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ASPECTS OF THE THEORY OF INFINITE DIMENSIONAL MANIFOLDS

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Abstract. The convenient setting for smooth mappings, holomorphic mappings, and real analytic mappings in infinite dimension is sketched. Infinite dimensional manifolds are discussed with special emphasis on smooth partitions of unity and tangent vectors as derivations. Manifolds of mappings and diffeomorphisms are treated. Finally the differential structure on the inductive limits of the groups $GL(n)$, $SO(n)$ and some of their homogeneus spaces is treated.

Introduction

The theory of infinite dimensional manifolds has already a long history. In the fifties and sixties smooth manifolds modeled on Banach spaces were investigated a lot. Here the starting point were the investigations of Marston Morse on the index of geodesics in Riemannian manifolds. He used Hilbert manifolds of curves in a Riemannian manifold.

Later on Fréchet manifolds were investigated from the point of view of topology: it was shown that under certain weak conditions they could be embedded as open subsets in the model space.

Then starting with a seminal short paper of J. Eells began the investigation of manifolds of mappings.

But all this became important for the mainstream of mathematics when loop groups and their Lie algebras - the Kac-Moody Lie algebras were used in Physics.

In this review paper we will try to find our own way through the field of infinite dimensional manifolds, and we will concentrate on the smooth manifolds, and on those which are not modeled on Banach spaces or Hilbert spaces - although the latter are very important as a technical mean to prove very important theorems like those leading to exotic $\mathbb{R}^4$'s.

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This paper is in final form and no version of it will appear elsewhere

The material presented in the later sections is from [Kriegl-Michor, Foundations of Global Analysis].

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1. Calculus of smooth mappings

1.1. The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces a whole flock of different theories were developed, each of them rather complicated and none really convincing. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. This was the original motivation for the development of a whole new field within general topology, convergence spaces.

Then in 1982, Alfred Frölicher and Andreas Kriegl presented independently the solution to the question for the right differential calculus in infinite dimensions. They joined forces in the further development of the theory and the (up to now) final outcome is the book [Frölicher-Kriegl, 1988].

In this section we will sketch the basic definitions and the most important results of the Frölicher-Kriegl calculus.

1.2. The c^∞-topology. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).
(3) $\{E_B \to E : B$ bounded absolutely convex in $E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

1.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(1) Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges.
(2) If B is bounded closed absolutely convex, then E_B is a Banach space.
(3) Any Lipschitz curve in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1 = c_2'$ (existence of antiderivative).

1.4. Lemma. *Let E be a locally convex space. Then the following properties are equivalent:*

(1) *E is c^∞-complete.*
(2) *If $f : \mathbb{R} \to E$ is scalarwise* Lip^k*, then f is* Lip^k*, for $k > 1$.*
(3) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.*
(4) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.*

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. f scalarwise C^∞ means that $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

This lemma says that on a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

1.5. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mapping from E to F.

For E and F finite dimensional this gives the usual notion of smooth mappings: this has been first proved in [Boman, 1967]. Constant mappings are smooth. Multilinear mappings are smooth if and only if they are bounded. Therefore we denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

1.6. Structure on $C^\infty(E, F)$. We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$.

1.7. Lemma. *For locally convex spaces E and F we have:*

(1) *If F is convenient, then also $C^\infty(E, F)$ is convenient, for any E. The space $L(E, F)$ is a closed linear subspace of $C^\infty(E, F)$, so it also is convenient.*
(2) *If E is convenient, then a curve $c : \mathbb{R} \to L(E, F)$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in F for all $x \in E$.*

1.8. Theorem. *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G)),$$

which is even a diffeomorphism.

Of course this statement is also true for c^∞-open subsets of convenient vector spaces.

1.9. Corollary. *Let all spaces be convenient vector spaces. Then the following canonical mappings are smooth.*

$$\begin{aligned}
&\mathrm{ev}: C^\infty(E,F)\times E \to F, \quad \mathrm{ev}(f,x)=f(x)\\
&\mathrm{ins}: E\to C^\infty(F,E\times F), \quad \mathrm{ins}(x)(y)=(x,y)\\
&(\quad)^\wedge : C^\infty(E,C^\infty(F,G))\to C^\infty(E\times F,G)\\
&(\quad)^\vee : C^\infty(E\times F,G)\to C^\infty(E,C^\infty(F,G))\\
&\mathrm{comp}: C^\infty(F,G)\times C^\infty(E,F)\to C^\infty(E,G)\\
&C^\infty(\quad,\quad): C^\infty(F,F')\times C^\infty(E',E)\to C^\infty(C^\infty(E,F),C^\infty(E',F'))\\
&\qquad (f,g)\mapsto (h\mapsto f\circ h\circ g)\\
&\prod : \prod C^\infty(E_i,F_i)\to C^\infty(\prod E_i,\prod F_i)
\end{aligned}$$

1.10. Theorem. *Let E and F be convenient vector spaces. Then the differential operator*

$$\begin{aligned}
&d: C^\infty(E,F)\to C^\infty(E,L(E,F)),\\
&df(x)v := \lim_{t\to 0}\frac{f(x+tv)-f(x)}{t},
\end{aligned}$$

exists and is linear and bounded (smooth). Also the chain rule holds:

$$d(f\circ g)(x)v = df(g(x))dg(x)v.$$

1.11. Remarks. Note that the conclusion of theorem 1.8 is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more.

If one wants theorem 1.8 to be true and assumes some other obvious properties, then the calculus of smooth functions is already uniquely determined.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E\times E'\to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

For Fréchet spaces smoothness in the sense described here coincides with the notion C^∞_c of [Keller, 1974]. This is the differential calculus used by [Michor, 1980], [Milnor, 1984], and [Pressley-Segal, 1986].

