Contributions to finite operator calculus in several variables

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Abstract: The following is a detailed development of Rota's finite operator calculus to the case of several variables. The main results are derived without recourse to the notion of shift invariance, which is investigated afterwards separately. In the last chapter operators invariant under a linear group action are investigated.

I want to thank J. Cigler for motivation and helpful discussions. This work is mainly based on [6] and Ciglers papers.

§1 Preliminaries

1.1 Multi indices: We consider the space N^n of all n - tupels $\alpha = (\alpha_1,\dots,\alpha_n)$ of non negative integers with the usual product order ($\alpha \leq \beta$ iff $\alpha_i \leq \beta_i$ for all i). Let $0 = (0,\dots,0)$ and $\varepsilon(i) = (0,\dots,1,\dots,0)$, where just the i'th coordinate is 1, and all other are zero. For $\alpha, \beta \in N^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_n}{\beta_n}$ with the usual conventions such as $0! = 1$ and $\binom{\alpha}{\beta} = 0$ if $\beta \leq \alpha$ does not hold. If $\beta \leq \alpha$, then $\binom{\alpha}{\beta} = \alpha!/\beta!(\alpha-\beta)!$. Furthermore we will use $(\alpha)_\beta = \beta!\binom{\alpha}{\beta} = \alpha_1(\alpha_1-1)\dots(\alpha_1-\beta_1+1)\dots\alpha_n\dots(\alpha_n-\alpha_n+1)$. If $x = (x_1,\dots,x_n)$ is an n - dimensional commuting variable we set $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and $(x)_\alpha = \alpha!\binom{x}{\alpha} = (x_1)_{\alpha_1}\dots(x_n)_{\alpha_n}$, where $(x_i)_{\alpha_i} = x_i(x_i-1)\dots(x_i-\alpha_i+1)$ are the one dimensional lower factorials.

1.2 We let $P_n = K[x]$ the polynomial ring in n commuting variables $x = (x_1,\dots,x_n)$ over a field K of characteristic 0. For $\alpha \in N^n$ the expressions x^α, $(x)_\alpha$, $\binom{x}{\alpha}$ denote elements of P_n. Any $f \in P_n$ has a unique representation in the form $f(x) = \sum_\alpha f_\alpha x^\alpha$, where all but finitely many $f_\alpha = 0$.

1.3 Clearly we have the binomial formula $(x + a)^\alpha = \sum_\beta \binom{\alpha}{\beta} a^\beta x^{\alpha-\beta}$ in P_n for all $\alpha \in N^n$ and each $a = (a_1,\dots,a_n)$ in K^n (or independent variables). We will need this in more general form:

Lemma: Let $A = (a_{ij})_{1 \leq i \leq n,\ 1 \leq j \leq m}$ be an $n \times m$ matrix whose entries are in K or independent variables. Then we have for each $\alpha \in N^n$:

$$\Big(\sum_{j=1}^{m} a_j\Big)^\alpha = \sum_{\substack{\beta = (\beta_{ij}) \in N^{nm} \\ |\beta_i| = \alpha_i}} \frac{\alpha!}{\beta!} A^\beta ,$$

where $a_j = (a_{1j},\dots,a_{nj})$, $\beta_i = (\beta_{i1},\dots,\beta_{im})$ N^m , $\beta! = \prod_{i,j} \beta_{ij}!$ and $A^\beta = \prod_{i,j} a_{ij}^{\beta_{ij}}$.

This lemma is just the binomial formula for longer sums for fixed i, multiplied together for all i. The proof is straightforward following these lines.

1.4 By $L(P_n)$ let us denote the algebra of K - linear mappings $P_n \longrightarrow P_n$. An element of $L(P_n)$ is called operator for short. If $Q_1,\dots,Q_n$ are pairwise commuting operators, we call the n - tupel $Q = (Q_1,\dots,Q_n)$ an operation. By Q^α we mean the operator $Q_1^{\alpha_1} \dots Q_n^{\alpha_n}$ for $\alpha \in N^n$.

In the next sections we collect some examples of operators and operations.

1.5 Let $g \in P_n$. Then $f \longmapsto f.g$ is an operator on P_n, called the multiplication operator $M(g)$ induced by g.
$M: P_n \longrightarrow L(P_n)$ is an isomorphism onto a commutative subalgebra of $L(P_n)$.

1.6 For any $1 \leqslant i \leqslant n$ let D_i or $\frac{\partial}{\partial x_i}$ be the (formal) partial differential operator on P_n in the direction x_i:
$\frac{\partial}{\partial x_i}(\sum f_\alpha x^\alpha) = \sum f_\alpha \alpha_i x^{\alpha-\varepsilon(i)}$. Clearly $\frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is an operation which we call $D = (D_1,\dots,D_n)$ if the "basis" $x_1,\dots,x_n$ is fixed.

1.7 Let $a(D) = \sum_\alpha a_\alpha D^\alpha$ be a formal power series in D. This defines an operator $a(D) \in L(P_n)$ by $a(D)f = \sum_\alpha a_\alpha (D^\alpha f)$ for $f \in P_n$. Since the degree of f is finite this is a finite sum.
We have the explicit formula:

$$(a(D)f)(x) = (\sum_\alpha a_\alpha D^\alpha)(\sum_\beta f_\beta x^\beta) = \sum_{\alpha,\beta} a_\alpha f_\beta D^\alpha x^\beta$$
$$= \sum_{\alpha,\beta} a_\alpha f_\beta (\beta)_\alpha x^{\beta-\alpha} = \sum_\mu (\sum_\alpha f_{\mu+\alpha} a_\alpha (\mu+\alpha)_\alpha) x^\mu .$$

This gives an algebra monomorphism onto a commutative subalgebra

$K[[D]] \longrightarrow L(P_n)$.

