Unbalanced Riemannian Metric Transport and the Wasserstein-Ebin Metric

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Abstract:

Work with M. Bauer and FX Viallard: We present weak Riemannian metrics on 5 different spaces related to the space of all Riemannian metrics which fit together via Riemannian submersions. Its geodesics are like the unbalanced version of the Brenier-Otto optimal transport with the most interesting version on the space of Riemannian metrics.

The spaces

Let (M, g_0) be a compact manifold equipped with a background Riemannian metric g_0 with $\rho_0 = \mathrm{vol}(g_0)$. Then let

- $\mathcal{C}(M) = M \times \mathbb{R}_{>0}$, the cone over M.
- $\mathrm{Dens}(M) = \Gamma_{>0}(|\Lambda^d|(T^*M))$ the space of all positive densities on M.
- $\mathrm{Diff}(M)$ the diffeomorphism group of M.
- $\mathrm{Met}(M)$ the space of all Riemannian metrics on M.
- $\mathrm{Gau}(TM) = GL(TM)$ the gauge group of TM; all vector bundle automorphisms of TM over Id_M.
- $\mathrm{Aut}(TM)$ the automorphism group of TM. We have a splitting exact sequence

  $$\{\mathrm{Id}_{TM}\} \longrightarrow \mathrm{Gau}(TM) \xrightarrow{i} \mathrm{Aut}(TM) \underset{T\varphi \leftarrow \varphi : j}{\overset{p : \bar\varphi \mapsto \varphi}{\rightleftarrows}} \mathrm{Diff}(M) \longrightarrow \{\mathrm{Id}_M\}$$

  The semidirect product $\mathrm{Diff}(M) \ltimes \mathrm{Gau}(TM) = \mathrm{Aut}(TM)$ is induced by $(\varphi, \alpha) \mapsto T\varphi \circ \alpha \in \mathrm{Aut}(TM)$ leading to

  $$(\varphi, \alpha)(\varphi', \alpha') = (\varphi \circ \varphi', (T\varphi')^{-1}.\alpha.T\varphi'.\alpha'),$$
  $$(\varphi, \alpha)^{-1} = (\varphi^{-1}, T\varphi.\alpha^{-1}.T\varphi^{-1}).$$

Theorem

Consider the projections π_i *defined via*

$$\pi_0 : \begin{cases} \mathrm{Aut}(\mathcal{C}(M)) \to \mathrm{Dens}(M) \\ (\varphi, \lambda) \mapsto \varphi_*(\lambda^2 \rho_0) \end{cases} \qquad \pi_1 : \begin{cases} \mathrm{Met}(M) \to \mathrm{Dens}(M) \\ g \mapsto \mathrm{vol}(g) \end{cases}$$

$$\pi_2 : \begin{cases} \mathrm{Aut}(TM) \to \mathrm{Aut}(\mathcal{C}(M)) \\ (\varphi, \alpha) \mapsto (\varphi, \sqrt{|\det \alpha^{-1}|}) \end{cases}$$

$$\pi_3 : \begin{cases} \mathrm{Aut}(TM) \to \mathrm{Met}(M) \\ (\varphi, \alpha) \mapsto \varphi_* \alpha_* g_0 = \varphi_*(\alpha^{-T} g_0 \alpha^{-1}) \end{cases}$$

$$\pi_4 : \begin{cases} \mathrm{Aut}(TM) \to \mathrm{Diff}(M) \times \mathrm{Met}(M) \\ (\varphi, \alpha) \mapsto (\varphi, \alpha_* g_0) = (\varphi, \alpha^{-T} g_0 \alpha^{-1}) \end{cases}$$

$$\pi_5 : \begin{cases} \mathrm{Diff}(M) \times \mathrm{Met}(M) \to \mathrm{Met}(M) \\ (\varphi, g) \mapsto \varphi_* g = (T\varphi)^{-T} g T\varphi^{-1} \end{cases}$$

Then the following is a commutative diagram of Riemannian submersions:

$$
\begin{tikzcd}
(\mathrm{Aut}(TM),\, G^{\mathrm{Aut}(TM)}) \arrow[r, "(\varphi,\alpha)\mapsto(\varphi,\alpha_* g_0)", "\pi_4"'] \arrow[d, "(\varphi,\alpha)\mapsto\left(\varphi,\frac{1}{\sqrt{|\det\alpha^{-1}|}}\right)"', "\pi_2"] \arrow[dr, "\pi_3", "(\varphi,\alpha)\mapsto\varphi_*\alpha_* g_0"'] & (\mathrm{Diff}(M)\times\mathrm{Met}(M), G^{W\times E}) \arrow[d, "\pi_5"', "(\varphi,g)\mapsto\varphi_* g"] \\
(\mathrm{Aut}(\mathcal{C}(M)),\, G^{\mathrm{Aut}(\mathcal{C}(M))}) \arrow[dr, "\pi_0", "(\varphi,\lambda)\mapsto\varphi_*(\lambda^2\rho_0)"'] & (\mathrm{Met}(M),\, G^{\mathrm{WE}}) \arrow[d, "\pi_1=\mathrm{vol}"] \\
 & (\mathrm{Dens}(M),\, G^{\mathrm{WFR}})
\end{tikzcd}
$$

