots \times GL_{e_k}$.

Let $\underline{X}$ and $\underline{Y}$ be (not necessarily reduced) affine GL_n-varieties and $\psi : \underline{Y} \to \underline{X}$ be a GL_n-equivariant map, i.e., the diagram

$$\begin{array}{ccc} \underline{Y} & \xrightarrow{\psi} & \underline{X} \\ g\cdot\downarrow & & \downarrow g\cdot \\ \underline{Y} & \xrightarrow{\psi} & \underline{X} \end{array}$$

is commutative.

Passing to the quotient varieties with $\bar{\psi}$ the induced map on the quotients, we have the diagram

$$\begin{array}{ccc} \underline{Y} & \xrightarrow{\psi} & \underline{X} \\ \pi_Y\downarrow & & \downarrow \pi_X \\ \underline{Y}/\!/GL_n & \xrightarrow{\bar{\psi}} & \underline{X}/\!/GL_n \end{array}$$

Let $\underline{Y} \ni y \overset{\psi}{\longmapsto} x = \psi(y) \in \underline{X}$. We assume that the following conditions hold

(i) ψ is etalé at y.

(ii) The GL_n-orbits $\mathcal{O}(y) \subset \underline{Y}$ and $\mathcal{O}(x) \subset \underline{X}$ are closed (this is the reason we have used the double slash notation for the quotients). That is, in the representation varieties we restrict to semi-simple representations.

(iii) The stabilizer subgroups are equal

$$\text{Stab}(x) = \text{Stab}(y) \subset GL_n .$$

In the case of representation varieties, for a semi-simple n-dim. representation M given by

$$M = M_1^{\oplus e_1} \oplus M_2^{\oplus e_2} \oplus \cdots \oplus M_k^{\oplus e_k}$$

where M_i are simple (or irreducible), this stabilizer subgroup is

$$GL(\alpha) = \begin{bmatrix} GL_{e_1} \otimes 1_{d_1} & & 0 \\ & \ddots & \\ 0 & & GL_{e_k} \otimes 1_{d_k} \end{bmatrix} \hookrightarrow GL_n$$

where $d_i = \dim. M_i$ (more on this point later). Hence the stabilizer subgroup is again reductive.

Theorem 23. (Luna's fundamental lemma)

Consider a GL_n-equivariant map $\underline{Y} \xrightarrow{\psi} \underline{X}$, $y \in \underline{Y}$, $x = \psi(y)$, and ψ is etalé at y. Suppose that the orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are closed and that ψ is injective on $\mathcal{O}(y)$. Then there is an affine open neighborhood of y, say $\mathcal{U} \hookrightarrow \underline{Y}$, such that

(1) $\mathcal{U} = \pi_Y^{-1}(\pi_Y \mathcal{U})$ and $\pi_Y(\mathcal{U}) = \mathcal{U}/\!/GL_n$.

(2) ψ is etalé on $\mathcal{U}$ and its image, $\psi(\mathcal{U})$ is affine.

(3) The induced morphism $\mathcal{U}/\!\!/GL_n \xrightarrow{\bar{\psi}|\mathcal{U}/\!\!/GL_n} \underline{X}/\!\!/GL_n$ is étale.

(4) The following diagram is commutative:

$$\begin{array}{ccc} Y \supset \mathcal{U} & \xrightarrow{\psi|\mathcal{U}} & \underline{X} \\ \pi_{\mathcal{U}} \downarrow & & \downarrow \pi_{\underline{X}} \\ \mathcal{U}/\!\!/GL_n & \xrightarrow{\bar{\psi}|\mathcal{U}/\!\!/GL_n} & \underline{X}/\!\!/GL_n \end{array}$$

(See LeBruyn as given in the references. ∎

Suppose H is a reductive subgroup of G acting on a variety V; then one defines an H-action on $G \times V$ via the map

$$h.(g,v) = (gh^{-1}, h.v)$$

and the corresponding quotient $G \times_H V := G \times V/H$ is the associated fiber bundle of the principal bundle $G(G/H, H)$; and it aquires a G-action via multiplication on the left in the first component. The quotient of $G \times_H V$ with respect to this G-action is shown to be

$$G \times_H V/G \cong V/H.$$

The Luna's slice theorem in invariant theory takes the following form when adopted to the situation under consideration.

