Regularity and Completeness of half Lie groups

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Abstract: Half Lie groups exist only in infinite dimensions: They are smooth manifolds and topological groups such that right translations are smooth. Main examples are Sobolev H^r-diffeomorphism groups of manifolds (these play an important role in shape analysis), or C^k-diffeomorphism groups, or semidirect products of a Lie group with kernel an infinite dimensional representation space (investigated by Marquis and Neeb). Here, we investigate mainly Banach half Lie groups, the groups of their C^k-elements which form a (weakened) scale of half Lie groups reminiscent of ILB-groups, extensions, and right invariant strong Riemannian metrics on them: Here surprisingly the full Hopf Rinov theorem holds which is wrong in general even for Hilbert manifolds.

Based on:

Martin Bauer, Philipp Harms, Peter W. Michor: Regularity and completeness of half-Lie groups. arxiv:2302.01631. J Europ Math Soc. https://doi.org/10.4171/JEMS/1587

T. Marquis and K.-H. Neeb. Half-Lie groups. *Transform. Groups*, 23(3):801–840, 2018.

K.-H. Neeb. On differentiable vectors for representations of infinite dimensional lie groups. *Journal of Functional Analysis*, 259(11):2814–2855, 2010.

Andreas Kriegl, Peter W. Michor: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, Volume: 53, Amer. Math. Soc., 1997

See also:
Wikipedia [https://en.wikipedia.org/wiki/Convenient_vector_space]

Half-Lie groups

A right (left) half-Lie group is a smooth manifold, mainly infinite dimensional, whose underlying topological space is a topological group, such that right (left) translations are smooth. We shall speak of Hilbert, Banach, Fréchet, etc. half-Lie groups to designate the nature of the modeling vector space. A homomorphism of half-Lie groups is a smooth group homomorphism.

Lie groups are right half-Lie groups and left half-Lie groups with jointly smooth multiplication, with smooth inversion if there is no implicit function theorem available. Every finite-dimensional half-Lie group is a Lie group by a result of Segal 1948. Every Banach half-Lie group with uniformly continuous multiplication is already a Banach Lie group. This can be seen as a solution of Hilbert's 5th problem in infinite dimensions, due to Birkhoff 1938 and Enflo 1969, also Benjamini 1998. Marquis and Neeb 2018 have collected a long list of examples of half-Lie groups. We next present two important special cases.

Diffeomorphism groups

The main motivating examples for the present investigation of half-Lie groups are diffeomorphism groups with finite regularity. These appear naturally in shape analysis and mathematical fluid dynamics. If (M, g) is a finite-dimensional compact Riemannian manifold or an open Riemannian manifold of bounded geometry, then the diffeomorphism group $\mathrm{Diff}_{H^s}(M)$ of Sobolev regularity $s > \dim(M)/2 + 1$ is a half-Lie group. Likewise, the groups $\mathrm{Diff}_{W^{s,p}}(M)$ for $s > \dim(M)/p + 1$, and $\mathrm{Diff}_{C^k}(M)$ for $1 \leq k < \infty$ and M compact are half-Lie groups. However, they are not Lie groups because left multiplication is non-smooth.

Representation theory

Let $\rho : G \to U(H)$ be a representation of a Banach Lie group G on an infinite dimensional Hilbert space H, which is continuous as a mapping $G \times H \ni (g, h) \mapsto \rho(g)h \in H$. Then the right semidirect product $G \ltimes H$ with operations

$$(g_1, h_1).(g_2, h_2) = (g_1 g_2, \rho(g_2^{-1})h_1 + h_2), \qquad (g, h)^{-1} = (g^{-1}, -\rho(g)h)$$

is a right half-Lie group. This class of examples has been studied in detail by Marquis and Neeb [MarquisNeeb18]. In their work, the roles of left and right translations are interchanged compared to ours, but this makes no difference as one may always pass to the group of inverses.

Differentiable elements

Let G be a (Banach) right half-Lie group. Then, $x \in G$ is called a C^k element if the left translations $\mu_x, \mu_x^{-1} : G \to G$ are C^k. The set of all C^k elements of G is denoted by G^k.

