Overview on Geometries of Shape Spaces, Diffeomorphism Groups, and Spaces of Riemannian Metrics

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Based on collaborations with: M. Bauer, M. Bruveris, P. Harms, D. Mumford

- A diagram of actions of diffeomorphism groups
- Riemannian geometries of spaces of immersions and shape spaces.
- A zoo of diffeomorphism groups on $\mathbb{R}^n$
- Right invariant Riemannian geometries on Diffeomorphism groups.
- Robust Infinite Dimensional Riemannian manifolds, Sobolev Metrics on Diffeomorphism Groups, and the Derived Geometry of Shape Spaces.

A diagram of actions of diffeomorphism groups.

$$
\begin{tikzcd}
\mathrm{Diff}(M) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] \arrow[dd, two heads, "\mathrm{Diff}(M,\mu)"] & & \mathrm{Imm}(M,N) \arrow[dl, "\text{needs } \bar{g}"] \arrow[dr, two heads, "\mathrm{Diff}(M)"'] & & \mathrm{Diff}_{\mathcal{A}}(N) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dd, "\text{r-acts}"] \\
& \mathrm{Met}(M) \arrow[dr, two heads, "\mathrm{Diff}(M)"'] & & B_i(M,N) \arrow[dl, "\text{needs } \bar{g}"] & \\
\mathrm{Vol}^1_+(M) & & \dfrac{\mathrm{Met}(M)}{\mathrm{Diff}(M)} & & \mathrm{Met}_{\mathcal{A}}(N)
\end{tikzcd}
$$

M compact , N pssibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar{g}$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities

$$
\begin{tikzcd}
\mathrm{Diff}(S^1) \arrow[rr, "\text{r-acts}"] \arrow[d, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] & & \mathrm{Imm}(S^1,\mathbb{R}^2) \arrow[dl, "\text{needs } \bar g"] \arrow[dr, "\mathrm{Diff}(S^1)"'] & & \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dd, "\text{r-acts}"] \\
\mathrm{Vol}_+(S^1) \arrow[d] & \mathrm{Met}(S^1) \arrow[dr, "\mathrm{Diff}(S^1)"] & & B_i(S^1,\mathbb{R}^2) \arrow[dl, "\text{needs } \bar g"'] & \\
\dfrac{\mathrm{Vol}_+(S^1)}{\mathrm{Diff}(S^1)} \arrow[r, "\int f d\theta", "="'] & \mathbb{R}_{>0} & \dfrac{\mathrm{Met}(S^1)}{\mathrm{Diff}(S^1)} \arrow[l, "="', "\int \sqrt{g} d\theta"] & & \mathrm{Met}(\mathbb{R}^2)
\end{tikzcd}
$$

$\mathrm{Diff}(S^1)$	Lie group of all diffeos on compact mf S^1
$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2),\ \mathcal{A} \in \{\mathcal{B}, H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to $\mathrm{Id}_{\mathbb{R}^2}$
$\mathrm{Imm}(S^1,\mathbb{R}^2)$	mf of all immersions $S^1 \to \mathbb{R}^2$
$B_i(S^1,\mathbb{R}^2) = \mathrm{Imm}/\mathrm{Diff}(S^1)$	shape space
$\mathrm{Vol}_+(S^1) = \left\{ f\, d\theta : f \in C^\infty(S^1,\mathbb{R}_{>0}) \right\}$	space of positive smooth probability densities
$\mathrm{Met}(S^1) = \left\{ g\, d\theta^2 : g \in C^\infty(S^1,\mathbb{R}_{>0}) \right\}$	space of metrics on S^1

The manifold of immersions

Let M be either S^1 or $[0, 2\pi]$.

$$\operatorname{Imm}(M, \mathbb{R}^2) := \{c \in C^\infty(M, \mathbb{R}^2) : c'(\theta) \neq 0\} \subset C^\infty(M, \mathbb{R}^2).$$

The tangent space of $\operatorname{Imm}(M, \mathbb{R}^2)$ at a curve c is the set of all vector fields along c:

$$T_c \operatorname{Imm}(M, \mathbb{R}^2) = \left\{ h : \begin{tikzcd} & T\mathbb{R}^2 \arrow[d, "\pi"] \\ M \arrow[ur, "h"] \arrow[r, "c"'] & \mathbb{R}^2 \end{tikzcd} \right\} \cong \{h \in C^\infty(M, \mathbb{R}^2)\}$$

Some Notation:

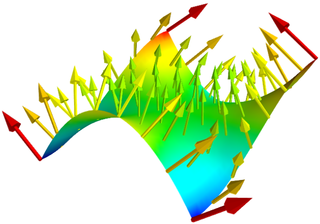
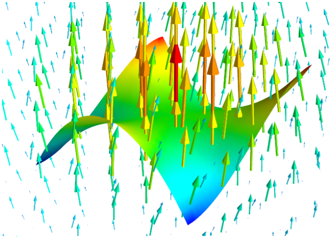
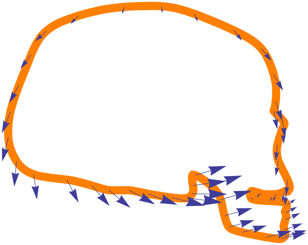
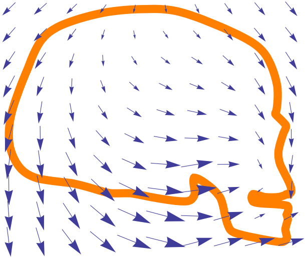
$$v(\theta) = \frac{c'(\theta)}{|c'(\theta)|}, \quad n(\theta) = iv(\theta), \quad ds = |c'(\theta)| d\theta, \quad D_s = \frac{1}{|c'(\theta)|} \partial_\theta$$

Inducing a metric on shape space

$$\begin{tikzcd} \operatorname{Imm}(M, N) \arrow[d, "\pi"] \\ B_i := \operatorname{Imm}(M, N)/\operatorname{Diff}(M) \end{tikzcd}$$

Every $\operatorname{Diff}(M)$-invariant metric "above" induces a unique metric "below" such that π is a Riemannian submersion.

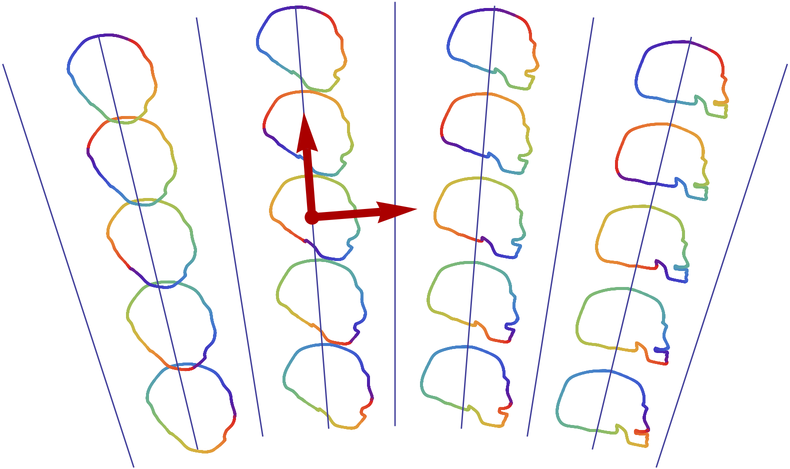
Inner versus Outer

The vertical and horizontal bundle

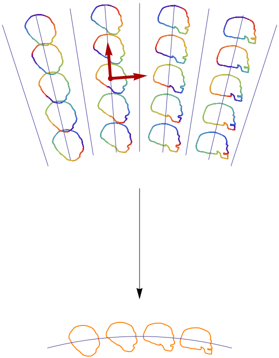
- $T\,\mathrm{Imm} = \mathrm{Vert} \bigoplus \mathrm{Hor}$.
- The vertical bundle is
$$\mathrm{Vert} := \ker T\pi \subset T\,\mathrm{Imm}\,.$$
- The horizontal bundle is
$$\mathrm{Hor} := (\ker T\pi)^{\perp, G} \subset T\,\mathrm{Imm}\,.$$
It might not be a complement - sometimes one has to go to the completion of $(T_f\,\mathrm{Imm}, G_f)$ in order to get a complement.

The vertical and horizontal bundle



Definition of a Riemannian metric

$$\begin{array}{c} \mathrm{Imm}(M,N) \\ \big\downarrow{\scriptstyle \pi} \\ B_i(M,N) \end{array}$$



1. Define a $\mathrm{Diff}(M)$-invariant metric G on Imm.
2. If the horizontal space is a complement, then $T\pi$ restricted to the horizontal space yields an isomorphism

$$(\ker T_f \pi)^{\perp, G} \cong T_{\pi(f)} B_i.$$

   Otherwise one has to induce the quotient metric, or use the completion.
3. This gives a metric on B_i such that $\pi : \mathrm{Imm} \to B_i$ is a *Riemannian submersion*.

Riemannian submersions

$$\begin{tikzcd} \mathrm{Imm}(M,N) \arrow[d, "\pi"] \\ B_i := \mathrm{Imm}(M,N)/\,\mathrm{Diff}(M) \end{tikzcd}$$

- Horizontal geodesics on $\mathrm{Imm}(M,N)$ project down to geodesics in shape space.
- O'Neill's formula connects sectional curvature on $\mathrm{Imm}(M,N)$ and on B_i.