Theorem 24. (Luna's slice theorem)

Let $\underline{X}$ be an affine GL_n-variety with quotient map $\underline{X} \xrightarrow{\pi_X} \underline{X}/\!\!/GL_n$. Let $x \in \underline{X}$ (the double slash notation indicates that either $O(x)$ is closed or else we consider $\overline{O(x)}$) and suppose $Stab(x) = H$ is reductive. Then there is a locally closed affine subscheme $\underline{S} \hookrightarrow \underline{X}$ containing x with the following properties

LS_1. (1) $\underline{S}$ is an affine H-variety.

(2) The action $GL_n \times \underline{S} \longrightarrow \underline{X}$ induces an étale GL_n-

equivariant morphism

$$GL_n \times_H \underline{S} \xrightarrow{\psi} \underline{X}$$

with affine image.

(3) The induced quotient map

$$(GL_n \times_H \underline{S})/\!\!/GL_n \cong \frac{\underline{S}}{H} \xrightarrow{\psi/\!\!/GL_n} \underline{X}/\!\!/GL_n$$

is étale.

(4) The diagram below is commutative

$$\begin{array}{ccc} GL_n \times_H \underline{S} & \xrightarrow{\psi} & \underline{X} \\ \downarrow & & \downarrow \pi_{\underline{X}} \\ \underline{S}/\!\!/H & \xrightarrow{\psi/\!\!/GL_n} & \underline{X}/\!\!/GL_n \end{array}$$

LS_2. If in addition we assume that $x \in \underline{X}$ is a smooth point of $\underline{X}$, then we can choose a slice $\underline{S}$ such that the following properties are satisfied

(1) $\underline{S}$ is smooth at x,

(2) there is an H-equivariant morphism

$$\underline{S} \xrightarrow{\phi} T_x^{sm}\underline{S} \cong N_x \qquad (\text{"sm" for small})$$

with $\phi(x) = 0$ and $Im(\phi)$ is affine.

(3) The induced morphism $\underline{S}/\!\!/H \xrightarrow{\phi/H} N_x/\!\!/H$ is étale.

(4) The diagram below is commutative

$$\begin{array}{ccc} GL_n \times_H \underline{S} & \xrightarrow{GL_n \times_H \phi} & GL_n \times_H N_x \\ \downarrow & & \downarrow \\ \underline{S}/\!\!/H & \xrightarrow{\phi/H} & N_x/\!\!/H \end{array}$$

∎

For the proof see Luna's cited article in the references of this chapter.

(4.9) The module structure of the normal space to the representation schemes

Let A be an affine k-algebra generated by m elements $\{a_1, \cdots, a_m\}$. The Cayley-Hamilton algebra A_n of degree n is then generated by m elements, i.e., there is a trace preserving algebra epimorphism $\mathbb{T}_n^m \xrightarrow{\psi^*} A_n$. This implies that we have a GL_n-equivariant closed embedding of affine schemes

$$\underline{Rep_n} A = \underline{Rep_n^{tr}} A_n \overset{\psi}{\hookrightarrow} \underline{Rep_n^{tr}} \mathbb{T}_n^m = M_n^m .$$

Consider a point ξ of the quotient scheme

$$\underline{ISS_n} A = \underline{Rep_n^{tr}} A_n /\!/ GL_n \;;$$

ξ determines an isomorphism class of a semi-simple n-dimensional representation of A, say

$$M_\xi = M_1^{\oplus e_1} \oplus M_2^{\oplus e_2} \oplus \cdots \oplus M_k^{\oplus e_k}$$

where M_i are distinct simple (i.e. IR) representations of A, e_i is the multiplicity of M_i and let $d_i = \dim . M_i$. The numbers (e_i, d_i) determine the representation type $\tau(\xi)$ of ξ (or of M_ξ); $\tau(\xi) = (e_1, d_1; e_2, d_2; \cdots; e_k, d_k)$.

Choosing a basis of the module M_ξ adopted to this decomposition we find a point $x = (X_1, \cdots, X_m)$ in the orbit $\mathcal{O}(M_\xi) \hookrightarrow M_n^m$ such that each of the $n \times n$ matrices X_i is of the form

$$X_i = \begin{bmatrix} m_1^{(i)} \otimes 1_{e_1} & & & 0 \\ & m_2^{(i)} \otimes 1_{e_2} & & \\ & & \ddots & \\ 0 & & & m_k^{(i)} \otimes 1_{e_k} \end{bmatrix}$$

where each $m_j^{(i)} \in M_{d_j}(k)$, the ring of $d_j \times d_j$ matrices over k.

