GAUGE THEORY FOR DIFFEOMORPHISM GROUPS

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ABSTRACT: We consider fibre bundles without structure group and develop the theory of connections, curvature, parallel transport, (nonlinear) frame bundle, the gauge group and it's action on the space of connections and the corresponding orbit space. We describe the classifying space and the rudiments of characteristic class theory and we give several self-duality conditions.

0. INTRODUCTION

Gauge theory is usually formulated in the context of a principal fibre bundle with structure group a compact Lie group, or some associated vector or fibre bundle. We start here with a fibre bundle $p: E \to M$ with standard fibre S without structure group, i.e. with structure group the whole diffeomorphism group Diff(S) of the standard fibre S. A (generalized) connection will be just a projection on the vertical bundle and it's curvature will be given with the help of the Frölicher-Nijenhuis bracket. The local theory of this point of view has been developped very far by Marco Modugno (1986), but he never considered parallel transport, which in general exists only (fibrewise) locally. If the standard fibre S is compact, parallel transport is globally defined, and on each fibre bundle there are connections with global parallel transport. We also give a semi-local description of connections via 'Christoffel forms' which allows to decide whether a given (generalized) connection is induced from a principal one for a given G-structure on the bundle.

Then we construct the holonomy group of a complete connection and also the holonomy Lie algebra. The holonomy group is a subgroup of the diffeomorphism group of some fibre; the holonomy Lie algebra is desribed by the curvature and the parallel transport, and if it turns out to be finite dimensional we can show that the holonomy group is a finite dimensional Lie group, the bundle bears a G-structure and the connection is induced from a principal one.

Next we construct the (nonlinear) frame bundle of a fibre bundle, whose standard fibre now has to be compact. This is a principal fibre bundle with structure group the diffeomorphism group of the standard

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fibre S, which is a Lie group modelled on nuclear Fréchet spaces. We use the differential geometry of manifolds of mappings and diffeomorphism groups as developped in Michor (1980): No Sobolev spaces are used. Any connection on the bundle E lifts to a principal connection on this frame bundle and the usual calculus of principal connections holds true. By lifting we see that any principal connection on this frame bundle admits a parallel transport (S compact is essential here).

The gauge group of the frame bundle coincides with the group of all diffeomorphism of E which respect fibres and cover the identity on M. This is a Lie group modelled on nuclear (LF)-spaces now; it acts smoothly on the nuclear (LF)-manifold of all connections on E and we take a brief glance on the space of orbits. I expect that it is stratified into smooth manifolds, each one corresponding to a conjugacy class of holonomy groups in the gauge group. Usual moduli spaces should be subsets of this corresponding to holonomy groups whose connected component of the identity sits in the conjugacy class of a fixed compact Lie group in the gauge group.

The classifying space of Diff(S) for compact S can be chosen as the manifold of all submanifolds of a Hilbert space of type S (a sort of nonlinear Grassmannian). It carries a universal S-bundle and even a universal connection (given by othogonal projection): Any connection on an S-bundle is pullback of this universal connection. Diff(S) contains the group of diffeomorphisms preserving some volume density as a smooth deformation retract, so on each bundle $E \to M$ we may choose a fibrewise volume form and connections respecting these volume forms. For these data the Chern-Weil construction works and gives a characteristic class in the second cohomology of M with twisted coefficients. These results have been obtained by Gerd Kainz in his doctoral dissertation (1988).

Finally we describe one type of Yang-Mills functional depending on a Riemannian metric on the base space M and a fibrewise symplectic structure, and we give several types of self-duality conditions.

Lots of open problems remain: The (infinite dimensional) differential geometry of the holonomy groups, the Ambrose Singer theorem if the holonomy Lie algebra is infite dimensional, the moduli space and it's stratification and the exploitation of self-duality.

1. CONNECTIONS ON FIBRE BUNDLES

1.1. Let (E,p,M,S) be a fibre bundle: E, M, S are smooth finite dimensional second countable manifolds, $p: E \to M$ is smooth and each x in M has an open neighbourhood U such that $p^{-1}(U) =: E|U$ is fibre respectingly diffeomorphic to $U \times S$. S is called the standard fibre.

The vertical bundle of (E,p,M,S), called $V(E)$, is the kernel of $Tp: TE \to TM$. It is a sub vector bundle of $TE \to E$.

A connection on (E,p,M,S) is just a fibre linear projection $\Phi: TE \to VE$; so Φ is a 1-form on E with values in the vertical bundle, $\Phi \in \Omega^1(E; VE)$. The kernel of Φ is called the horizontal subspace $\ker\Phi$. Clearly for each u in E the mapping $T_u p: (\ker\Phi)_u \to T_{p(u)}M$ is a linear isomorphism, so $(Tp, \pi_E): \ker\Phi \to TM \times_M E$ is a diffeomorphism, fibre linear in the first coordinate, whose inverse $\chi: TM \times_M E \to \ker\Phi \to TE$ is called the horizontal lifting. Clearly the connection Φ can equivalently be described by giving a horizontal sub vector bundle of $TE \to E$ or by specifying the horizontal lifting χ satisfying $(Tp,\pi_E)\,\chi = Id$ on $TM \times_M E$.

1.2. The curvature of a connetion $\Phi \in \Omega^1(E;VE)$ is most conveniently expressed with the help of the Frölicher-Nijenhuis bracket: see the appendix. The Frölicher-Nijenhuis bracket of Φ with itself turns out to be $[\Phi,\Phi](X,Y) = 2\,\Phi[(Id - \Phi)X, (Id - \Phi)Y]$ for vector fields X, Y in $\mathfrak{X}(E)$. This formula shows that $[\Phi,\Phi]$ is the obstruction against integrability of the horizontal bundle $\ker\Phi$. We put $R := [\Phi,\Phi]$ in $\Omega^2(E, VE)$, a 2-form on E with values in the vertical bundle, and we call R the curvature. It's relation with the usual curvature on principal bundles is explained in section 2.

By the graded Jacobi identity we have $[R,\Phi] = \frac{1}{2}[[\Phi,\Phi],\Phi] = 0$; this is the form of the Bianchi identity in our setting.

An immediate consequence of the naturality of the Frölicher-Nijenhuis bracket (see appendix) is the following: Let $f: N \to M$ be a smooth mapping. Consider the pullback bundle (f^*E, f^*p, N, S) :

$$\begin{array}{ccc} f^*E & \xrightarrow{p^*f} & E \\ \big\downarrow{\scriptstyle f^*p} & & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array}$$

Since p^*f is fibrewise a diffeomorphism, $(f^*\Phi)_u(X) = T_u(p^*f)^{-1}\Phi\, T_u(p^*f)X$ is well defined for X in $T_u(f^*E)$ and a connection Φ on E and we get in this way a connection $f^*\Phi$ on f^*E. Clearly $f^*\Phi$ and Φ are p^*f-related and so their curvatures are also p^*f-related.

1.3. Let us suppose that a connection Φ on (E,p,M,S) has zero curvature. Then the horizontal bundle $\ker\Phi$ is integrable: each point u of E lies on a leaf $L(u)$ such that $T_v(L(u)) = (\ker\Phi)_v$ for each v in $L(u)$. Then $p|L(u)\colon L(u) \to M$ is locally a diffeomorphism. But in general it is neither surjective nor is it a covering of it's image.

1.4. <u>Local description</u> of a connection Φ on (E,p,M,S). Let us fix a fibre bundle atlas (U_α,ψ_α), i.e. (U_α) is an open cover of M and $\psi_\alpha\colon E|U_\alpha \to U_\alpha \times S$ is a diffeomorphism respecting all fibres over U_α. Then $\psi_\alpha \psi_\beta^{-1}(x,s) = (x,\psi_{\alpha\beta}(x,s))$, where $\psi_{\alpha\beta}\colon(U_\alpha \cap U_\beta)\times S \to S$ is smooth and $\psi_{\alpha\beta}(x,.)$ is a diffeomorphism of S for each x in $U_{\alpha\beta} := U_\alpha \cap U_\beta$. We will later consider the mappings $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(S)$; their differentiability will be a subtle question. $(\psi_{\alpha\beta})$ satisfies the usual cocycle condition for transition functions. Given a connection Φ on E we consider now the connection $(\psi_\alpha^{-1})^*\Phi$ in $\Omega^1(U_\alpha \times S,\ U_\alpha \times TS)$, which is described by $((\psi_\alpha^{-1})^*\Phi)(\xi_x,\eta_y) = -\Gamma^\alpha(\xi_x,y) + \eta_y$ for ξ_x in T_xU_α and η_y in T_yS, and where $(0_x,\ \Gamma^\alpha(\xi_x,y)) := -\ T(\psi_\alpha)\ \Phi\ (T(\psi_\alpha)^{-1}(\xi_x,0_y))$. We consider Γ^α as an element of $\Omega^1(U_\alpha,\ \mathfrak{X}(S))$, a 1-form on U_α with values in the infinite dimensional Lie algebra $\mathfrak{X}(S)$ of all vector fields on the standard fibre. The Γ^α are called the <u>Christoffel forms</u> of the connection Φ with respect to the bundle atlas (U_α,ψ_α). A short computation shows that they transform as follows: $T_y(\psi_{\beta\alpha}(x,.))\ \Gamma^\alpha(\xi_x,y) = \Gamma^\beta(\xi_x,\psi_{\beta\alpha}(x,y)) - T_x(\psi_{\beta\alpha}(.,y))\xi_x$. The curvature of the connection $(\psi_\alpha^{-1})^*\Phi$ is given by (using 1.2) $(\psi_\alpha^{-1})^*R = \frac{1}{2}\ (\psi_\alpha^{-1})^*[\Phi,\Phi] = \frac{1}{2}\ [(\psi_\alpha^{-1})^*\Phi,\ (\psi_\alpha^{-1})^*\Phi]$ and a short calculation yields $((\psi_\alpha^{-1})^*R)((\xi_1,\eta_1),(\xi_2,\eta_2)) = d\Gamma^\alpha(\xi_1,\xi_2) + [\Gamma^\alpha(\xi_1),\Gamma^\alpha(\xi_2)]$, where $d\Gamma^\alpha$ is the usual exteriour derivative of the 1-form Γ^α with values in $\mathfrak{X}(S)$ and the bracket is the one of $\mathfrak{X}(S)$. This is the Maurer-Cartan formula which in this setting appears only in the level of local description.