By the inverse function theorem, the C^k property of μ_x^{-1} follows from the C^k property of μ_x, provided that μ_x has an invertible derivative at some (and hence any) point. However, we do not know this. For this reason, we require μ_x^{-1} to be C^k. Next, we will show that the set of C^k elements in a Banach half-Lie group G is again a Banach half-Lie group, provided that G carries a right-invariant local addition, i.e.:
A smooth map $\tau : TG \supseteq V \to G$, defined on an open neighborhood V of the 0-section in TG, such that $\tau(0_x) = x$ for all $x \in G$ and $(\pi_G, \tau) : V \to G \times G$ is a diffeomorphism onto its open range.
τ is called right-invariant if $T\mu^y(V) = V$ and $\tau \circ T\mu^y = \mu^y \circ \tau$ holds for all $y \in G$.

Theorem (differentiable elements). *For any Banach right half-Lie group G carrying a right-invariant local addition, the following statements hold:*

(a) *For any $k \in \mathbb{N}$, G^k is a Banach half-Lie group.*

(b) *The tangent space $T_e G^k$ is the set of all $X \in T_e G$ such that the right-invariant vector field $R_X : G \ni x \mapsto T_e\mu^x(X) \in TG$ is C^k.*

(c) *The inclusion $G^k \to G$ is smooth.*

(d) *For any $\ell \in \mathbb{N}$, $G^{k+\ell}$ is a subset of $(G^k)^\ell$.*

Corollary. *For any Banach half-Lie group G carrying a right-invariant local addition, the set $G^\infty = \cap_{k\in\mathbb{N}} G^k$ of smooth elements in G is an ILB (in a weak sense) manifold and a Lie group.*

Corollary. *Let G be a Banach half-Lie group carrying a right-invariant local addition. For any $k \in \mathbb{N}$, the Lie bracket*

$$[\cdot,\cdot] : T_eG^{k+1} \times T_eG^{k+1} \to T_eG^k$$

is well defined in the following three equivalent ways:

(a) *Any vectors $X, Y \in T_eG^{k+1}$ extend uniquely to right-invariant C^1 vector fields $R_X, R_Y \in \mathfrak{X}_{C^1}(G^k)^{G^k}$, and $[X,Y] := dR_Y(e)(X) - dR_X(e)(Y)$, where the right-hand side is interpreted in a chart around $e \in G^k$.*

(b) *The derivation $f \mapsto R_X R_Y f - R_Y R_X f(e)$ on smooth real-valued functions defined near $e \in G^k$ is the derivative in the direction of a vector in T_eG^k, which is denoted by $[X,Y]$.*

(c) *The vector field R_X has a C^1 flow $\mathrm{Fl}^{R_X} : \mathbb{R} \times G^k \to G^k$, and consequently, the derivative $[X,Y] := -\partial_t|_0 (\mathrm{Fl}_t^{R_X})^* R_Y(e) \in T_eG^k$ exists.*

Regular half-Lie groups

Let G be a Banach right half-Lie group, and let $\mathcal{F}$ be a subset of $L^1_{\text{loc}}(\mathbb{R}, T_eG)$. Then, G is called $\mathcal{F}$-regular if for all $X \in \mathcal{F}$, there exists a unique solution $g \in W^{1,1}_{\text{loc}}(\mathbb{R}, G)$ of the differential equation

$$\partial_t g(t) = T_e\mu^{g(t)}X(t), \qquad g(0) = e.$$

This solution will be denoted by $\text{Evol}(X)$, and its evaluation at $t = 1$ by $\text{evol}(X)$. For $\mathcal{F} = C^\infty$ we speak of regularity of G.

Any Banach Lie group G, and in particular every finite dimensional Lie group, is regular. On half-Lie groups G the fields R_X is not smooth, even if X is smooth. However, R_X is a time-dependent C^k vector field on G^k. This is used in the following:

Corollary *Let G be a Banach right half-Lie group carrying a right-invariant local addition. Then, for any $k \in \mathbb{N}_{\geq 1} \cup \infty$, the half Lie group G^k is regular.*

Tools for the proof of the theorem

Let U, V, W be open in Banach spaces E, F, G.