It is now easy to see that the stabilizer subgroup $Stab(x) \subset GL_n$ is the set of those invertible matrices, $g \in GL_n$

which commute with each X_i, $i = 1, \cdots, m$. That is, $Stab(x)$ is the multiplicative group of units of the centralizer in $M_n(k)$ of the algebra generated by X_i's; the algebra generated by X_i's is

$$\begin{bmatrix} M_{d_1}(k) \otimes 1_{e_1} & & & 0 \\ & M_{d_2}(k) \otimes 1_{e_2} & & \\ & & \ddots & \\ 0 & & & M_{d_k}(k) \otimes 1_{e_k} \end{bmatrix} \hookrightarrow M_n(k)$$

and the centralizer of this algebra in $M_n(k)$ is

$$\begin{bmatrix} 1_{d_1} \otimes M_{e_1}(k) & & & 0 \\ & 1_{d_2} \otimes M_{e_2}(k) & & \\ & & \ddots & \\ 0 & & & 1_{d_k} \otimes M_{e_k}(k) \end{bmatrix} \hookrightarrow M_n(k)$$

and the group of units of this subalgebra is

$$\begin{bmatrix} 1_{d_1} \otimes GL_{e_1} & & \\ & 1_{d_2} \otimes GL_{e_2} & \\ & & \ddots \\ & & & 1_{d_k} \otimes GL_{e_k} \end{bmatrix} \hookrightarrow GL_n \qquad \leftarrow (63)$$

$$\therefore \; Stab(x) \cong GL_{e_1} \times \cdots \times GL_{e_k} \qquad \leftarrow (64)$$

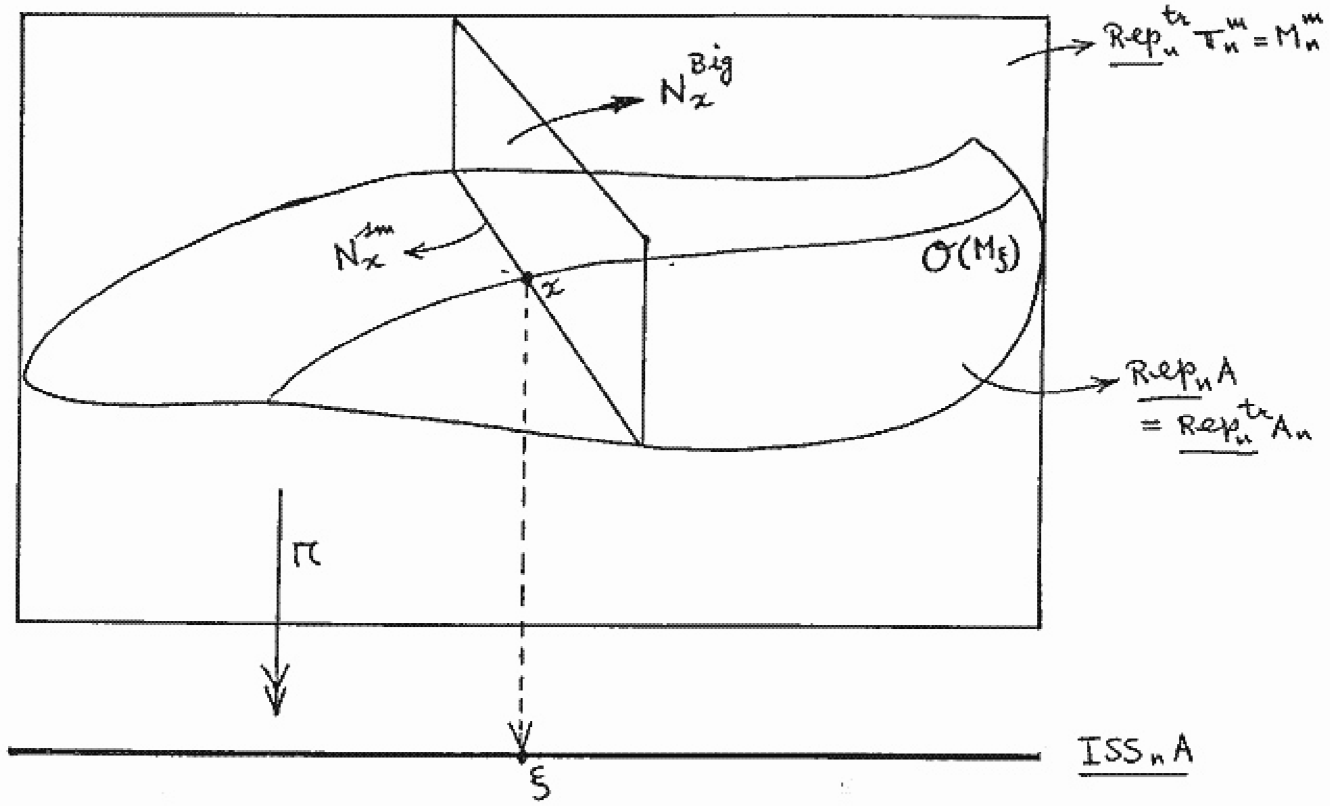
with the embedding $Stab(x) \hookrightarrow GL_n$ given by (63).