1.5. THEOREM (Parallel transport): Let Φ be a connection on a fibre bundle (E,p,M,S) and let $c:(a,b) \to M$ be a smooth curve with 0 in (a,b), $c(0) = x$. Then there is a neighbourhood U of $E_x \times \{0\}$ in $E_x \times (a,b)$ and a smooth mapping $Pt_c: U \to E$ such that:

1. $p(Pt(c,u_x,t)) = c(t)$ if defined and $Pt(c,u_x,0) = u_x$.
2. $\Phi(\frac{d}{dt} Pt(c,u_x,t)) = 0_{c(t)}$ if defined.
3. If $f: (\bar{a},\bar{b}) \to (a,b)$ is smooth with 0 in $(\bar{a},\bar{b})$, then $Pt(c,u_x,f(t)) = Pt(c\circ f, Pt(c,u_x,f(0)),t)$, if defined.
4. U is maximal for properties 1 and 2.
5. In a certain sense Pt depends smoothly on c, u_x and t.

1^{st} proof: In local bundle coordinates $\Phi(\frac{d}{dt} Pt(c,u_x,t)) = 0$ is an ordinary differential equation, nonlinear, with initial condition $Pt(c,u_x,0) = u_x$. So there is a maximally defined local solution curve which is unique. All further properties are consequences of uniqueness.

2^{nd} proof: Consider the pullback bundle $(c^*E, c^*p, (a,b),S)$ and the pullback connection $c^*\Phi$ on it. It's curvature is zero, since the horizontal bundle is 1-dimensional. By 1.3 the horizontal foliation exists and the parallel transport just follows a leaf. Map it back into E.

3^{rd} proof: Consider a fibre bundle atlas (U_α,ψ_α) as in 1.4. Then $\psi_\alpha(Pt(c,\psi_\alpha^{-1}(x,y),t) = (c(t),\gamma(y,t))$ and $\gamma(y,t)$ is the evolution operator ('flow') of the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t),.)$ on S. Clearly local solutions exist and all properties follow. qed.

1.6. A connection Φ on (E,p,M,S) is called a <u>complete connection</u>, if the parallel transport Pt_c along any smooth curve $c: (a,b) \to M$ is defined on the whole of $E_{c(0)} \times (a,b)$. If the standard fibre is compact, then any connection on (E,p,M,S) is complete: this follows immediately from the third proof of 1.5. The following result is due to S. Halperin (1987):

THEOREM: Each fibre bundle (E,p,M,S) admits complete connections. <u>This is true</u>

Proof: In the following lemma we will show that there are complete Riemannian metrics g on E and g_1 on M such that $p: E \to M$ is a Riemannian submersion. Let the connection Φ be just orthonormal projection $TE \to VE$ with respect to g. Then the horizontal lift mapping $\chi(.,u): T_{p(u)}M \to T_uE$ is an isometry, thus the arc lenght of $Pt(c,u,.)$ equals that of c and thus $\lim_{t\to s} Pt(c,u,t)$ exists if $c(s)$ is defined and we may continue.

1.7. LEMMA: Let (E,p,M,S) be a fibre bundle. Then there are complete Riemannian metrics g on E and g_1 on M such that $p: E \to M$ is a Riemannian submersion.

Proof: Let $\dim M = m$. Let (U_α, ψ_α) be a fibre bundle atlas as in 1.4. By topological dimension theory (see Nagata 1965) the open cover (U_α) of M admits a refinement such that any $m+1$ members have empty intersection. Let (U_α) itself have this property. Choose a smooth partition of unity (f_α) subordinated to (U_α). Then the sets $V_\alpha := \{x: f_\alpha(x) > \frac{1}{m+1}\} \subset U_\alpha$ are still an open cover of M since $\Sigma f_\alpha(x) = 1$ and at most m of the $f_\alpha(x)$ can be nonzero. Assume that each V_α is connected. Then we choose an open cover (W_α) of M such that $\overline{W_\alpha} \subset V_\alpha$.

Now let g_1 and g_2 be complete Riemannian metrics on M and S respectively (see Nomizu-Ozeki 1961 or J. Morrow 1970). Then $g_1|U_\alpha \times g_2$ is a metric on $U_\alpha \times S$ and we consider the metric $g := \Sigma f_\alpha\, \psi_\alpha^*(g_1|U_\alpha \times g_2)$ on E. We claim that g is a complete metric on E. So let B be a closed g-bounded set in E. Then $p(B)$ is g_1-bounded in M since $p: (E,g) \to (M,g_1)$ is a Riemannian submersion (see local considerations below), and has thus compact closure in M since g_1 is complete. Then $B'_\alpha := \overline{p(B)} \cap \overline{W_\alpha}$ is compact in V_α and clearly $B''_\alpha := p^{-1}(B'_\alpha) \cap B$ is closed and g-bounded and it suffices to show that each B''_α is compact since there are only finitely many of them and they cover B. Now we consider $B_\alpha := \psi_\alpha(B''_\alpha) \subset V_\alpha \times S$. A little thought shows that B_α is still bounded with respect to the metric $g_\alpha := (\psi_\alpha^{-1})^* g$. We obtain $(g_\alpha)_{(x,y)} = (g_1)_x \times (\sum_\beta f_\beta(x)\, \psi_{\beta\alpha}(x,.)^* g_2)_y$. Let $r :=$ g_α-diameter of B_α in $V_\alpha \times S$, choose $R > (m+1)^{1/2}(r+1)$ and choose (x_0,y_0) in B_α. Let $D = D(y_0,R)$ be the open disc with center y_0 and g_2-radius R in S. Since g_2 is complete, D has compact closure in S. We claim that $B_\alpha \subset B'_\alpha \times D$ which then implies that B_α is compact and finishes the proof. Suppose not and let (x,y) be in $B_\alpha \setminus B'_\alpha \times D$. Then for any piecewise smooth curve $c = (c_1,c_2): [0,1] \to V_\alpha \times D$ from (x_0,y_0) to (x,y) we have g_2-lenght $(c_2) = \int_0^1 |c_2'(t)|_{g_2} dt \geq R$. But then g_α-lenght $(c) = \int_0^1 |c'(t)|_{g_\alpha} dt =$

$$= \int_0^1 \left(|c_1'(t)|^2_{g_1} + \sum_\beta f_\beta(c_1(t))\, (\psi_{\beta\alpha}(c_1(t),.)^* g_2)(c_2'(t),c_2'(t)) \right)^{1/2} dt \geq$$

$\geq \int_0^1 (f_\alpha(c_1(t))\, g_2(c_2'(t),c_2'(t)))^{1/2}dt \geq (m+1)^{-1/2}\int_0^1 |c_2'(t)|_{g_2}\, dt \geq$
$\geq (m+1)^{-1/2}\, R > r+1.$

So (x,y) cannot be in B_α either. qed.

2. ASSOCIATED BUNDLES AND INDUCED CONNECTIONS

2.1. Let G be a Lie group which might be infinite dimensional like $Diff(S)$. Let us consider a principal bundle (P,p,M,G) with structure group G: so there is a bundle atlas $(U_\alpha, \psi_\alpha : P|U_\alpha \to U_\alpha \times G)$ such that the transition functions $\psi_{\alpha\beta}: U_{\alpha\beta}\times G \to G$ are given by $\psi_{\alpha\beta}(x,g) = \tilde{\psi}_{\alpha\beta}(x).g$, where $\tilde{\psi}_{\alpha\beta}: U_{\alpha\beta} \to G$ are smooth. Let $r: P\times G \to G$ be the principal right action, let $\mathfrak{g}$ be the Lie algebra of G and let $\zeta: \mathfrak{g} \to \mathfrak{X}(P)$ be the fundamental vector field mapping, i.e. $\zeta(X)(u) = T_e(r_u).X$ for X in $\mathfrak{g}$ and u in P, where $r_u(g) = r(u,g)$. We also write $r^g: P \to P$ for $r^g(u) = r(u,g)$.

A connection Φ in $\Omega^1(P;\, VP)$ is called a principal connection, if $(r^g)^*\Phi = \Phi$ for all g in G, i.e. $T(r^g)^{-1}\Phi\, T(r^g) = \Phi$. Then the curvature $R = \frac{1}{2}[\Phi,\Phi]$ is also (r^g)-related to itself (see the appendix). Since $X \to \zeta(X)(u)$ is an isomorphism of $\mathfrak{g}$ onto V_uP, there is a unique ω in $\Omega^1(P;\mathfrak{g})$ such that $\Phi(\xi) = \zeta(\omega(\xi))$ for ξ in TP and by the usual properties of fundamental vector fields we have $(r^g)^*\omega = Ad(g^{-1})\,\omega$ and clearly $\omega(\zeta_X) = X$, so ω is the usual description of a principal connection. On the other hand there is ρ in $\Omega^2(P;\mathfrak{g})$ such that $\zeta(\rho(\xi,\eta)) = R(\xi,\eta)$ for ξ,η in T_uP.

LEMMA: $\rho = -d\omega - \frac{1}{2}[\omega,\omega]_{\mathfrak{g}}$ (Maurer-Cartan formula).

Proof: Clearly both sides vanish on fundamental vector fields and coincide on horizontal vector fields. qed.

2.2. Now we describe the Christoffel forms Γ^α from 1.4 for a principal connection $\Phi = \zeta\omega$ with respect to a principal bundle atlas (U_α,ψ_α). Here $T(\psi_\alpha)\,\Phi\,(T(\psi_\alpha^{-1})\xi_x,\, 0_g) = (0_x,\, -\Gamma^\alpha(\xi_x,g))$ by 1.4, where Γ^α is in $\Omega^1(U_\alpha,\mathfrak{X}(G))$. Let $s_\alpha: U_\alpha \to E|U_\alpha$ be given by $s_\alpha(x) = \psi_\alpha^{-1}(x,e)$, and let $\omega_\alpha = s_\alpha^*\omega$ in $\Omega^1(U_\alpha,\, \mathfrak{g})$, the usual physicists description of a connection. Then it turns out that $\Gamma^\alpha(\xi_x,g) = -\, T_e(\lambda_g)\, Ad(g^{-1})\, \omega_\alpha(\xi_x)$, where λ_g denotes left translation.

2.3. Now let $\ell: G \times S \to S$ be a left action. Then the associated bundle $P[S]$ is given as the orbit space $\frac{G \times S}{G}$, where G acts on $P \times S$ by $(u,s)g = (ug, g^{-1}s)$. The differentiable structure is most conveniently (since G might be infinite dimensional) described by: choose a cocycle of transition functions $(\tilde{\psi}_{\alpha\beta}: U_{\alpha\beta} \to G)$ for P associated to a principal bundle atlas and let them act on S via ℓ. Then glue the the $U_\alpha \times S$ to get the associated bundle. We have a smooth map $q: P \times S \to P[S]$, which is itself a principal bundle projection, and $q_u: \{u\} \times S \to (P[S])_{p(u)}$ is a diffeomorphism for each u in P.