- Jet composition $\bullet : J^k(V,W) \times_V J^k(U,V) \to J^k(U,W)$ satisfies $\|\tau \bullet \sigma\| \le (1+\|\tau\|)(1+\|\sigma\|^k)$ and $\|\tilde\tau \bullet \tilde\sigma - \tau \bullet \sigma\| \le \|\tilde\tau - \tau\|(1+\|\tilde\sigma\|^k)$.
- Jet evaluation $\odot : J^k(U,V) \times_U TU \to J^{k-1}(TU, TV)$, given by $j^k_x f \odot \xi = j^{k-1}_\xi Tf$, satisfies $\|\sigma \odot \xi\| \le \|\xi\| + (k+1)\|\sigma\| + \|\xi\|\|\sigma\|$ and $\|\tilde\sigma \odot \tilde\xi - \sigma \odot \xi\| \le \|\tilde\sigma - \sigma\|(k + \|\tilde\xi\|) + \|\tilde\xi - \xi\|(1+\|\sigma\|)$.
- Jet inversion $(\cdot)^{-1} : J^k(U,V)^{\mathrm{inv}} \to J^k(V,U)^{\mathrm{inv}}$ is continuous.
- For any $k \in \mathbb{N}$ and right-invariant k-times differentiable vector field X, the k-jet at $y \in G$ is uniquely determined by the k-jet at $e \in G$ as follows: $j^k_y X = (j^{k+1}_e \mu^y \odot X(e)) \bullet j^k_e X \bullet j^k_y \mu^{y^{-1}}$.
- Let X and Y be right-invariant vector C^k-fields on G. Then, for y in a chart at $e \in G$, one has $\|j^k_y X - j^k_y Y\| \le \|j^k_e X - j^k_e Y\| p(\|j^k_e X\|, \|j^k_e Y\|, \|j^{k+1}_e \mu^y\|, \|j^k_y \mu^{y^{-1}}\|)$ for a polynomial p dep. only on k, in the norm induced from $T_e G$ via the chart.

More tools

- The space $\mathfrak{X}_{C^k}(G)^G$ of right invariant C^k-vector fields is Banach with respect to the norm $\|X\| := \|X(e)\|_{T_eG} + \cdots + \|d^kX(e)\|_{L^{(k)}(T_eG,\ldots,T_eG;T_eG)}$.
- Let G carry a right-invariant local addition. Then, for $k \in \mathbb{N}$, the space $\mathrm{Diff}_{C^k}(G)^G$ of right invariant C^k-diffeomorphisms of G is a Banach manifold. For any $g \in \mathrm{Diff}_{C^k}(G)^G$, the pull-back $g_* : \mathrm{Diff}_{C^k}(G)^G \ni f \mapsto f \circ g \in \mathrm{Diff}_{C^k}(G)^G$ is smooth. Moreover, the evaluation map $\mathrm{ev}_e : \mathrm{Diff}_{C^k}(G)^G \to G$ is smooth.
- Let G carry a right-invariant local addition. Then, for any $k \in \mathbb{N}$, the space $\mathrm{Diff}_{C^k}(G)^G$ is a topological group.
- Let G carry a right-invariant local addition. Then, the set $J_e^k(G,G)^G = \{j_e^k f : f \in \mathrm{Diff}_{C^k}(G)^G\}$ is a submanifold of $J_e^k(G,G)$, and $\mathrm{Diff}_{C^k}(G)^G$ is diffeomorphic to $J_e^k(G,G)^G$ via the map $j_e^k : \mathrm{Diff}_{C^k}(G)^G \ni f \mapsto j_e^k f \in J_e^k(G,G)^G$.

Riemannian metrics

Theorem. *Let g be a weak Riemannian metric on a convenient manifold M. Then the following are equivalent:*

(a) *g is a strong Riemannian metric on M.*
(b) *M is a Hilbert manifold and $g^\vee : TM \to T^*M$ is surjective.*
(c) *M is a Hilbert manifold and $g^\vee : TM \to T^*M$ is a vector bundle isomorphism.*

Theorem *Let G be a Banach half-Lie group with a right-invariant Riemannian metric. Assume that left-translation by any $x \in G$ is Lipschitz continuous with respect to the geodesic distance d, i.e.,*

$$|\mu_x| := \inf \{ C \in \mathbb{R}_+ : d(xx_0, xx_1) \le C d(x_0, x_1), \forall x_0, x_1 \in G \} < \infty \,.$$

Then the group elements with vanishing geodesic distance to the identity form a normal subgroup.
If the Riemannian metric is strong then the geodesic distance is always non-degenerate, i.e., the normal subgroup of elements with vanishing geodesic distance is the trivial subgroup.