where the corresponding Riemannian metrics are given as follows:

$$G^{\mathrm{Aut}(TM)}_{(\varphi,\alpha)}((\delta\varphi,\delta\alpha),(\delta\varphi,\delta\alpha)) =$$

$$= \int \Big(g_0(\delta\varphi,\delta\varphi) + \tfrac{d\Lambda}{4}\,\mathrm{Tr}(g_0\alpha^{-1}\delta\alpha g_0^{-1}\delta\alpha^\top\alpha^{-\top})\Big)\,|\det(\alpha)|^{-1}\,\mathrm{vol}(g_0)$$

$$G^{W\times E}_{(\varphi,g)}((\delta\varphi,h),(\delta\varphi,h)) = \int_M g_0(\delta\varphi,\delta\varphi)\,\mathrm{vol}(g)$$

$$+ \tfrac{d\Lambda}{16}\int_M \mathrm{tr}(g^{-1}hg^{-1}h)\,\mathrm{vol}(g)$$

$$G^{\mathrm{Aut}(\mathcal{C}(M))}_{\varphi,\lambda}((\delta\varphi,\delta\lambda),(\delta\varphi,\delta\lambda)) = \int_M \left(\lambda^2 g_0(\delta\varphi,\delta\varphi) + \Lambda(\delta\lambda)^2\right)\mathrm{vol}(g_0)$$

$$G^{\mathrm{WE}}_g(\delta g,\delta g) = \inf_{v,h}\int_M g_0(v,v)\,\mathrm{vol}(g) + \tfrac{d\Lambda}{16}\int_M \mathrm{tr}(g^{-1}hg^{-1}h)\,\mathrm{vol}(g)$$

$$\text{subject to:}\quad \delta g = -\mathcal{L}_v g + h$$

$$G^{\mathrm{WFR}}_\rho(\delta\rho,\delta\rho) = \inf_{v,f}\int_M g_0(v,v)\rho + \tfrac{\Lambda}{4}\int_M f^2\rho$$

$$\text{subject to:}\quad \delta\rho = -\mathcal{L}_v\rho + f\rho.$$

The second summand in the integral in the definition of the metric $G^{\mathrm{Aut}(TM)}$ comes from the left invariant Riemannian metric on $GL(T_xM)$ given by

$$\begin{aligned} G_A^{GL(T_xM),g_0}(X,X) &= \operatorname{tr}(A^{-1}Xg_0^{-1}(A^{-1}X)^Tg_0) \\ &= \operatorname{tr}(A^{-1}Xg_0^{-1}X^TA^{-T}g_0), \end{aligned}$$

where $A^{-T} = (A^{-1})^T = (A^T)^{-1} : T_x^*M \to T_x^*M$.

Attention: We do not yet have a proof that the Wasserstein-Ebin metric G^{WE} on $\mathrm{Met}(M)$ induces a non-degenerate metric via the infimum of length of curves connecting two points.

Lemma

Let $\varphi(t)$ be a smooth curve of diffeomorphisms on a manifold M locally defined for each t, with $f_0 = \mathrm{Id}_M$. We consider the two time dependent vector fields

$$X(t)(x) := (T_x\varphi(t))^{-1}\partial_t\varphi(t)(x), \qquad Y(t)(x) := (\partial_t\varphi(t))(\varphi(t)^{-1}(x)).$$

Then $T(\varphi(t)).X(t) = \partial_t\varphi(t) = Y(t) \circ \varphi(t)$, so $X(t)$ and $Y(t)$ are $\varphi(t)$-related. Moreover, for any tensor field K on M we have:

$$\partial_t\varphi(t)^*K = \varphi(t)^*\mathcal{L}_{Y(t)}K = \mathcal{L}_{X(t)}\,\varphi(t)^*K\,.$$
$$\partial_t\varphi(t)_*K = \partial_t(\varphi(t)^{-1})^*K = -\varphi(t)_*\mathcal{L}_{X(t)}K = -\mathcal{L}_{Y(t)}\,\varphi(t)_*K\,.$$