Note that TG is again a Lie group, $TG = \mathfrak{g} \circledS G$ is the semidirect product in the right trivialisation, where G acts on $\mathfrak{g}$ via Ad. Then (TP, Tp, TM, TG) is again a principal bundle, $T\ell: TG \times TS \to TS$ is again a left action, and $T(P[S,\ell]) = TP[TS, T\ell]$ is again an associated bundle. It's vertical bundle $V(P[S,\ell]) = P[TS, T\ell \circ (0_G \times Id_{TS}) =: T_2\ell]$ is associated to P itself. In particular $VP = V(P[G,\lambda]) = P[TG, T_2\lambda]$, and $VP/G = P[\mathfrak{g}, Ad]$. All this is easily checked by considering the different actions of the cocycles of transition functions.

2.4. Now we want to induce connections. Let $\Phi = \zeta\omega$ be a principal connection on P. We will induce a connection $\bar{\Phi}$ on the associated bundle $P[S]$ with the help of the following diagram:

$$\begin{array}{ccc} TP \times TS & \xrightarrow{\Phi \times Id} & TP \times TS \\ \downarrow Tq & & \downarrow Tq \\ T(P[S]) = TP[TS, T_2\ell] & \overset{\bar{\Phi}}{\dashrightarrow} & T(P[S]) \\ & \searrow \quad TM \quad \swarrow & \end{array}$$

Clearly $\Phi \times Id$ is TG-equivariant, so it induces $\bar{\Phi}$, which turns out to be a projection onto the vertical bundle, the induced connection. Note that $\Phi \times Id$ and $\bar{\Phi}$ are q-related, so $R \times 0 = \frac{1}{2}[\Phi \times Id, \Phi \times Id] = \frac{1}{2}[\Phi, \Phi] \times 0$ and $\bar{R} = \frac{1}{2}[\bar{\Phi}, \bar{\Phi}]$ are also q-related: The curvature of an induced connection is the induced curvature.

2.5. Now for the converse. Let $E = P[S]$ be an associated bundle. Let us suppose that the action $\ell: G \times S \to S$ is infinitesimally effective, i.e. $\zeta: \mathfrak{g} \to \mathfrak{X}(S)$ is injective, where $\zeta(X)(s) = T_e(\ell^s).X$ for X in $\mathfrak{g}$ and s in S is the fundamental vector field mapping.

THEOREM: In this situation, a connection Φ in $\Omega^1(P[S], V(P[S]))$ is induced by a (necessarily unique) pricipal connection on P if and only if for some bundle atlas $(U_\alpha, \psi_\alpha: P[S]|U_\alpha \to U_\alpha \times S)$ belonging to the G-structure (so the transition functions have values in G and act on S via ℓ) the Christoffel forms Γ^α of $\bar{\Phi}$ take values in $\mathfrak{X}_{fund}(S)$, the Lie algebra of fundamental vector fields.

Indication of proof: If the Christoffel forms have values in $\mathfrak{X}_{fund}(S)$ then $\Gamma^\alpha(\xi_x) = - \zeta\omega_\alpha(\xi_x)$ for unique ω_α in $\Omega^1(U_\alpha,\mathfrak{g})$, since $\zeta: \mathfrak{g} \to \mathfrak{X}_{fund}(S)$ is an isomorphism. These ω_α transform as required:

$\omega_\alpha = Ad(\tilde{\psi}_{\beta\alpha}{}^{-1})\omega_\beta + \tilde{\psi}_{\beta\alpha}{}^{-1}\, d\tilde{\psi}_{\beta\alpha}$. So they give a principal connection on P. The converse is a computation.

3. HOLONOMY GROUPS

3.1. Let (E,p,M,S) be a fibre bundle with complete connection Φ and let M be connected. We fix a base point x_0 in M and identify E_{x_0} with the standard fibre S. For each closed piecewise smooth curve $c: [0,1] \to M$ through x_0 the parallel transport $Pt(c,.,1) =: Pt(c,1)$ (pieced together over the smooth parts) is a diffeomorphism of S; all these diffeomorphisms form the group $Hol(\Phi,x_0)$, the holonomy group of Φ at x_0, a subgroup of the diffeomorphism group Diff(S). If we consider only those piecewise smooth curves which are homotopic to zero we get a subgroup $Hol_0(\Phi,x_0)$, called the restricted holonomy group of Φ at x_0.

3.2. For a complete connection Φ let $\chi: TM \times_M E \to TE$ be the horizontal lifting and let R be the curvature. For any x in M and X_x in T_xM the horizontal lift $\chi(X_x) = \chi(X_x,.): E_x \to TE$ is a vector field along E_x. For X_x, Y_x in T_xM we consider $R(\chi X_x,\chi Y_x)$ in $\mathfrak{X}(E_x)$. Now choose any piecewise smooth curve c from x_0 to x, consider the diffeomorphism $Pt(c,1)$: $S = E_{x_0} \to E_x$ and the pullback $Pt(c,1)^* R(\chi X_x,\chi Y_x)$ in $\mathfrak{X}(S)$. Let us denote the closed linear subspace generated by all these vector fields (for all x in M, X_x, Y_x in T_xM and curves c from x_0 to x) in $\mathfrak{X}(S)$ with the compact C^∞-topology by the symbol $hol(\Phi,x_0)$ and let us call it the holonomy Lie algebra of Φ at x_0.

LEMMA: $hol(\Phi,x_o)$ is a Lie subalgebra of (S).

Proof: Let X be in $\mathfrak{X}(M)$ and let $Fl^{\chi X}_t$ denote the local flow of the horizontal lift χX of X. It restricts to parallel transport along any of the flow lines of X in M. Then we have that

$$\frac{d}{dt}\Big|_0 (Fl^{\chi X}_s)^* \, (Fl^{\chi Y}_t)^* \, (Fl^{\chi X}_{-s})^* \, (Fl^{\chi Z}_z)^* \, R(\chi U,\chi V)|E_{x_o} =$$

$$= (Fl^{\chi X}_s)^* \, [\chi Y, \, (Fl^{\chi X}_{-s})^* \, (Fl^{\chi Z}_z)^* \, R(\chi U,\chi V)]|E_{x_o} =$$

$$= [(Fl^{\chi X}_s)^* \, \chi Y, (Fl^{\chi Z}_z)^* \, R(\chi U,\chi V)]|E_{x_o}$$

is in $hol(\Phi,x_o)$. By the Jacobi identity and some manipulation the result follows. qed.

3.3. The theorem of Ambrose and Singer now admits the following generalisation.

THEOREM: Let Φ be a connection on the fibre bundle (E,p,M,S) with compact standard fibre S and let M be connected. Suppose that for some x_o in M the holonomy Lie algebra $hol(\Phi,x_o)$ is finite dimensional. Then there is a principal bundle (P,p,M,G) with finite dimensional structure group G, an irreducible connection ω on it and a smooth action of G on S such that the Lie algebra $\mathfrak{g}$ of G equals the holonomy algebra $hol(\Phi,x_o)$, the fibre bundle E is isomorphic to the associated bundle $P[S]$ and Φ is the connection induced by ω. G is isomorphic to the holonomy group $Hol(\Phi,x_o)$. P and ω are unique up to isomorphism.

Proof: Let us identify E_{x_o} and S. Then $\mathfrak{g} := hol(\Phi,x_o)$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(S)$, and since S is compact, there is a finite dimensional connected Lie group G_o of diffeomorphisms of S with Lie algebra $\mathfrak{g}$.

Claim 1: G_o equals $Hol_o(\Phi,x_o)$, the restricted holonomy group.

Let f be in $Hol_o(\Phi,x_o)$, then $f = Pt(c,1)$ for a piecewise smooth closed curve through x_o. Since parallel transport is essentially invariant under reparametrisation (1.5.3), we can replace c by $c \circ g$, where g is smooth and flat at each corner of c. So we may assume that c is smooth. Then c is homotopic to zero, so by approximation we may assume that there is a smooth homotopy $H: \mathbb{R}^2 \to M$ with $H_1|I = c$ and $H_o|I = x_o$. Then $Pt(H_t,1)=f_t$ is a curve in $Hol_o(\Phi,x_o)$ which is smooth as a mapping $\mathbb{R} \times S \to S$.

Claim 2: $(\frac{d}{dt} f_t) \circ f_t^{-1} =: Z_t$ is in $\mathfrak{g}$ for all t.

To prove claim 2 we consider the pullback bundle $H^*E \to \mathbb{R}^2$ with the induced connection $H^*\Phi$. It is sufficient to prove claim 2 there. Let $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$ be the constant vector fields on $\mathbb{R}^2$, so $[X,Y] = 0$. Then $Pt(c,s) = Fl^{\chi X}_s$ and so on. Put $f_{t,s} = Fl^{\chi X}_{-s}\ Fl^{\chi Y}_{-t}\ Fl^{\chi X}_{s}\ Fl^{\chi Y}_{t}: S \to S$, so $f_{t,1} = f_t$. Then

$$(\frac{d}{dt} f_{t,s})\, f_{t,s}^{-1} = -(Fl^{\chi X}_s)^*\chi Y + (Fl^{\chi X}_s)^*\,(Fl^{\chi Y}_t)^*\,(Fl^{\chi X}_{-s})\chi Y \quad \text{in } \mathfrak{X}(S) \text{ and}$$

$$(\frac{d}{dt} f_t)\, f_t^{-1} = \int_0^1 \frac{d}{ds}((\frac{d}{dt} f_{t,s})\, f_{t,s}^{-1})ds = \int_0^1 (-(Fl^{\chi X}_s)^*[\chi X,\chi Y] +$$
$$+ (Fl^{\chi X}_s)^*[\chi X,\ (Fl^{\chi Y}_t)^*\ (Fl^{\chi X}_{-s})^*\chi Y] - (Fl^{\chi X}_s)^*\ (Fl^{\chi Y}_t)^*\ (Fl^{\chi X}_{-s})^*[\chi X,\chi Y]\ )\ ds.$$

Since $[X,Y] = 0$ we have $[\chi X,\chi Y] = \Phi[\chi X,\chi Y] = R(\chi X,\chi Y)$, and also we have

$$(Fl^{\chi X}_t)^*\chi Y = \chi((Fl^{\chi X}_t)^*\chi Y) + \Phi((Fl^{\chi X}_t)^*\chi Y) = \chi Y + \int_0^t \frac{d}{dt}\Phi(Fl^{\chi X}_t)^*\chi Y\ dt =$$
$$= \chi Y + \int_0^t \Phi(Fl^{\chi X}_t)^*[\chi X,\chi Y]\ dt = \chi Y + \int_0^t (Fl^{\chi X})^* R(\chi X,\chi Y)\ dt.$$

Thus all parts of the integrand above are in $\mathfrak{g}$ and so $(\frac{d}{dt} f_t)\ f_t^{-1}$ is in $\mathfrak{g}$ for all t and claim 2 is proved.