We can introduce an integral vector $\alpha = (e_1, \cdots, e_k)$ and write

$$Stab(x) \cong GL(\alpha) := GL_{e_1} \times \cdots \times GL_{e_k} .$$

Clearly a different choice of the point $x \in \mathcal{O}(M_\xi)$ gives a subgroup of GL_n conjugated to $Stab(x)$.

We now compute the normal space N_x^{big} to the orbit $\mathcal{O}(M_\xi)$ in $M_n^m = \underline{Rep}_n^{tr}\, \mathbb{T}_n^m$. The following figure shows the normal spaces N_x^{sm} and N_x^{Big}:



$$N_x^{sm} := \frac{T_x \underline{Rep}_n A}{T_x \mathcal{O}(M_\xi)} \lhd N_x^{Big} := \frac{T_x M_n^m}{T_x \mathcal{O}(M_\xi)} \qquad \leftarrow (65)$$

We have already seen that the normal space N_x^{sm} can be identified with the self extensions $Ext_A^1(M_\xi, M_\xi)$, and we wish to give a quiver description of this space.

The ideal is to describe the $GL(\alpha)$-module structure of N_x^{Big} = normal space to the orbit $\mathcal{O}(M_\xi)$ in M_n^m and then to identify the direct summand N_x^{sm}.

We know that all derivations of $M_n(k)$ are inner and hence each inner derivation is specified by an element $R \in M_n(k)$; so $Der(M_n(k)) = M_n(k)$. It then follows

that

$$Der(M_n^m) = Der(\underbrace{M_n \oplus \cdots \oplus M_n}_{m\text{-fold}}) = Der(M_n) \oplus \cdots \oplus Der(M_n) =$$
$$= M_n \oplus \cdots \oplus M_n = M_n^m . \implies T_x M_n^m = M_n^m ;$$

(this is rather clear for M_n^m is a vector space).

Moreover, because $\mathcal{O}(x) \cong G/Stab(x)$, $G \twoheadrightarrow \mathcal{O}(x)$ is surjective and hence the tangent map corresponding to the dominant morphism $G \twoheadrightarrow \mathcal{O}(x) \hookrightarrow \overline{\mathcal{O}(x)}$ is surjective. Hence if $x = (X_1, \dots, X_m) \in \mathcal{O}(M_\xi)$, the tangent space $T_x(\mathcal{O}(M_\xi))$ in M_n^m to the orbit is equal to the image of the linear map

$$gl_n = M_n(k) \longrightarrow M_n(k) \oplus \cdots \oplus M_n(k) = T_x M_n^m$$
$$R \longmapsto ([R, X_1], \dots, [R, X_m])$$

The kernel of this map is the centralizer of the subalgebra generated by the X_i's ; so we have an exact sequence of $GL(\alpha)$ ($= Stab(x)$) - modules

$$0 \to Lie\, GL(\alpha) = gl(\alpha) \to gl_n = M_n \longrightarrow T_x \mathcal{O}(x) \longrightarrow 0 \qquad \leftarrow (66)$$

(Corresponding to the exact sequence of groups (or a principal bundle)
$$0 \to Stab(x) \to GL_n \to GL_n/Stab(x) \cong \mathcal{O}(x) \to 0).$$

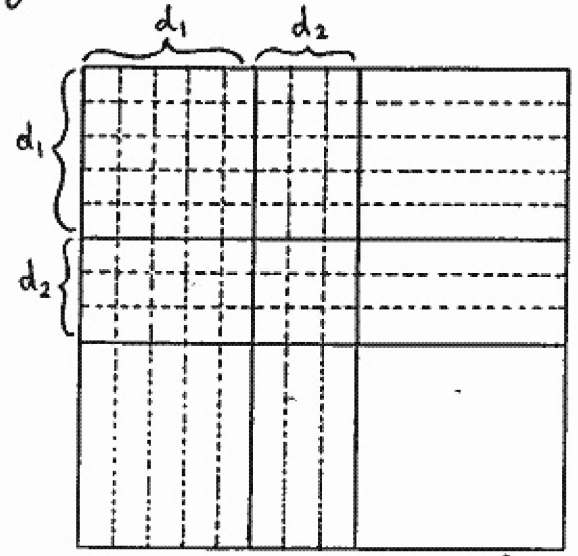
Because $GL(\alpha)$ is a reductive group, every $GL(\alpha)$-module is completely reducible, and so the sequence splits. Moreover, because $M_n^m = T_x M_n^m$, the normal space in M_n^m to the orbit $\mathcal{O}(M_\xi)$ is isomorphic, as a $GL(\alpha)$-module to