[Micheli, M, Mumford, Izvestija 2013]

L^2 metric

$$G_c^0(h,k) = \int_M \langle h(\theta), k(\theta)\rangle ds.$$

Problem: The induced geodesic distance vanishes.

$$\begin{tikzcd}
\mathrm{Diff}(S^1) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"'] & & \mathrm{Imm}(S^1,\mathbb{R}^2) \arrow[dl, "\text{needs } \bar{g}"] \arrow[dr, "\mathrm{Diff}(S^1)"'] & & \mathrm{Diff}_c(\mathbb{R}^2) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts (LDDMM)}"] \\
& \mathrm{Met}(S^1) & & B_i(S^1,\mathbb{R}^2) &
\end{tikzcd}$$

Movies about vanishing: $\mathrm{Diff}(S^1)$ $\mathrm{Imm}(S^1,\mathbb{R}^2)$

[MichorMumford2005a,2005b], [BauerBruverisHarmsMichor2011,2012]

The simplest (L^2-) metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$

$$G_c^0(h,k) = \int_{S^1} \langle h,k\rangle ds = \int_{S^1} \langle h,k\rangle |c_\theta|\, d\theta$$

Geodesic equation

$$\boxed{c_{tt} = -\frac{1}{2|c_\theta|}\partial_\theta\Big(\frac{|c_t|^2\, c_\theta}{|c_\theta|}\Big) - \frac{1}{|c_\theta|^2}\langle c_{t\theta}, c_\theta\rangle c_t.}$$

A relative of Burger's equation.
Conserved momenta for G^0 along any geodesic $t \mapsto c(\quad, t)$:

$$\boxed{\begin{array}{ll} \langle v, c_t\rangle |c_\theta|^2 \ \in \mathfrak{X}(S^1) & \text{reparam. mom.} \\ \displaystyle\int_{S^1} c_t ds \ \in \mathbb{R}^2 & \text{linear moment.} \\ \displaystyle\int_{S^1} \langle Jc, c_t\rangle ds \in \mathbb{R} & \text{angular moment.} \end{array}}$$

Horizontal Geodesics for G^0 on $B_i(S^1, \mathbb{R}^2)$

$\langle c_t, c_\theta \rangle = 0$ and $c_t = an = aJ\frac{c_\theta}{|c_\theta|}$ for $a \in C^\infty(S^1, \mathbb{R})$. We use functions a, $s = |c_\theta|$, and κ, only holonomic derivatives:

$$s_t = -a\kappa s, \quad a_t = \tfrac{1}{2}\kappa a^2,$$
$$\kappa_t = a\kappa^2 + \frac{1}{s}\left(\frac{a_\theta}{s}\right)_\theta = a\kappa^2 + \frac{a_{\theta\theta}}{s^2} - \frac{a_\theta s_\theta}{s^3}.$$

We may assume $s|_{t=0} \equiv 1$. Let $v(\theta) = a(0,\theta)$, the initial value for a. Then
$\frac{s_t}{s} = -a\kappa = -2\frac{a_t}{a}$, so $\log(sa^2)_t = 0$, thus
$s(t,\theta)a(t,\theta)^2 = s(0,\theta)a(0,\theta)^2 = v(\theta)^2$,
a conserved quantity along the geodesic. We substitute $s = \frac{v^2}{a^2}$ and $\kappa = 2\frac{a_t}{a^2}$ to get

$$
\begin{aligned}
&a_{tt} - 4\frac{a_t^2}{a} - \frac{a^6 a_{\theta\theta}}{2v^4} + \frac{a^6 a_\theta v_\theta}{v^5} - \frac{a^5 a_\theta^2}{v^4} = 0,\\
&a(0,\theta) = v(\theta),
\end{aligned}
$$

a nonlinear hyperbolic second order equation. Note that wherever $v = 0$ then also $a = 0$ for all t. So substitute $a = vb$. The outcome is

$$
\boxed{
\begin{aligned}
&(b^{-3})_{tt} = -\frac{v^2}{2}(b^3)_{\theta\theta} - 2vv_\theta(b^3)_\theta - \frac{3vv_{\theta\theta}}{2}b^3,\\
&b(0,\theta) = 1.
\end{aligned}
}
$$

This is the codimension 1 version where
Burgers' equation is the codimension 0 version.

Weak Riem. metrics on $\mathrm{Emb}(M, N) \subset \mathrm{Imm}(M, N)$.

Metrics on the space of immersions of the form:

$$G_f^P(h, k) = \int_M \bar{g}(P^f h, k) \operatorname{vol}(f^* \bar{g})$$

where $\bar{g}$ is some fixed metric on N, $g = f^* \bar{g}$ is the induced metric on M, $h, k \in \Gamma(f^* TN)$ are tangent vectors at f to $\mathrm{Imm}(M, N)$, and P^f is a positive, selfadjoint, bijective (scalar) pseudo differential operator of order $2p$ depending smoothly on f. Good example: $P^f = 1 + A(\Delta^g)^p$, where Δ^g is the Bochner-Laplacian on M induced by the metric $g = f^* \bar{g}$. Also P has to be $\mathrm{Diff}(M)$-invariant: $\varphi^* \circ P_f = P_{f \circ \varphi} \circ \varphi^*$.

Elastic metrics on plane curves

Here $M = S^1$ or $[0, 1\pi]$, $N = \mathbb{R}^2$. The elastic metrics on $\mathrm{Imm}(M, \mathbb{R}^2)$ is

$$G_c^{a,b}(h,k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds,$$

with

$$\begin{aligned} P_c^{a,b}(h) = &- a^2 \langle D_s^2 h, n\rangle n - b^2 \langle D_s^2 h, v\rangle v \\ &+ (a^2 - b^2)\kappa\big(\langle D_s h, v\rangle n + \langle D_s h, n\rangle v\big) \\ &+ (\delta_{2\pi} - \delta_0)\big(a^2 \langle n, D_s h\rangle n + b^2 \langle v, D_s h\rangle v\big). \end{aligned}$$

Sobolev type metrics

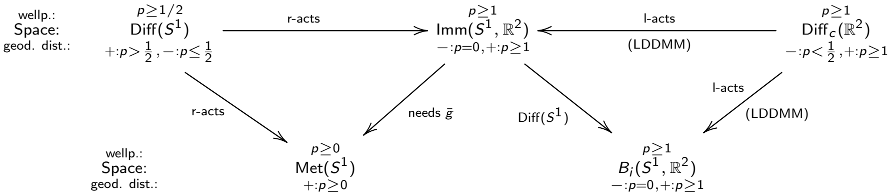
Advantages of Sobolev type metrics:

1. Positive geodesic distance
2. Geodesic equations are well posed
3. Spaces are geodesically complete for $p > \frac{\dim(M)}{2} + 1$.

   [Bruveris, M, Mumford, 2014] for plane curves. A remark in [Ebin, Marsden, 1970], and [Bruveris, Meyer, 2014] for diffeomorphism groups.

Problems:

1. Analytic solutions to the geodesic equation?
2. Curvature of shape space with respect to these metrics?
3. Numerics are in general computational expensive



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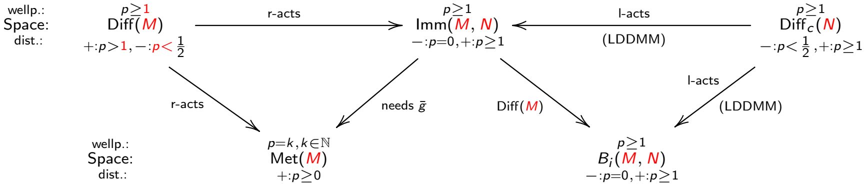


Geodesic equation.

The geodesic equation for a Sobolev-type metric G^P on immersions is given by

$$\begin{aligned}\nabla_{\partial_t} f_t =& \frac{1}{2} P^{-1}\Big(\operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2.Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ & \qquad - \bar{g}(Pf_t, f_t).\operatorname{Tr}^g(S)\Big) \\ & - P^{-1}\Big((\nabla_{f_t} P)f_t + \operatorname{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big)Pf_t\Big).\end{aligned}$$

The geodesic equation written in terms of the momentum for a Sobolev-type metric G^P on Imm is given by:

$$\begin{cases} p = Pf_t \otimes \operatorname{vol}(f^*\bar{g}) \\ \nabla_{\partial_t} p = \frac{1}{2}\big(\operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ \qquad\qquad - \bar{g}(Pf_t, f_t)\operatorname{Tr}^{f^*\bar{g}}(S)\big) \otimes \operatorname{vol}(f^*\bar{g}) \end{cases}$$

Wellposedness

Assumption 1: $P, \nabla P$ and $\operatorname{Adj}(\nabla P)^\perp$ are smooth sections of the bundles

$$\begin{array}{ccc} L(T\mathrm{Imm}; T\mathrm{Imm}) & L^2(T\mathrm{Imm}; T\mathrm{Imm}) & L^2(T\mathrm{Imm}; T\mathrm{Imm}) \\ \downarrow & \downarrow & \downarrow \\ \mathrm{Imm} & \mathrm{Imm} & \mathrm{Imm}, \end{array}$$

respectively. Viewed locally in trivializ. of these bundles,

$P_f h, \quad (\nabla P)_f(h,k), \quad \big(\operatorname{Adj}(\nabla P)_f(h,k)\big)^\perp$ are pseudo-differential operators of order $2p$ in h, k separately. As mappings in f they are non-linear, and we assume they are a composition of operators of the following type:

(a) Local operators of order $l \leq 2p$, i.e., nonlinear differential operators $A(f)(x) = A(x, \hat\nabla^l f(x), \hat\nabla^{l-1} f(x), \ldots, \hat\nabla f(x), f(x))$

(b) Linear pseudo-differential operators of degrees l_i,

such that the total (top) order of the composition is $\leq 2p$.

Assumption 2: For each $f \in \mathrm{Imm}(M,N)$, the operator P_f is an elliptic pseudo-differential operator of order $2p$ for $p > 0$ which is positive and symmetric with respect to the H^0-metric on Imm, i.e.

$$\int_M \bar g(P_f h, k)\,\mathrm{vol}(g) = \int_M \bar g(h, P_f k)\,\mathrm{vol}(g) \qquad \text{for } h, k \in T_f\mathrm{Imm}.$$

Theorem [Bauer, Harms, M, 2011] *Let $p \geq 1$ and $k > \dim(M)/2+1$, and let P satisfy the assumptions.*
Then the geodesic equation has unique local solutions in the Sobolev manifold Imm^{k+2p} of H^{k+2p}-immersions. The solutions depend smoothly on t and on the initial conditions $f(0,\ .\ )$ and $f_t(0,\ .\ )$. The domain of existence (in t) is uniform in k and thus this also holds in $\mathrm{Imm}(M,N)$. Moreover, in each Sobolev completion Imm^{k+2p}, the Riemannian exponential mapping $\exp^P$ exists and is smooth on a neighborhood of the zero section in the tangent bundle, and $(\pi, \exp^P)$ is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in $\mathrm{Imm}^{k+2p} \times \mathrm{Imm}^{k+2p}$. All these neighborhoods are uniform in $k > \dim(M)/2+1$ and can be chosen H^{k_0+2p}-open, for $k_0 > \dim(M)/2+1$. Thus both properties of the exponential mapping continue to hold in $\mathrm{Imm}(M,N)$.

Sobolev metrics of order ≥ 2 on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ are complete

Theorem. [Bruveris, M, Mumford, 2014] *Let $n \geq 2$ and the metric G on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ be given by*

$$G_c(h,k) = \int_{S^1} \sum_{j=0}^{n} a_j \langle D_s^j h, D_s^j k \rangle \,\mathrm{d}s\,,$$

with $a_j \geq 0$ and $a_0, a_n \neq 0$. Given initial conditions $(c_0, u_0) \in T\,\mathrm{Imm}(S^1, \mathbb{R}^2)$ the solution of the geodesic equation

$$\partial_t \left(\sum_{j=0}^{n} (-1)^j \,|c'|\, D_s^{2j} c_t \right) = -\frac{a_0}{2}\,|c'|\, D_s\left(\langle c_t, c_t\rangle v\right)$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{2k-1} (-1)^{k+j} \frac{a_k}{2}\,|c'|\, D_s\left(\langle D_s^{2k-j} c_t, D_s^j c_t\rangle v\right)\,.$$

for the metric G with initial values (c_0, u_0) exists for all time.