1.8 For fixed $a = (a_1,\dots,a_n) \in K^n$ we have the shift by a, given by $(E_a f)(x) = f(x+a)$. For any monom x^α, $\alpha \in N^n$, we have

$$E_a x^\alpha = (x+a)^\alpha = \sum_\beta \binom{\alpha}{\beta} a^\beta x^{\alpha-\beta} = \sum_\beta \frac{a^\beta}{\beta!} (\alpha)_\beta x^{\alpha-\beta}$$
$$= \Big(\sum_\beta \frac{a^\beta}{\beta!} D^\beta\Big) x^\alpha = \exp(a_1D_1 + \dots + a_nD_n) x^\alpha = e^{\langle a,D\rangle} x^\alpha ,$$

where $\langle a,D\rangle = \langle (a_1,\dots,a_n),(D_1,\dots,D_n)\rangle = a_1D_1 + \dots + a_nD_n$ is the usual formal inner product.
So in particular we have $E_a \in K[[D]]$.

1.9 Let us interpret for the moment the x_i as the coordinate functionals of the running point $x \in K^n$ with respect to the standard basis $e_1,\dots,e_n$ of K^n. If $a_1,\dots,a_n$ is another basis with coordinate functionals $y_1,\dots,y_n$, then there is an invertible matrix $A = (A_{ij})$ over K such that $a_j = \sum_i A_{ij} e_i$.
If $B = (B_{ij})$ is the inverse matrix, then in turn

$$e_i = \sum_j B_{ji} a_j , \quad x_j = \sum_i A_{ji} y_i \text{ and } y_j = \sum_i B_{ji} x_i .$$

If $f \in P_n$, $f(x) = \sum_\alpha f_\alpha x^\alpha$, let us interpret f as a polynomial mapping on K^n, expressed in the coordinate functions x_i. If we express in the coordinate functions y_i we get

$$f(x) = \sum_\alpha f_\alpha x^\alpha = \sum_\alpha f_\alpha \Big(\sum_j A_{1j} y_j , \dots , \sum_j A_{nj} y_j\Big)$$
$$= \sum_\alpha f_\alpha \sum_{\substack{\beta = (\beta_{ij}) \in N^{nn} \\ |\beta_i| = \alpha_i}} \frac{\alpha!}{\beta!} \prod_{i,j} (A_{ij} y_j)^{\beta_{ij}} \qquad \text{by 1.3}$$
$$= \sum_\alpha f_\alpha \sum_{\substack{\beta = (\beta_{ij}) \\ |\beta_i| = \alpha_i}} \frac{\alpha!}{\beta!} y^{\Sigma \beta_i} A^\beta$$

with the same conventions as in lemma 1.3.

1.10 Consider a linear mapping $\bar{A}: K^n \longrightarrow K^n$ whose matrix with respect to the standard basis is $A = (A_{ij})$. If $x = (x_1,\dots,x_n)$

then $\bar{A}(x)$ has the coordinates $(\sum_j A_{1j} x_j , \ldots , \sum_j A_{nj} x_j)$.
If $f \in P_n$, $f(x) = \sum_\alpha f_\alpha x^\alpha$, then $f \circ \bar{A} \in P_n$ and we have

$$(f \circ \bar{A})(x) = \sum_\alpha f_\alpha \, ((\sum_j A_{ij} x_j)_i)^\alpha$$

$$= \sum_\alpha f_\alpha \sum_{\substack{\beta = (\beta_{ij}) \in N^{nn} \\ |\beta_i| = \alpha_i}} \frac{\alpha!}{\beta!} A^\beta x^{\sum_i \beta_i}$$

as the computation in 1.9 shows. $\bar{A}^*: P_n \longrightarrow P_n$, given by $f \longmapsto f \circ \bar{A}$, is an operator, even a ring homomorphism.

1.11 Now let $P: K^n \longrightarrow K^n$ be a polynomial mapping, i.e. $P = (p_1, \ldots, p_n)$, $p_i = \sum_\alpha p_{i\alpha} x^\alpha \in P_n$. For $f \in P_n$ we get again a polynomial $f \circ P$; $f \longmapsto f \circ P$ is an operator $P^* \in L(P_n)$, even a ring homomorphism. Let $f(x) = \sum_\beta f_\beta x^\beta$, then we have

$$(f \circ P)(x) = \sum_\beta f_\beta P(x)^\beta = \sum_\beta f_\beta \left(\sum_\alpha p_{1\alpha} x^\alpha, \ldots, \sum_\alpha p_{n\alpha} x^\alpha \right)^\beta .$$

Here α runs only formally through all of N^n , above some bound everything is zero. So we may apply lemma 1.3 and the above equals

$$\sum_\beta f_\beta \sum_{\substack{\mu = (\mu_{i\alpha}) \in N^{n \times N^n} \\ |\mu_i| = \sum_\alpha \mu_{i\alpha} = \beta_i}} \frac{\beta!}{\mu!} p^\mu x^{\sum_{i,\alpha} \alpha \mu_{i\alpha}}$$

$$= \sum_\nu \Big(\sum_\beta f_\beta \sum_{\substack{\mu = (\mu_{i\alpha}) \in N^{n \times N^n} \\ \sum_\alpha \mu_{i\alpha} = \beta_i \\ \sum_{j,\alpha} \alpha_i \mu_{j\alpha} = \nu_i}} \frac{\beta!}{\mu!} p^\mu \Big) x^\nu .$$