Now claim 1 can be shown as follows: There is a unique smooth curve $g(t)$ in G_σ satisfying $Z_t.g(t) = \frac{d}{dt}g(t)$ and $g(0) = Id$; via the action of G_o on S $g(t)$ is a curve of diffeomorphisms on S, generated by the time dependent vector field Z_t, so $g(t) = f_t$ and $f = f_1$ is in G_0.

Step 3: Now we make $Hol(\Phi,x_o)$ into a Lie group which we call G, by taking $Hol_o(\Phi,x_o)$ as its connected component of identity. Then $Hol(\Phi,x_o)/G_o$ is a countable group, since the fundamental group $\pi_1(M)$ is countable (by Morse theory M is homotopy equivalent to a countable CW-complex).

Step 4: Construction of a cocycle of transition functions with values in G. Let $(U_\alpha,\ u_\alpha: U_\alpha \to \mathbb{R}^n)$ be a locally finite smooth atlas of M such that each $u_\alpha: U_\alpha \to \mathbb{R}^n$ is a diffeomorphism onto $\mathbb{R}^n$. Put $x_\alpha = u_\alpha^{-1}(0)$. Choose smooth curves $c_\alpha: [0,1] \to M$ with $c_\alpha(0) = x_o$, $c_\alpha(1) = x_\alpha$. For each x in U_α let $c^x_\alpha: [0,1] \to M$ be the smooth curve $t \to u_\alpha^{-1}(t.u_\alpha(x))$, then c^x_α connects x_α and x and the mapping $(x,t) \to c^x_\alpha(t)$ is smooth $U_\alpha \times [0,1] \to M$.

Now we define a fibre bundle atlas $(U_\alpha,\psi_\alpha: E|U_\alpha \to U_\alpha \times S)$ by $\psi_\alpha^{-1}(x,s) = Pt(c^x_\alpha,1)\ Pt(c_\alpha,1)\ s$. Clearly ψ_α is smooth. Let us investigate the transition functions:

$$\psi_\beta\ \psi_\alpha^{-1}(x,s) = (x,\ Pt(c_\alpha,1)^{-1}\ Pt(c^x_\alpha,1)^{-1}\ Pt(c^x_\beta,1)\ Pt(c_\beta,1)\ s)$$
$$= (x,\ Pt(c_\beta.c^x_\beta.(c^x_\beta)^{-1}.(c_\beta)^{-1},4)\ s)$$
$$=:(x,\ \psi_{\beta\alpha}(x)\ s),\ \text{where } \psi_{\beta\alpha}: U_{\beta\alpha} \to G.$$

Clearly $\psi_{\beta\alpha}: U_{\beta\alpha} \times S \to S$ is smooth and consequently (see section 4) $\psi_{\beta\alpha}: U_{\beta\alpha} \to \mathrm{Diff}(S)$ is smooth and also $U_{\beta\alpha} \to G$. $(\psi_{\alpha\beta})$ is a cocycle of transition functions and we use it to glue a principal bundle with structure group G over M which we call (P,p,M,G). It is clear that $P \times S/G$ equals (E,p,M,S).

Step 5: Lifting the connection Φ to P. For this we have to compute the Christoffel symbols of Φ with respect to the bundle atlas of step 4. For this we need more sophisticated methods. Since it is heuristically obvious we omit this part of the proof. qed.

3.4. Remarks: I expect that theorem 3.3 is also true if S is not compact but the connection Φ is supposed to be complete. The problem here is to show that the holonomy Lie algebra consists of complete vector fields. By a theorem of Palais (1957) it suffices to find complete vector fields which generate the holonomy Lie algebra.

It would be interesting to find the relation between $\mathrm{Hol}(\Phi,x_0)$ and $\mathrm{hol}(\Phi,x_0)$ if the latter is not supposed to be finite dimensional (for complete connections). Se 4.6 below for some curious fact.

4. THE (NONLINEAR) FRAME BUNDLE OF A FIBRE BUNDLE

4.1. Let (E,p,M,S) be a fibre bundle. Let $(U_\alpha, \psi_\alpha: E|U_\alpha \to U_\alpha \times S)$ be a fibre bundle atlas as described in 1.4. Then $\psi_\alpha \psi_\beta^{-1}(x,s) = (x,\psi_{\alpha\beta}(x,s))$ for smooth mappings $\psi_{\alpha\beta}: U_{\alpha\beta} \times S \to S$ such that $\psi_{\alpha\beta}(x,.)$ is in $\mathrm{Diff}(S)$.

For second countable smooth manifolds Y, Z the space $C^\infty(Y,Z)$ is an infinite dimensional smooth manifold modelled on nuclear (LF)-spaces (see Michor 1980). In general we have $C^\infty(U_{\alpha\beta},C^\infty(S,S)) \subseteq C^\infty(U_{\alpha\beta} \times S,S)$ with equality if and only if S is compact (see Michor 1980, 11.9). The image in the right hand side consists of these maps f which satisfy: for any compact set K in $U_{\alpha\beta}$ there is a compact set K' in S such that $f(x_1,s) = f(x_2,s)$ for all x_1, x_2 in the same connected component of K and for all s in $S \setminus K'$. So $\psi_{\alpha\beta}: U_{\alpha\beta} \to \mathrm{Diff}(S)$ is smooth if and only if (for $U_{\alpha\beta}$ connected) $\psi_{\alpha\beta}(U_{\alpha\beta})$ is contained in a left coset of the open and closed subgroup $\mathrm{Diff}_c(S)$ of diffeomorphisms with compact support. This is in principle true only for bundles which are "trivial near fibrewise infinity" or have " discrete structure group near fibrewise infinity".

To avoid these topological difficulties we will assume for the rest of this section that the standard fibre S is compact. Then $\psi_{\alpha\beta}: U_{\alpha\beta} \to \mathrm{Diff}(S)$ is smooth without any restrictions and $(\psi_{\alpha\beta}: U_{\alpha\beta} \to \mathrm{Diff}(S))$ form the cocycle of transition functions for (E,p,M,S).

4.2. Now we define the nonlinear frame bundle of (E,p,M,S) as follows. Let $\mathrm{Diff}\{S,E\} := \bigcup_x \mathrm{Diff}(S,E_x)$ where the union is over all x in M, and give it the differentiable structure which one gets by applying the functor $\mathrm{Diff}(S,.)$ to the cocyle of transition functions $(\psi_{\alpha\beta})$. Then the resulting cocycle of $\mathrm{Diff}\{S,E\}$ gives it the structure of a smooth principal bundle over M with structure group $\mathrm{Diff}(S)$. The principal action is just composition from the right.

4.3. Consider now the smooth action $\mathrm{Diff}(S) \times S \xrightarrow{ev} S$ and the associated bundle as in the following diagram:

$$\begin{array}{ccc} \mathrm{Diff}\{S,E\} \times S & \longrightarrow & \dfrac{\mathrm{Diff}\{S,E\} \times S}{\mathrm{Diff}(S)} = \mathrm{Diff}\{S,E\}[S, ev] \\ \downarrow ev & \nearrow & \\ E & \cong & \end{array}$$

Clearly the associated bundle gives us back the bundle (E,p,M,S).

4.4. Let Φ in $\Omega^1(E;VE)$ be a connection on E. We want to realise Φ as a principal connection on $\mathrm{Diff}\{S,E\}$. For that we must get a good picture of the tangent space of $\mathrm{Diff}\{S,E\}$. With the methods of Michor 1980 it is easy to see that $T\,\mathrm{Diff}\{S,E\} = \bigcup_{x \text{ in } M} \{ f \text{ in } C^\infty(S,TE|E_x): Tp\circ f = \text{one point in } T_xM \text{ and } \pi_E\circ f \text{ is in } \mathrm{Diff}(S,E_x)\}$. Given the connection Φ we consider $\omega(f) = T(\pi_E\circ f)^{-1}\circ\Phi\circ f : S \to TE \to VE \to TS$ for f in $T\,\mathrm{Diff}\{S,E\}$. Then $\omega(f)$ is a vector field on S and $\omega: T\,\mathrm{Diff}\{S,E\} \to \mathfrak{X}(S)$ is fibre linear, so it is a one form on $\mathrm{Diff}\{S,E\}$ with values in the Lie algebra $\mathfrak{X}(S)$ of the structure group $\mathrm{Diff}(S)$ (with the negative of the usual bracket).

LEMMA: ω in $\Omega^1(\mathrm{Diff}\{S,E\}; \mathfrak{X}(S))$ is a principal connection, and the induced connection on $E = \mathrm{Diff}\{S,E\}[S,ev]$ coincides with Φ.

Proof: This follows directly from 2.5. But we also give direct proof. The fundamental vector field $\zeta(X)$ on $\mathrm{Diff}\{S,E\}$ for X in $\mathfrak{X}(S)$ is given by $\zeta(X)(g) = Tg\circ X$. Then $\omega(\zeta(X)(g)) = Tg^{-1}\circ\Phi\circ Tg\,X = X$, since $Tg\circ X$ has vertical values. So Φ reproduces fundamental vector fields.

Let h be in Diff(S) and denote by r^h the principal right action. Then
$((r^h)^*\omega)(f) = \omega(T(r^h)\, f) = \omega(f\circ h) = T(\pi_E\circ f\circ h)^{-1}\circ\Phi\circ f\circ h = Th^{-1}\circ\omega(f)\circ h =$
$= \mathrm{Ad}_{\mathrm{Diff}(S)}(h^{-1})\,\omega(f).$ qed.

4.5. THEOREM: Let (E,p,M,S) be a fibre bundle with compact standard fibre S. Then connections on E and principal connections on Diff{S,E} correspond to each other bijectively, and their curvatures are related as in 2.4. Each principal connection on Diff{S,E} → M admits a smooth global parallel transport. The holonomy groups and restricted holonomy groups are equal as subgroups of Diff(S).

Proof: This follows directly from section 2. Each connection on E is complete since S is compact, and the lift to Diff{S,E} of its parallel transport is the (global) parallel transport of the lift of the connection, so the two last assertions follow. qed.