Recall: $ds = |c'|d\theta$ is arc-length measure, $D_s = \frac{1}{|c'|}\partial_\theta$ is the derivative with respect to arc-length, $v = c'/|c'|$ is the unit length tangent vector to c and $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbb{R}^2$.

Thus if G is a Sobolev-type metric of order at least 2, so that

$$\int_{S^1} (|h|^2 + |D_s^2 h|^2) ds \leq C \, G_c(h,h),$$

then the Riemannian manifold $(\operatorname{Imm}(S^1, \mathbb{R}^2), G)$ is geodesically complete. If the Sobolev-type metric is invariant under the reparameterization group $\operatorname{Diff}(S^1)$, also the induced metric on shape space $\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$ is geodesically complete.

The proof of this theorem is surprisingly difficult.

The elastic metric

$$G_c^{a,b}(h,k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds,$$

$$\boxed{\begin{aligned} c_t &= u \in C^\infty(\mathbb{R}_{>0} \times M, \mathbb{R}^2) \\ P(u_t) &= P(\frac{1}{2} H_c(u,u) - K_c(u,u)) \\ &= \frac{1}{2}(\delta_0 - \delta_{2\pi})\big(\langle D_s u, D_s u\rangle v + \tfrac{3}{4}\langle v, D_s u\rangle^2 v \\ &\qquad - 2\langle D_s u, v\rangle D_s u - \tfrac{3}{2}\langle n, D_s u\rangle^2 v\big) \\ &\qquad + D_s\big(\langle D_s u, D_s u\rangle v + \tfrac{3}{4}\langle v, D_s u\rangle^2 v \\ &\qquad - 2\langle D_s u, v\rangle D_s u - \tfrac{3}{2}\langle n, D_s u\rangle^2 v\big) \end{aligned}}$$

Note: Only a metric on Imm/transl.

Representation of the elastic metrics

Aim: Represent the class of elastic metrics as the pullback metric of a flat metric on $C^\infty(M, \mathbb{R}^2)$, i.e.: find a map

$$R : \mathrm{Imm}(M, \mathbb{R}^2) \mapsto C^\infty(M, \mathbb{R}^n)$$

such that

$$G_c^{a,b}(h, k) = R^* \langle h, k \rangle_{L^2} = \langle T_c R.h, T_c R.k \rangle_{L^2}.$$

[YounesMichorShahMumford2008] [SrivastavaKlassenJoshiJermyn2011]

The R transform on open curves

Theorem

The metric $G^{a,b}$ is the pullback of the flat L^2 metric via the transform R:

$$R^{a,b}: \operatorname{Imm}([0,2\pi],\mathbb{R}^2) \to C^\infty([0,2\pi],\mathbb{R}^3)$$
$$R^{a,b}(c) = |c'|^{1/2}\left(a\begin{pmatrix} v \\ 0\end{pmatrix} + \sqrt{4b^2-a^2}\begin{pmatrix}0\\1\end{pmatrix}\right) \quad .$$

The metric $G^{a,b}$ is flat on open curves, geodesics are the preimages under the R-transform of geodesics on the flat space $\operatorname{im} R$ *and the geodesic distance between $c, \overline{c} \in \operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{trans}$ is given by the integral over the pointwise distance in the image* $\operatorname{Im}(R)$*.*
The curvature on $B([0,2\pi],\mathbb{R}^2)$ is non-negative.

[BauerBruverisMarslandMichor2014]

The R transform on open curves II

Image of R is characterized by the condition:

$$(4b^2 - a^2)(R_1^2(c) + R_2^2(c)) = a^2 R_3^2(c)$$

Define the flat cone

$$C^{a,b} = \{q \in \mathbb{R}^3 : (4b^2 - a^2)(q_1^2 + q_2^2) = a^2 q_3^2, q_3 > 0\}.$$

Then $\operatorname{Im} R = C^\infty(S^1, C^{a,b})$. The inverse of R is given by:

$$\begin{aligned} &R^{-1} : \operatorname{im} R \to \operatorname{Imm}([0, 2\pi], \mathbb{R}^2)/\operatorname{trans} \\ &R^{-1}(q)(\theta) = p_0 + \frac{1}{2ab} \int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta\,. \end{aligned}$$

The R transform on closed curves I

Characterize image using the inverse:

$$R^{-1}(q)(\theta) = p_0 + \frac{1}{2ab}\int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta\,.$$

$R^{-1}(q)(\theta)$ is closed iff

$$F(q) = \int_0^{2\pi} |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta = 0$$

A basis of the orthogonal complement $\left(T_q\mathscr{C}^{a,b}\right)^\perp$ is given by the two gradients $\operatorname{grad}^{L^2} F_i(q)$

The R transform on closed curves II

Theorem

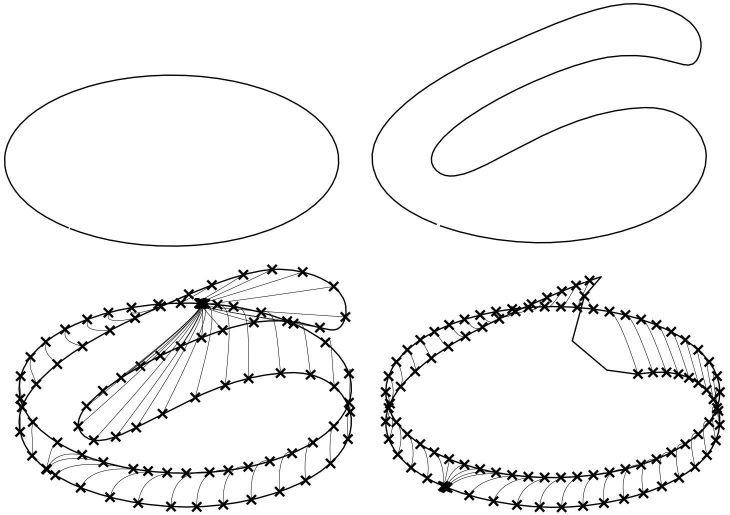
The image $\mathscr{C}^{a,b}$ of the manifold of closed curves under the R-transform is a codimension 2 submanifold of the flat space $\mathrm{Im}(R)_{open}$. A basis of the orthogonal complement $\left(T_q\mathscr{C}^{a,b}\right)^{\perp}$ is given by the two vectors

$$U_1(q) = \frac{1}{\sqrt{q_1^2+q_2^2}}\begin{pmatrix} 2q_1^2+q_2^2 \\ q_1q_2 \\ 0 \end{pmatrix} + \frac{2}{a}\sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 0 \\ q_1 \end{pmatrix},$$

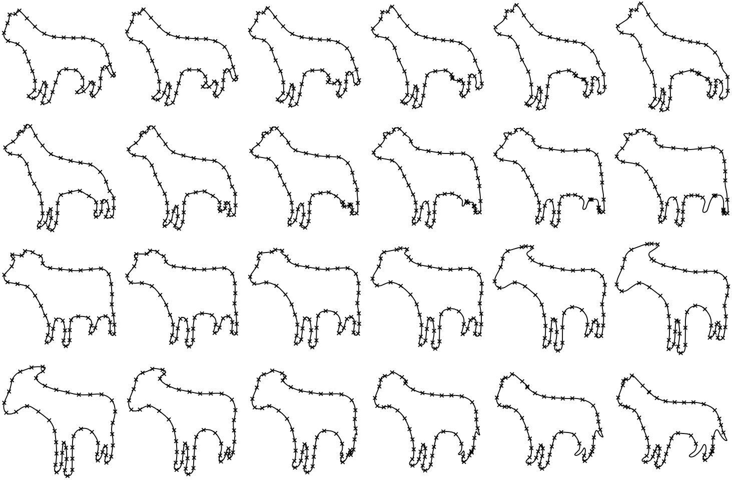
$$U_2(q) = \frac{1}{\sqrt{q_1^2+q_2^2}}\begin{pmatrix} q_1q_2 \\ q_1^2+2q_2^2 \\ 0 \end{pmatrix} + \frac{2}{a}\sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 0 \\ q_2 \end{pmatrix}.$$

[BauerBruverisMarslandMichor2012]

compress and stretch

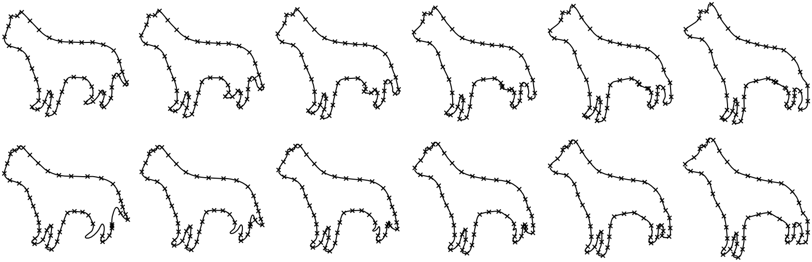


A geodesic Rectangle



Non-symmetric distances

			$I_1 \to I_2$			$I_2 \to I_1$		
I_1	I_2	# iterations	# points	distance	# iterations	# points	distance	%diff
cat	cow	28	456	7.339	33	462	8.729	15.9
cat	dog	36	475	8.027	102	455	10.060	20.2
cat	donkey	73	476	12.620	102	482	12.010	4.8
cow	donkey	32	452	7.959	26	511	7.915	0.6
dog	donkey	15	457	8.299	10	476	8.901	6.8
shark	airplane	63	491	13.741	40	487	13.453	2.1



An example of a metric space with strongly negatively curved regions

$$\begin{tikzcd}
\mathrm{Diff}(S^2) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"'] & & \mathrm{Imm}(S^2, \mathbb{R}^3) \arrow[dl, "\text{needs } \bar{g}"] \arrow[dr, "\mathrm{Diff}(S^2)"'] & & \mathrm{Diff}_c(\mathbb{R}^2) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \\
& \mathrm{Met}(S^2) & & B_i(S^2, \mathbb{R}^3) &
\end{tikzcd}$$

$$G_f^{\Phi}(h,k) = \int_M \Phi(f)\overline{g}(h,k)\,\mathrm{vol}(g)$$

[BauerHarmsMichor2012]

Non-vanishing geodesic distance

The pathlength metric on shape space induced by G^{Φ} separates points if one of the following holds:

- $\Phi \geq C_1 + C_2 \|\mathrm{Tr}^g(S)\|^2$ with $C_1, C_2 > 0$ or
- $\Phi \geq C_3 \,\mathrm{Vol}$

This leads us to consider $\Phi = \Phi(\mathrm{Vol}, \|\mathrm{Tr}^g(S)\|^2)$.
Special cases:

- G^A-metric: $\Phi = 1 + A\|\mathrm{Tr}^g(S)\|^2$
- Conformal metrics: $\Phi = \Phi(\mathrm{Vol})$