§2 Basic sequences and delta operators

2.1 Definition: An admissible sequence $p = (p_\alpha)_{\alpha \in N^n}$ is a sequence of polynomials $p_\alpha \in P_n$ such that p_α is of degree $|\alpha|$ and for any $m \in N$ the set $\{p_\alpha : |\alpha| \leq m\}$ is K - linearly independent in P_n.
It is clear that then $\{p_\alpha : |\alpha| \leq m\}$ is a K - basis of the space of all polynomials of degree $\leq$ m by an dimension argument.
So $\{p_\alpha : \alpha \in N^n\}$ is a K - basis of P_n and any $f \in P_n$ has a unique representation of the form $f = \sum_\alpha a_\alpha p_\alpha$.
The notion of admissible sequence is the generalisation of the so called Sheffer sequences in [6], leaving away the condition of shift invariance.

2.2 Let $p = (p_\alpha)$ be an admissible sequence, then for $1 \leq i \leq n$ we have an operator $T_i = T(p)_i \in L(P_n)$ defined by
$T_i (\sum_\alpha a_\alpha p_\alpha) = \sum_\alpha a_\alpha p_{\alpha+\varepsilon(i)}$.
Clearly $T = T(p) = (T_1,\dots,T_n)$ is an operation: we call it the admissible operation for the sequence $p = (p_\alpha)$. We have the following formulas: $T(p)^\beta (\sum_\alpha a_\alpha p_\alpha) = \sum_\alpha a_\alpha p_{\alpha+\beta}$,
$p_\alpha = T(p)^\alpha (p_0)$.
Examples: $\underline{x} = (x^\alpha)$ is an admissible sequence, $T(\underline{x})_i = M(x_i)$.
The following is a construction principle: for $1 \leq i \leq n$ let $(p_{im}(t))_{m \in N}$ be a sequence of polynomials in one variable t such that p_{im} is exactly of degree m for each i and $p_{io} \neq 0$.
Then $p_\alpha(x_1,\dots,x_n) = p_{1\alpha_1}(x_1)\, p_{2\alpha_2}(x_2) \dots p_{n\alpha_n}(x_n)$ is an admissible sequence.

2.3 Remark: If $p = (p_\alpha)$ is an admissible sequence then for each $m \in N$ the homogeneous parts of degree m of p_α for $|\alpha| = m$ constitute a basis of the space of all homogeneous polynomials of degree m. This space has dimension $\binom{m+n-1}{m}$.

2.4 Proposition: Let $p = (p_\alpha)$ be an admissible sequence and let $T = (T_1,\dots,T_n)$ be the admissible operation for p. Then there exists a unique operation $P = (P_1,\dots,P_n)$ such that $P_i(p_o) = 0$ and $P_i\, T_j - T_j\, P_i = \delta_{ij}\, \mathrm{Id}$, for $1 \leq i,j \leq n$. If f is of degree m in P_n, then $P_i(f)$ is of degree m-1.

Proof: The idea is from Cigler [1].

If there is P_i with $P_i\, T_i - T_i\, P_i = \mathrm{Id}$ then for $m \geq 1$ we have $P_i\, T_i^m = T_i^m\, P_i + m\, T_i^{m-1}$. This is seen by induction. Now we use that all the P_i's commute and $P_i(p_o) = 0$: for $\alpha \in N^n$

$$P_i(p_\alpha) = P_i(T^\alpha p_o) = P_i\, T_1^{\alpha_1} \dots T_n^{\alpha_n}\, p_o$$
$$= T_1^{\alpha_1} \dots T_{i-1}^{\alpha_{i-1}}\, (P_i\, T_i^{\alpha_i})\, T_{i+1}^{\alpha_{i+1}} \dots T_n^{\alpha_n}\, p_o$$
$$= T^\alpha P_i\, p_o + \alpha_i\, T^{\alpha-\varepsilon(i)}\, p_o \; = \alpha_i\, p_{\alpha-\varepsilon(i)}.$$

So we got a formula for P_i;this proves uniqueness:

(1) $P_i(p_\alpha) = \alpha_i\, p_{\alpha-\varepsilon(i)}$.

Now we take this formula for definition, then each $P_i \in L(P_n)$ and a straightforward computation shows that $P = (P_1,\dots,P_n)$ is an operation and satisfies $P_i\, T_j - T_j\, P_i = \delta_{ij}\, \mathrm{Id}$. The degree condition is clear from the formula. qed.

2.5 If $p = \underline{x} = (x^\alpha)$, then $P = D = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$. This motivates the following definition:

Definition: If $p = (p_\alpha)$ is an admissible sequence then the operation $P = (P_1,\dots,P_n)$ uniquely given by 2.4 is called the delta operation for the sequence p. We have the following formula:

(1) $P^\alpha\, p_\beta \;=\; (\beta)_\alpha\, p_{\beta-\alpha}$.

The name delta operation shold indicate that P acts on p as the differential operation $D = \frac{\partial}{\partial x}$ acts on $\underline{x}$.