Hopf-Rinow

Theorem. *Let G be a connected half Lie group equipped with a right invariant strong Riemannian metric g, and let $d : G \times G \to \mathbb{R}_+$ be the induced geodesic distance on G. Then the following completeness properties hold for (G, g):*

(a) *the space (G, d) is a complete metric space, i.e., every d-Cauchy sequences converge in G;*

(b) *the exponential map $\exp_e^g : T_eG \to G$ is defined on all T_eG;*

(c) *the exponential map $\exp^g : TG \to G$ is defined on all of TG;*

(d) *the space (G, g) is geodesically complete, i.e., every geodesic is maximally definable on all of $\mathbb{R}$.*

If also G is L^2-regular and for each $x \in G$ the sets $\mathcal{A}_x := \{\xi \in L^2([0,1], T_eG) : \mathrm{evol}(\xi) = x\} \subset L^2([0,1], T_eG)$ are weakly closed. Then

(e) *the space (G, g) is geodesically convex, i.e., any two points in G can be connected by a geodesic of minimal length.*

Extensions

Let N and G be right half Lie groups. A right half Lie group E is called a *smooth extension* of G over N if we have a short exact sequence of smooth group homomorphisms

$$e \to N \xrightarrow{i} E \xrightarrow{p} G \to e$$

such that i and p are smooth, and both of the following two conditions is satisfied:

1. p admits a local smooth section s near e (equivalently near any point), and i is initial [KM97, 27.11]
2. i admits a local smooth retraction r near e (equivalently near any point), and p is final [KM97, 27.15].

If E is a Lie group, by $s(p(x)).i(r(x)) = x \in E$ the two conditions are equivalent, and then E is locally diffeomorphic to $N \times G$ via (r, p) with local inverse $(i \circ \mathrm{pr}_1).(s \circ \mathrm{pr}_2)$.

Not every smooth exact sequence of even Lie groups admits local sections as required above. Let, for example, N be a closed linear subspace in a convenient vector space E which is not a direct summand, and let G be E/N. Then the tangent mapping at 0 of a local smooth splitting would make N a direct summand.

Two smooth extensions are defined to be *equivalent* if there exists a diffeomorphic isomorphism φ fitting commutatively into the diagram

$$\begin{tikzcd}
e \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "\varphi"'] & G \arrow[r] \arrow[d, equal] & e \\
e \arrow[r] & N \arrow[r, "i'"] & E' \arrow[r, "p'"] & G \arrow[r] & e.
\end{tikzcd}$$

Note that if a smooth homomorphism φ exists, then it is an isomorphism, and if additionally E is a Banach Lie group, then φ has a smooth inverse.

Split extensions

A smooth extension is called *split* if there exists a section $s : G \to E$ of p which is smooth near e and is a homomorphism of groups. Automatically, s is then globally smooth. Then, one gets a right action $\rho : N \times G \to N$ by $i\rho(n,g) = s(g^{-1}).i(n).s(g)$ describing a semidirect product. The action $n \mapsto \rho(n,g)$ is smooth for all g if the extension satifies condition (C) of the following two conditions on an extension.

(C) For each $x \in E$, conjugation in E induces a smooth automorphism of the normal subgroup N; i.e., $n \mapsto x.i(n).x^{-1}$ induces a smooth map in $\mathrm{Aut}(N)$.

(L) The normal subgroup N is a Lie group.

Semidirect products over group representations (see above) satisfy both conditions (C) and (L).

Theorem (Semi-direct products)

Let $\rho : N \times G \to N$ be the right action of a Banach right half-Lie group G on a Banach right half-Lie group N. Assume that ρ is continuous, and $\rho(\cdot, g) : N \to N$ is a smooth automorphism, for all $g \in G$.

(a) *The semidirect product $G \ltimes N$ with group operations*

$$(g, n).(g', n') = (gg', \rho(n, g')n'), \qquad (g, n)^{-1} = (g^{-1}, \rho(n^{-1}, g^{-1}))$$

is a Banach right half-Lie group.