Geodesic equation on shape space $B_i(M, \mathbb{R}^n)$, with $\Phi = \Phi(\mathrm{Vol}, \mathrm{Tr}(L))$

$$
\begin{aligned}
f_t &= a.\nu, \\
a_t &= \frac{1}{\Phi}\Big[\frac{\Phi}{2} a^2 \operatorname{Tr}(L) - \frac{1}{2}\operatorname{Tr}(L)\int_M (\partial_1\Phi) a^2 \operatorname{vol}(g) - \frac{1}{2} a^2 \Delta(\partial_2\Phi) \\
&\qquad + 2a g^{-1}(d(\partial_2\Phi), da) + (\partial_2\Phi)\|da\|^2_{g^{-1}} \\
&\qquad + (\partial_1\Phi) a \int_M \operatorname{Tr}(L).a \operatorname{vol}(g) - \frac{1}{2}(\partial_2\Phi)\operatorname{Tr}(L^2) a^2\Big]
\end{aligned}
$$

Sectional curvature on B_i

Chart for B_i centered at $\pi(f_0)$ so that $\pi(f_0) = 0$ in this chart:

$$a \in C^\infty(M) \longleftrightarrow \pi(f_0 + a.\nu^{f_0}).$$

For a linear 2-dim. subspace $P \subset T_{\pi(f_0)}B_i$ spanned by a_1, a_1, the sectional curvature is defined as:

$$k(P) = -\frac{G^\Phi_{\pi(f_0)}\big(\mathcal{R}_{\pi(f_0)}(a_1, a_2)a_1, a_2\big)}{\|a_1\|^2\|a_2\|^2 - G^\Phi_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) = G_0^\Phi(R_0(a_1, a_2)a_1, a_2) = \\
\frac{1}{2}d^2G_0^\Phi(a_1, a_1)(a_2, a_2) + \frac{1}{2}d^2G_0^\Phi(a_2, a_2)(a_1, a_1) \\
- d^2G_0^\Phi(a_1, a_2)(a_1, a_2) \\
+ G_0^\Phi(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0^\Phi(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)).
\end{aligned}$$

Sectional curvature on B_i for $\Phi = \mathrm{Vol}$

$$k(P) = -\frac{\mathcal{R}_0(a_1, a_2, a_1, a_2)}{\|a_1\|^2 \|a_2\|^2 - G^{\Phi}_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

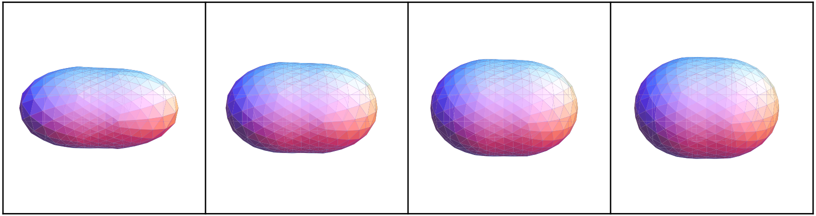
$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) &= -\frac{1}{2}\,\mathrm{Vol}\int_M \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}}\,\mathrm{vol}(g) \\
&+ \frac{1}{4\,\mathrm{Vol}}\overline{\mathrm{Tr}(L)^2}\left(\overline{a_1^2}.\overline{a_2^2} - \overline{a_1.a_2}^2\right) \\
&+ \frac{1}{4}\left(\overline{a_1^2}.\overline{\mathrm{Tr}(L)^2 a_2^2} - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}(L)^2 a_1.a_2} + \overline{a_2^2}.\overline{\mathrm{Tr}(L)^2 a_1^2}\right) \\
&- \frac{3}{4\,\mathrm{Vol}}\left(\overline{a_1^2}.\overline{\mathrm{Tr}(L)a_2}^2 - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}(L)a_1}.\overline{\mathrm{Tr}(L)a_2} + \overline{a_2^2}.\overline{\mathrm{Tr}(L)a_1}^2\right) \\
&+ \frac{1}{2}\left(\overline{a_1^2}.\overline{\mathrm{Tr}^g((da_2)^2)} - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}^g(da_1.da_2)} + \overline{a_2^2}\overline{\mathrm{Tr}^g((da_1)^2)}\right) \\
&- \frac{1}{2}\left(\overline{a_1^2}.\overline{a_2^2.\,\mathrm{Tr}(L^2)} - 2.\overline{a_1.a_2}.\overline{a_1.a_2.\,\mathrm{Tr}(L^2)} + \overline{a_2^2}.\overline{a_1^2.\,\mathrm{Tr}(L^2)}\right).
\end{aligned}$$

Sectional curvature on B_i for $\Phi = 1 + A\,\mathrm{Tr}(L)^2$

$$k(P) = -\frac{\mathcal{R}_0(a_1, a_2, a_1, a_2)}{\|a_1\|^2\|a_2\|^2 - G^{\Phi}_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) &= \int_M A(a_1\Delta a_2 - a_2\Delta a_1)^2 \,\mathrm{vol}(g)\\
&+ \int_M 2A\,\mathrm{Tr}(L) g_2^0\big((a_1 da_2 - a_2 da_1)\otimes(a_1 da_2 - a_2 da_1), s\big)\,\mathrm{vol}(g)\\
&+ \int_M \frac{1}{1 + A\,\mathrm{Tr}(L)^2}\Big[-4A^2 g^{-1}\big(d\,\mathrm{Tr}(L), a_1 da_2 - a_2 da_1\big)^2\\
&- \Big(\frac{1}{2}\big(1 + A\,\mathrm{Tr}(L)^2\big)^2 + 2A^2\,\mathrm{Tr}(L)\Delta(\mathrm{Tr}(L)) + 2A^2\,\mathrm{Tr}(L^2)\,\mathrm{Tr}(L)^2\Big)\cdot\\
&\quad\cdot \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}} + (2A^2\,\mathrm{Tr}(L)^2)\|da_1\wedge da_2\|^2_{g_0^2}\\
&+ (8A^2\,\mathrm{Tr}(L)) g_2^0\big(d\,\mathrm{Tr}(L)\otimes(a_1 da_2 - a_2 da_1), da_1\wedge da_2\big)\Big]\,\mathrm{vol}(g)
\end{aligned}$$

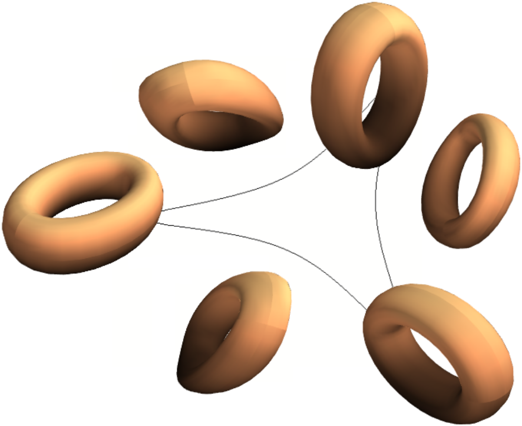
Negative Curvature: A toy example



Movies: Ex1: $\Phi = 1 + .4\,\mathrm{Tr}(L)^2$ Ex2: $\Phi = e^{Vol}$ Ex3: $\Phi = e^{Vol}$

Another toy example

$$G_f^{\Phi}(h,k) = \int_{\mathbb{T}^2} \overline{g}((1+\Delta)h,k)\,\mathrm{vol}(g) \text{ on } \mathrm{Imm}(\mathbb{T}^2,\mathbb{R}^3):$$



[BauerBruveris2011]

A Zoo of diffeomorphism groups on $\mathbb{R}^n$

For suitable convenient vector space $\mathcal{A}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ let $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ be the group of all diffeomorphisms of $\mathbb{R}^n$ of the form $\mathrm{Id} + f$ for $f \in \mathcal{A}(\mathbb{R}^n)^n$ with $\det(I_n + df(x)) \geq \varepsilon > 0$.

Theorem. *The sets of diffeomorphisms* $\mathrm{Diff}_c(\mathbb{R}^n)$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *and* $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms*

$$\mathrm{Diff}_c(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{H^\infty}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n) \,.$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$. *Similarly for suitable Denjoy-Carleman spaces of ultradifferentiable functions both of Roumieu and Beurling type:*

$$\mathrm{Diff}_{\mathcal{D}^{[M]}}(\mathbb{R}^n) \rightarrowtail \mathrm{Diff}_{\mathcal{S}^{[M]}_{[L]}}(\mathbb{R}^n) \rightarrowtail \mathrm{Diff}_{W^{[M],p}}(\mathbb{R}^n) \rightarrowtail \mathrm{Diff}_{W^{[M],q}}(\mathbb{R}^n) \rightarrowtail \mathrm{Diff}_{\mathcal{B}^{[M]}}(\mathbb{R}^n).$$

We require that the $M = (M_k)$ *is log-convex and has moderate growth, and that also* $C_b^{(M)} \supseteq C^\omega$ *in the Beurling case.*

[M,Mumford,2013], partly [B.Walter,2012]; for Denjoy-Carleman ultradifferentiable diffeomorphisms [Kriegl, M, Rainer 2014].

Right invariant Riemannian geometries on Diffeomorphism groups.

For $M = N$ the space $\mathrm{Emb}(M, M)$ equals the *diffeomorphism group of* M. An operator $P \in \Gamma\big(L(T\,\mathrm{Emb}; T\,\mathrm{Emb})\big)$ that is invariant under reparametrizations induces a right-invariant Riemannian metric on this space. Thus one gets the geodesic equation for right-invariant Sobolev metrics on diffeomorphism groups and well-posedness of this equation. The geodesic equation on $\mathrm{Diff}(M)$ in terms of the momentum p is given by

$$\begin{cases} \quad p = Pf_t \otimes \mathrm{vol}(g), \\ \nabla_{\partial_t} p = -Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \otimes \mathrm{vol}(g). \end{cases}$$

Note that this equation is not right-trivialized, in contrast to the equation given in [Arnold 1966]. The special case of theorem now reads as follows:

Theorem. [Bauer, Harms, M, 2011] *Let* $p \geq 1$ *and* $k > \frac{\dim(M)}{2} + 1$ *and let* P *satisfy the assumptions.*
The initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold Diff^{k+2p} *of* H^{k+2p}*-diffeomorphisms. The solutions depend smoothly on* t *and on the initial conditions* $f(0, \,.\,)$ *and* $f_t(0, \,.\,)$*. The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathrm{Diff}(M)$*.*
Moreover, in each Sobolev completion Diff^{k+2p}*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Diff}^{k+2p} \times \mathrm{Diff}^{k+2p}$*. All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* H^{k_0+2p}*-open, for* $k_0 > \dim(M)/2 + 1$*. Thus both properties of the exponential mapping continue to hold in* $\mathrm{Diff}(M)$*.*

Arnold's formula for geodesics on Lie groups: Notation

Let G be a regular convenient Lie group, with Lie algebra $\mathfrak{g}$. Let $\mu : G \times G \to G$ be the group multiplication, μ_x the left translation and μ^y the right translation, $\mu_x(y) = \mu^y(x) = xy = \mu(x, y)$.