If $a(P) = \sum_\alpha a_\alpha P^\alpha$ is a formal power series in $P = (P_1,\dots,P_n)$ then this gives an operator by $a(P)f = \sum_\alpha a_\alpha (P^\alpha f)$, since $P^\alpha f$ is 0 if α is big enough. This defines an algebra monomorphism $K[[P]] \longrightarrow L(P_n)$ onto a commutative subalgebra of $L(P_n)$. Exactly as in 1.7 we have the explicit formula

(2) $$a(P)f = (\sum_\alpha a_\alpha P^\alpha)(\sum_\beta f_\beta p_\beta) = \sum_{\alpha,\beta} a_\alpha f_\beta (\beta)_\alpha p_{\beta-\alpha} = \sum_\mu (\sum_\alpha f_{\mu+\alpha} a_\alpha (\mu+\alpha)_\alpha) p_\alpha .$$

2.6 If $p = (p_\alpha)$ is an admissible sequence we may define an inner product in P_n, the p - <u>inner product</u>, by defining it on the basis p_α : $\langle p_\alpha , p_\beta \rangle_p = \alpha!\, \delta_{\alpha\beta}$. By linear extension: $\langle \sum_\alpha f_\alpha p_\alpha , \sum_\beta g_\beta p_\beta \rangle_p = \sum_\alpha f_\alpha g_\alpha \alpha!$.
For any $f \in P_n$ we get $f = \sum_\alpha \langle f , p_\alpha \rangle_p \frac{1}{\alpha!} p_\alpha$.

<u>Lemma</u>: <u>If</u> $p = (p_\alpha)$ <u>is an admissible sequence</u>, $T = (T_1,\dots,T_n)$ <u>is the admissible operation for</u> p <u>and</u> $P = (P_1,\dots,P_n)$ <u>is the delta operation for</u> p, <u>then</u> P <u>is the adjoint to</u> T <u>via the</u> p - <u>inner product</u> $\langle \, , \, \rangle_p$, <u>i.e.</u> $\langle T_i f, g \rangle_p = \langle f, P_i g \rangle_p$ <u>for all</u> $f,g \in P_n$, <u>or</u>, <u>equivalently</u>, $\langle T^\alpha f, g \rangle_p = \langle f, P^\alpha g \rangle_p$ <u>for all</u> $\alpha \in N^n$.

The proof is a straightforward computation which we omit.

2.7 Let us denote by $A_0 : P_n \longrightarrow K$ the linear functional which associates the constant term $f(0)$ to $f \in P_n$.

<u>Lemma</u>: <u>Let</u> $p = (p_\alpha)$ <u>be an admissible sequence and let</u> $P = (P_1,\dots,P_n)$ <u>be the delta operation for</u> p. <u>Then the matrix</u> $(A_0(P^\alpha p_\beta))_{0 \leq |\alpha| \leq m,\ 0 \leq |\beta| \leq m} \in GL(\binom{m+n}{m}, K)$ <u>for all</u> $m \in N$. <u>Here</u> $\binom{m+n}{m}$ <u>is the dimension of the space of all polynomials of degree</u> $\leq m$.

Proof: The dimension formula can be seen by induction. We want to compute the determinant of the matrix considered and start with the following remarks:

If $\alpha \nleq \beta$ then $P^\alpha p_\beta = (\beta)_\alpha \, p_{\beta-\alpha} = 0$.

If $\alpha = \beta$ then $P^\alpha p_\beta = \alpha! \, p_o$.

Let π be a non trivial permutation of $\{\alpha : 0 \leq |\alpha| \leq m\}$. Choose α, $0 \leq |\alpha| \leq m$, such that $\alpha \neq \pi(\alpha)$ and $|\alpha|$ is minimal for that. Then either $\alpha \nleq \pi(\alpha)$ and $P^\alpha p_{\pi(\alpha)} = 0$ or $\alpha < \pi(\alpha)$, but then $\pi^{-1}(\alpha) \nleq \alpha$ (otherwise $\pi^{-1}(\alpha) < \alpha$ and so $|\pi^{-1}(\alpha)| < |\alpha|$ which contradicts the minmality of $|\alpha|$), so $P^{\pi^{-1}(\alpha)} p_\alpha = P^{\pi^{-1}(\alpha)} p_{\pi(\pi^{-1}(\alpha))} = 0$. Thus $\prod_{|\alpha| \leq m} A_o(P^\alpha p_{\pi(\alpha)}) = 0$ if $\pi \neq \mathrm{Id}$, so the determinant of the matrix is just $\prod_{|\alpha| \leq m} A_o(P^\alpha p_\alpha) = \prod_{|\alpha| \leq m} \alpha! \, p_o \neq 0$. qed.

2.8 Lemma: If $p = (p_\alpha)$ is an admissible sequence and $P = (P_1, \ldots, P_n)$ is the delta operation for p, then for any admissible sequence $q = (q_\alpha)$ and for any $m \in \mathbb{N}$ the matrix
$$(A_o(P^\alpha q_\beta))_{0 \leq |\alpha| \leq m,\ 0 \leq |\beta| \leq m} \in GL(\tbinom{m+n}{m}, K)$$

Proof: $\{p_\alpha : |\alpha| \leq m\}$ and $\{q_\alpha : |\alpha| \leq m\}$ are bases of the space of all polynomials of degree $\leq m$. Thus there is an invertible $\binom{m+n}{m} \times \binom{m+n}{m}$ - matrix $A = (a_{\alpha\beta})_{|\alpha| \leq m, |\beta| \leq m}$ over K such that $q_\beta = \sum_\alpha a_{\alpha\beta} \, p_\alpha$. But then
$$(A_o(P^\alpha q_\beta))_{|\alpha| \leq m, |\beta| \leq m} = (\sum_\mu A_o(P^\alpha p_\mu) \, a_{\mu\beta})_{|\alpha| \leq m, |\beta| \leq m}$$
$= (A_o(P^\alpha p_\mu))_{\alpha,\mu} \cdot (a_{\mu\beta})_{\mu,\beta}$ is the product of two invertible matrices. qed.