(b) *The set of C^k elements in $G \ltimes N$ is given by $(G \ltimes N)^k = G^k \ltimes N^{k,\rho}$, where*

$$N^{k,\rho} = \left\{ n \in N^k : \rho(n, g')n' \text{ is } C^k \text{ in } (g', n') \in G \ltimes N \right\}$$
$$= (\{e\} \times N) \cap (G \ltimes N)^k.$$

(c) *The set $N^{k,\rho}$ is a Banach manifold.*

Non-split extensions

For non split extensions we require both conditions (C) and (L). So N is assumed to a Lie group; in contrast, E and G are merely right half Lie groups. Then, a smooth extension is in particular a principal N-bundle $E \to G$, and existence of a continuous section implies that the bundle is trivial. So we may choose a (not continuous) section $s : G \to E$ with $s(e) = e$ which is smooth near e, on an open e-neighborhood $U \subset G$ say, which induces mappings

$$\begin{aligned} \alpha : G \to \mathrm{Aut}(N), \qquad & \alpha^x(n) = s(x)^{-1}.n.s(x), \\ f : G \times G \to N, \qquad & f(x, y) = s(xy)^{-1} s(x) s(y), \end{aligned}$$

where $\mathrm{Aut}(N)$ is the group of smooth group automorphisms of N; Note that here we need condition (C). The mapping $\alpha : N \times G \to N$ is continuous on $U \times N$, and f is continuous near $U \times U$. We cannot do better since E is a right half Lie group only. By the definition of α and by associativity, we have the following properties:

$$
\begin{aligned}
\alpha^x \circ \alpha^y &= \mathrm{conj}_{f(y,x)^{-1}} \circ \alpha^{yx}, \\
f(xy,z)\alpha^z(f(x,y)) &= f(x,yz)f(y,z), \\
f(e,e) = f(x,e) &= f(e,x) = e,
\end{aligned}
$$

where $\mathrm{conj}_h(n) = hnh^{-1}$ is conjugation by h, an inner automorphism. Thus, α induces a group anti-homomorphism $\bar{\alpha} : G \to \mathrm{Aut}(N)/\,\mathrm{Int}(N)$ where $\mathrm{Int}(N)$ is the normal subgroup of all inner automorphisms in $\mathrm{Aut}(N)$. In terms of (α, f) the group structure on E is given by

$$
\begin{aligned}
s(x)m.s(y)n &= s(x)s(y)s(y)^{-1}ms(y)n = s(xy)f(x,y)\alpha^y(m)n, \\
(s(x)m)^{-1} &= s(x^{-1})\alpha^{x^{-1}}(m^{-1})f(x,x^{-1})^{-1}
\end{aligned}
$$

Since E is a right half Lie group this implies first that $(x,m) \mapsto f(x,y)\alpha^y(m)n \in N$ is smooth near (e,e); since N is a Lie group and $m \mapsto \alpha^y(m)$ is smooth we conclude that $x \mapsto f(x,y)$ *is smooth on* U.

Reconstruction of the half Lie group structure

The half Lie group structure on E can be recontructed from the extension data (α, f) with the local smoothness assumptions near e from above as follows: Choose $e \in V \subset U$ open with $V^{-1} = V$ and $V.V \subset U$, and let $\tilde{V} := p^{-1}(V)$. We then have: $\alpha : U \to \mathrm{Aut}(N)$ is smooth, and $f : V \times V \to N$ is continuous and $x \mapsto f(x, y)$ is smooth and the group multiplication is continuous on $\tilde{V} \times \tilde{V} \to \tilde{U}$ and right translations are smooth $\mu^y : \tilde{V} \to \tilde{U}$ for all $y \in \tilde{V}$. We then use $(\tilde{V}.x, \mu^{x^{-1}} : \tilde{V}.x \to \tilde{V})_{x \in E}$ as atlas for E. The chart changes are $\mu^{y^{-1}} \circ \mu^x = \mu^{x.y^{-1}} : (\tilde{V}.x \cap \tilde{V}.y).x^{-1} = \tilde{V} \cap (\tilde{V}.y.x^{-1}) \to \tilde{V} \cap (\tilde{V}.x.y^{-1})$, so they are smooth. The resulting smooth manifold structure on E has the property that $p : E \to G$ and $i : N \to E$ are smooth, the group structure maps μ and ν are continuous with smooth left translations. Moreover E is Hausdorff: Either $p(x) = p(y)$ and then we can separate them already in one chart $x.\tilde{V} = p^{-1}(p(x).V)$, or we can separate them with open sets of the form $p^{-1}(U_1)$ and $p^{-1}(U_2)$.