Let $L, R : \mathfrak{g} \to \mathfrak{X}(G)$ be the left- and right-invariant vector field mappings, given by $L_X(g) = T_e(\mu_g).X$ and $R_X = T_e(\mu^g).X$, resp. They are related by $L_X(g) = R_{\mathrm{Ad}(g)X}(g)$. Their flows are given by

$$\begin{aligned}
\mathrm{Fl}_t^{L_X}(g) &= g.\exp(tX) = \mu^{\exp(tX)}(g), \\
\mathrm{Fl}_t^{R_X}(g) &= \exp(tX).g = \mu_{\exp(tX)}(g).
\end{aligned}$$

The right Maurer–Cartan form $\kappa = \kappa^r \in \Omega^1(G, \mathfrak{g})$ is given by $\kappa_x(\xi) := T_x(\mu^{x^{-1}}) \cdot \xi$.
The left Maurer–Cartan form $\kappa^l \in \Omega^1(G, \mathfrak{g})$ is given by $\kappa_x(\xi) := T_x(\mu_{x^{-1}}) \cdot \xi$.

κ^r *satisfies the left Maurer-Cartan equation* $d\kappa - \frac{1}{2}[\kappa, \kappa]_{\mathfrak{g}}^{\wedge} = 0$, where $[\quad,\quad]_{\wedge}$ denotes the wedge product of $\mathfrak{g}$-valued forms on G induced by the Lie bracket. Note that $\frac{1}{2}[\kappa, \kappa]_{\wedge}(\xi, \eta) = [\kappa(\xi), \kappa(\eta)]$.
κ^l *satisfies the right Maurer-Cartan equation* $d\kappa + \frac{1}{2}[\kappa, \kappa]_{\mathfrak{g}}^{\wedge} = 0$.

Geodesics of a Right-Invariant Metric on a Lie Group

Let $\gamma = \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive-definite bounded (weak) inner product. Then

$$\gamma_x(\xi, \eta) = \gamma\big(T(\mu^{x^{-1}}) \cdot \xi,\ T(\mu^{x^{-1}}) \cdot \eta\big) = \gamma\big(\kappa(\xi),\, \kappa(\eta)\big)$$

is a right-invariant (weak) Riemannian metric on G. Denote by $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ the mapping induced by γ, and by $\langle \alpha, X \rangle_{\mathfrak{g}}$ the duality evaluation between $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.
Let $g : [a, b] \to G$ be a smooth curve. The velocity field of g, viewed in the right trivializations, coincides with the right logarithmic derivative

$$\delta^r(g) = T(\mu^{g^{-1}}) \cdot \partial_t g = \kappa(\partial_t g) = (g^* \kappa)(\partial_t).$$

The energy of the curve $g(t)$ is given by

$$E(g) = \frac{1}{2} \int_a^b \gamma_g(g', g')dt = \frac{1}{2} \int_a^b \gamma\big((g^*\kappa)(\partial_t), (g^*\kappa)(\partial_t)\big)\, dt.$$

Thus the curve $g(0,t)$ is critical for the energy if and only if

$$\check{\gamma}(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check{\gamma}((g^*\kappa)(\partial_t)) = 0.$$

In terms of the right logarithmic derivative $u : [a,b] \to \mathfrak{g}$ of $g : [a,b] \to G$, given by $u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}}) \cdot g'(t)$, the geodesic equation has the expression

$$\boxed{\partial_t u = -\,\check{\gamma}^{-1}\,\mathrm{ad}(u)^*\,\check{\gamma}(u)} \tag{1}$$

Thus the geodesic equation exists in general if and only if $\mathrm{ad}(X)^*\check{\gamma}(X)$ is in the image of $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$, i.e.

$$\mathrm{ad}(X)^*\check{\gamma}(X) \in \check{\gamma}(\mathfrak{g}) \tag{2}$$

for every $X \in \mathfrak{X}$. Condition (2) then leads to the existence of the Christoffel symbols. [Arnold 1966] has the more restrictive condition $\mathrm{ad}(X)^*\check{\gamma}(Y) \in \check{\gamma} \in \mathfrak{g}$. The geodesic equation for the *momentum* $p := \gamma(u)$:

$$p_t = -\,\mathrm{ad}(\check{\gamma}^{-1}(p))^*p.$$

A covariant formula for curvature and its relations to O'Neill's curvature formulas.

Mario Micheli in his 2008 thesis derived the the coordinate version of the following formula for the sectional curvature expression, which is valid for **closed** 1-forms α, β on a Riemannian manifold (M, g), where we view $g : TM \to T^*M$ and so g^{-1} is the dual inner product on T^*M. Here $\alpha^\sharp = g^{-1}(\alpha)$.

$$\boxed{\begin{aligned}
&g\big(R(\alpha^\sharp, \beta^\sharp)\alpha^\sharp, \beta^\sharp\big) = \\
&-\tfrac{1}{2}\alpha^\sharp\alpha^\sharp(\|\beta\|^2_{g^{-1}}) - \tfrac{1}{2}\beta^\sharp\beta^\sharp(\|\alpha\|^2_{g^{-1}}) + \tfrac{1}{2}(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g^{-1}(\alpha, \beta) \\
&\qquad\qquad\big(\text{last line } = -\alpha^\sharp\beta([\alpha^\sharp, \beta^\sharp]) + \beta^\sharp\alpha([\alpha^\sharp, \beta^\sharp])\big) \\
&-\tfrac{1}{4}\|d(g^{-1}(\alpha, \beta))\|^2_{g^{-1}} + \tfrac{1}{4}g^{-1}\big(d(\|\alpha\|^2_{g^{-1}}), d(\|\beta\|^2_{g^{-1}})\big) \\
&+\tfrac{3}{4}\big\|[\alpha^\sharp, \beta^\sharp]\big\|^2_g
\end{aligned}}$$

Mario's formula in coordinates

Assume that $\alpha = \alpha_i dx^i, \beta = \beta_i dx^i$ where the coefficients α_i, β_i are *constants*, hence α, β are closed.
Then $\alpha^\sharp = g^{ij}\alpha_i\partial_j, \beta^\sharp = g^{ij}\beta_i\partial_j$ and we have:

$$\boxed{\begin{aligned}&4g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)\\ &= (\alpha_i\beta_k - \alpha_k\beta_i)\cdot(\alpha_j\beta_l - \alpha_l\beta_j)\cdot\\ &\cdot\Big(2g^{is}(g^{jt}g^{kl}_{,t})_{,s} - \tfrac{1}{2}g^{ij}_{,s}g^{st}g^{kl}_{,t} - 3g^{is}g^{kp}_{,s}g_{pq}g^{jt}g^{lq}_{,t}\Big)\end{aligned}}$$

Covariant curvature and O'Neill's formula, finite dim.

Let $p : (E, g_E) \to (B, g_B)$ be a Riemannian submersion:
For $b \in B$ and $x \in E_b := p^{-1}(b)$ the g_E-orthogonal splitting

$$T_x E = T_x(E_{p(x)}) \oplus T_x(E_{p(x)})^{\perp, g_E} =: T_x(E_{p(x)}) \oplus \mathrm{Hor}_x(p).$$
$$T_x p : (\mathrm{Hor}_x(p), g_E) \to (T_b B, g_B)$$

is an isometry. A vector field $X \in \mathfrak{X}(E)$ is decomposed as $X = X^{\mathrm{hor}} + X^{\mathrm{ver}}$ into horizontal and vertical parts. Each vector field $\xi \in \mathfrak{X}(B)$ can be uniquely lifted to a smooth horizontal field $\xi^{\mathrm{hor}} \in \Gamma(\mathrm{Hor}(p)) \subset \mathfrak{X}(E)$.

Semilocal version of Mario's formula, force, and stress

Let (M, g) be a robust Riemannian manifold, $x \in M$, $\alpha, \beta \in g_x(T_x M)$. Assume we are given local smooth vector fields X_α and X_β such that:

1. $X_\alpha(x) = \alpha^\sharp(x), \quad X_\beta(x) = \beta^\sharp(x)$,
2. Then $\alpha^\sharp - X_\alpha$ is zero at x hence has a well defined derivative $D_x(\alpha^\sharp - X_\alpha)$ lying in $\mathrm{Hom}(T_x M, T_x M)$. For a vector field Y we have $D_x(\alpha^\sharp - X_\alpha).Y_x = [Y, \alpha^\sharp - X_\alpha](x) = \mathcal{L}_Y(\alpha^\sharp - X_\alpha)|_x$. The same holds for β.
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$,
4. $[X_\alpha, X_\beta] = 0$.

Locally constant 1-forms and vector fields will do. We then define:

$$\mathcal{F}(\alpha, \beta) := \tfrac{1}{2} d(g^{-1}(\alpha, \beta)), \qquad \text{a 1-form on } M \text{ called the } \textit{force},$$

$$\begin{aligned}\mathcal{D}(\alpha, \beta)(x) &:= D_x(\beta^\sharp - X_\beta).\alpha^\sharp(x) \\ &= d(\beta^\sharp - X_\beta).\alpha^\sharp(x), \quad \in T_x M \text{ called the } \textit{stress}.\end{aligned}$$

$$\Longrightarrow \mathcal{D}(\alpha, \beta)(x) - \mathcal{D}(\beta, \alpha)(x) = [\alpha^\sharp, \beta^\sharp](x)$$

Then in the notation above:

$$
\boxed{
\begin{aligned}
&g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)(x) = R_{11}+R_{12}+R_2+R_3\\
&R_{11} = \tfrac{1}{2}\Big(\mathcal{L}^2_{X_\alpha}(g^{-1})(\beta,\beta) - 2\mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}(g^{-1})(\alpha,\beta)\\
&\qquad\qquad + \mathcal{L}^2_{X_\beta}(g^{-1})(\alpha,\alpha)\Big)(x)\\
&R_{12} = \langle\mathcal{F}(\alpha,\alpha),\mathcal{D}(\beta,\beta)\rangle + \langle\mathcal{F}(\beta,\beta),\mathcal{D}(\alpha,\alpha)\rangle\\
&\qquad\qquad - \langle\mathcal{F}(\alpha,\beta),\mathcal{D}(\alpha,\beta)+\mathcal{D}(\beta,\alpha)\rangle\\
&R_2 = \Big(\|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} - \big\langle\mathcal{F}(\alpha,\alpha)),\mathcal{F}(\beta,\beta)\big\rangle_{g^{-1}}\Big)(x)\\
&R_3 = -\tfrac{3}{4}\|\mathcal{D}(\alpha,\beta)-\mathcal{D}(\beta,\alpha)\|^2_{g_x}
\end{aligned}
}
$$

Diffeomorphism groups

Let N be a manifold. We consider the following regular Lie groups:

$\mathrm{Diff}(N)$, the group of all diffeomorphisms of N if N is compact.
$\mathrm{Diff}_c(N)$, the group of diffeomorphisms with compact support.