A prevalent opinion in contemporary mathematics is, that for infinite dimensional calculus each serious application needs its own foundation. By a serious application one obviously means some application of a hard inverse function theorem. These theorems can be proved, if by assuming enough a priori estimates one creates enough Banach space situation for some modified iteration procedure to converge. Many authors try to build their platonic idea of an a priori estimate into their differential calculus. We think that this makes the calculus inapplicable and hides the origin of the a priori estimates. We believe, that the calculus itself should be as easy to use as possible, and that all further assumptions (which most often come from ellipticity of some nonlinear partial differential equation of geometric origin) should be treated separately, in a setting depending on the specific problem. We are sure that in this sense the Frölicher-Kriegl calculus as presented here and its holomorphic and real analytic offsprings in sections 2 and 3 below are universally usable for most applications.

We believe that the recent development of the theory of locally convex spaces missed its original aim: the development of calculus. It laid too much emphasis on (locally convex) topologies and ignored and denigraded the work of Hogbe-Nlend, Colombeau and collaborators on the original idea of Sebastião e Silva, that bornologies are the right concept for this kind of functional analysis.

2. Calculus of holomorphic mappings

2.1. Along the lines of thought of the Frölicher-Kriegl calculus of smooth mappings, in [Kriegl-Nel, 1985] the cartesian closed setting for holomorphic mappings was developed. The right definition of this calculus was already given by [Fantappié, 1930 and 1933]. We will now sketch the basics and the main results. It can be shown that again convenient vector spaces are the right ones to consider. Here we will start with them for the sake of shortness.

2.2. Let E be a complex locally convex vector space whose underlying real space is convenient – this will be called convenient in the sequel. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and let us denote by $\mathcal{H}(\mathbb{D}, E)$ the space of all mappings $c : \mathbb{D} \to E$ such that $\lambda \circ c : \mathbb{D} \to \mathbb{C}$ is holomorphic for each continuous complex-linear functional λ on E. Its elements will be called the holomorphic curves.

If E and F are convenient complex vector spaces (or c^∞-open sets therein), a mapping $f : E \to F$ is called *holomorphic* if $f \circ c$ is a holomorphic curve in F for each holomorphic curve c in E. Obviously f is holomorphic if and only if $\lambda \circ f : E \to \mathbb{C}$ is holomorphic for each complex linear continuous functional λ on F. Let $\mathcal{H}(E, F)$ denote the space of all holomorphic mappings from E to F.

2.3. Theorem (Hartog's theorem). *Let E_k for $k = 1, 2$ and F be complex convenient vector spaces and let $U_k \subset E_k$ be c^∞-open. A mapping $f : U_1 \times U_2 \to F$ is holomorphic if and only if it is separately holomorphic (i. e. $f(\quad, y)$ and $f(x, \quad)$ are holomorphic for all $x \in U_1$ and $y \in U_2$).*

This implies also that in finite dimensions we have recovered the usual definition.

2.4 Lemma. *If $f : E \supset U \to F$ is holomorphic then $df : U \times E \to F$ exists, is holomorphic and $\mathbb{C}$-linear in the second variable.*

A multilinear mapping is holomorphic if and only if it is bounded.

2.5 Lemma. *If E and F are Banach spaces and U is open in E, then for a mapping $f : U \to F$ the following conditions are equivalent:*

(1) *f is holomorphic.*
(2) *f is locally a convergent series of homogeneous continuous polynomials.*
(3) *f is $\mathbb{C}$-differentiable in the sense of Fréchet.*

2.6 Lemma. *Let E and F be convenient vector spaces. A mapping $f : E \to F$ is holomorphic if and only if it is smooth and its derivative is everywhere $\mathbb{C}$-linear.*

An immediate consequence of this result is that $\mathcal{H}(E,F)$ is a closed linear subspace of $C^\infty(E_\mathbb{R},F_\mathbb{R})$ and so it is a convenient vector space if F is one, by 1.7. The chain rule follows from 1.10. The following theorem is an easy consequence of 1.8.

2.7 Theorem. *The category of convenient complex vector spaces and holomorphic mappings between them is cartesian closed, i. e.*

$$\mathcal{H}(E\times F,G)\cong\mathcal{H}(E,\mathcal{H}(F,G)).$$

An immediate consequence of this is again that all canonical structural mappings as in 1.9 are holomorphic.

3. Calculus of real analytic mappings

3.1. In this section we sketch the cartesian closed setting to real analytic mappings in infinite dimension following the lines of the Frölicher-Kriegl calculus, as it is presented in [Kriegl-Michor, 1990]. Surprisingly enough one has to deviate from the most obvious notion of real analytic curves in order to get a meaningful theory, but again convenient vector spaces turn out to be the right kind of spaces.

3.2. Real analytic curves. Let E be a real convenient vector space with dual E'. A curve $c:\mathbb{R}\to E$ is called *real analytic* if $\lambda\circ c:\mathbb{R}\to\mathbb{R}$ is real analytic for each $\lambda\in E'$. It turns out that the set of these curves depends only on the bornology of E.

In contrast a curve is called *topologically real analytic* if it is locally given by power series which converge in the topology of E. They can be extended to germs of holomorphic curves along $\mathbb{R}$ in the complexification $E_\mathbb{C}$ of E. If the dual E' of E admits a Baire topology which is compatible with the duality, then each real analytic curve in E is in fact topologically real analytic for the bornological topology on E.

3.3. Real analytic mappings. Let E and F be convenient vector spaces. Let U be a c^∞-open set in E. A mapping $f:U\to F$ is called *real analytic* if and only if it is smooth (maps smooth curves to smooth curves) and maps real analytic curves to real analytic curves.