2.9 For $m \in \mathbb{N}$ let $\mathfrak{M}(m)$ be the space of all $\binom{m+n}{m} \times \binom{m+n}{m}$ - matrices $A = (a_{\alpha\beta})_{|\alpha| \leq m, |\beta| \leq m}$ over K such that $a_{\alpha\beta} = 0$ if $|\alpha| > |\beta|$.

Lemma: $\mathfrak{M}(m)$ is a subalgebra of the algebra of all $\binom{m+n}{m}\times\binom{m+n}{m}$ - matrices and $\mathfrak{M}(m) \cap GL(\binom{m+n}{m},K)$ is a subgroup of $GL(\binom{m+n}{m},K)$ (i.e. if $A \in \mathfrak{M}(m)$ and A is invertible then $A^{-1} \in \mathfrak{M}(m)$), which we denote by $\mathfrak{G}(m)$.

Proof: In a suitable order of $\{\alpha : |\alpha| \leq m\}$ $\mathfrak{M}(m)$ appears as an algebra of "staircased upper triangular matrices".
$\mathfrak{M}(m)$ is clearly a linear space, we have to show that it is closed under multiplication: let $A,B \in \mathfrak{M}(m)$, $A = (a_{\alpha\beta})$, $B = (b_{\alpha\beta})$. Then $A.B = (\sum_{\gamma} a_{\alpha\gamma} b_{\gamma\beta})_{\alpha,\beta}$. If $|\alpha| > |\beta|$ then there is no $\gamma \in N^n$ with $|\alpha| \leq |\gamma|$ and $|\gamma| \leq |\beta|$, i.e. no γ such that both $a_{\alpha\gamma} \neq 0$ and $b_{\gamma\beta} \neq 0$. Thus $\sum_{\gamma} a_{\alpha\gamma} b_{\gamma\beta} = 0$ and $A.B \in \mathfrak{M}(m)$.
Now let $A = (a_{\alpha\beta}) \in \mathfrak{M}(m)$ be invertible. For α,β let $A(\alpha,\beta)$ be the $(\binom{m+n}{m}-1)$ $(\binom{m+n}{m}-1)$ - matrix obtained from A by deleting the α - th row and the β- column. If $A^{-1} = (c_{\alpha\beta})$ then $c_{\alpha\beta} = (\pm 1) \det A(\beta,\alpha)/ \det A$. We have

$$\det A(\beta,\alpha) = \sum_{\substack{\pi \in \mathrm{Perm}\{\gamma : |\gamma| \leq m\} \\ \pi(\beta) = \alpha}} (\pm 1) \prod_{\substack{|\delta| \leq m \\ \delta \neq \beta}} a_{\delta,\pi(\delta)} .$$

If $|\alpha| > |\beta|$ and π is such a permutation with $\pi(\beta) = \alpha$ then there some δ with $m \geq |\delta| > |\beta|$ and $|\pi(\delta)| \leq |\beta|$. But then $|\delta| > |\beta| \geq |\pi(\delta)|$, so $a_{\delta,\pi(\delta)} = 0$, so $\det A(\beta,\alpha) = 0$ and $A^{-1} \in \mathfrak{M}(m)$. qed.

2.10 We consider now the set $\mathfrak{M}$ consisting of all (infinite) matrices $A = (a_{\alpha\beta})_{\alpha,\beta \in N^n}$ such that $a_{\alpha\beta} = 0$ if $|\alpha| > |\beta|$.
We define multiplication in $\mathfrak{M}$ by $A.B = (\sum_{\gamma} a_{\alpha\gamma} b_{\gamma\beta})_{\alpha,\beta}$.
It is easily seen that each sum is actually a finite one and that for each $m \in N$ we have $(A.B)_m = A_m . B_m$ if we denote by A_m the $\binom{m+n}{m}\times\binom{m+n}{m}$ - matrix $(a_{\alpha\beta})_{|\alpha| \leq m, |\beta| \leq m}$ in $\mathfrak{M}(m)$.
The method of proof of 2.9 shows that if $A \in \mathfrak{M}$ is such that $A_m \in \mathfrak{G}(m)$ for each m there is a matrix B (with $B_m = A_m^{-1}$) in $\mathfrak{M}$ with $B.A = A.B = Id$. We define $\mathfrak{G}$ to be the subset of all these

matrices. $\mathcal{G}$ is a topological group, even metrizable. In fact $\mathfrak{M}$ is the inverse limit of all the $\mathfrak{M}(m)$'s over the projection maps $\mathfrak{M}(m) \longrightarrow \mathfrak{M}(m')$ $(m \geqslant m')$ given by deleting all entries $a_{\alpha\beta}$ with $|\alpha| > m'$ or $|\beta| > m'$. Likewise $\mathcal{G}$ is the inverse limit of all the groups $\mathcal{G}(m)$.

Now we have all the results and concepts necessary to give a reasonable definition of an abstract delta operation.

2.11 Definition: A delta operation on P_n is an operation $R = (R_1, \dots, R_n)$ satisfying the following properties:

1. $R_i(c) = 0$ for each constant $c \in K$ and all i.
2. If $f \in P_n$ has degree m , then $R_i(f)$ has degree $\leqslant m-1$.
3. For some admissible sequence $q = (q_\alpha)$ $(A_0(R^\alpha q_\beta))_{\alpha,\beta} \in \mathcal{G}$.

Remark: In view of 2.10 condition 3 means that $(A_0(R^\alpha q_\beta))_{|\alpha| \leqslant m, |\beta| \leqslant m} \in \mathcal{G}(m)$ for each m. The method of proof of 2.8 shows that if 3 holds for one admissible sequence then it holds for all.