$$N_x^{Big} \cong \underbrace{M_n \oplus \cdots \oplus M_n}_{(m-1)\text{-fold}} \oplus gl(\alpha) \qquad \leftarrow (67)$$

with the action of $GL(\alpha)$ on N_x^{Big} is given by simultaneous conjugation (here we consider $GL(\alpha) \hookrightarrow GL_n$ as in (63)).

The link with quivers come from the following considerations:

The action of $GL(\alpha) \hookrightarrow GL_n$ (embedding given by (63)) by conjugation on M_n has the following block structure



(i) for each $1 \le i \le k$ we have d_i^2 copies of $GL(\alpha)$ - modules M_{e_i} on which GL_{e_i} acts by conjugation and other factors are trivial;
(ii) for all $1 \le i, j \le k$ we have $d_i d_j$ copies of $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ act via

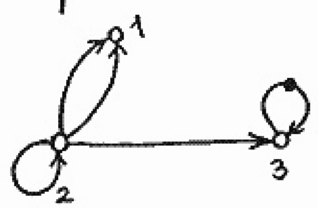
$$g \cdot m = g_i m g_j^{-1}, \quad \forall m \in M_{e_i \times e_j}, \ \forall g = (g_i, g_j) \in GL_{e_i} \times GL_{e_j}$$

and other factors of $GL(\alpha)$ act trivially. ∎

These $GL(\alpha)$ components are precisely the modules appearing in the representation space of quivers as we shall see now. We make a short deviation into quivers.

Def. (Marked quiver)

A marked quiver $Q^\bullet$ is a quiver such that some of its loops may acquire a marking; e.g.,



where the loop at the vertex v_3 is marked whereas that in the vertex v_2 is ordinary. ∎

Def. A representation of a marked quiver $Q^\bullet$ is a representation of the underlying quiver Q (i.e., the quiver obtained by forgetting the marks) such that the matrices corresponding to marked loops have zero trace. ∎

Fixing a dimension vector α we can write

$$\mathrm{Rep}_\alpha Q^\bullet = \{ V \in \mathrm{Rep}_\alpha Q \mid tr(V_a) = 0, \text{ if } a \text{ is a marked loop} \} \quad \leftarrow (68)$$

clearly $\mathrm{Rep}_\alpha Q^\bullet$ is a subvariety of the affine variety $\mathrm{Rep}_\alpha Q$. Moreover, since the path algebra kQ is smooth, by proposition 19, the corresponding representation variety $\mathrm{Rep}_\alpha Q$ is a smooth $GL(\alpha)$-variety and $\mathrm{Rep}_\alpha Q^\bullet$ is a smooth subvariety of $\mathrm{Rep}_\alpha Q$.

Def. The quiver trace algebra $\mathbb{T}Q$

Let R be the polynomial algebra over k in variables t_p where p is a word in arrows $a_j \in Q$, and is determined only up to a cyclic permutations.

Consider the algebra $\mathbb{T}Q := R \otimes_k kQ$. This is the same algebra obtained by adjoining formally all traces to the path algebra kQ. Clearly all $t_p = 0$ unless p is an oriented cycle, because if p is a word in arrows of Q then a cyclic permutation of p is zero in kQ; so we only retain those variables t_p where p is an oriented cycle in Q. We define a formal trace map on $\mathbb{T}Q$ by

$$tr(p) = t_p$$

if p is an oriented cycle in Q and $tr(p) = 0$ otherwise. ∎

Def. The Cayley-Hamilton algebra $\mathbb{T}_\alpha Q$

For a fixed dimension vector $\alpha = (d_1, \dots, d_k)$ with

$\sum_{i=1}^{k} d_i = n$ we define $\mathbb{T}_\alpha Q$ to be the quotient algebra

$$\mathbb{T}_\alpha Q = \mathbb{T}Q \big/ (\chi_a^{(n)}(a),\ tr(v_i) - d_i) \qquad \leftarrow (69)$$

where $\chi_a^{(n)}(a)$ are the C.H. relation of degree n, $a \in \mathbb{T}Q$. $\mathbb{T}_\alpha Q$ is a C.H. algebra of degree n with a decomposition of unity $1 = e_1 + \cdots + e_k$ into orthogonal idempotents s.t., $tr(e_i) = d_i$. ∎