The converse construction

Given a Lie group N and a right half Lie group G, we consider pairs (α, f) of mappings such that

$$\begin{aligned}
\alpha : G \to \mathrm{Aut}(N) \quad & \text{with } \hat{\alpha} : N \times G \to N \text{ continuous near } \{e\} \times N \\
f : G \times G \to N \quad & \text{continuous near } (e, e) \text{ and} \\
& x \mapsto f(x, y) \text{ smooth near } e \text{ for each } y \text{ near } e \\
& \text{with the properties}
\end{aligned}$$

$$\begin{aligned}
\alpha^x \circ \alpha^y &= \mathrm{conj}_{f(x,y)^{-1}} \circ \alpha^{yx}, \\
f(e, e) = f(x, e) &= f(e, y) = e, \\
e &= f(xy, z)^{-1} f(x, yz) f(y, z) \alpha^z(f(x, y)^{-1})
\end{aligned}$$

Theorem. *In the above notation, the following assertions hold:*

- *Every such pair (α, f) defines a smooth right half Lie group extension E of G over N, given by the set $E = G \times N$, with the group structure*
$$(x, m).(y, n) = (xy, f(x, y)\alpha^y(m)n),$$
$$(x, m)^{-1} = (x^{-1}, \alpha^{x^{-1}}(m^{-1})f(x, x^{-1})^{-1}).$$
The topology and the manifold structure on the extension then is the one extended by right translations from a suitable neigborhood of e, as described above. Up to isomorphism, every extension of G over N can be so obtained.
- *Two data (α, f) and (α_1, f_1) define equivalent extensions if there exists a mapping $b : G \to N$ (smooth near e) such that*
$$\alpha_1^x = \mathrm{conj}_{b(x)^{-1}} \circ \alpha^x,$$
$$f_1(x, y) = b(xy)^{-1} f(x, y)\alpha^y(b(x))b(y).$$
The induced smooth isomorphism $E \to E_1$ between the extensions defined by (α, f) and (α_1, f_1) is given by $(x, n) \mapsto (x, b(x)n)$.

Theorem, continued.

- *A datum (α, f) describes a splitting extension (a semidirect product) if and only if it is equivalent to a datum (α_1, f_1), where f_1 is constant $= e$. This is the case if and only if there exists a map $b : G \to N$ (smooth near e) with*

$$f(x,y) = b(xy)^{-1}\alpha^{y}(b(x))f(x,y)b(y). \tag{1}$$

Note that for such a pair $(\alpha_1, f_1 = e)$ the map α_1 must be a homomorphism and thus is continuous everywhere.

Smooth extension and C^r-elements

Let $e \to N \xrightarrow{i} E \xrightarrow{p} G \to e$ be a smooth extension of right half Lie groups. In general, the tangent bundle of a half Lie group is not a group since we cannot differentiate the multiplication μ. But we get a sequence of vector bundles carrying right actions of the base half Lie groups

$$
\begin{tikzcd}
 & TN \arrow[r, "Ti"] \arrow[d, "\pi_N"'] & TE \arrow[r, "Tp"] \arrow[d, "\pi_E"'] & TG \arrow[d, "\pi_G"'] & \\
e \arrow[r] & N \arrow[r, "i"] & E \arrow[r, "p"] & G \arrow[r] & e
\end{tikzcd}
$$

• *which is fiberwise exact in the sense that* $T_z p : T_z E \to T_{p(z)} G$ *is onto with kernel* $\ker(T_z p) = T\mu^z(T_e i(T_e N))$ *and* $T_n i : T_n N \to T_{i(n)} N$ *is injective with* $\operatorname{im}(T_n i) = \ker(T_{i(n)} p)$ *for all* $z \in E$ *and* $n \in N$.