Let $C^\omega(U,F)$ denote the space of all real analytic mappings. We equip the space $C^\omega(U,\mathbb{R})$ of all real analytic functions with the initial topology with respect to the families of mappings

$$C^\omega(U,\mathbb{R})\xrightarrow{c^*}C^\omega(\mathbb{R},\mathbb{R}),\ \text{for all } c\in C^\omega(\mathbb{R},U)$$
$$C^\omega(U,\mathbb{R})\xrightarrow{c^*}C^\infty(\mathbb{R},\mathbb{R}),\ \text{for all } c\in C^\infty(\mathbb{R},U),$$

where $C^\infty(\mathbb{R},\mathbb{R})$ carries the topology of compact convergence in each derivative separately as in section 1, and where $C^\omega(\mathbb{R},\mathbb{R})$ is equipped with the final locally convex topology with respect to the embeddings (restriction mappings) of all spaces of holomorphic mappings from a neighborhood V of $\mathbb{R}$ in $\mathbb{C}$ mapping $\mathbb{R}$ to $\mathbb{R}$, and each of these spaces carries the topology of compact convergence.

Furthermore we equip the space $C^\omega(U,F)$ with the initial topology with respect to the family of mappings

$$C^\omega(U,F)\xrightarrow{\lambda_*}C^\omega(U,\mathbb{R}),\ \text{for all } \lambda\in F'.$$

It turns out that this is again a convenient space.

3.4. Theorem. *In the setting of 3.3 a mapping $f : U \to F$ is real analytic if and only if it is smooth and is real analytic along each affine line in E.*

3.5. Theorem. *The category of convenient spaces and real analytic mappings is cartesian closed. So the equation*

$$C^\omega(U, C^\omega(V, F)) \cong C^\omega(U \times V, F)$$

is valid for all c^∞-open sets U in E and V in F, where E, F, and G are convenient vector spaces.

This implies again that all structure mappings as in 1.9 are real analytic. Furthermore the differential operator

$$d : C^\omega(U, F) \to C^\omega(U, L(E, F))$$

exists, is unique and real analytic. Multilinear mappings are real analytic if and only if they are bounded. Powerful real analytic uniform boundedness principles are available.

4. Infinite dimensional manifolds

4.1. In this section we will concentrate on two topics: Smooth partitions of unity, and several kinds of tangent vectors.

4.2. In the usual way we define manifolds by gluing c^∞-open sets in convenient vector spaces via smooth (holomorphic, real analytic) diffeomorphisms. Then we equip them with the identification topology with respect to the c^∞-topologies on all modeling spaces. We require some properties from this topology, like Hausdorff and regular (which here is not a consequence of Hausdorff).

Mappings between manifolds are smooth (holomorphic, real analytic), if they have this property when composed which any chart mappings.

4.3. Lemma. *A manifold M is metrizable if and only if it is paracompact and modeled on Fréchet spaces.*

4.4. Lemma. *For a convenient vector space E the set $C^\infty(M, E)$ of smooth E-valued functions on a manifold M is again a convenient vector space. Likewise for the real analytic and holomorphic case.*

4.5. Theorem. *If M is a smooth manifold modeled on convenient vector spaces admitting smooth bump functions and $\mathcal{U}$ is a locally finite open cover of M, then there exists a smooth partition of unity $\{\varphi_U : U \in \mathcal{U}\}$ with carr/supp$(\varphi_U) \subset U$ for all $U \in \mathcal{U}$.*

If M is in addition paracompact, then this is true for every open cover $\mathcal{U}$ of M.

Convenient vector spaces which are nuclear admit smooth bump functions.

4.6. The tangent spaces of a convenient vector space E. Let $a \in E$. A *kinematic tangent vector* with foot point a is simply a pair (a, X) with $X \in E$. Let $T_aE = E$ be the space of all kinematic tangent vectors with foot point a. It consists of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \to E$ with $c(0) = a$, which explains the choice of the name kinematic.

For each open neighborhood U of a in E (a, X) induces a linear mapping $X_a : C^\infty(U, \mathbb{R}) \to \mathbb{R}$ by $X_a(f) := df(a)(X_a)$, which is continuous for the convenient vector space topology on $C^\infty(U,\mathbb{R})$, and satisfies $X_a(f\cdot g) = X_a(f)\cdot g(a) + f(a)\cdot X_a(g)$, so it is a *continuous derivation over* ev_a. The value $X_a(f)$ depends only on the germ of f at a.

An *operational tangent vector* of E with foot point a is a bounded derivation $\partial : C^\infty_a(E,\mathbb{R}) \to \mathbb{R}$ over ev_a. Let D_aE be the vector space of all these derivations. Any $\partial \in D_aE$ induces a bounded derivation $C^\infty(U,\mathbb{R}) \to \mathbb{R}$ over ev_a for each open neighborhood U of a in E. So the vector space D_aE is a closed linear subspace of the convenient vector space $\prod_U L(C^\infty(U,\mathbb{R}),\mathbb{R})$. We equip D_aE with the induced convenient vector space structure. Note that the spaces D_aE are isomorphic for all $a \in E$.

Example. Let $Y \in E''$ be an element in the bidual of E. Then for each $a \in E$ we have an operational tangent vector $Y_a \in D_aE$, given by $Y_a(f) := Y(df(a))$. So we have a canonical injection $E'' \to D_aE$.

Example. Let $\ell : L^2(E;\mathbb{R}) \to \mathbb{R}$ be a bounded linear functional which vanishes on the subset $E' \otimes E'$. Then for each $a \in E$ we have an operational tangent vector $\partial^2_\ell|_a \in D_aE$ given by $\partial^2_\ell|_a(f) := \ell(d^2f(a))$, since

$$\begin{aligned}
&\ell(d^2(fg)(a)) = \\
&\quad = \ell(d^2f(a)g(a) + df(a)\otimes dg(a) + dg(a)\otimes df(a) + f(a)d^2g(a)) \\
&\quad = \ell(d^2f(a))g(a) + 0 + f(a)\ell(d^2g(a)).
\end{aligned}$$

4.7. Lemma. *Let $\ell \in L^k_{sym}(E;\mathbb{R})'$ be a bounded linear functional which vanishes on the subspace*

$$\sum_{i=1}^{k-1} L^i_{sym}(E;\mathbb{R}) \vee L^{k-i}_{sym}(E;\mathbb{R})$$

of decomposable elements of $L^k_{sym}(E;\mathbb{R})$. Then ℓ defines an operational tangent vector $\partial^k_\ell|_a \in D_aE$ for each $a \in E$ by

$$\partial^k_\ell|_a(f) := \ell(d^kf(a)).$$

The linear mapping $\ell \mapsto \partial^k_\ell|_a$ is an embedding onto a topological direct summand $D^{(k)}_aE$ of D_aE. Its left inverse is given by $\partial \mapsto (\Phi \mapsto \partial((\Phi \circ diag)(a+\quad)))$. The sum $\sum_{k>0} D^{(k)}_aE$ in D_aE is a direct one.