4.6 Remark on the holonomy Lie algebra: Let M be connected, let $\rho = -d\omega - \frac{1}{2}[\omega,\omega]_{\mathfrak{X}(S)}$ be the usual $\mathfrak{X}(S)$-valued curvature of the lifted connection ω on Diff{S,E}. Then we consider the $\mathbb{R}$-linear span of all elements $\rho(\xi_f,\eta_f)$ in $\mathfrak{X}(S)$, where ξ_f, η_f in $T_f\mathrm{Diff}\{S,E\}$ are arbitrary (horizontal) tangent vectors, and we call this span hol(ω). Then by the Diff(S)-equivariance of ρ the vector space hol(ω) is an ideal in the Lie algebra $\mathfrak{X}(S)$.

LEMMA: Let $f: S \to E_{x_0}$ be any diffeomorphism in $\mathrm{Diff}\{S,E\}_{x_0}$. Then $f_*: \mathfrak{X}(S) \to \mathfrak{X}(E_{x_0})$ induces an isomorphism between hol(ω) and the $\mathbb{R}$-linear span of all $g^*R(\chi X,\chi Y)$, X,Y in T_xM, x in M and $g: E_{x_0} \to E_x$ any diffeomorphism.

The proof is obvious. Note that the closure of $f_*(\mathrm{hol}(\omega))$ is (a priori) larger than $\mathrm{hol}(\Phi,x_0)$ of section 3. Which one is the right holonomy Lie algebra?

5. GAUGE THEORY

We fix the setting of section 4. In particular S is supposed to be compact.

5.1. With some abuse of notation we consider the bundle

$\mathrm{Diff}\{E,E\} := \bigcup_{x\in M} \mathrm{Diff}(E_x,E_x)$, which bears the smooth structure described by the cocycle of transition functions $\mathrm{Diff}(\psi_{\alpha\beta}^{-1},\psi_{\alpha\beta}) = (\psi_{\alpha\beta})_* (\psi_{\beta\alpha})^*$, where $(\psi_{\alpha\beta})$ is a cocycle of transition functions for (E,p,M,S).

LEMMA: The assiciated bundle Diff{S,E}[Diff(S), conj] is isomorphic to the fibre bundle Diff{E,E}.

Proof: The mapping A: Diff{S,E} x Diff(S) → Diff{E,E}, given by $A(f,g) = f\circ g\circ f^{-1}: E_x \to S \to S \to E_x$ for f in $\mathrm{Diff}(S,E_x)$, is Diff(S)-invariant, so it factors to a smooth mapping Diff{S,E}[Diff(S)] → Diff{E,E}. It is bijective and admits locally (over M) smooth inverses, so it is a diffeomorphism. qed.

5.2. The <u>gauge group</u> Gau(E) of the bundle (E,p,M,S) is by definition the space of all principal bundle automorphisms of (Diff{S,E},p,M,Diff(S)) which cover the identity on M. The differential geometry for the infinite dimensional spaces involved is given in (Michor 1980), so the usual reasoning gives that Gau(E) equals the space of smooth sections of the associated bundle Diff{S,E}[Diff(S),conj], which by 3.1 equals the space of sections of the bundle Diff{E,E} → M. We equip it with the topology and differentiable structure as a space of smooth sections: since only the image space is infinite dimensional, this makes no difficulties. Since the fibre S is compact we see from a local (on M) application of the exponential law (Michor 1980, 11.9) that Γ(Diff{E,E} → M) ↪ Diff(E) is an embedding of a splitting closed submanifold.

THEOREM: The gauge group Gau(E) = Γ(Diff{E,E}) is a splitting closed subgroup of Diff(E), if S is compact. It has an exponential map (which is not surjective on any neighbourhood of the identity). Its Lie algebra consists of all vertical vector fields with compact support on E, with the negative of the usual Lie bracket.

5.3. Remark: If S is not compact, then we may circumvent the nonlinear frame bundle, and we may define the gauge group Gau(E) directly as the splitting closed subgroup of Diff(E) which consists of all fibre respecting diffeomorphisms of E which cover the identity on M. Contrary to the finite dimensional case with noncompact structure group G Gau(E) is then not discrete. Its Lie algebra consists of all vertical vector fields on

E with compact support. For simplicity's sake we stick to compact fibre in this paper.

5.4. The space of connections or gauge potentials. Let $J^1(E)$ be the bundle of 1-jets of sections of $E \to M$. Then $J^1E \to E$ is an affine bundle, in fact $J^1E = \{\ell \text{ in } L(T_xM, T_uE): Tp\circ\ell = Id_{T_xM}, u \text{ in } E \text{ with } p(u) = x\}$. Then a section γ in $\Gamma(J^1E)$ is just a horizontal lift mapping $TM \times_M E \to TE$ which is fibre linear over E. By the discussion of 1.1 γ then describes a connection and we may view $\Gamma(J^1E)$ as the space of connections of E. The next result follows easily from (Michor 1980, §11).

THEOREM: The action of the gauge group Gau(E) on the space of connections $\Gamma(J^1E)$ is smooth.

5.5. We will now give a different description of the action . We view now a connection Φ as always as a linear fibre projection $\Phi: TE \to VE$, so $Conn(E) := \{\Phi \text{ in } \Omega^1(E;TE): \Phi\circ\Phi = \Phi, Im(\Phi) = VE\}$, which is a closed splitting affine subspace of $\Omega^1(E;TE)$. The natural isomorphism Conn(E) = $\Gamma(J^1E)$ is also one for the topology and the differentiable structures. Then the action of f in $Gau(E) \subseteq Diff(E)$ on a connection Φ is given by $f_*\Phi = (f^{-1})^*\Phi = Tf\circ\Phi\circ Tf^{-1}$. Now it is very easy to describe the infinitesimal action. Let X be a vertical vector field with compact support on E and consider its global flow Fl^X_t. Then $\frac{d}{dt}|_0(Fl^X_t)^*\Phi = [X,\Phi]$, the Frölicher-Nijenhuis bracket (see appendix). The tangent space of Conn(E) at Φ is $T_\Phi Conn(E) = \{\Psi \text{ in } \Omega^1(E;VE): \Psi|VE = 0\}$. The infinitesimal orbit at Φ in Conn(E) is then $\{[X,\Phi], X \text{ in } \mathfrak{X}_c^{vert}(E)\}$.

The isotropy subgroup of a connection Φ is $\{f \text{ in } Gau(E): f_*\Phi = \Phi\}$. Clearly this is just the group of those f which respect the horizontal bundle $\ker\Phi$. The most interesting object is of course the orbit space Conn(E)/Gau(E). I am convinced that it is stratified into smooth manifolds, each one corresponding to a conjugacy class of holonomy groups in Diff(S). Those strata whose holonomy groups are (up to conjugacy) contained in a fixed compact group (like SU(2)) acting on S I expect to be diffeomorphic to the strata of usual gauge theory.

6. CLASSIFYING SPACE FOR Diff(S) AND A CHARACTERISTIC CLASS.

This section is based on the doctoral dissertation of Gerd Kainz (1988).

6.1. Let $\mathbb{R}^\infty = \oplus \mathbb{R}$ be the countable direct sum of copies of $\mathbb{R}$, the space of finite sequences, with the direct sum topology. Let S be a compact manifold. By a slight generalisation of (Michor 1980, §13) the space $\mathrm{Emb}(S, \mathbb{R}^\infty)$ of smooth embeddings $S \to \mathbb{R}$ is an open submanifold of the smooth manifold $C^\infty(S, \mathbb{R}^\infty)$ and it is also the total space of a smooth principal bundle with structure group Diff(S), acting by composition from the right. The base space $B(S, \mathbb{R}^\infty) := \mathrm{Emb}(S, \mathbb{R}^\infty)/\mathrm{Diff}(S)$ is a smooth manifold, modelled on nuclear (LF)-spaces. Its topology is Lindelöf and the model spaces admit smooth partitions of unity, so also $B(S, \mathbb{R}^\infty)$ admits smooth partitions of unity. $B(S, \mathbb{R}^\infty)$ can be viewed as the space of all submanifolds of $\mathbb{R}^\infty$ diffeomorphic to S, a nonlinear analogue of the infinite dimensional Grassmanian.

Furthermore the total space $\mathrm{Emb}(S, \mathbb{R}^\infty)$ is smoothly contractible. Ramadas (1982) constructed the continuous homotopy $A: \ell^2 \times I \to \ell^2$ through isometries given by $A_0 = \mathrm{Id}$, $A_t(a_0,a_1,a_2,\dots) =$
$= (a_0,a_1,\dots,a_{n-2}, a_{n-1}\cos\theta_n(t), a_{n-1}\sin\theta_n(t), a_n\cos\theta_n(t), a_n\sin\theta_n(t),$
$a_{n+1}\cos\theta_n(t), a_{n+1}\sin\theta_n(t),\dots)$ for $\frac{1}{n+1} \leqslant t \leqslant \frac{1}{n}$, where
$\theta_n(t) = n((n+1)t-1)\frac{\pi}{2}$. Then $A_{1/2}(a_0,a_1,a_2,\dots) = (a_0,0,a_1,0,a_2,0,\dots)$ is in ℓ^2_{even} and $A_1(a_0,a_1,\dots) = (0,a_0,0,a_1,0,a_2,0,\dots)$ is in ℓ^2_{odd}. This same homotopy is defined also on $\mathbb{R}^\infty$ and it turns out that it is even smooth there in the Frölicher-Kriegl calculus (Frölicher-Kriegl 1988). So given two embeddings e_1 and e_2 in $\mathrm{Emb}(S, \mathbb{R}^\infty)$ we first deform e_1 through embeddings to e_1' in $\mathrm{Emb}(S, \mathbb{R}^\infty_{\mathrm{even}})$, and e_2 to e_2' in $\mathrm{Emb}(S, \mathbb{R}^\infty_{\mathrm{odd}})$, and then we connect them by $t.e_1' + (1-t).e_2'$.

By the general theory of classifying spaces $B(S, \mathbb{R}^\infty)$ is then a classifying space for Diff(S)-principal bundles over finite dimensional manifolds.