2.12 Theorem: Let $R = (R_1, \dots, R_n)$ be a delta operation on P_n. For any sequence of constants $(c_\alpha)_{\alpha \in N^n}$ with $c_0 \neq 0$ there is a unique admissible sequence $p = (p_\alpha)$ with $A_0\, p_\alpha = c_\alpha$ such that R is just the delta operation for p (i.e. $R^\alpha p_\beta = (\beta)_\alpha\, p_{\beta-\alpha}$).

Proof: If there is such an admissible sequence p then there exists an infinite matrix $A = (a_{\alpha\beta})_{\alpha,\beta \in N^n}$ such that the following conditions (1) - (4) are fulfilled:

(1) $A = (a_{\alpha\beta}) \in \mathcal{G}$.

(2) $p_\alpha(x) = \sum_\mu a_{\mu\alpha} x^\mu$.

By (2) alone A is uniquely determined and an element of $\mathcal{G}$ since p is an admissible sequence. (1) and (2) are equivalent to the fact that p is an admissible sequence.

(3) $a_{o\alpha} = c_\alpha$ for all α .

This is just the initial condition $A_o\, p_\alpha = c_\alpha$.

Now for any $\lambda \in N^n$ let us consider the linear functional A_λ: $P_n \longrightarrow K$, given by $A_\lambda(\sum_\alpha f_\alpha x^\alpha) = f_\lambda$, i.e. the λ- th coordinate functional of the basis $\underline{x} = (x^\lambda)$ of P_n.

Our main concern is $R^\alpha p_\beta = (\beta)_\alpha\, p_{\beta-\alpha}$, i. e. condition

(4) $R^\alpha(\sum_\nu a_{\nu\beta}\, x^\nu) = (\beta)_\alpha \sum_\mu a_{\mu,\beta-\alpha}\, x^\mu$, or

(4') $\sum_\nu a_{\nu\beta}\, R^\alpha x^\nu = \sum_\mu (\beta)_\alpha\, a_{\mu,\beta-\alpha}\, x^\mu$, or

(4'') $\sum_\nu a_{\nu\beta}\, A_\lambda(R^\alpha x^\nu) = (\beta)_\alpha\, a_{\lambda,\beta-\alpha}$ for all α,β,λ .

We need a

<u>Sublemma</u>: $\sum_\lambda (A_o(R^\mu x^\lambda))(A_\lambda(R^\alpha x^\nu)) = A_o(R^{\alpha+\mu} x^\nu)$ for all α,ν,μ.

Proof of the sublemma: $\sum_\lambda (A_o(R^\mu x^\lambda))(A_\lambda(R^\alpha x^\nu))$

$= A_o R^\mu (\sum_\lambda (A_\lambda R^\alpha x^\nu)\, x^\lambda) = A_o R^\mu R^\alpha x^\nu = A_o R^{\alpha+\mu} x^\nu$.

Now we show that the infinite system of equations (3),(4) has a unique solution $A = (a_{\alpha\beta})$ in $\mathcal{Y}$. We can then define the admissible sequence p by (2) and the theorem follows.

For that we look at the reduced system:

(3) $a_{o\alpha} = c_\alpha$

(4'', $\lambda = 0$) $\sum_\nu a_{\nu\beta}\, A_o(R^\alpha x^\nu) = (\beta)_\alpha\, a_{o,\beta-\alpha}$.

which is equivalent to

(5) $\sum_\nu A_o(R^\alpha x^\nu)\, a_{\nu\beta} = (\beta)_\alpha\, c_{\beta-\alpha}$ for all α,β .

Since R is a delta operation, by 2.11.3 (and the following remark) $(A_o(R^\alpha x^\nu))_{\alpha,\nu} \in \mathcal{Y}$. Also $((\beta)_\alpha\, c_{\beta-\alpha})_{\alpha,\beta} \in \mathcal{Y}$, since each entry with $\alpha \nleq \beta$ is zero, so for each m the projection into $\mathcal{M}(m)$ is better than of upper triangular form and the determinant is just the product of the diagonal elements which are all $\neq 0$ (cf. the proof of 2.7 where we had the same situation). So (5) is just an equation in $\mathcal{Y}$:

$(A_o(R^\alpha x^\nu))_{\alpha,\nu} \cdot A = ((\beta)_\alpha\, c_{\beta-\alpha})$, which clearly has a unique solution A in the group $\mathcal{G}$. This A fulfills (1) and (3), (4'', $\lambda = 0$) (these two are equivalent to (5)). It remains to show that (4'') holds for all λ.

It is easily seen that $(A_\lambda(R^\alpha x^\mu))_{\lambda,\alpha,\mu}$ is an element of $\mathfrak{M}$ if one of its indices is fixed.

Thus $(\sum_\mu A_\lambda(R^\alpha x^\mu) \, a_{\mu\beta})_{\lambda,\alpha} \in \mathfrak{M}$, furthermore $(A_0(R^\mu x^\lambda))_{\mu,\lambda} \in \mathcal{G}$ and we have

$$
\begin{aligned}
&(A_0(R^\mu x^\lambda))_{\mu,\lambda} \cdot (\sum_\mu a_{\mu\beta} A_\lambda(R^\alpha x^\mu))_{\lambda,\alpha} \\
&= \sum_\mu a_{\mu\beta} (\sum_\lambda A_0(R^\mu x^\lambda) A_\lambda(R^\alpha x^\mu)) \\
&= \sum_\mu a_{\mu\beta} A_0(R^{\mu+\alpha} x^\mu) && \text{by the sublemma} \\
&= (\beta)_{\mu+\alpha} c_{\beta-\mu-\alpha} && \text{by (5)} \\
&= (\beta)_\alpha (\beta-\alpha)_\mu c_{(\beta-\alpha)-\mu} \\
&= (\beta)_\alpha \sum_\lambda a_{\lambda,\beta-\alpha} A_0(R^\mu x^\lambda) && \text{by (5) again} \\
&= (A_0(R^\mu x^\lambda))_{\mu,\lambda} \cdot ((\beta)_\alpha a_{\lambda,\beta-\alpha})_{\lambda,\alpha} .
\end{aligned}
$$

Putting away the invertible matrix $(A_0(R^\mu x^\lambda))$, the result (4'') follows. qed.