4.8. Lemma. *If E is an infinite dimensional Hilbert space, all operational tangent space summands $D^{(k)}_0E$ are not zero.*

4.9. Definition. A convenient vector space is said to have the *(bornological) approximation property* if $E'\otimes E$ is dense in $L(E,E)$ in the bornological locally convex topology.

The following spaces have the bornological approximation property: $\mathbb{R}^{(\mathbb{N})}$, nuclear Fréchet spaces, nuclear (LF) spaces.

4.10 Theorem. *Let E be a convenient vector space which has the approximation property. Then we have $D_aE = D^{(1)}_aE \cong E''$. So if E is in addition reflexive, each operational tangent vector is a kinematic one.*

4.11. The kinematic tangent bundle TM of a manifold M is constructed by gluing all the kinematic tangent bundles of charts with the help of the kinematic tangent mappings (derivatives) of the chart changes. $TM \to M$ is a vector bundle and $T : C^\infty(M,N) \to C^\infty(TM,TN)$ is well defined and has the usual properties.

4.12. The operational tangent bundle DM of a manifold M is constructed by gluing all operational tangent spaces of charts. Then $\pi_M : DM \to M$ is again a vector bundle which contains the kinematic tangent bundle TM as a splitting subbundle. Also for each $k \in \mathbb{N}$ the same gluing construction as above gives us tangent bundles $D^{(k)}M$ which are splitting sub bundles of DM. The mappings $D^{(k)} : C^\infty(M,N) \to C^\infty(D^{(k)}M, D^{(k)}N)$ are well defined for all k (including no k) and have the usual properties.

Note that for manifolds modeled on reflexive spaces having the bornological approximation property the operational and the kinematic tangent bundles coincide.

5. Manifolds of mappings

5.1. Theorem (Manifold structure of $C^\infty(M,N)$). *Let M and N be smooth finite dimensional manifolds, let M be compact. Then the space $C^\infty(M,N)$ of all smooth mappings from M to N is a smooth manifold, modeled on spaces $C^\infty(f^*TN)$ of smooth sections of pullback bundles along $f : M \to N$ over M.*

A careful description of this theorem (but without the help of the Frölicher-Kriegl calculus) can be found in [Michor, 1980]. We include a proof of this result here because the result is important and the proof is much simpler now.

Proof. Choose a smooth Riemannian metric on N. Let $\exp : TN \supseteq U \to N$ be the smooth exponential mapping of this Riemannian metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_N, \exp) : U \to N \times N$ is a smooth diffeomorphism onto an open neighborhood V of the diagonal.

For $f \in C^\infty(M,N)$ we consider the pullback vector bundle

$$\begin{array}{ccccc} M \times_N TN & = & f^*TN & \xrightarrow{\pi_N^* f} & TN \\ & & {\scriptstyle f^*\pi_N}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ & & M & \xrightarrow{f} & N. \end{array}$$

Then $C^\infty(f^*TN)$ is canonically isomorphic to $C_f^\infty(M,TN) := \{h \in C^\infty(M,TN) : \pi_N \circ h = f\}$ via $s \mapsto (\pi_N^* f) \circ s$ and $(Id_M, h) \leftarrow h$. We consider the space $C_c^\infty(f^*TN)$ of all smooth sections with compact support and equip it with the inductive limit topology

$$C_c^\infty(f^*TN) = \operatorname*{inj\,lim}_K C_K^\infty(f^*TN),$$

where K runs through all compact sets in M and each of the spaces $C_K^\infty(f^*TN)$ is equipped with the topology of uniform convergence (on K) in all derivatives separately. Now let

$$\begin{gathered} U_f := \{g \in C^\infty(M,N) : (f(x), g(x)) \in V \text{ for all } x \in M, g \sim f\}, \\ u_f : U_f \to C_c^\infty(f^*TN), \\ u_f(g)(x) = (x, \exp_{f(x)}^{-1}(g(x))) = (x, ((\pi_N, \exp)^{-1} \circ (f,g))(x)). \end{gathered}$$

Here $g \sim f$ means that g equals f off some compact set. Then u_f is a bijective mapping from U_f onto the set $\{s \in C_c^\infty(f^*TN) : s(M) \subseteq f^*U\}$, whose inverse is given by $u_f^{-1}(s) = \exp \circ (\pi_N^* f) \circ s$, where we view $U \to N$ as fiber bundle. The set $u_f(U_f)$ is open in $C_c^\infty(f^*TN)$ for the topology described above.

Now we consider the atlas $(U_f, u_f)_{f \in C^\infty(M,N)}$ for $C^\infty(M,N)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C_c^\infty(g^*TN)$ by

$$\begin{aligned}(u_f \circ u_g^{-1})(s) &= (Id_M, (\pi_N, \exp)^{-1} \circ (f, \exp \circ (\pi_N^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s),\end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)})))$ is a smooth diffeomorphism $\tau_g : g^*TN \supseteq g^*U \to (g \times Id_N)^{-1}(V) \subseteq M \times N$ which is fiber respecting over M.

Smooth curves in $C_c^\infty(f^*TN)$ are just smooth sections of the bundle $pr_2^* f^*TN \to \mathbb{R} \times M$, which have compact support in M locally in$\mathbb{R}$. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and obviously maps smooth curves to smooth curves, therefore it is also smooth.