6.2. We will now give a more detailed description of the classifying process. Let us again consider the smooth action $\mathrm{Diff}(S) \times S \xrightarrow{ev} S$ as in 3.3 and the associated fibre bundle $\mathrm{Emb}(S, \mathbb{R}^\infty)[S,ev] = \mathrm{Emb}(S, \mathbb{R}^\infty) \times_{\mathrm{Diff}(S)} S$ which we call $E(S, \mathbb{R}^\infty)$, a smooth fibre bundle over $B(S, \mathbb{R}^\infty)$ with standard fibre S. Remembering the interpretation of $B(S, \mathbb{R}^\infty)$ as the nonlinear

Grassmannian, we van visualize $E(S, \mathbb{R}^\infty)$ as the "universal S-bundle" over $B(S, \mathbb{R}^\infty)$ as follows: $E(S, \mathbb{R}^\infty) = \{(N,x) \text{ in } B(S, \mathbb{R}^\infty) \times \mathbb{R}^\infty : x \in N\}$. Then $T\,E(S, \mathbb{R}^\infty)$ consists of all (N,ξ,x,v), where N is in $B(S, \mathbb{R}^\infty)$, ξ is a vector field along and normal to N in $\mathbb{R}^\infty$, x is in N and v is a tangent vector at x of $\mathbb{R}^\infty$ with $v^\perp = \xi(x)$. This follows from the description of the principal fibre bundle $Emb(S, \mathbb{R}^\infty) \to B(S, \mathbb{R}^\infty)$ given in (Michor 1980, §13). Obviously the vertical bundle $V\,E(S, \mathbb{R}^\infty)$ consists of all (N,x,v), where N is in $B(S, \mathbb{R}^\infty)$, x is in N and $v \in T_xN$. The orthonormal projection $\mathbb{R}^\infty \to T_xN$ defines a connection Φ_0: $T\,E(S, \mathbb{R}^\infty) \to V\,E(S, \mathbb{R}^\infty)$, given by $\Phi_0(N,\xi,x,v) = (N,x,v - \xi(x)) = (N,x,$ tangential component of $v)$.

THEOREM: $(E(S, \mathbb{R}^\infty), p, B(S, \mathbb{R}^\infty), S)$ is a universal bundle for S-bundles and Φ_0 is a universal connection, i.e. for each bundle (E,p,M,S) and connection Φ in $\Omega^1(E;VE)$ there is a smooth (classifying) map $f: M \to B(S, \mathbb{R}^\infty)$ such that (E,Φ) is isomorphic to $(f^*E(S, \mathbb{R}^\infty), f^*\Phi_0)$. Homotopic maps pull back isomorphic S-bundles and conversely (the homotopy can be chosen smooth). The pulled back connection stays the same if and only if $i(T_{(x,t)}H.(0_x,\frac{\partial}{\partial t}))R_0 = 0$ for all x in M and t in I, where H is the homotopy and R_0 is the curvature of Φ_0.

Proof: Choose a Riemannian metric g_1 on the vector bundle $VE \to E$, another one g_2 on M and consider $(Tp|\ker\Phi)^*g_2 \oplus g_1$, a Riemannian metric on E for which the horizontal and the vertical spaces are orthogonal. By the theorem of Nash there is an isometric embedding $h: E \to \mathbb{R}^N$ for N large enough. We then embed $\mathbb{R}^N$ into $\mathbb{R}$ and consider $f: M \to B(S, \mathbb{R}^\infty)$, given by $f(x) = h(E_x)$. Then

$$\begin{array}{ccc} E & \xrightarrow{(f,h)} & E(S, \mathbb{R}^\infty) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & B(S, \mathbb{R}^\infty) \end{array}$$

is fibrewise a diffeomorphism, so $f^*E(S, \mathbb{R}^\infty) = E$. Since also $T(f,h)$ maps horizontal and vertical vectors to othogonal vectors, it pulls back Φ_0 to Φ .
Now let H be a continuous homotopy $M \times I \to B(S, \mathbb{R}^\infty)$. Then we may approximate H by smooth maps with the same H_0 and H_1, if they are smooth (see Bröcker-Jänich 1973, the infinite dimensionality of $B(S, \mathbb{R}^\infty)$ does not disturb). Then we consider the bundle $H^*E(S, \mathbb{R}^\infty) \to M \times I$, equipped with the connection $H^*\Phi_0$, whose curvature is H^*R_0. Let ∂_t be the vector field

∂_t on $M \times I$. Paralleltransport along the lines $t \to (x,t)$ with respect to $H^*\Phi_o$ is given by the flow of the horizontal lift $(H^*\chi_o)(\partial_t)$ of ∂_t. Let us compute its action on $H^*\Phi_o$:

$$\frac{d}{dt}(Fl_t^{(H^*\chi_o)(\partial_t)})^* H^* \Phi_o = \frac{d}{dt} H^*(Fl_t^{\chi_o(TH(0,\partial_t)})^* \Phi_o = H^*[\chi_o TH(o,\partial_t), \Phi_o]$$
$$= - H^* i(\chi_o\ TH(0,\partial_t))R_o \qquad \text{qed.}$$

6.3. THEOREM (Ebin-Marsden 1970): Let S be a compact orientable manifold, let μ be a positive volume form on S with total mass 1. Then $Diff(S)$ splits topologically and smoothly , $Diff(S)$ = $Diff(S) = Diff_\mu(S) \times Vol(S)$, where $Diff_\mu(S)$ is the Fréchet Lie group of all μ-preserving diffeomorphisms and $Vol(S)$ is the space of all volumes of total mass 1 on S.

6.4. A consequence of theorem 6.3 is that the classifying spaces of $Diff(S)$ and $Diff_\mu(S)$ are homotopically equivalent. The space $B_1(S, \mathbb{R}^\infty)$ of all submanifolds of $\mathbb{R}^\infty$ of type S and volume 1 (in the volume form induced from the inner product on $\mathbb{R}^\infty$) is a closed submanifold of codimension 1 (by the Nash-Moser-Hamilton inverse function theorem) and clearly it is the classifying space for $Diff_\mu(S)$-bundles. This can be seen as in 6.2; the total space is $Emb_\mu(S, \mathbb{R}^\infty)$, the space of all embeddings $e: S \to \mathbb{R}^\infty$ with $vol(e^*\langle\ ,\ \rangle) = \mu$. This gives a principal fibre bundle with structure group $Diff_\mu(S)$. The associated bundle $Emb_\mu(S, \mathbb{R}^\infty)[S,ev] = (Emb_\mu(S, \mathbb{R}^\infty) \times S)/Diff_\mu(S)$ can again be viewed as $\{(N,x)$ in $B_1(S, \mathbb{R}^\infty) \times \mathbb{R}^\infty$ with x in $N\}$ and it is the total space of a bundle $E_\mu(S, \mathbb{R}^\infty)$ over $B_1(S, \mathbb{R}^\infty)$ with fibre type S and a volume form of mass 1 on each fibre, which depends smoothly on the base.

6.5. Bundles with fibre volumes: Let (E,p,M,S) be a fibre bundle and let μ be a section of the real line bundle $\Lambda^n V^*E \to E$, where $n = \dim S$, such that for each x in M the n-form $\mu_x := \mu|E_x$ is a positive volume form of total mass 1. Such a form will be called a <u>fibre volume</u> on E. Such forms exist on any oriented bundle with compact oriented fibre type. We may plug n vertical vector fields X_i from $\Gamma(VE)$ into μ to get a function $\mu(X_1,\dots,X_n)$ on E. But we cannot treat μ as a differential form on E.

LEMMA: Let μ_o be a fixed volume form on S and let μ be a fibre volume on E. Then there is a bundle atlas (U_α,ψ_α) on E such that $(\psi_\alpha|E_x)^*\mu_o = \mu_x$ for all x in U_α.

Proof: Take any bundle atlas (U_α,ψ'_α). By theorem 6.3 there is $\varphi_{\alpha,x}$ in Diff(S) such that $(\varphi_{\alpha,x})^*(\psi'^{-1}_\alpha|\{x\}\times S)^*\mu_x = \mu_o$, and $\varphi_{\alpha,x}$ depends smoothly on x. Then $\psi_\alpha = (\mathrm{Id}\times\varphi_{\alpha,x})\ \psi'_\alpha$ is the desired bundle chart. qed.

6.6. Let Φ be a connection on (E,p,M,S) and let χ: $TM\times_M E \to TE$ be its horizontal lift. Then for X in $\mathfrak{X}(M)$ the flow $Fl^{\chi X}_t$ of its horizontal lift respects fibres, so for the fibre volume μ on E the pullback $(Fl^{\chi X}_t)^*\mu$ makes sense and is again a fibre volume on E. We can now define a kind of Lie derivative by $\tilde{L}_{\chi X}\mu := \frac{d}{dt}|_0(Fl^{\chi X}_t)^*\mu$.

Let j: VE $\to$ TE be the embedding of the vertical bundle and consider $j^*: \Omega^k(E) \to \Gamma(\Lambda^k V^*E)$. Likewise we consider $\Phi^*: \Gamma(\Lambda^k V^*E) \to \Omega^k(E)$. These are algebraic mappings, algebra homomorphisms for the wedge product, and satisfy $j^*\circ\Phi^* = \mathrm{Id}$. $\Phi^*\circ j^*$ is a projection onto the image of Φ^*. With these operators we may write $\tilde{L}_{\chi X}\mu = j^*\ L_{\chi X}\ \Phi^*\ \mu = j^*\ i_{\chi X}\ d\ \Phi^*\ \mu$, where $L_{\chi X}$ is the usual Lie derivative. Thus $\tilde{L}_{\chi X}\mu$ is $C^\infty(M)$-linear in X.

We may also regard $\tilde{L}$ as a linear connection on the vector bundle $\bigsqcup_{x\ \mathrm{in}\ M}\Omega(E_x) \to M$, whose fibre is the nuclear Frechet space $\Omega(S)$.

<u>Definition</u>: The connection Φ is called μ-<u>respecting</u>, if $\tilde{L}_{\chi X}\mu = 0$ for all X in (M).

LEMMA: 1. A connection Φ is μ-respecting if and only if for some (any) bundle atlas (U_α,ψ_α) as in lemma 6.5 the corresponding Christoffel forms Γ^α take values in the Lie algebra $\mathfrak{X}_{\mu_o}(S)$ of divergence free vector fields ξ $(L_\xi\mu_o = 0)$.

2. On E there exist many μ-respecting connections.

3. If Φ is μ-respecting and R is its curvature, then for X_x, Y_x in T_xM the form $i(R(\chi X_x,\chi Y_x))\mu_x$ is closed in $\Omega^{n-1}(E_x)$.

Proof: 1. Put $\Phi^\alpha := (\psi_\alpha^{-1})^*\Phi$, a connection on $U_\alpha\times S$. Its horizontal lift is given by $\chi^\alpha(X) = (X,\Gamma^\alpha(X))$. Then $\tilde{L}_{\chi^\alpha(X)}\mu_o = L_{\Gamma^\alpha(X)}\mu_o$ and this is zero for all X in $\mathfrak{X}(U_\alpha)$ if and only if Γ^α is in $\Omega^1(U_\alpha, \mathfrak{X}_{\mu_o}(S))$.

2. By 1 μ-respecting connections exist locally and since $\tilde{L}_{\chi X}\mu$ is $C^\infty(M)$-linear in X we may glue them via a partition of unity on M.