Similar to what we did earlier for the Cayley-Hamilton algebras, one can establish the following

Theorem 25. (1) $\mathbb{T}_\alpha Q$ is the algebra of $GL(\alpha)$-equivariant maps from $Rep_\alpha Q$ to $M_n(k)$, i.e.,

$$\mathbb{T}_\alpha Q = M_n\left(k[Rep_\alpha Q]\right)^{GL(\alpha)} ;$$

(2) The quiver necklace algebra

$$\mathbb{N}_\alpha Q = k[Rep_\alpha Q]^{GL(\alpha)}$$

is generated by traces along all oriented cycles in the quiver Q of length bounded by n^2+1. ∎
(See Le Bruyn in the references of this chapter).

These results can be extended to marked quivers $Q^\bullet$ as follows. Let $\{l_1, l_2, \ldots, l_m\}$ be the marked loops in $Q^\bullet$ and define

$$\mathbb{N}_\alpha Q^\bullet = \mathbb{N}_\alpha Q \big/ (tr(l_1), \ldots, tr(l_m)) ,$$

$$\mathbb{T}_\alpha Q^\bullet = \mathbb{T}_\alpha Q \big/ (tr(l_1), \ldots, tr(l_m)) .$$

Then $\mathbb{T}_\alpha Q^\bullet$ is a smooth C.H. algebra; it is the algebra of $GL(\alpha)$-equivariant maps from $Rep_\alpha Q^\bullet$ to $M_n(k)$; i.e.,

$$\mathbb{T}_\alpha Q^\bullet = M_n\left(k[Rep_n Q^\bullet]\right)^{GL(\alpha)} .$$

The algebra $\mathbb{N}_\alpha Q^\bullet$ is the algebra of $GL(\alpha)$-polynomial

invariants of $Rep_\alpha Q^\bullet$,

$$\mathbb{N}_\alpha Q^\bullet = k[Rep_n Q^\bullet]^{GL(\alpha)}$$

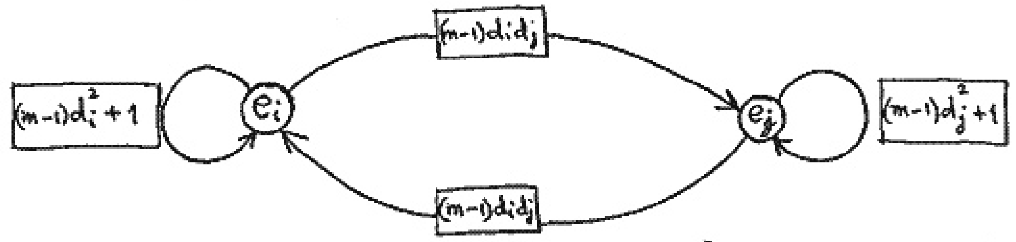
and it is generated by the traces along oriented cycles in the quiver Q of length bounded by n^2+1. ∎

We now go back to (67) and the $GL(\alpha)$-module structure of

$$N_x^{Big} = \underbrace{M_n \oplus \cdots \oplus M_n}_{(m-1)\ \text{factors}} \oplus\ gl(\alpha)$$

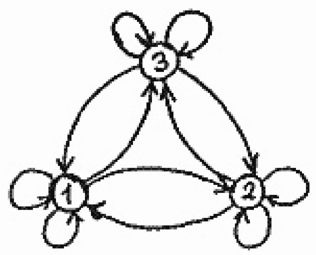
explained after (67). That argument justifies the following result:

Theorem 26. Let ξ be a representation of type $\tau = (e_1, d_1; \cdots; e_k, d_k)$ and let $\alpha = (e_1, \ldots, e_k)$. Then, the $GL(\alpha)$-module structure of the normal space N_x^{Big} in $Rep_n^{tr} \mathbb{T}_n^m = M_n^m$ to the orbit of the semi-simple n-dimensional representation $\mathcal{O}(M_\xi)$ is isomorphic to $Rep_\alpha Q_\xi$ where the quiver Q_ξ has k vertices and the subquiver on any two vertices v_i, v_j, $1 \leq i,j \leq k$, has the following form



That is, in each vertex v_i there are $(m-1)d_i^2+1$ loops and there are $(m-1)d_i d_j$ arrows from the vertex v_i to v_j for all $1 \leq i,j \leq k$. ∎