Finally we put the identification topology from this atlas onto the space $C^\infty(M,N)$, which is obviously finer than the compact open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the smooth structure does not depend on the choice of the smooth Riemannian metric on N. □

5.2. Theorem (C^ω-manifold structure of $C^\omega(M,N)$). *Let M and N be real analytic manifolds, let M be compact. Then the space $C^\omega(M,N)$ of all real analytic mappings from M to N is a real analytic manifold, modeled on spaces $C^\omega(f^*TN)$ of real analytic sections of pullback bundles along $f : M \to N$ over M.*

The proof can be found in [Kriegl-Michor, 1990]. It is a variant of the above proof, using a real analytic Riemannian metric.

5.3. Theorem (C^ω-manifold structure on $C^\infty(M,N)$). *Let M and N be real analytic manifolds, with M compact. Then the smooth manifold $C^\infty(M,N)$ is even a real analytic manifold.*

Proof. For a fixed real analytic exponential mapping on N the charts (U_f, u_f) from 5.1 for $f \in C^\omega(M,N)$ form a smooth atlas for $C^\infty(M,N)$, since $C^\omega(M,N)$ is dense in $C^\infty(M,N)$ by [Grauert, 1958, Proposition 8]. The chart changings $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ are real analytic: this follows from a careful description of the set of real analytic curves into $C^\infty(f^*TN)$. See again [Kriegl-Michor, 1990, 7.7] for more details. □

5.4 Remark. If M is not compact, $C^\omega(M,N)$ is dense in $C^\infty(M,N)$ for the Whitney-C^∞-topology by [Grauert, 1958, Prop. 8]. This is not the case for the topology used in 5.1 in which $C^\infty(M,N)$ is a smooth manifold. The charts U_f for $f \in C^\omega(M,N)$ do not cover $C^\infty(M,N)$.

5.5. Theorem. *Let M and N be smooth manifolds. Then the two infinite dimensional smooth vector bundles $TC^\infty(M,N)$ and $C^\infty(M,TN)$ over $C^\infty(M,N)$ are canonically isomorphic. The same assertion is true for $C^\omega(M,N)$, if M is compact.*

5.6. Theorem (Exponential law). *Let $\mathcal{M}$ be a (possibly infinite dimensional) smooth manifold, and let M and N be finite dimensional smooth manifolds.*

Then we have a canonical embedding

$$C^\infty(\mathcal{M}, C^\infty(M,N)) \subseteq C^\infty(\mathcal{M}\times M, N),$$

where we have equality if and only if M is compact.

If M and N are real analytic manifolds with M compact we have

$$C^\omega(\mathcal{M}, C^\omega(M,N)) = C^\omega(\mathcal{M}\times M, N)$$

for each real analytic (possibly infinite dimensional) manifold.

5.7. Corollary. *If M is compact and M, N are finite dimensional smooth manifolds, then the evaluation mapping* $\operatorname{ev}: C^\infty(M,N)\times M \to N$ *is smooth.*

If P is another compact smooth manifold, then the composition mapping $\operatorname{comp}: C^\infty(M,N)\times C^\infty(P,M)\to C^\infty(P,N)$ *is smooth.*

In particular $f_*: C^\infty(M,N)\to C^\infty(M,N')$ and $g^*: C^\infty(M,N)\to C^\infty(P,N)$ are smooth for $f\in C^\infty(N,N')$ and $g\in C^\infty(P,M)$.

The corresponding statement for real analytic mappings is also true.

5.8. Theorem (Diffeomorphism groups). *For a smooth manifold M the group* $\operatorname{Diff}(M)$ *of all smooth diffeomorphisms of M is an open submanifold of $C^\infty(M,M)$, composition and inversion are smooth.*

The Lie algebra of the smooth infinite dimensional Lie group $\operatorname{Diff}(M)$ *is the convenient vector space $C^\infty_c(TM)$ of all smooth vector fields on M with compact support, equipped with the negative of the usual Lie bracket. The exponential mapping* $\operatorname{Exp}: C^\infty_c(TM)\to \operatorname{Diff}^\infty(M)$ *is the flow mapping to time 1, and it is smooth.*

For a compact real analytic manifold M the group $\operatorname{Diff}^\omega(M)$ *of all real analytic diffeomorphisms is a real analytic Lie group with Lie algebra $C^\omega(TM)$ and with real analytic exponential mapping.*

5.9. Remarks. The group $\operatorname{Diff}(M)$ of smooth diffeomorphisms does not carry any real analytic Lie group structure by [Milnor, 1984, 9.2], and it has no complexification in general, see [Pressley-Segal, 1986, 3.3]. The mapping

$$Ad\circ \operatorname{Exp}: C^\infty_c(TM)\to \operatorname{Diff}(M)\to L(C^\infty(TM), C^\infty(TM))$$

is not real analytic, see [Michor, 1983, 4.11].

For $x\in M$ the mapping $ev_x\circ \operatorname{Exp}: C^\infty_c(TM)\to \operatorname{Diff}(M)\to M$ is not real analytic since $(ev_x\circ \operatorname{Exp})(tX) = Fl^X_t(x)$, which is not real analytic in t for general smooth X.

The exponential mapping $\operatorname{Exp}: C^\infty_c(TM)\to \operatorname{Diff}(M)$ is in a very strong sense not surjective: In [Grabowski, 1988] it is shown, that $\operatorname{Diff}(M)$ contains an arcwise connected free subgroup on $2^{\aleph_0}$ generators which meets the image of Exp only at the identity.

The real analytic Lie group $\operatorname{Diff}^\omega(M)$ is *regular* in the sense of [Milnor, 1984. 7.6], who weakened the original concept of [Omori, 1982]. This condition means that the mapping associating the evolution operator to each time dependent vector field on M is smooth. It is even real analytic, compare the proof of theorem 5.9.