3. Choose local vector fields X, Y on M extending X_x and Y_x such that $[X,Y] = 0$. Then $[\chi X,\chi Y] = R(\chi X,\chi Y)$, which we denote $R(X,Y)$ for short (see the proof of 3.3). We then have for the embedding $j_x: E_x \to E$:

$$d(i_{R(X,Y)}\mu|E_x) = d\, j_x^*\, i([\chi X,\chi Y])\, \Phi^*\mu = j_x^*\, d\, i([\chi X,\chi Y])\, \Phi^*\mu =$$
$$= j_x^*\, L_{[\chi X,\chi Y]}\Phi^*\mu = j_x^*\, [L_{\chi X},L_{\chi Y}]\Phi^*\mu = 0. \qquad \text{qed.}$$

6.7. So we may consider the de Rham cohomology class $[i(R(X,Y))\mu|E_x]$ in $H^{n-1}(E_x)$ for X, Y in $\mathfrak{X}(M)$. $H^{n-1}(p) := \bigsqcup_{x \text{ in } M} H^{n-1}(E_x) \to M$ is a vector bundle, described by the cocycle of transition functions $U_{\alpha\beta} \to GL(H^{n-1}(S))$, $x \to H^{n-1}(\psi_{\beta\alpha}(x,.))$, which are locally constant. So $H^{n-1}(p) \to M$ admits a unique flat linear connection ∇ respecting the resulting discrete structure group, and the induced covariant exteriour derivative $d_\nabla: \Omega^k(M; H^{n-1}(p)) \to \Omega^{k+1}(M; H^{n-1}(p))$ defines the de Rham cohomology of M with values in the twisted coefficient domain $H^{n-1}(p)$. We may view $[i_R\mu]: X_x,Y_x \to [i(R(X_x,Y_x))\mu_x]$ in $H^{n-1}(E_x)$ as an element of $\Omega^2(M; H^{n-1}(p))$.

LEMMA: 1. If γ in $\Gamma(\Lambda^k V^*E)$ induces a section $[\gamma]: M \to H^k(p)$, then $\nabla_X[\gamma] = [\tilde{L}_{\chi X}\gamma]$.

2. If Φ is a μ-respecting connection on E then $d_\nabla[i_R\mu] = 0$.

Proof: Choose a bundle atlas (U_α,ψ_α) as in lemma 6.5. Then the parallel sections of the vector bundle $H^k(p)$ for the unique flat connection respecting the discrete structure group are exactly those which in each bundle chart $U_\alpha \times H^k(S)$ are given by the (locally) constant maps $U_{\alpha\beta} \to H^k(S)$. Obviously $\nabla_X[\gamma] = [\tilde{L}_{\chi X}\gamma]$ defines a connection, and in $U_\alpha \times \Omega^k(S)$ we have $\tilde{L}_{\chi^\alpha(X)}\gamma = \tilde{L}_{(X,\Gamma^\alpha(X))}\gamma = d\gamma(X) + L_{\Gamma^\alpha(X)}\gamma$, where d is the exteriour derivative on U_α of $\Omega^k(S)$-valued maps. But if $[\gamma]$ is parallel, so constant, then $L_{\Gamma^\alpha(X)}\gamma = d\, i_{\Gamma^\alpha(X)} + i_{\Gamma^\alpha(X)}0$ is exact anyhow, so $d\gamma(X)(x)$ is exact. So 1 follows.

2. Let X_0, X_1, X_2 in $\mathfrak{X}(M)$. Then we have

$$d_\nabla[i_R\mu](X_0,X_1,X_2) = \Sigma_{\text{cyclic}}\big(\nabla_{X_0}([i_R\mu](X_1,X_2)) - [i_R\mu]([X_0,X_1],X_2)\big)$$
$$= \Sigma_{\text{cyclic}}[\, \tilde{L}_{\chi X_0}(i_{R(\chi X_1,\chi X_2)}\mu) - i_{R(\chi[X_0,X_1],\chi X_2)}\mu\,]$$
$$= \Sigma_{\text{cyclic}}[\, i([\chi X_0,R(\chi X_1,\chi X_2)])\mu + i_{R(\chi X_1,\chi X_2)}\tilde{L}_{\chi X_0}\mu - \dots\,]$$
$$= [\Sigma_{\text{cyclic}}\, i([\chi X_0,R(\chi X_1,\chi X_2)] + 0 - R(\chi[X_0,X_1],\chi X_2))\mu]\ ,$$

where we used $[\mathcal{L}_{\chi X}, i_Z] = i_{[\chi X, Z]}$ for a vertical vector field Z.
Now we need the Bianchi identity $[R,\Phi] = 0$. Writing out the global formula for it from the appendix for horizontal vector fields χX_i we get:
$0 = [R,\Phi](\chi X_0, \chi X_1, \chi X_2) = \Sigma_{cyclic}(-\Phi([R(\chi X_0,\chi X_1),\chi X_2]) - R([\chi X_0,\chi X_1],\chi X_2))$
From this it follows that $d_\nabla[i_R\mu] = 0$. qed.

6.8. Definition: So we may define the Kainz-class $k(E)$ of the bundle (E,p,M,S) with fibre volume as the class (for μ-respecting connection)

$$k(E) = [[i_R\mu]_{H^{n-1}(p)}]_{H^2(M;H^{n-1}(p))}.$$

LEMMA: The class $k(E)$ is independent of the choice of the μ-respecting connection Φ on E.

Proof: Let Φ, Φ' both be μ-respecting connections, let χ, χ' be their horizontal liftings. Put $\Psi := \chi' - \chi$: $TM \times_M E \to VE$, which is fibre linear over E, then $\chi_t := \chi + t\Psi = (1-t)\chi + t\chi'$ is a curve of horizontal lifts which preserve μ. In fact $L_{\Psi X}\mu = 0$ (ordinary Lie derivative) for any X in $\mathfrak{X}(M)$. Let Φ_t be the connection corresponding to χ_t and let R_t be its curvature. Then we have $R_t(X,Y)$ $(= R_t(\chi_t X, \chi_t Y)) = \Phi_t[\chi_t X, \chi_t Y] =$

$$= [\chi_t X, \chi_t Y] - \chi_t[X,Y] = R(X,Y) + t[\chi X, \Psi Y] + t[\Psi X, \chi Y] + t^2[\Psi X, \Psi Y] - t\Psi[X,Y].$$

$$\frac{d}{dt} R_t(X,Y) = [\chi X, \Psi Y] + [\Psi X, \chi Y] + 2t[\Psi X, \Psi Y] - \Psi[X,Y].$$

$$\frac{d}{dt} i(R_t(X,Y))\mu = i_{[\chi X,\Psi Y]}\mu - i_{[\chi Y,\Psi X]}\mu + 2t\, i_{[\Psi X,\Psi Y]}\mu - i_{\Psi[X,Y]}\mu.$$

$$= [\mathcal{L}_{\chi X}, i_{\Psi Y}]\mu - [\mathcal{L}_{\chi Y}, i_{\Psi X}]\mu + 2t[L_{\Psi X}, i_{\Psi Y}]\mu - i_{\Psi[X,Y]}\mu$$

$$= \mathcal{L}_{\chi X}\, i_{\Psi Y}\mu - \mathcal{L}_{\chi Y}\, i_{\Psi X}\mu - i_{\Psi[X,Y]}\mu + 2t(0 + d\, i_{\Psi X}\, i_{\Psi Y}\mu).$$

Now we use 6.8.1 to compute the differential of the 1-form $[i_\Psi\mu]$:

$$d_\nabla[i_\Psi\mu](X,Y) = \nabla_X[i_{\Psi Y}\mu] - \nabla_Y[i_{\Psi X}\mu] - [i_{\Psi[X,Y]}\mu]$$

$$= [\, \mathcal{L}_{\chi X}\, i_{\Psi Y}\mu - \mathcal{L}_{\chi Y}\, i_{\Psi X}\mu - i_{\Psi[X,Y]}\mu\,].$$

So we get $\frac{d}{dt}[i_{R_t}\mu] = d_\nabla[i_\Psi\mu]$ in the space $\Omega^2(M; H^{n-1}(p))$. qed.

6.9. There is another interpretation of the Kainz class $k(E)$: Consider the Serre spectral sequence of the fibre bundle $E \to M$. Its E_2-term is just the cohomology of M with twisted coefficients in $H^*(p)$. Then we start with the element in $H^0(M; H^n(p))$ given by the fibre volume μ and consider its second differential in $H^2(M; H^{n-1}(p))$. This is exactly the Kainz class. So it does not depend on the choice of the fibre volume μ, only on the orientation of the bundle it defines. I have not been able to find a pure differential geometric proof of this fact.

7. SELF DUALITY

7.1. Let again (E,p,M,S) be a fibre bundle with compact standard fibre S. Let ω be a fibrewise symplectic form on E, so that ω_x in $\Omega^2(E_x)$ is a symplectic form on the fibre E_x for each x in M and depends smoothly on x. So as in 6.5 we may plug two vertical vector fields X, Y into ω to get a function $\omega(X,Y)$ on E.

If $\dim S = 2n$, let $(\omega^n)_x = \omega_x \wedge \ldots \wedge \omega_x$ be the Liouville volume form of ω_x on E_x. Let us suppose that each fibre E_x has total mass 1 for this fibre volume: multiply ω by a suitable positive smooth function on M.

I do not know whether lemma 6.5 remains true for ω.

7.2. A connection Φ on E is again called ω-respecting, if $L_{\chi X}\omega :=$ $= \frac{d}{dt}|_0 (Fl^{\chi X}_t)^*\omega = 0$ for all X in $\mathfrak{X}(M)$.

LEMMA: If $\dim H^2(S;\mathbb{R}) = 1$ then there are ω-respecting connections on E and they form a convex set in Conn(E).

Proof: Let us first investigate the trivial situation. So let $E = M \times S$, $\omega: M \to Z^2(S)$ is smooth and ω_x is a symplectic form on S for each x in M. Since $\int \omega_x{}^n = 1$ for each x and $\alpha \to \alpha^n$ is a covering map $H^2(S) = \mathbb{R} \to$ $\to H^{2n}(S) = \mathbb{R}$ (outside 0), we see that the cohomology class $[\omega_x]$ in $H^2(S)$ is locally constant, so $\omega: M \to \omega_{x_0} + B^2(S)$ if M is connected. Thus $d^M\omega: TM \to B^2(S)$ is a 1-form on M with values in $B^2(S)$. Let $G: \Omega^k(S) \to \Omega^k(S)$ be the Green operator of Hodge theory for some Riemann-metric on S and define Γ in $\Omega^1(M,\mathfrak{X}(S))$ by $i(\Gamma(X_x))\omega_x = -\, d^*\, G\,(d^M\omega)_x(X_x)$. Since ω_x is non degenerate, Γ is uniquely detemined by this proceedure and we have $L_{\Gamma(X_x)}\,\omega_x = d^S\, i_{\Gamma(X_x)}\,\omega_x + 0 = -\, d\, d^*\, G\,(d^M\omega)_x(X_x) =$

$$= -\,(d^M\omega)_x(X_x), \text{ since it is in } B^2(S).$$

Also $\tilde{L}_{\chi X}\omega = \tilde{L}_{(X,\Gamma(X))}\omega = (d^M\omega)(X) + L_{\Gamma(X)}$, so Γ is the Christoffel form of an ω-respecting connection and $(\tilde{L}_{\chi X}\omega)_x$ depends only on X_x.