π_0 is a Riemannian submersion

Let $(\varphi, \lambda) \in \mathrm{Aut}(\mathcal{C}(M))$ with $\pi_0(\varphi, \lambda) = \varphi_*(\lambda^2 \rho_0) = \rho$, and let $Y := \delta\varphi \circ \varphi^{-1} \in \mathfrak{X}(M)$. By the lemma we have:

$$\begin{aligned} T_{(\varphi,\lambda)}\pi_0.(\delta\varphi, \delta\lambda) &= -\mathcal{L}_Y \varphi_*(\lambda^2\rho_0) + \varphi_*(2\lambda.\delta\lambda.\rho_0) \\ &= -\mathcal{L}_Y\rho + \varphi_*(2\tfrac{\delta\lambda}{\lambda}).\varphi_*(\lambda^2\rho_0) \\ &= -\mathcal{L}_Y\rho + \varphi_*(2\tfrac{\delta\lambda}{\lambda}).\rho = -\mathcal{L}_Y\rho + f\rho, \text{ where } f = \varphi_*(2\tfrac{\delta\lambda}{\lambda}). \end{aligned}$$

For fixed $(\varphi, \lambda) \in \mathrm{Aut}(\mathcal{C}(M))$ with $\pi_0(\varphi, \lambda) = \varphi_*(\lambda^2\rho_0) = \rho$, i.e., $\rho_0 = \frac{1}{\lambda^2}\varphi^*\rho$, we have to find the infimum of the $G^{\mathrm{Aut}(\mathcal{C}(M))}_{\varphi,\lambda}$-norm of all $(\delta\varphi, \delta\lambda) \in T_{(\varphi,\lambda)}\mathrm{Aut}(\mathcal{C}(M))$ with

$$\boxed{-\mathcal{L}_Y\rho + f\rho = T_{(\varphi,\lambda)}\pi_0.(\delta\varphi, \delta\lambda) = \delta\rho \quad \text{Continuity Equation}}$$

where $f = \varphi_*(2\frac{\delta\lambda}{\lambda}) \in C^\infty(M, \mathbb{R}_{>0})$.

Fix $(\varphi,\lambda)\in \operatorname{Aut}(\mathcal{C}(M))$ with $\pi_0(\varphi,\lambda)=\varphi_*(\lambda^2\rho_0)=\rho$, i.e., $\rho_0=\frac{1}{\lambda^2}\varphi^*\rho$. Search the inf over all $(\delta\varphi,\delta\lambda)\in T_{(\varphi,\lambda)}\operatorname{Aut}(\mathcal{C}(M))$ with $-\mathcal{L}_Y\rho+f\rho=T_{(\varphi,\lambda)}\pi_0.(\delta\varphi,\delta\lambda)=\delta\rho$, where $Y:=\delta\varphi\circ\varphi^{-1}\in\mathfrak{X}(M)$ and $f=\varphi_*(2\frac{\delta\lambda}{\lambda})\in C^\infty(M,\mathbb{R}_{>0})$, of

$$
\begin{aligned}
&G^{\operatorname{Aut}(\mathcal{C}(M))}_{\varphi,\lambda}((\delta\varphi,\delta\lambda),(\delta\varphi,\delta\lambda))=\int_M\big(\lambda^2 g_0(\delta\varphi,\delta\varphi)+\Lambda(\delta\lambda)^2\big)\rho_0\\
&=\int_M g_0(\delta\varphi,\delta\varphi)\lambda^2\rho_0+\Lambda\int_M\delta\lambda^2\rho_0\\
&=\int_M g_0(\delta\varphi,\delta\varphi)\varphi^*\rho+\Lambda\int_M\frac{\delta\lambda^2}{\lambda^2}\varphi^*\rho\\
&=\int_M\varphi^*\Big(g_0(\delta\varphi\circ\varphi^{-1},\delta\varphi\circ\varphi^{-1})\rho\Big)+\frac{\Lambda}{4}\int_M\varphi^*(f^2\rho)\\
&=\int_M g_0(Y,Y)\rho+\frac{\Lambda}{4}\int_M f^2\rho\quad\text{subject to }-\mathcal{L}_Y\rho+f\rho=\delta\rho\\
&=G^{\mathrm{WFR}}_\rho(\delta\rho,\delta\rho)
\end{aligned}
$$

This is independent of the choice of (φ,λ) which satisfy $\pi_0(\varphi,\lambda)=\varphi_*(\lambda^2\rho_0)=\rho$.