5.10. Theorem. *Let M and N be smooth manifolds. Then the diffeomorphism group $\operatorname{Diff}(M)$ acts smoothly from the right on the smooth manifold $\operatorname{Imm}(M,N)$ of all smooth immersions $M \to N$, which is an open subset of $C^\infty(M,N)$.*

Then the space of orbits $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ is Hausdorff in the quotient topology.

Let $\operatorname{Imm}_{\text{free}}(M,N)$ be set of all immersions, on which $\operatorname{Diff}(M)$ acts freely. Then this is open in $C^\infty(M,N)$ and is the total space of a smooth principal fiber bundle

$$\operatorname{Imm}_{\text{free}}(M,N) \to \operatorname{Imm}_{\text{free}}(M,N)/\operatorname{Diff}(M).$$

In particular the space $\operatorname{Emb}(M,N)$ of all smooth embeddings is the total space of smooth principal fiber bundle.

This is proved in [Cervera-Mascaro-Michor, 1989], where also the existence of smooth transversals to each orbit is shown and the stratification of the orbit space into smooth manifolds is given.

5.11. Theorem (Principal bundle of embeddings). *Let M and N be real analytic manifolds with M compact. Then the set $Emb^\omega(M,N)$ of all real analytic embeddings $M \to N$ is an open submanifold of $C^\omega(M,N)$. It is the total space of a real analytic principal fiber bundle with structure group $\operatorname{Diff}^\omega(M)$, whose real analytic base manifold is the space of all submanifolds of N of type M.*

See [Kriegl-Michor, 1990], section 6.

5.12. Theorem (Classifying space for $\operatorname{Diff}(M)$). *Let M be a compact smooth manifold. Then the space $Emb(M,\ell^2)$ of smooth embeddings of M into the Hilbert space ℓ^2 is the total space of a smooth principal fiber bundle with structure group $\operatorname{Diff}(M)$ and smooth base manifold $B(M,\ell^2)$, which is a classifying space for the Lie group $\operatorname{Diff}(M)$. It carries a universal $\operatorname{Diff}(M)$-connection.*

In other words:

$$(\operatorname{Emb}(M,\ell^2) \times_{\operatorname{Diff}(M)} M \to B(M,\ell^2)$$

classifies fiber bundles with typical fiber M and carries a universal (generalized) connection.

See [Michor, 1988, section 6].

6. Manifolds for algebraic topology

6.1 Convention. In this section the space $\mathbb{R}^{(\mathbb{N})}$ of all finite sequences with the direct sum topology will be denoted by $\mathbb{R}^\infty$ following the common usage in algebraic topology. It is a convenient vector space.

We consider on it the weak inner product $\langle x,y\rangle := \sum x_i y_i$, which is bilinear and bounded, therefore smooth. It is called weak, since it is non degenerate in the following sense: the associated linear mapping $\mathbb{R}^\infty \to (\mathbb{R}^\infty)' = \mathbb{R}^{\mathbb{N}}$ is injective, but far from being surjective. We will also use the weak Euclidean distance $|x| := \sqrt{\langle x,x\rangle}$, whose square is a smooth function.

6.2. Example: The sphere S^∞.

The *sphere* S^∞ is the set $\{x \in R^\infty : \langle x,x\rangle = 1\}$. This is the usual infinite dimensional sphere used in algebraic topology, the topological inductive limit of $S^n \subset S^{n+1} \subset \dots$.

We show that S^∞ is a smooth manifold by describing an explicit smooth atlas for S^∞, the *stereographic atlas.* Choose $a \in S^\infty$ ("south pole"). Let

$$\begin{aligned} U_+ &:= S^\infty \setminus \{a\}, \qquad u_+ : U_+ \to \{a\}^\perp, \qquad u_+(x) = \tfrac{x-\langle x,a\rangle a}{1-\langle x,a\rangle}, \\ U_- &:= S^\infty \setminus \{-a\}, \qquad u_- : U_- \to \{a\}^\perp, \qquad u_-(x) = \tfrac{x-\langle x,a\rangle a}{1+\langle x,a\rangle}. \end{aligned}$$

From an obvious drawing in the 2-plane through 0, x, and a it is easily seen that u_+ is the usual stereographic projection. We also get

$$u_+^{-1}(y) = \tfrac{|y|^2-1}{|y|^2+1}a + \tfrac{2}{|y|^2+1}y \qquad \text{for } y \in \{a\}^\perp \setminus \{0\}$$

and $(u_- \circ u_+^{-1})(y) = \frac{y}{|y|^2}$. The latter equation can directly be seen from the drawing using "Strahlensatz".

The two stereographic charts above can be extended to charts on open sets in $\mathbb{R}^\infty$ in such a way that S^∞ becomes a splitting submanifold of $\mathbb{R}^\infty$:

$$\begin{aligned} \tilde{u}_+ &: \mathbb{R}^\infty \setminus [0,+\infty)a \to a^\perp + (-1,+\infty)a \\ \tilde{u}_+(z) &:= u_+(\frac{z}{|z|}) + (|z|-1)a. \end{aligned}$$

Since the model space $\mathbb{R}^\infty$ of S^∞ has the bornological approximation property by 4.9, and is reflexive, by 4.10 the operational tangent bundle of S^∞ equals the kinematic one: $DS^\infty = TS^\infty$.

We claim that TS^∞ is diffeomorphic to $\{(x,v) \in S^\infty \times \mathbb{R}^\infty : \langle x,v\rangle = 0\}$.

The $X_x \in T_xS^\infty$ are exactly of the form $c'(0)$ for a smooth curve $c : \mathbb{R} \to S^\infty$ with $c(0) = x$ by 4.11. Then $0 = \frac{d}{dt}|0\langle c(t), c(t)\rangle = 2\langle x, X_x\rangle$. For $v \in x^\perp$ we use $c(t) = \cos(|v|t)x + \sin(|v|t)\frac{v}{|v|}$.