So there are local solutions and and we may use partition of unity on M to glue the horizontal lifts to get a global ω-respecting connection. Likewise the last assertion follows. qed.

7.3. Let Φ be an ω-respecting connection on (E,p,M,S). Let $R = R(\Phi)$ be the curvature of Φ. Then $i(R(\chi X_x,\chi Y_x))\omega_x$ is in $\Omega^1(E_x)$ and $i(R(\chi X_x,\chi Y_x))\,\omega_x^n$ is in $\Omega^{2n-1}(E_x)$ and as in 6.7 we may prove that both are closed forms, so $[i(R(\chi X_x,\chi Y_x))\omega_x]$ is in $H^1(E_x)$ and the class of the second form is in $H^{2n-1}(E_x)$

Let us also fix an auxiliary Riemannian metric on the base manifold M and let us consider the associated $*$-operator $*: \Omega^k(M) \to \Omega^{m-k}(M)$, where $m = \dim M$. Let us suppose that the dimension if M is 4, let us consider $[i_R\omega]$ in $\Omega^2(M;\, H^1(p))$ and $*[i_R\omega^n]$ in $\Omega^2(M;\, H^{2n-1}(p))$. Both are closed forms on M (before applying $*$), since the proof of 6.8 applies without change. Let $D_x: H^{2n-1}(E_x) \to H^1(E_x)$ be the Poincaré-duality operator which induces a duality operator $D: H^{2n-1}(p) \to H^1(p)$ for the twisted coefficient domains.

Definition: An ω-respecting connection Φ is called self dual or anti self dual if $D * [i_R\omega^n] = [i_R\omega]$ or $= - [i_R\omega]$ respectively.

7.4. For the self duality notion of 7.3 there is also a Yang-Mills functional, if M is compact: Let Φ be any connection (not necessarily ω-respecting. For X_i in T_xM consider the following 4-form.
$\varphi(\Phi)(X_1,\dots,X_4) = \frac{1}{4}\,\mathrm{Alt} \int_{E_x} i(R(\chi X_1,\chi X_2)\omega_x \wedge_S *(i_R\omega^n)_x(X_3,X_4)$, where Alt is the alternator of the indices of the X_i. The Yang-Mills functional is then $F(\Phi) = \int_M \varphi(\Phi)$.

A typical example of a fibre bundle with compact symplectic fibres is the union of coadjoint orbits of a fixed type of a compact Lie group. Related to this are open subsets of Poisson manifolds. ω-respecting connections exist if all orbits have second Betti number 1. The Yang-Mills functional exists even without this requirement and also the self-duality requirement of 7.3 can be formulated without it.

7.5. Now we scetch another notion of self duality: We consider a fibrewise contact structure on the fibre bundle (E,p,M,S) with compact standard fibre S. So let α_x be a contact structure on E_x for each x in M, depending smoothly on x. In more detail: α_x is in $\Omega^1(E_x)$ and $\alpha_x \wedge d\alpha_x^{\,n}$ is a positive volume form on E_x, where $\dim S = 2n+1$ now. A connection Φ on E is again called α-respecting, if $L_{\chi X}\alpha = 0$ for each vector field on M. If R is the curvature of an α-respecting connection then $i(R(\chi X,\chi Y))\alpha$ is fibrewise locally constant. If S is connected, then it is a closed 2-form on M with real coefficients and defines a cohomology class in $H^2(M; \mathbb{R})$. But it is not clear that α-respecting connections exist.

A. APPENDIX: THE FRÖLICHER NIJENHUIS BRACKET

We include here a short review of those properties of the Frölicher Nijenhuis bracket that we need. References are (Michor, 1987) and (Frölicher-Nijenhuis 1956).

A.1. The space $\Omega(M) = \bigoplus \Omega^k(M)$ of differential forms on a manifold M is a $\mathbb{Z}$-graded commutative algebra. A linear mapping $D: \Omega(M) \to \Omega(M)$ is called of degree k if $D(\Omega^h(M)) \to \Omega^{h+k}(M)$, and it is a graded derivation of degree k if $D(\varphi \wedge \psi) = D\varphi \wedge \psi + (-1)^{hk} \varphi \wedge D\psi$ for φ in $\Omega^h(M)$. The space Der $\Omega(M)$ = $= \bigoplus \mathrm{Der}_k \Omega(M)$ of all graded derivations is a graded Lie algebra with the graded commutator $[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ as bracket.
A derivation D is called algebraic, if $D|\Omega^0(M) = 0$; then the derivation property says that it is of tensorial character, uniquely determinend by $D|\Omega^1(M)$, which can be viewed as a section of $\Lambda^{k+1}T^*M \times TM \to M$. If K denotes this section, we write $D = i(K)$ and we have:

LEMMA: For K in $\Omega^{k+1}(M;TM)$, φ in $\Omega^h(M)$ and X_i in $\mathfrak{X}(M)$ we have

$$i(K)\varphi\,(X_1,\dots,X_{k+h}) = \\ = \frac{1}{(k+1)!\,(h-1)!} \sum \mathrm{sign}\sigma\, \varphi(K(X_{\sigma 1},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots).$$

A.2. The exteriour derivative d is a graded derivation which is not algebraic. In view of the well known equation $L_X = d\, i_X + i_X\, d$ we put $L_K = L(K) := [i(K),d]$ for K in $\Omega^k(M;TM)$ and call it the Lie derivation along K. Then $L(Id_{TM}) = d$.

LEMMA: Any derivation D in $\mathrm{Der}_k\Omega(M)$ can be written uniquely in the form $D = L(K) + i(P)$ for K in $\Omega^k(M;TM)$ and P in $\Omega^{k+1}(M;TM)$. D is algebraic if and only if $K = 0$, and $P = 0$ if and only if $[D,d] = 0$.

A.3. Definition: For K_j in $\Omega^{k_j}(M;TM)$ we clearly have $[[L(K_1),L(K_2)],d] = 0$, so $[L(K_1),L(K_2)] = L([K_1,K_2])$ for a unique $[K_1,K_2]$ in $\Omega^{k_1+k_2}(M;TM)$, which is called the Frölicher Nijenhuis bracket of K_1, K_2. For vector fields it coincides with the Lie bracket and it gives a graded Lie algebra structure to $\Omega(M;TM) = \oplus \Omega^k(M;TM)$.

A.4. THEOREM: 1. For K in $\Omega^k(M;TM)$ and P in $\Omega^{p+1}(M;TM)$ we have

$$[L(K),i(P)] = i([K,P]) - (-1)^{kp}L(i(P)K).$$

2. For K_j in $\Omega^{k_j}(M;TM)$ and φ in $\Omega^q(M)$ we have

$$[\varphi\wedge K_1,K_2] = \varphi\wedge[K_1,K_2] - (-1)^{(q+k_1)k_2}L(K_2)\varphi\wedge K_1 + (-1)^{q+k_1}d\varphi\wedge i(K_1)K_2.$$

3. For X,Y in $\mathfrak{X}(M)$, φ in $\Omega^q(M)$, ψ in $\Omega(M)$ we have

$$[\varphi\otimes X,\psi\otimes Y] = \varphi\wedge\psi\otimes[X,Y] + \varphi\wedge L_X\psi\otimes Y - L_Y\varphi\wedge\psi\otimes X + (-1)^q(d\varphi\wedge i_X\psi\otimes Y + i_Y\varphi\wedge d\psi\otimes X).$$

A.5. THEOREM: For K in $\Omega^k(M;TM)$ and P in $\Omega^p(M;TM)$ we have:

$$\begin{aligned}
&[K,P](X_1,\dots,X_{k+p}) = \\
&= \frac{1}{k!\,p!}\sum \operatorname{sign}\sigma\,[K(X_{\sigma1},\dots,X_{\sigma k}),P(X_{\sigma(k+1)},\dots,X_{\sigma(k+p)})] \\
&+ \frac{1}{k!\,(p-1)!}\sum \operatorname{sign}\sigma\,P([K(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots) \\
&+ \frac{(-1)^{kp}}{(k-1)!\,p!}\sum \operatorname{sign}\sigma\,K([P(X_{\sigma1},\dots,X_{\sigma p}),X_{\sigma(p+1)}],X_{\sigma(p+2)},\dots) \\
&+ \frac{(-1)^{k-1}}{(k-1)!(p-1)!\,2!}\sum \operatorname{sign}\sigma\,P(K([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(k+2)},\dots) \\
&+ \frac{(-1)^{(k-1)p}}{(k-1)!(p-1)!\,2!}\sum \operatorname{sign}\sigma\,K(P([X_{\sigma1},X_{\sigma2}],X_{\sigma3},\dots),X_{\sigma(p+2)},\dots)
\end{aligned}$$

A.5. Let $f:M\to N$ be a smooth mapping between smooth second countable manifolds. Two vector valued differential forms K in $\Omega^k(M;TM)$ and K' in $\Omega^k(N;TN)$ are called f-related, if for all X_i in T_xM we have $K'_{f(x)}(T_xf.X_1,\dots,T_xf.X_k) = T_xf.K_x(X_1,\dots,X_k)$. If this is the case then the pullback operator $f^*: \Omega(N)\to\Omega(M)$ intertwines $i(K)$, $i(K')$ and also intertwines $L(K)$, $L(K')$.

LEMMA: If K_j and K'_j are f-related for $j = 1,2$, then their Frölicher Nijenhuis brackets $[K_1,K_2]$ and $[K'_1,K'_2]$ are also f-related.

A.6. If X is a vector field on M, then $(Fl^X_t)^*: \Omega^k(M;TM)\to\Omega^k(M;TM)$ makes sense locally.

LEMMA: $\frac{d}{dt}(Fl^X_t)^* K = (Fl^X_t)^*[X,K]$ for the Frölicher Nijenhuis bracket.

A.7. Let me finally remark that the Frölicher Nijenhuis bracket is not the only natural bilinear concomitant between vector valued differential forms. See Kolař-Michor, 1987, for the determination of all these concomitants. There are no surprising ones.

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