• *A right invariant local addition* $\tau^E : TE \supset V \to E$ *respects* $i(N)$ *if we have* $\tau^E(T(i(N)) \cap V) \subset i(N)$. *Then* τ^E *induces right invariant local additions* $\tau^N : TN \supset i^{-1}(V) \to N$ *and* $\tau^G : TG \supset p(V) \to G$ *satisfying* $p \circ \tau^E = \tau^G \circ Tp$ *and* $i \circ \tau^N = \tau^N \circ Ti$. Namely, we get τ^N by assumption and τ^G.

For each $r \in \mathbb{Z}_{>0}$ we have a functor $G \mapsto G^r$ from the category of half Lie groups and smooth homomorphisms to the category of groups. Thus we get the sequence of the groups of C^r-elements $e \to N^r \xrightarrow{i} E^r \xrightarrow{p} G^r \to e$, which in general is not exact. If E admits a right invariant local addition respecting $i(N)$, then this is a sequence of half Lie groups and smooth homomorphisms.

• Let $x \in E^r$, so $\mu_x : E \to E$ is C^r. Then $p \circ \mu_x = \mu_{p(x)} \circ p : E \to G$ is C^r and since p is final, $\mu_{p(x)} : G \to G$ is C^r. Thus $p(x) \in G^r$.

• If $n \in i^{-1}(E^r) \subset N$, i.e, $i(n) \in E^r \cap i(N)$ then $i \circ \mu_n = \mu_{i(n)} \circ i : N \to E$ is C^r and since i is initial, $\mu_n : N \to N$ is C^r. Thus $i^{-1}(E^k) \subseteq N^r$.

• *In the setting of non split extensions (so N is a Lie group and all conjugations in E are smooth on N) we have*

$$
\begin{aligned}
i^{-1}(E^r) &= N^{r,\alpha} = \{m \in N : \alpha(m,\cdot) : G \to N \text{ is } C^k \text{ near } e \in G\} \\
E^r/(i(N) \cap E^r) &\subseteq G^{r,f} = \{x \in G^r : f(x,\cdot) : G \to N \text{ is } C^r \text{ near } e \in G\}\,.
\end{aligned}
$$

In general, $G^{r,f}$ is not a group.

Theorem. *In this setting (so N is a Lie group and all conjugations in E are smooth on N), and if E admits a right invariant local addition respecting $i(N)$, the following sequence is is again a smooth extension. An extension datum (α, f) induces by restriction a corresponding extension datum.*

$$
\begin{tikzcd}
e \arrow[r] & i^{-1}(E^r) = N^{r,\alpha} \arrow[r, "i"] \arrow[d, hook] & E^r \arrow[r, "p"] & E^r/N^{r,\alpha} \arrow[r] \arrow[d, hook] & e \\
& N & & G^{r,f} \arrow[r, hook] & G^r
\end{tikzcd}
$$

Examples of extensions

For a smooth principal bundle $q : P \to M$ over a (for simplicity) compact Riemannian manifold M with finite dimensional structure group G with principal right action $\rho : P \times G \to P$ we let

$$\begin{aligned} \mathrm{Aut}_{C^k}(P) &= \{f \in \mathrm{Diff}_{C^k}(P) : \rho^g \circ f = f \circ \rho^g \text{ for all } g \in G\}, \\ \mathrm{Gau}_{C^k}(P) &= \{f \in \mathrm{Aut}_{C^k}(P) : p(f) = \mathrm{Id}_M \in \mathrm{Diff}_{C^k}(M)\}, \end{aligned}$$

where $p : \mathrm{Aut}_{C^k}(P) \to \mathrm{Diff}_{C^k}(M)$ is defined by $q \circ f = p(f) \circ q$. This is well-defined as $q \circ f : P \to P \to M$ is constant on the fibers and thus factors in the above way. For $k = \infty$ we will also write $\mathrm{Aut}(P) = \mathrm{Aut}_{C^\infty}(P)$ and $\mathrm{Gau}(P) = \mathrm{Gau}_{C^\infty}(P)$ and for $l > \dim(P)/2 + 1$ we will also consider the H^l versions $\mathrm{Aut}_{H^l}(P)$ and $\mathrm{Gau}_{C^l}(P)$.