If (N, g) is a *Riemannian manifold of bounded geometry*, we also may consider:

$\mathrm{Diff}_{\mathcal{S}}(N)$, the group of all diffeos which fall rapidly to the identity.
$\mathrm{Diff}_{H^\infty}(N)$, the group of all diffeos which are modelled on the space $\Gamma_{H^\infty}(TM)$, the intersection of all Sobolev spaces of vector fields.

The Lie algebras are the spaces $\mathfrak{X}_{\mathcal{A}}(N)$ of vector fields, where $\mathcal{A} \in \{C_c^\infty, \mathcal{S}, H^\infty\}$, with the negative of the usual bracket as Lie bracket.

Riemann metrics on Diff(N).

The concept of robust Riemannian manifolds, and also the reproducing Hilbert space approach in Chapter 12 of [Younes 2010] leads to:
We construct a right invariant weak Riemannian metric by assuming that we have a Hilbert space $\mathcal{H}$ together with two bounded injective linear mappings

$$\mathfrak{X}_{\mathcal{S}}(N) = \Gamma_{\mathcal{S}}(TN) \xrightarrow{j_1} \mathcal{H} \xrightarrow{j_2} \Gamma_{C^2_b}(TN) \tag{1}$$

where $\Gamma_{C^2_b}(TN)$ is the Banach space of all C^2 vector fields X on N which are globally bounded together with $\nabla^g X$ and $\nabla^g \nabla^g X$ with respect to g, such that $j_2 \circ j_1 : \Gamma_{\mathcal{S}}(TN) \to \Gamma_{C^2_b}(TN)$ is the canonical embedding. We also assume that j_1 has dense image.

Dualizing the Banach spaces in equation (1) and using the canonical isomorphisms between $\mathcal{H}$ and its dual $\mathcal{H}'$ – which we call L and K, we get the diagram:

$$
\begin{tikzcd}
\Gamma_{\mathcal{S}}(TN) \arrow[d, hook, "j_1"] & \Gamma_{\mathcal{S}'}(T^*N) \\
\mathcal{H} \arrow[r, shift left, "L"] \arrow[d, hook, "j_2"] & \mathcal{H}' \arrow[l, shift left, "K"] \arrow[u, hook, "j_1'"'] \\
\Gamma_{C_b^2}(TN) & \Gamma_{M^2}(T^*N) \arrow[u, hook, "j_2'"']
\end{tikzcd}
$$

Here $\Gamma_{\mathcal{S}'}(T^*N)$, the space of *1-co-currents*, is the dual of the space of smooth vector fields $\Gamma_{\mathcal{S}}(TN) = \mathfrak{X}_{\mathcal{S}}(N)$. It contains the space $\Gamma_{\mathcal{S}}(T^*N \otimes \mathrm{vol}(N))$ of smooth measure valued cotangent vectors on N, and also the bigger subspace of second derivatives of finite measure valued 1-forms on N, written as $\Gamma_{M^2}(T^*N)$ which is part of the dual of $\Gamma_{C_b^2}(TN)$. In what follows, we will have many momentum variables with values in these spaces.

In the case (called the *standard case* below) that $N = \mathbb{R}^n$ and that

$$\langle X, Y\rangle_L = \int_{\mathbb{R}^n} \langle (1 - A\Delta)^l X, Y\rangle \, dx$$

we have

$$L(x,y) = \Big(\frac{1}{(2\pi)^n} \int_{\xi\in\mathbb{R}^n} e^{i\langle \xi, x-y\rangle} (1 + A|\xi|^2)^l d\xi \Big) \sum_{i=1}^{n} (du^i|_x \otimes dx) \otimes (du^i|_y \otimes dy)$$

where $d\xi$, dx and dy denote Lebesque measure, where (u^i) are linear coord. on $\mathbb{R}^n$. Here $\mathcal{H}$ consists of Sobolev H^l vector fields.

$$K(x,y) = K_l(x-y) \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i},$$

$$K_l(x) = \frac{1}{(2\pi)^n} \int_{\xi\in\mathbb{R}^n} \frac{e^{i\langle \xi, x\rangle}}{(1 + A|\xi|^2)^l} d\xi$$

where K_l is given by a classical Bessel function which is C^{2l}.

The geodesic equation on $\mathrm{Diff}_{\mathcal{S}}(N)$

According to [Arnold 1966], slightly generalized as explained above: Let $\varphi : [a,b] \to \mathrm{Diff}_{\mathcal{S}}(N)$ be a smooth curve. In terms of its right logarithmic derivative $u : [a,b] \to \mathfrak{X}_{\mathcal{S}}(N)$,
$u(t) := \varphi^*\kappa(\partial_t) = T_{\varphi(t)}(\mu^{\varphi(t)^{-1}}) \cdot \varphi'(t) = \varphi'(t) \circ \varphi(t)^{-1}$, the geodesic equation is

$$\boxed{L(u_t) = -\operatorname{ad}(u)^* L(u)}$$

Condition for the existence of the geodesic equation:

$$X \mapsto K(\operatorname{ad}(X)^* L(X))$$

is bounded quadratic $\mathfrak{X}_{\mathcal{S}}(N) \to \mathfrak{X}_{\mathcal{S}}(N)$.

The Lie algebra of $\mathrm{Diff}_{\mathcal{S}}(N)$ is the space $\mathfrak{X}_{\mathcal{S}}(N)$ of all rapidly decreasing smooth vector fields with Lie bracket the negative of the usual Lie bracket $\operatorname{ad}_X Y = -[X,Y]$.

Using *Lie derivatives*, the computation of ad_X^* is especially simple. Namely, for any section ω of $T^*N \otimes \mathrm{vol}$ and vector fields $\xi, \eta \in \mathfrak{X}_{\mathcal{S}}(N)$, we have:

$$\int_N (\omega, [\xi, \eta]) = \int_N (\omega, \mathcal{L}_\xi(\eta)) = -\int_N (\mathcal{L}_\xi(\omega), \eta),$$

hence $\operatorname{ad}_\xi^*(\omega) = +\mathcal{L}_\xi(\omega)$.

Thus the Hamiltonian version of the geodesic equation on the smooth dual $L(\mathfrak{X}_{\mathcal{S}}(N)) \subset \Gamma_{C^2}(T^*N \otimes \mathrm{vol})$ becomes

$$\boxed{\partial_t \alpha = -\operatorname{ad}^*_{K(\alpha)} \alpha = -\mathcal{L}_{K(\alpha)} \alpha,}$$

or, keeping track of everything,

$$\boxed{\begin{aligned} \partial_t \varphi &= u \circ \varphi, \\ \partial_t \alpha &= -\mathcal{L}_u \alpha \\ u = K(\alpha) &= \alpha^\sharp, \quad \alpha = L(u) = u^\flat. \end{aligned}} \tag{1}$$

Landmark space as homogeneus space of solitons

A *landmark* $q = (q_1, \dots, q_N)$ is an N-tuple of distinct points in $\mathbb{R}^n$; so $\mathrm{Land}^N \subset (\mathbb{R}^n)^N$ is open. Let $q^0 = (q^0_1, \dots, q^0_N)$ be a fixed standard template landmark. Then we have the the surjective mapping

$$\begin{aligned}
&\mathrm{ev}_{q^0} : \mathrm{Diff}(\mathbb{R}^n) \to \mathrm{Land}^N,\\
&\varphi \mapsto \mathrm{ev}_{q^0}(\varphi) = \varphi(q^0) = (\varphi(q^0_1), \dots, \varphi(q^0_N)).
\end{aligned}$$

The fiber of ev_{q^0} over a landmark $q = \varphi_0(q^0)$ is

$$\begin{aligned}
&\{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q\}\\
&= \varphi_0 \circ \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q^0\}\\
&= \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q) = q\} \circ \varphi_0;
\end{aligned}$$

The tangent space to the fiber is

$$\{X \circ \varphi_0 : X \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n), X(q_i) = 0 \text{ for all } i\}.$$

A tangent vector $Y \circ \varphi_0 \in T_{\varphi_0} \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is $G^L_{\varphi_0}$-perpendicular to the fiber over q if

$$\int_{\mathbb{R}^n} \langle LY, X \rangle \, dx = 0 \quad \forall X \text{ with } X(q) = 0.$$

If we require Y to be smooth then $Y = 0$. So we assume that $LY = \sum_i P_i.\delta_{q_i}$, a distributional vector field with support in q. Here $P_i \in T_{q_i}\mathbb{R}^n$. But then

$$Y(x) = L^{-1}\Big(\sum_i P_i.\delta_{q_i}\Big) = \int_{\mathbb{R}^n} K(x-y) \sum_i P_i.\delta_{q_i}(y) \, dy$$

$$= \sum_i K(x - q_i).P_i$$

$$T_{\varphi_0}(\mathrm{ev}_{q^0}).(Y \circ \varphi_0) = Y(q_k)_k = \sum_i (K(q_k - q_i).P_i)_k$$

Now let us consider a tangent vector $P = (P_k) \in T_q \operatorname{Land}^N$. Its horizontal lift with footpoint φ_0 is $P^{\text{hor}} \circ \varphi_0$ where the vector field P^{hor} on $\mathbb{R}^n$ is given as follows: Let $K^{-1}(q)_{ki}$ be the inverse of the $(N \times N)$-matrix $K(q)_{ij} = K(q_i - q_j)$. Then

$$
\begin{aligned}
P^{\text{hor}}(x) &= \sum_{i,j} K(x - q_i) K^{-1}(q)_{ij} P_j \\
L(P^{\text{hor}}(x)) &= \sum_{i,j} \delta(x - q_i) K^{-1}(q)_{ij} P_j
\end{aligned}
$$

Note that P^{hor} is a vector field of class H^{2l-1}.

The Riemannian metric on Land^N induced by the g^L-metric on $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is

$$\begin{aligned} g^L_q(P,Q) &= G^L_{\varphi_0}(P^{\mathrm{hor}}, Q^{\mathrm{hor}}) \\ &= \int_{\mathbb{R}^n} \langle L(P^{\mathrm{hor}}), Q^{\mathrm{hor}} \rangle \, dx \\ &= \int_{\mathbb{R}^n} \Big\langle \sum_{i,j} \delta(x-q_i) K^{-1}(q)_{ij} P_j, \\ &\qquad\qquad \sum_{k,l} K(x-q_k) K^{-1}(q)_{kl} Q_l \Big\rangle \, dx \\ &= \sum_{i,j,k,l} K^{-1}(q)_{ij} K(q_i-q_k) K^{-1}(q)_{kl} \langle P_j, Q_l \rangle \\ g^L_q(P,Q) &= \sum_{k,l} K^{-1}(q)_{kl} \langle P_k, Q_l \rangle. \end{aligned} \tag{1}$$

The geodesic equation in vector form is:

$$\boxed{\begin{aligned}&\ddot{q}_n = \\&-\frac{1}{2}\sum_{k,i,j,l} K^{-1}(q)_{ki}\,\mathrm{grad}\,K(q_i - q_j)(K(q)_{in} - K(q)_{jn}) \\&\qquad\qquad K^{-1}(q)_{jl}\langle \dot{q}_k, \dot{q}_l\rangle \\&+\sum_{k,i} K^{-1}(q)_{ki}\Big\langle \mathrm{grad}\,K(q_i - q_n), \dot{q}_i - \dot{q}_n\Big\rangle \dot{q}_k\end{aligned}}$$

The geodesic equation on $T^*\mathrm{Land}^N(\mathbb{R}^n)$

.