Example. Let $m=2$, $n=3$ and the representation type $\tau = (1,1; 1,1; 1,1)$, (notice that $n = \sum_i d_i$, $3 = 1+1+1$); i.e. M_ξ is the direct sum of three distinct one-dimensional representations. Then the quiver Q_ξ is



The following discussion establishes our next theorem. We say that A is smooth at $\xi \in \underline{ISS_n A}$ if the representation variety $\underline{Rep_n^{tr}} A_n$ ($= \underline{Rep_n} A$) is smooth at M_ξ. Before we can apply the Luna's slice theorem we have to specify the normal space N_x^{sm} to $O(M_\xi)$ in $\underline{Rep_n^{tr}} A_n$. We have the GL_n-equivariant embeddings

$$O(M_\xi) \hookrightarrow \underline{Rep_n^{tr}} A_n \hookrightarrow \underline{Rep_n^{tr}} \mathbb{T}_n^m = M_n^m ,$$

and the corresponding embeddings of the tangent spaces in x

$$T_x O(M_\xi) \hookrightarrow T_x \underline{Rep_n^{tr}} A_n \hookrightarrow T_x M_n^m .$$

Because $GL(\alpha)$ is reductive (and N_x^{big} is a $GL(\alpha)$-module) we obtain for the normal spaces to the orbit, as given by

$$N_x^{sm} = \frac{T_x \underline{Rep_n^{tr}} A_n}{T_x O(M_\xi)} \lhd N_x^{Big} = \frac{T_x M_n^m}{T_x O(M_\xi)}$$

a direct sum decomposition as $GL(\alpha)$-modules. We already know the isotypical decomposition of N_x^{Big} as the $GL(\alpha)$-module, $Rep_\alpha Q_\xi$ (given by theorem $\underline{26}$); this allows us to control N_x^{sm}: we only have to observe that arrows in Q_ξ correspond to simple $GL(\alpha)$-module whereas a loop at vertex v_i decomposes as $GL(\alpha)$-module into simples

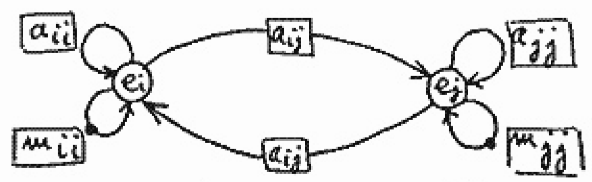
$$M_{e_i} = M^\circ_{e_i} \oplus k_{triv}$$

where k_{triv} is the 1-dimensional simple rep. with trivial $GL(\alpha)$-action and $M^\circ_{e_i}$ is the space of zero trace matrices in M_{e_i} (notice that any square matrix M can be written as $M = (M - tr(M)\mathbb{1}) + (tr(M))\mathbb{1} = M^\circ + (tr(M))\mathbb{1}$).
Any $GL(\alpha)$-submodule of N_x^{Big} can be represented by a marked quiver according to the following rules

(i) a loop at vertex v_i corresponds to the $GL(\alpha)$-module M_{e_i} on which GL_{e_i} acts by conjugation and the other factors act trivially;
(ii) a marked loop at vertex v_i correspond to the simple $GL(\alpha)$-module $M^\circ_{e_i}$ on which GL_{e_i} act by conjugation and the other factors act trivially;
(iii) an arrow from the vertex v_i to vertex v_j correspond to the simple $GL(\alpha)$-module $M_{e_i \times e_j}$ on which $GL_{e_i} \times GL_{e_j}$ act via $g \cdot m = g_i m g_j^{-1}$ and other factors act trivially.

Combining this with the previous result that the normal space is the space of self-extensions $Ext_A^1(M_\xi, M_\xi)$ or trace preserving self-extensions $Ext_B^{tr}(M_\xi, M_\xi)$ when B is a C.H. algebra of degree n trace generated by m elements, we have the following result.

Theorem 27. Consider the marked quiver Q° on k vertices such that the full marked subquiver on any two vertices $v_i \neq v_j$ has the form



where the numbers in the diagram satisfy
$a_{ij} \leq (m-1) d_i d_j$,
$a_{ii} + m_{ii} \leq (m-1) d_i^2 + 1$.

then

(1) Let A be an affine k-algebra generated by m-elements; let M_ξ be an n-dim. semi-simple representation of A of representation type $\tau(\xi) = (e_1, d_1; \cdots; e_k, d_k)$ and let $\alpha = (e_1, \cdots, e_k)$. Then the normal space N_x^{sm} at $x \in \mathcal{O}(M_\xi)$ with respect to the representation space $\underline{Rep}_n A$ is isomorphic to the $GL(\alpha)$-module of quiver representations $Rep_\alpha Q_\xi$ of the above type with