The construction of S^∞ works for any positive definite bounded bilinear form on any convenient vector space.

6.3. Example. The Grassmannians and the Stiefel manifolds.
The *Grassmann manifold* $G(k,\infty;\mathbb{R})$ is the set of all k-dimensional linear subspaces of the space of all finite sequences $\mathbb{R}^\infty$.

The *Stiefel manifold* $O(k,\infty;\mathbb{R})$ *of orthonormal k-frames* is the set of all linear isometries $\mathbb{R}^k \to \mathbb{R}^\infty$, where the latter space is again equipped with the standard weak inner product described at the beginning of 6.2.

The *Stiefel manifold* $GL(k,\infty;\mathbb{R})$ *of all k-frames* is the set of all injective linear mappings $\mathbb{R}^k \to \mathbb{R}^\infty$.

There is a canonical *transposition mapping* $(\quad)^t : L(\mathbb{R}^k,\mathbb{R}^\infty) \to L(\mathbb{R}^\infty,\mathbb{R}^k)$ which is given by

$$A^t : \mathbb{R}^\infty \xrightarrow{incl} \mathbb{R}^\mathbb{N} = (\mathbb{R}^\infty)' \xrightarrow{A'} (\mathbb{R}^k)' = \mathbb{R}^k$$

and satisfies $\langle A^t(x), y\rangle = \langle x, A(y)\rangle$. The transposition mapping is bounded and linear, so it is real analytic.

Then we have

$$GL(k,\infty) = \{A \in L(\mathbb{R}^k,\mathbb{R}^\infty) : A^t \circ A \in GL(k)\},$$

since $A^t \circ A \in GL(k)$ if and only if $\langle Ax, Ay\rangle = \langle A^t Ax, y\rangle = 0$ for all y implies $x = 0$, which is equivalent to A injective. So in particular $GL(k,\infty)$ is open in $L(\mathbb{R}^k, \mathbb{R}^\infty)$. The Lie group $GL(k)$ acts freely from the right on the space $GL(k,\infty)$. Two elements of $GL(k,\infty)$ lie in the same orbit if and only if they have the same image in $\mathbb{R}^\infty$. We have a surjective mapping $\pi : GL(k,\infty) \to G(k,\infty)$, given by $\pi(A) = A(\mathbb{R}^k)$, where the inverse images of points are exactly the $GL(k)$-orbits.

Similarly we have

$$O(k,\infty) = \{A \in L(\mathbb{R}^k, \mathbb{R}^\infty) : A^t \circ A = Id_k\}.$$

Now the Lie group $O(k)$ of all isometries of $\mathbb{R}^k$ acts freely from the right on the space $O(k,\infty)$. Two elements of $O(k,\infty)$ lie in the same orbit if and only if they have the same image in $\mathbb{R}^\infty$. The projection $\pi : GL(k,\infty) \to G(k,\infty)$ restricts to a surjective mapping $\pi : O(k,\infty) \to G(k,\infty)$ and the inverse images of points are now exactly the $O(k)$-orbits.

6.4. Lemma (Iwasawa decomposition). *Let $T(k;\mathbb{R})$ be the group of all upper triangular $k \times k$-matrices with positive entries on the main diagonal. Then each $B \in GL(k,\infty)$ can be written in the form $B = p(B) \circ q(B)$, with unique $p(B) \in O(k,\infty)$ and $q(B) \in T(k)$. The mapping $q : GL(k,\infty) \to T(k)$ is real analytic.*

6.5. Theorem. *The following are a real analytic principal fiber bundles:*

$$\begin{gathered}(O(k,\infty;\mathbb{R}), \pi, G(k,\infty;\mathbb{R}), O(k,\mathbb{R})),\\ (GL(k,\infty;\mathbb{R}), \pi, G(k,\infty;\mathbb{R}), GL(k,\mathbb{R})),\\ (GL(k,\infty;\mathbb{R}), p, O(k,\infty;\mathbb{R}), T(k;\mathbb{R})).\end{gathered}$$

The last one is trivial.

The embeddings $\mathbb{R}^n \to \mathbb{R}^\infty$ induce real analytic embeddings, which respect the principal right actions of all the structure groups

$$\begin{gathered}O(k,n) \to O(k,\infty),\\ GL(k,n) \to GL(k,\infty),\\ G(k,n) \to G(k,\infty).\end{gathered}$$

All these cones are inductive limits in the category of real analytic (and smooth) manifolds.

6.6. Theorem. *The following manifolds are real analytically diffeomorphic to the homogeneous spaces indicated:*

$$\begin{gathered}GL(k,\infty) \cong GL(\infty) \Big/ \begin{pmatrix} Id_k & L(\mathbb{R}^k, \mathbb{R}^{\infty-k}) \\ 0 & GL(\infty-k)\end{pmatrix}\\ O(k,\infty) \cong O(\infty)/Id_k \times O(\infty-k)\\ G(k,\infty) \cong O(\infty)/O(k) \times O(\infty-k).\end{gathered}$$

The universal vector bundle $(E(k,\infty), \pi, G(k,\infty), \mathbb{R}^k)$ is defined as the associated bundle

$$\begin{aligned}E(k,\infty) &= O(k,\infty)[\mathbb{R}^k]\\ &= \{(Q,x) : x \in Q\} \subset G(k,\infty) \times \mathbb{R}^\infty.\end{aligned}$$

The tangent bundle of the Grassmannian is then given by

$$TG(k,\infty) = L(E(k,\infty), E(k,\infty)^\perp).$$

6.7 Theorem. *The principal bundle $(O(k,\infty),\pi,G(k,\infty))$ is classifying for finite dimensional principal $O(k)$-bundles and carries a universal real analytic $O(k)$-connection $\omega\in\Omega^1(O(k,\infty),\mathfrak{o}(k))$.*

This means: For each finite dimensional smooth or real analytic principal $O(k)$-bundle $P\to M$ with principal connection ω_P there is a smooth or real analytic mapping $f:M\to G(k,\infty)$ such that the pullback $O(k)$-bundle $f^*O(k,\infty)$ is isomorphic to P and the pullback connection $f^*\omega=\omega_P$ via this isomorphism.