Example Proof: π_5 is a Riemannian submersion

It is obviously a submersion.

$$\begin{aligned}
\pi_5(\varphi, g) &= \varphi_* g = T^*\varphi^{-1} \circ g \circ T\varphi^{-1} =: \bar{g} \\
T_{(\varphi,g)}\pi_5(\delta\varphi, h) &= -\mathcal{L}_Y \varphi_* g + \varphi_* h \quad \text{where } Y = \delta\varphi \circ \varphi^{-1} = \varphi_* X \\
\pi_5^{-1}(\bar{g}) &= \{(\varphi, g) \in \mathrm{Diff}(M) \times \mathrm{Met}(M) : \varphi_* g = \bar{g}\} \\
&= \{(\varphi, \varphi^* \bar{g}) : \varphi \in \mathrm{Diff}(M)\} \\
T_{(\varphi,\varphi^*\bar{g})}\pi_5^{-1}(\bar{g}) &= \{(Y \circ \varphi, \varphi^* \mathcal{L}_Y \bar{g}) : Y \in \mathfrak{X}(M)\}
\end{aligned}$$

Note the appearence of the continuity equation in $T\pi_5$.

We do not have access to a horizontal bundle, thus we proceed as follows. $\pi_5 : \mathrm{Diff}(M) \times \mathrm{Met}(M) \to \mathrm{Met}(M)$ is a Riemannian submersion in the sense that $T_{(\varphi,g)}\pi_5 : T_\varphi \mathrm{Diff}(M) \times T_g \mathrm{Met}(M) \to T_{\varphi_* g} \mathrm{Met}(M)$ is a norm-quotient mapping independently of φ.

For $g = \varphi^* \bar{g}$ and $k = T_{(\varphi,g)}\pi_5(\delta\varphi, h) = -\mathcal{L}_Y \varphi_* g + \varphi_* h = -\mathcal{L}_Y \bar{g} + \varphi_* h \in T_{\bar{g}} \operatorname{Met}(M)$ we have

$$
\begin{aligned}
\|k\|^2_{G^{WE}_{\bar{g}}} &= G^{WE}_{\bar{g}}(k,k) = G^{WE}_{\varphi_* g}(T_{(\varphi,g)}\pi_5(\delta\varphi, h), T_{(\varphi,g)}\pi_5(\delta\varphi, h)) \\
&= \inf\Big\{ \int_M g_0(Y,Y)\operatorname{vol}(\bar{g}) + \tfrac{d\Lambda}{4}\int_M \operatorname{tr}(\bar{g}^{-1}\bar{h}\bar{g}^{-1}\bar{h})\operatorname{vol}(\bar{g}) : \\
&\qquad Y \in \mathfrak{X}(M), \bar{h} \in T_{\bar{g}} \operatorname{Met}(M) \text{ with } k = -\mathcal{L}_Y \bar{g} + \bar{h}\Big\} \\
&= \inf\Big\{ \int_M g_0(Y \circ \varphi, Y \circ \varphi)\operatorname{vol}(g) + \tfrac{d\Lambda}{4}\int_M \operatorname{tr}(g^{-1}hg^{-1}h)\operatorname{vol}(g) : \\
&\qquad Y \in \mathfrak{X}(M), h = \varphi^*\bar{h} \in T_g \operatorname{Met}(M) \text{ with } k = -\mathcal{L}_Y \bar{g} + \bar{h}\Big\}
\end{aligned}
$$

where in the end we applied φ^* to both integrands.

This should be equal to

$$\begin{aligned}\inf\Big\{\|(Y\circ\varphi,h)\|^2_{G^{W\times E}_{(\varphi,g)}} &: (Y\circ\varphi,h)\in T_\varphi \operatorname{Diff}(M)\times T_g \operatorname{Met}(M)\\ &h\in T_g\operatorname{Met}(M),\, Y\in\mathfrak{X}(M) \text{ with } T_{(\varphi,\varphi^*\bar g)}\pi_5(Y\circ\varphi,h)=k\Big\}\\ =\inf\Big\{\int_M g_0(Y\circ\varphi,Y\circ\varphi)\operatorname{vol}(g) &+ \tfrac{d\Lambda}{16}\int_M \operatorname{tr}(g^{-1}hg^{-1}h)\operatorname{vol}(g) :\\ &h\in T_g\operatorname{Met}(M),\, Y\in\mathfrak{X}(M) \text{ with } -\mathcal{L}_Y\bar g+\varphi_* h=k\Big\}\end{aligned}$$

independently of φ, which is the case since both integrals do not change by applying ψ^* for each fixed φ and g separately, independently of φ. Since squared norms are homogeneous of order 2, that is, $\|tk\|^2=t^2\|k\|^2$, it also follows that the induced norm on the image by such a Riemannian submersion actually comes from a Riemannian metric. Unfortunately, the horizontal lifts for these Riemannian submersions are not simple: They may lie in a completion of each fiber. $\square$

Thank you for listening.