The cotangent bundle $T^*\mathrm{Land}^N(\mathbb{R}^n) = \mathrm{Land}^N(\mathbb{R}^n) \times ((\mathbb{R}^n)^N)^* \ni (q, \alpha)$. We shall treat $\mathbb{R}^n$ like scalars; $\langle \quad , \quad \rangle$ is always the standard inner product on $\mathbb{R}^n$.
The metric looks like

$$(g^L)_q^{-1}(\alpha, \beta) = \sum_{i,j} K(q)_{ij} \langle \alpha_i, \beta_j \rangle,$$
$$K(q)_{ij} = K(q_i - q_j).$$

The energy function

$$E(q,\alpha) = \tfrac{1}{2}(g^L)_q^{-1}(\alpha,\alpha) = \tfrac{1}{2}\sum_{i,j} K(q)_{ij}\langle\alpha_i,\alpha_j\rangle$$

and its Hamiltonian vector field (using $\mathbb{R}^n$-valued derivatives to save notation)

$$H_E(q,\alpha) = \sum_{i,k=1}^{N}\Big(K(q_k - q_i)\alpha_i\frac{\partial}{\partial q_k} + \operatorname{grad} K(q_i - q_k)\langle\alpha_i,\alpha_k\rangle\frac{\partial}{\partial\alpha_k}\Big).$$

So the geodesic equation is the flow of this vector field:

$$\boxed{\begin{aligned}\dot{q}_k &= \sum_i K(q_i - q_k)\alpha_i\\ \dot{\alpha}_k &= -\sum_i \operatorname{grad} K(q_i - q_k)\langle\alpha_i,\alpha_k\rangle\end{aligned}}$$

Stress and Force

$$\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha_i, \quad \alpha^\sharp = \sum_{i,k} K(q_k - q_i)\langle \alpha_i, \tfrac{\partial}{\partial q^k}\rangle$$

$$\mathcal{D}(\alpha, \beta) := \sum_{i,j} dK(q_i - q_j)(\alpha_i^\sharp - \alpha_j^\sharp)\Big\langle \beta_j, \frac{\partial}{\partial q_i}\Big\rangle, \quad \text{the } \textit{stress.}$$

$$\mathcal{D}(\alpha, \beta) - \mathcal{D}(\beta, \alpha) = (D_{\alpha^\sharp}\beta^\sharp) - D_{\beta^\sharp}\alpha^\sharp = [\alpha^\sharp, \beta^\sharp], \quad \textit{Lie bracket.}$$

$$\mathcal{F}_i(\alpha, \beta) = \frac{1}{2}\sum_k \operatorname{grad} K(q_i - q_k)(\langle \alpha_i, \beta_k\rangle + \langle \beta_i, \alpha_k\rangle)$$

$$\mathcal{F}(\alpha, \beta) := \sum_i \langle \mathcal{F}_i(\alpha, \beta), dq_i\rangle = \frac{1}{2}\, d\, g^{-1}(\alpha, \beta) \quad \text{the } \textit{force}.$$

The geodesic equation on $T^*\,\mathrm{Land}^N(\mathbb{R}^n)$ then becomes

$$\boxed{\begin{aligned} \dot{q} &= \alpha^\sharp \\ \dot{\alpha} &= -\mathcal{F}(\alpha, \alpha) \end{aligned}}$$

Curvature via the cotangent bundle

From the semilocal version of Mario's formula for the sectional curvature expression for constant 1-forms α, β on landmark space, where $\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha_i$, we get directly:

$$\boxed{\begin{aligned}
&g^L\big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\big) = \\
&= \big\langle \mathcal{D}(\alpha,\beta) + \mathcal{D}(\beta,\alpha), \mathcal{F}(\alpha,\beta)\big\rangle \\
&\quad - \big\langle \mathcal{D}(\alpha,\alpha), \mathcal{F}(\beta,\beta)\big\rangle - \big\langle \mathcal{D}(\beta,\beta), \mathcal{F}(\alpha,\alpha)\big\rangle \\
&\quad - \tfrac{1}{2}\sum_{i,j}\Big(d^2K(q_i - q_j)(\beta_i^\sharp - \beta_j^\sharp, \beta_i^\sharp - \beta_j^\sharp)\langle\alpha_i,\alpha_j\rangle \\
&\qquad\qquad - 2d^2K(q_i - q_j)(\beta_i^\sharp - \beta_j^\sharp, \alpha_i^\sharp - \alpha_j^\sharp)\langle\beta_i,\alpha_j\rangle \\
&\qquad\qquad + d^2K(q_i - q_j)(\alpha_i^\sharp - \alpha_j^\sharp, \alpha_i^\sharp - \alpha_j^\sharp)\langle\beta_i,\beta_j\rangle\Big) \\
&\quad - \|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} + g^{-1}\big(\mathcal{F}(\alpha,\alpha), \mathcal{F}(\beta,\beta)\big). \\
&\quad + \tfrac{3}{4}\|[\alpha^\sharp,\beta^\sharp]\|^2_g
\end{aligned}}$$

Bundle of embeddings over the differentiable Chow variety.

Let M be a compact connected manifold with $\dim(M) < \dim(N)$. The smooth manifold $\mathrm{Emb}(M, N)$ of all embeddings $M \to N$ is the total space of a smooth principal bundle with structure group $\mathrm{Diff}(M)$ acting freely by composition from the right hand side.

The quotient manifold $B(M, N)$ can be viewed as the space of all submanifolds of N of diffeomorphism type M; we call it the *differentiable Chow manifold* or the *non-linear Grassmannian*.

$B(M, N)$ is a smooth manifold with charts centered at $F \in B(M, N)$ diffeomorphic to open subsets of the Frechet space of sections of the normal bundle $TF^{\perp, g} \subset TN|_F$.

Let $\ell : \mathrm{Diff}_{\mathcal{S}}(N) \times B(M, N) \to B(M, N)$ be the smooth left action. In the following we will consider just one open $\mathrm{Diff}_{\mathcal{S}}(N)$-orbit $\ell(\mathrm{Diff}_{\mathcal{S}}(N), F_0)$ in $B(M, N)$.

The induced Riemannian cometric on $T^*B(M,N)$

We follow the procedure used for $\mathrm{Diff}_{\mathcal{S}}(N)$. For any $F \subset N$, we decompose $\mathcal{H}$ into:

$$\begin{aligned} \mathcal{H}_F^{\text{vert}} &= j_2^{-1}\big(\{X \in \Gamma_{C_b^2}(TN) : X(x) \in T_xF, \text{ for all } x \in F\}\big) \\ \mathcal{H}_F^{\text{hor}} &= \text{perpendicular complement of } \mathcal{H}_F^{\text{vert}} \end{aligned}$$

It is then easy to check that we get the diagram:

$$\begin{tikzcd} \Gamma_{\mathcal{S}}(TN) \arrow[r, hook, "j_1"] \arrow[d, two heads, "\text{res}"] & \mathcal{H} \arrow[r, hook, "j_2"] \arrow[d, two heads] & \Gamma_{C_b^2}(TN) \arrow[d, two heads, "\text{res}"] \\ \Gamma_{\mathcal{S}}(\mathrm{Nor}(F)) \arrow[r, hook, "j_1^f"] & \mathcal{H}_F^{\text{hor}} \arrow[r, hook, "j_2^f"] & \Gamma_{C_b^2}(\mathrm{Nor}(F)). \end{tikzcd}$$

Here $\mathrm{Nor}(F) = TN|_F / TF$.

As this is an orthogonal decomposition, L and K take $\mathcal{H}_F^{\mathrm{vert}}$ and $\mathcal{H}_F^{\mathrm{hor}}$ into their own duals and, as before we get:

$$\begin{tikzcd}
\Gamma_{\mathcal{S}}(\mathrm{Nor}(F)) \arrow[d, hook, "j_1"] & \Gamma_{\mathcal{S}'}(\mathrm{Nor}^*(F)) \\
\mathcal{H}_F^{\mathrm{hor}} \arrow[r, shift left, "L_F"] \arrow[d, hook, "j_2"] & (\mathcal{H}_F^{\mathrm{hor}})' \arrow[l, shift left, "K_F"] \arrow[u, hook, "j_1'"'] \\
\Gamma_{C_b^2}(\mathrm{Nor}(F)) & \Gamma_{M^2}(\mathrm{Nor}^*(F)) \arrow[u, hook, "j_2'"']
\end{tikzcd}$$

K_F is just the restriction of K to this subspace of $\mathcal{H}'$ and is given by the kernel:

$$K_F(x_1, x_2) := \text{image of } K(x_1, x_2) \in \mathrm{Nor}_{x_1}(F) \otimes \mathrm{Nor}_{x_2}(F)), \quad x_1, x_2 \in F.$$

This is a C^2 section over $F \times F$ of $\mathrm{pr}_1^* \mathrm{Nor}(F) \otimes \mathrm{pr}_2^* \mathrm{Nor}(F)$.

We can identify $\mathcal{H}_F^{\text{hor}}$ as the closure of the image under K_F of measure valued 1-forms supported by F and with values in $\text{Nor}^*(F)$. A dense set of elements in $\mathcal{H}_F^{\text{hor}}$ is given by either taking the 1-forms with finite support or taking smooth 1-forms. In the smooth case, fix a volume form κ on M and a smooth covector $\xi \in \Gamma_{\mathcal{S}}(\text{Nor}^*(F))$. Then $\xi.\kappa$ defines a horizontal vector field h like this:

$$h(x_1) = \int_{x_2 \in F} \big| K_F(x_1, x_2) \big| \xi(x_2).\kappa(x_2) \big\rangle$$

The horizontal lift h^{hor} of any $h \in T_F B(M, N)$ is then:

$$h^{\text{hor}}(y_1) = K(L_F h)(y_1) = \int_{x_2 \in F} \big| K(y_1, x_2) \big| L_F h(x_2) \big\rangle, \quad y_1 \in N.$$

Note that all elements of the cotangent space $\alpha \in \Gamma_{\mathcal{S}'}(\text{Nor}^*(F))$ can be pushed up to N by $(j_F)_*$, where $j_F : F \hookrightarrow N$ is the inclusion, and this identifies $(j_F)_*\alpha$ with a 1-co-current on N.