Theorem

The following is a smooth extension of regular Lie groups:

$$\{\mathrm{Id}_P\} \longrightarrow \mathrm{Gau}(P) \xrightarrow{i=\mathrm{incl}} \mathrm{Aut}(P) \xrightarrow{p} \mathrm{Diff}^P(M) \longrightarrow \{\mathrm{Id}_M\}$$

For the corresponding counterpart of mappings with finite regularity we obtain extensions of half-Lie groups admitting local sections as in and smooth conjugations as in:

$$\{\mathrm{Id}_P\} \longrightarrow \mathrm{Gau}_{C^k}(P) \xrightarrow{i=\mathrm{incl}} \mathrm{Aut}_{C^k}(P) \xrightarrow{p} \mathrm{Diff}^P_{C^k}(M) \longrightarrow \{\mathrm{Id}_M\}$$

$$\{\mathrm{Id}_P\} \longrightarrow \mathrm{Gau}_{H^k}(P) \xrightarrow{i=\mathrm{incl}} \mathrm{Aut}_{H^k}(P) \xrightarrow{p} \mathrm{Diff}^P_{H^k}(M) \longrightarrow \{\mathrm{Id}_M\}$$

General fiber bundles

For a finite dimensional, compact, fiber bundle $q : E \to M$ we let

$$\mathrm{Diff}^0_{C^k,\mathrm{fiber}}(E) := \left\{ f \in \mathrm{Diff}^0_{C^k}(E) : q \circ f = p(f) \circ q \right\}, \tag{2}$$

$$\mathrm{Diff}^0_{C^k,\mathrm{fiber}}(E)_{\mathrm{Id}_M} := \left\{ f \in \mathrm{Diff}^0_{C^k,\mathrm{fiber}}(E) : p(f) = \mathrm{Id}_M \right\} \tag{3}$$

where $p : \mathrm{Diff}^0_{C^k,\mathrm{fiber}}(E) \to \mathrm{Diff}^0_{C^k}(M)$ is uniquely determined by the defining relation $q \circ f = p(f) \circ q$.
For $k = \infty$ we will also write $\mathrm{Diff}^0_{C^\infty,\mathrm{fiber}}(E) = \mathrm{Diff}^0_{\mathrm{fiber}}(E)$ and $\mathrm{Diff}^0_{C^\infty,\mathrm{fiber}}(E)_{\mathrm{Id}_M} = \mathrm{Diff}^0_{\mathrm{fiber}}(E)_{\mathrm{Id}_M}$ and for $l > \dim(P)/2 + 1$ we will also consider the H^l versions $\mathrm{Diff}^0_{H^l,\mathrm{fiber}}(E)$ and $\mathrm{Diff}^0_{H^l,\mathrm{fiber}}(E)_{\mathrm{Id}_M}$.

Theorem

The following is a smooth extension of Lie groups:

$$\{\mathrm{Id}_E\} \longrightarrow \mathrm{Diff}^0_{fiber}(E)_{\mathrm{Id}_M} \xrightarrow{i=\mathrm{incl}} \mathrm{Diff}^0_{fiber}(E) \xrightarrow{p} \mathrm{Diff}^0(M) \longrightarrow \{\mathrm{Id}_M\}$$

For the corresponding counterpart of mappings with finite regularity we obtain extensions of half-Lie groups admitting local sections:

$$\{\mathrm{Id}_E\} \to \mathrm{Diff}^0_{C^k,fiber}(E)_{\mathrm{Id}_M} \xrightarrow{i} \mathrm{Diff}^0_{C^k,fiber}(E) \xrightarrow{p} \mathrm{Diff}^0_{C^k,}(M) \to \{\mathrm{Id}_M\}$$

$$\{\mathrm{Id}_E\} \to \mathrm{Diff}^0_{H^k,fiber}(E)_{\mathrm{Id}_M} \xrightarrow{i} \mathrm{Diff}^0_{H^k,fiber}(E) \xrightarrow{p} \mathrm{Diff}^0_{H^k,}(M) \to \{\mathrm{Id}_M\}$$

In contrast to the above, the kernels $\mathrm{Diff}^0_{C^k,fiber}(E)_{\mathrm{Id}_M}$ *and* $\mathrm{Diff}^0_{H^k,fiber}(E)_{\mathrm{Id}_M}$ *are only half-Lie groups, and conjugation is only continuous.*

Thank you for listening.