2.13 <u>Definition</u>: If $R = (R_1,\dots,R_n)$ is a delta operation, then we call the unique admissible sequence $r = (r_\alpha)$ with $r_0(0) = 1$ and $r_\alpha(0) = 0$ for $\alpha \neq 0$ and $R^\alpha r_\beta = (\beta)_\alpha r_{\beta-\alpha}$ the <u>basic sequence</u> for R.

For a delta operation R we have an algebra monomorphism $K[[R]] \longrightarrow L(P_n)$ onto a commutative subalgebra, given by $(\sum_\alpha a_\alpha R^\alpha) \longmapsto (f \longmapsto \sum_\alpha a_\alpha (R^\alpha f))$. Compare 2.5; formula 2.5.2 is here valid too.

We also note the following

<u>Corollary</u>: If R is a delta operation, then the following strenghened version of 2.11.2 is valid: if $f \in P_n$ is of degree m then $R_i f$ is of degree m-1.

2.14 Lemma: Let $p = (p_\alpha)$ be a basic sequence. For any other basic sequence $q = (q_\alpha)$ there is a unique matrix $a = (a_{\alpha\beta})$ such that:

1. $a \in \mathcal{G}$.
2. $a_{0\alpha} = \delta_{0\alpha}$ for all α.
3. $q_\alpha = \sum_\beta a_{\beta\alpha} p_\beta$, $q = p.a$ for short.

Proof: $\{p_\alpha\}$ and $\{q_\alpha\}$ are both bases of P_n respecting the filtration by degree, so there is an element $a \in \mathcal{G}$ with $q = p.a$. Condition 2 just expresses the fact that $p_\alpha(0) = \delta_{0\alpha}$, $q_\alpha(0) = \delta_{0\alpha}$

qed.

2.15 Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ consisting of all elements $a = (a_{\alpha\beta})$ with $a_{0\alpha} = \delta_{0\alpha}$, then $\mathcal{G}_0$ acts freely and transitively on the set of all basic sequences. Likewise the subgroup $\mathcal{G}_1$ of $\mathcal{G}$ consisting of all elements $a = (a_{\alpha\beta})$ with $a_{00} \neq 0$ acts freely and transitively on the set of all admissible sequences. The unique element a of 2.14 could be called the matrix of connection constants from the basic sequence p to the basic sequence q.

2.16 Corollary: Let R and Q be delta operations with basic sequences r and q respectively. Then there is a unique matrix $a = (a_{\alpha\beta})$ such that:

1. $a \in \mathcal{G}_0$
2. $q = r.a$
3. $\sum_\gamma a_{\gamma\beta} Q^\alpha r_\gamma = \sum_\mu (\beta)_\alpha \, a_{\mu,\beta-\alpha} r_\mu$ for all α, β.
4. $\sum_\gamma a_{\gamma\beta} A_0(Q^\alpha r_\gamma) = \delta_{\alpha\beta} \, \alpha!$ for all α, β.

Remark: 1. and 2. are just a reformulation of 2.15. The whole statement is theorem 2.12 recasted for the fixed initial sequence $c_\alpha = \delta_{0\alpha}$ and with $\underline{x}$ replaced by r and R replaced by Q.

3 and 4 are restatements of 2.12.4' and 2.12.5 in this new situation. The corollary can be proved by going through the proof of 2.12 again with the obvious changes (A_λ should be replaced by $A_\lambda^{(r)}$, the coordinate functional for the basis (r_λ) in the sublemma).

2.17 Assume the data from 2.16. Let us denote $J = (\alpha!\,\delta_{\alpha\beta})_{\alpha,\beta} \in \mathcal{G}_0$ then 2.16.4 reads as follows:

1. $(A_0(Q^\alpha r_\beta))_{\alpha,\beta} \cdot a = J$,

so we have

2. $q = r.a = r \cdot (A_0(Q^\alpha r_\beta))_{\alpha,\beta}^{-1} \cdot J$

and by symmetry

3. $r = q \cdot (A_0(R^\alpha q_\beta))_{\alpha,\beta}^{-1} \cdot J$

but we have also by 1

4. $r = q \cdot a^{-1} = q \cdot J^{-1} \cdot (A_0(Q^\alpha r_\beta))_{\alpha,\beta}$.

From 3 and 4 we get

5. $(A_0(Q^\alpha r_\beta))_{\alpha,\beta} = J \cdot (A_0(R^\alpha q_\beta))_{\alpha,\beta}^{-1} \cdot J$.

As an application we put formulas back into 1:

6. $a = J^{-1} \cdot (A_0(R^\alpha q_\beta))_{\alpha,\beta}$, i.e. $a_{\alpha\beta} = \frac{1}{\alpha!} A_0(R^\alpha q_\beta)$.