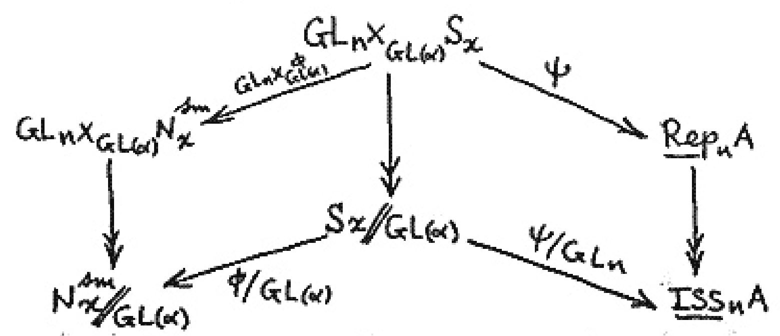
- $a_{ii} = \dim_k Ext^1_A(V_i, V_i)$ and $m_{ii} = 0$ for all $1 \le i \le k$,
- $a_{ij} = \dim_k Ext^1_A(V_i, V_j)$ for all $1 \le i \ne j \le n$.

(2) Let B be a C.H. algebra of degree n, trace generated by m elements, let M_ξ be a trace preserving n-dimensional semi-simple representation of B of type $\tau(\xi) = (e_1, d_1; \cdots, e_k, d_k)$ and let $\alpha = (e_1, \cdots, e_k)$. Then the normal space N_x^{tr} at $x \in \mathcal{O}(M_\xi)$ with respect to the trace preserving representation space $\underline{Rep}_n^{tr} B$ is isomorphic to the $GL(\alpha)$-module $\underline{Rep_\alpha Q^\bullet_\xi}$ of the above type, with

$$a_{ij} = \dim_k Ext^{tr}_B(V_i, V_j) \quad \text{for all } 1 \le i \ne j \le k,$$

and the (marked) vertex loops further determine the structure of $Ext^{tr}_B(M_\xi, M_\xi)$. ∎

Under the assumptions of the theorem, the étale slice result enables us with a slice $S_x \xrightarrow{\phi} N_x^{sm}$ and the commutative diagram



where the vertical maps are the quotient maps, and all the diagonal maps are étale and the upper ones are GL_n-equivariant.

Hence, the GL_n-local structure of the representation variety $\underline{Rep}_n A = \underline{Rep}_n^{tr} A_n$ in a neighborhood of the orbit of x is the same as that of the associated fiber bundle $GL_n \times_{GL(\alpha)} N_x^{sm}$ in a neighborhood of the orbit of $(\mathbb{1}_n, 0)$. Further, the local structure of the quotient scheme $\underline{ISS}_n A$ in a neighborhood of ξ is the same as that of the quotient variety of the marked quiver representations (because $N_x^{sm} \cong Rep_\alpha Q^\bullet_\xi$, $N_x^{sm} /\!/ GL(\alpha) \cong Rep_\alpha Q^\bullet_\xi /\!/ GL(\alpha)$ the quotient variety of marked quiver representations) in a neighborhood of the trivial representation $\overline{0}$. ∎

THE END

References of Chapter 4

(1) Lieven Le Bruyn. "Non-Commutative geometry" as given in chapters 1 and 2.

(2) H.P. KRAFT. "Geometric methods in representation theory" Springer LNM 944 (1982).

(3) C. Procesi. "A formal inverse to the Cayley-Hamilton theorem". J. Alg. 107, 63-74 (1987)

(4) C. Procesi. "Finite dimensional representations of algebras" Israel J. Math. Vol. 19 (1974)

(5) C. Procesi. "The invariant theory of n×n matrices" Adv. in Math. 19 (1976) 306-381.

(6) A. Schofield. "General representations of quivers" Proc. London Math. Soc. 65 (1992) 46-64.

(7) D. Luna. "Slices étale" Bull. Soc. math. France Mémorie 33, 1973, 81-105

(8) P. Gabriel & A.V. Roiter. "Representations of finite-dimensional algebras" Springer-Verlag (1997)

(9) R.S. Pierce. "Associative algebras" GTM 88 (1982), Springer-Verlag.

(10) P.J. Hilton & U. Stammbach. "A course in Homological algebra" GTM (4) Springer-Verlag.

(11) N. Jacobson. "Basic algebra II"

(12) S. Lang. "Fundamentals of differential geometry" GTM (191) Springer-Verlag.

(13) Y.A. Drozd & V.V. Kirichenko. "finite dim. algebras" Springer-Verlag