6.8. The Lie group $GL(\infty,\mathbb{R})$. The canonical embeddings $\mathbb{R}^n\to\mathbb{R}^{n+1}$ onto the first n coordinates induce injections $GL(n)\to GL(n+1)$. The inductive limit is

$$GL(\infty):=\varinjlim_n GL(n)$$

in the category of sets. Since each $GL(n)$ also injects into $L(\mathbb{R}^\infty,\mathbb{R}^\infty)$ we can visualize $GL(\infty)$ as the set of all $\mathbb{N}\times\mathbb{N}$-matrices which are invertible and differ from the identity in finitely many entries only.

We also consider the Lie algebra $\mathfrak{gl}(\infty)$ of all $\mathbb{N}\times\mathbb{N}$-matrices with only finitely many nonzero entries, which is isomorphic to $\mathbb{R}^{(\mathbb{N}\times\mathbb{N})}$, and we equip it with this convenient vector space structure. Then $\mathfrak{gl}(\infty)=\varinjlim_n\mathfrak{gl}(n)$ in the category of real analytic mappings.

Claim. $\mathfrak{gl}(\infty)=L(\mathbb{R}^{\mathbb{N}},\mathbb{R}^{(\mathbb{N})})$ *as convenient vector spaces. Composition is a bounded bilinear mapping on* $\mathfrak{gl}(\infty)$.

6.9. Theorem. *$GL(\infty)$ is a real analytic Lie group modeled on $\mathbb{R}^\infty$, with Lie algebra $\mathfrak{gl}(\infty)$ and is the inductive limit of the Lie groups $GL(n)$ in the category of real analytic manifolds. The exponential mapping is well defined, is real analytic and a local real analytic diffeomorphism onto a neighborhood of the identity. The Campbell-Baker-Hausdorff formula gives a real analytic mapping near 0 and expresses the multiplication on $GL(\infty)$ via* exp. *The determinant* $\det: GL(\infty)\to\mathbb{R}\setminus 0$ *is a real analytic homomorphism. We have a real analytic left action $GL(\infty)\times\mathbb{R}^\infty\to\mathbb{R}^\infty$, such that $\mathbb{R}^\infty\setminus 0$ is one orbit, but the injection $GL(\infty)\hookrightarrow L(\mathbb{R}^\infty,\mathbb{R}^\infty)$ does not generate the topology.*

Proof. Since the exponential mappings are compatible with the inductive limits all these assertions follow from the inductive limit property. □

6.10. Theorem. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(\infty)$. Then there is a smoothly arcwise connected splitting Lie subgroup G of $GL(\infty)$ whose Lie algebra is $\mathfrak{g}$. The exponential mapping of $GL(\infty)$ restricts to that of G, which is local diffeomorphism near zero. The Campbell Baker Hausdorff formula gives a real analytic mapping near 0 and has the usual properties, also on G.*

Proof. Let $\mathfrak{g}_n:=\mathfrak{g}\cap\mathfrak{gl}(n)$, a finite dimensional Lie subalgebra of $\mathfrak{g}$. Then $\bigcup\mathfrak{g}_n=\mathfrak{g}$. The convenient structure $\mathfrak{g}=\varinjlim_n\mathfrak{g}_n$ coincides with the structure inherited as a complemented subspace, since $\mathfrak{gl}(\infty)$ carries the finest locally convex structure.

So for each n there is a connected Lie subgroup $G_n\subset GL(n)$ with Lie algebra $\mathfrak{g}_n$. Since $\mathfrak{g}_n\subset\mathfrak{g}_{n+1}$ we have $G_n\subset G_{n+1}$ and we may consider $G:=\bigcup_n G_n\subset GL(\infty)$. Each $g\in G$ lies in some G_n and may be connected to Id via a smooth curve there, which is also smooth curve in G, so G is smoothly arcwise connected.

All mappings $\exp|\mathfrak{g}_n:\mathfrak{g}_n\to G_n$ are local real analytic diffeomorphisms near 0, so $\exp:\mathfrak{g}\to G$ is also a local real analytic diffeomorphism near zero onto an open neighborhood of the identity in G. The rest is clear. □

6.11. Examples.
The Lie group $SO(\infty, \mathbb{R})$ is the inductive limit

$$SO(\infty) := \varinjlim_n SO(n) \subset GL(\infty).$$

It is the connected Lie subgroup of $GL(\infty)$ with the Lie algebra $\mathfrak{o}(\infty) = \{X \in \mathfrak{gl}(\infty) : X^t = -X\}$ of skew elements. Obviously we have

$$SO(\infty) = \{A \in GL(\infty) : \langle Ax, Ay \rangle = \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{R}^\infty \text{ and } \det(A) = 1\}.$$

The Lie group $O(\infty)$ is the inductive limit

$$\begin{aligned} O(\infty) &:= \varinjlim_n O(n) \subset GL(\infty) \\ &= \{A \in GL(\infty) : \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^\infty\}. \end{aligned}$$

It has two connected components, that of the identity is $SO(\infty)$.

The Lie group $SL(\infty)$ is the inductive limit

$$\begin{aligned} SL(\infty) &: = \varinjlim_n SL(n) \subset GL(\infty) \\ &= \{A \in GL(\infty) : \det(A) = 1\}. \end{aligned}$$

It is the connected Lie subgroup with Lie algebra $\mathfrak{sl}(\infty) = \{X \in \mathfrak{gl}(\infty) : \operatorname{Trace}(X) = 0\}$.

6.12. We stop here to give examples. Of course this method is also applicable for the complex versions of the most important homogeneous spaces. This will be treated elsewhere.

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