Finally the induced homogeneous weak Riemannian metric on $B(M,N)$ is given like this:

$$\begin{aligned}\langle h,k\rangle_F &= \int_N (h^{\text{hor}}(y_1), L(k^{\text{hor}})(y_1)) = \int_{y_1\in N} (K(L_F h))(y_1), (L_F k)(y_1)) \\ &= \int_{(y_1,y_2)\in N\times N} (K(y_1,y_2), (L_F h)(y_1)\otimes (L_F k)(y_2)) \\ &= \int_{(x_1,x_1)\in F\times F} \Big\langle L_F h(x_1) \Big| K_F(x_1,x_2) \Big| L_F h(x_2) \Big\rangle\end{aligned}$$

With this metric, the projection from $\mathrm{Diff}_{\mathcal{S}}(N)$ to $B(M,N)$ is a submersion.

The inverse co-metric on the smooth cotangent bundle $\bigsqcup_{F\in B(M,N)} \Gamma(\mathrm{Nor}^*(F)\otimes \mathrm{vol}(F)) \subset T^*B(M,N)$ is much simpler and easier to handle:

$$\boxed{\langle \alpha, \beta\rangle_F = \iint_{F\times F} \big\langle \alpha(x_1)\big| K_F(x_1,x_2)\big|\beta(x_1)\big\rangle.}$$

It is simply the restriction to the co-metric on the Hilbert sub-bundle of $T^*\,\mathrm{Diff}_{\mathcal{S}}(N)$ defined by $\mathcal{H}'$ to the Hilbert sub-bundle of subspace $T^*B(M,N)$ defined by $\mathcal{H}'_F$.

Because they are related by a submersion, the geodesics on $B(M,N)$ are the horizontal geodesics on $\mathrm{Diff}_{\mathcal{S}}(N)$. We have two variables: a family $\{F_t\}$ of submanifolds in $B(M,N)$ and a time varying momentum $\alpha(t,\cdot) \in \mathrm{Nor}^*(F_t) \otimes \mathrm{vol}(F_t)$ which lifts to the horizontal 1-co-current $(j_{F_t})_*(\alpha(t,\cdot)$ on N. Then the horizontal geodesic on $\mathrm{Diff}_{\mathcal{S}}(N)$ is given by the same equations as before:

$$\boxed{\begin{aligned}
\partial_t(F_t) &= \mathrm{res}_{\mathrm{Nor}(F_t)}(u(t,\cdot)) \\
u(t,x) &= \int_{(F_t)_y} \big|K(x,y)\big|\alpha(t,y)\big\rangle \in \mathfrak{X}_{\mathcal{S}}(N) \\
\partial_t\left((j_{F_t})_*(\alpha(t,\cdot)\right) &= -\mathcal{L}_{u(t,\cdot)}((j_{F_t})_*(\alpha(t,\cdot)).
\end{aligned}}$$

This is a complete description for geodesics on $B(M,N)$ but it is not very clear how to compute the Lie derivative of $(j_{F_t})_*(\alpha(t,\cdot)$. One can unwind this Lie derivative via a torsion-free connection, but we turn to a different approach which will be essential for working out the curvature of $B(M,N)$.

Auxiliary tensors on $B(M, N)$

For $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let B_X be the infinitesimal action on $B(M, N)$ given by $B_X(F) = T_{\mathrm{Id}}(\ell^F)X$ with its flow $\mathrm{Fl}_t^{B_X}(F) = \mathrm{Fl}_t^X(F)$. We have $[B_X, B_Y] = B_{[X,Y]}$.
$\{B_X(F) : X \in \mathfrak{X}_{\mathcal{S}}(N)\}$ equals the tangent space $T_F B(M, N)$.

Note that $B(M, N)$ is naturally submanifold of the vector space of m-currents on N:

$$B(M, N) \hookrightarrow \Gamma_{\mathcal{S}'}(\Lambda^m T^* N), \quad \text{via } F \mapsto \left(\omega \mapsto \int_F \omega\right).$$

Any $\alpha \in \Omega^m(N)$ is a linear coordinate on $\Gamma_{\mathcal{S}'}(TN)$ and this restricts to the function $B_\alpha \in C^\infty(B(M, N), \mathbb{R})$ given by $B_\alpha(F) = \int_F \alpha$.
If $\alpha = d\beta$ for $\beta \in \Omega^{m-1}(N)$ then

$$B_\alpha(F) = B_{d\beta}(F) = \int_F j_F^* d\beta = \int_F dj_F^* \beta = 0$$

by Stokes' theorem.

For $\alpha \in \Omega^m(N)$ and $X \in \mathfrak{X}_{\mathcal{S}}(N)$ we can evaluate the vector field B_X on the function B_α:

$$\begin{aligned} B_X(B_\alpha)(F) &= dB_\alpha(B_X)(F) = \partial_t|_0 B_\alpha(Fl_t^X(F)) \\ &= \int_F j_F^* \mathcal{L}_X \alpha = B_{\mathcal{L}_X(\alpha)}(F) \\ \text{as well as } &= \int_F j_F^*(i_X d\alpha + d i_X \alpha) = \int_F j_F^* i_X d\alpha = B_{i_X(d\alpha)}(F) \end{aligned}$$

If $X \in \mathfrak{X}_{\mathcal{S}}(N)$ is tangent to F along F then $B_X(B_\alpha)(F) = \int_F \mathcal{L}_{X|_F} j_F^* \alpha = 0$.

More generally, a pm-form α on N^k defines a function $B_\alpha^{(p)}$ on $B(M,N)$ by $B_\alpha^{(p)}(F) = \int_{F^p} \alpha$.

For $\alpha \in \Omega^{m+k}(N)$ we denote by B_α the k-form in $\Omega^k(B(M,N))$ given by the skew-symmetric multi-linear form:

$$(B_\alpha)_F(B_{X_1}(F), \dots, B_{X_k}(F)) = \int_F {j_F}^*(i_{X_1 \wedge \dots \wedge X_k} \alpha).$$

This is well defined: If one of the X_i is tangential to F at a point $x \in F$ then ${j_F}^*$ pulls back the resulting m-form to 0 at x.

Note that any smooth cotangent vector a to $F \in B(M,N)$ is equal to $B_\alpha(F)$ for some closed $(m+1)$-form α. Smooth cotangent vectors at F are elements of $\Gamma_{\mathcal{S}}(F, \mathrm{Nor}^*(F) \otimes \Lambda^m T^*(F))$.

Likewise, a $2m+k$ form $\alpha \in \Omega^{2m+k}(N^2)$ defines a k-form on $B(M,N)$ as follows: First, for $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let $X^{(2)} \in \mathfrak{X}(N^2)$ be given by

$$X^{(2)}_{(n_1,n_2)} := (X_{n_1} \times 0_{n_2}) + (0_{n_1} \times X_{n_2})$$

Then we put

$$(B^{(2)}_\alpha)_F(B_{X_1}(F), \dots, B_{X_k}(F)) = \int_{F^2} {j_{F^2}}^*(i_{X_1^{(2)} \wedge \dots \wedge X_k^{(2)}} \alpha).$$

This is just B applied to the submanifold $F^2 \subset N^2$ and to the special vector fields $X^{(2)}$.

Using this for $p = 2$, we find that for any two m-forms α, β on N, the inner product of B_α and B_β is given by:

$$g_B^{-1}(B_\alpha, B_\beta) = B^{(2)}_{\langle \alpha | K | \beta \rangle}.$$

We have

$$\begin{aligned} i_{B_X} B_\alpha &= B_{i_X \alpha} \\ dB_\alpha &= B_{d\alpha} \quad \text{for any } \alpha \in \Omega^{m+k}(N) \\ \mathcal{L}_{B_X} B_\alpha &= B_{\mathcal{L}_X \alpha} \end{aligned}$$

Force and Stress

Moving to curvature, fix F. Then we claim that for any two smooth co-vectors a, b at F, we can construct not only two closed $(m+1)$-forms α, β on N as above but also two commuting vector fields X_α, X_β on N in a neighborhood of F such that:

1. $B_\alpha(F) = a$ and $B_\beta(F) = b$,
2. $B_{X_\alpha}(F) = a^\sharp$ and $B_{X_\beta}(F) = b^\sharp$
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$
4. $[X_\alpha, X_\beta] = 0$

The force is

$$2\mathcal{F}(\alpha, \beta) = d(\langle B_\alpha, B_\beta \rangle) = d\left(B^{(2)}_{\langle \alpha | K | \beta \rangle}\right) = B^{(2)}_{d(\langle \alpha | K | \beta \rangle)}.$$

The stress $\mathcal{D} = \mathcal{D}_N$ on N can be computed as:

$$\mathcal{D}(\alpha, \beta, F)(x) = (\text{restr. to Nor}(F))\left(-\int_{y \in F} \left| \mathcal{L}_{X^{(2)}_\alpha}(x, y) K(x, y) \right| \beta(y) \right\rangle\right)$$

The curvature

Finally, the semilocal Mario formula and some computations lead to:

$$\langle R_{B(M,N)}(B^\sharp_\alpha, B^\sharp_\beta)B^\sharp_\beta, B^\sharp_\alpha\rangle(F) = R_{11} + R_{12} + R_2 + R_3$$

$$\begin{aligned}
R_{11} &= \frac{1}{2}\iint_{F\times F}\Big(\big\langle\beta\big|\mathcal{L}_{X^{(2)}_\alpha}\mathcal{L}_{X^{(2)}_\alpha}K\big|\beta\big\rangle + \big\langle\alpha\big|\mathcal{L}_{X^{(2)}_\beta}\mathcal{L}_{X^{(2)}_\beta}K\big|\alpha\big\rangle \\
&\qquad - 2\big\langle\alpha\big|\mathcal{L}_{X^{(2)}_\alpha}\mathcal{L}_{X^{(2)}_\beta}K\big|\beta\big\rangle\Big) \\
R_{12} &= \int_F\Big(\langle\mathcal{D}(\alpha,\alpha,F),\mathcal{F}(\beta,\beta,F)\rangle + \langle\mathcal{D}(\beta,\beta,F),\mathcal{F}(\alpha,\alpha,F)\rangle \\
&\qquad - \langle\mathcal{D}(\alpha,\beta,F)+\mathcal{D}(\beta,\alpha,F),\mathcal{F}(\alpha,\beta,F)\rangle\Big) \\
R_2 &= \|\mathcal{F}(\alpha,\beta,F)\|^2_{K_F} - \big\langle\mathcal{F}(\alpha,\alpha,F)),\mathcal{F}(\beta,\beta,F)\big\rangle_{K_F} \\
R_3 &= -\frac{3}{4}\|\mathcal{D}(\alpha,\beta,F)-\mathcal{D}(\beta,\alpha,F)\|^2_{L_F}
\end{aligned}$$

Thank you for listening