2.18 Formula 2.17.6 is not very deep, we may derive it directly using

<u>Proposition</u> (Taylor formula): <u>Let</u> R <u>be a delta operation with basic sequence</u> r. <u>Then for any</u> $f \in P_n$ <u>we have</u>

$$f = \sum_\alpha (A_0\, R^\alpha f) \frac{1}{\alpha!} r_\alpha \quad .$$

<u>Proof</u>: We have $A_0(R^\alpha r_\beta) = \alpha!\,\delta_{\alpha\beta}$, so we get $r_\beta = \sum_\alpha \frac{1}{\alpha!}(A_0\, R^\alpha r_\beta)\, r_\alpha$. Since $\sum_\alpha \frac{r_\alpha}{\alpha!} A_0\, R^\alpha$ is an operator and (r_α) is a basis we have $\sum_\alpha \frac{r_\alpha}{\alpha!} A_0\, R^\alpha = \mathrm{Id}$. qed.

Now 2.17.6 is clear: $\sum_\alpha a_{\alpha\beta}\, r_\alpha = q_\beta = \sum_\alpha \frac{r_\alpha}{\alpha!} A_0(R^\alpha q_\beta)$; now use that (r_α) is a basis.

2.19 Lemma: Let $t = (t_1, \ldots, t_n)$ be a n - dimensional commutative variable, $a_i(t) = \sum_\alpha a_{i\alpha} t^\alpha \in K[[t]]$ for $1 \leq i \leq n$ and $b(t) = \sum_\beta b_\beta t^\beta \in K[[t]]$. Suppose that $a_{io} = 0$ for all i. Then we have for $a(t) = (a_1(t), \ldots, a_n(t))$:

$$b(a(t)) = \sum_\beta b_\beta (a(t))^\beta = \sum_\gamma \Big(\sum_\beta b_\beta \sum_{\substack{\lambda = (\lambda_{i\alpha}) \in N^{n \times N^n} \\ \sum_\alpha \lambda_{i\alpha} = \beta_i \\ \sum_{i,\alpha} \lambda_{i\alpha} \cdot \alpha = \gamma}} \frac{\beta!}{\lambda!} A^\lambda \Big) t^\gamma ,$$

where $A = (a_{i\alpha})$, $A^\lambda = \prod_{i,\alpha} a_{i\alpha}^{\lambda_{i\alpha}}$ and $\lambda! = \prod \lambda_{i\alpha}!$.

Proof: See 1.11: if we truncate all $a_i(t)$ and $b(t)$ at a certain degree m, we compose just polynomial mappings and can apply the formula of 1.11 (which we derived using 1.3). This gives all the monomials of $b(a(t))$ up to degree m. Since m is arbitrary the formula is valid. An alternative proof can be given using lemma 1.3 for formal power series (instead of finite sums), where it is valid too, and straightforward computation (as in 1.11). qed.

2.20 Theorem: Let $R = (R_1, \ldots, R_n)$ be a delta operation and $Q_1, \ldots, Q_n \in K[[R]]$ with representations $Q_i = \sum_\alpha a_{i\alpha} R^\alpha = a_i(R)$. The operation $Q = (Q_1, \ldots, Q_n)$ is a delta operation if and only if $a_{io} = 0$ for all i and $(a_{i,\varepsilon(j)})_{i,j} \in GL(n,K)$. In this case we have $K[[R]] = K[[Q]]$.

Proof: Let us first suppose that Q is a delta operation. Then by 2.11.1 $0 = Q_i(1) = \sum_\alpha a_{i\alpha} R^\alpha(1) = a_{io}$.

Now let $p = (p_\alpha)$ be the basic sequence for R. Then by 2.11.3 $(A_o(Q^\alpha p_\beta))_{\alpha,\beta} \in \mathcal{G}$, so the following matrix is invertible:

$$(A_o(Q^\alpha p_\beta))_{|\alpha| \leq 1, |\beta| \leq 1} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & (a_{i,\varepsilon(j)})_{i,j} \end{array} \right)$$

and the condition is satisfied.

Now let us suppose that conversely the two conditions are satisfied. We have to check 2.11.1-3.

1. $Q_i(c) = \sum_\alpha a_{i\alpha} R^\alpha(c) = a_{io} c = 0.$

2. If $f \in P_n$ has degree m, then $Q_i(f) = \sum_\alpha a_{i\alpha} (R^\alpha f)$, all $R^\alpha f$ have degree $\leq m-1$, so $Q_i(f)$ has degree $\leq m-1$.

3. We claim that for the basic sequence $p = (p_\alpha)$ of R the matrix $(A_o(Q^\alpha p_\beta))_{\alpha,\beta} \in \mathcal{Y}$. Let $a(R) = (a_1(R),\dots,a_n(R))$, then the constant term of a(R) is zero and the linear term (with respect to R) is $(\sum_i a_{1,\varepsilon(i)} R^{\varepsilon(i)},\dots,\sum_i a_{n,\varepsilon(i)} R^{\varepsilon(i)})$ with an invertible matrix $a_{i,\varepsilon(j)}$. By the implicit function theorem for formal power series (cf. [8], p. 137) the formal power series is invertible with respect to composition, i.e. there is a formal power series $b(R) = (b_1(R),\dots,b_n(R)) \in K[[R]]^n$ such that $b(a(R)) = R$ and $a(b(R)) = R$. Let $b_i(R) = \sum_\alpha b_{i\alpha} R^\alpha$.

Now $(A_o(R^\beta p_\gamma))_{\beta,\gamma} \in \mathcal{Y}$. We truncate at $m \in N$: $b(Q) = R$, so

$$(A_o(R^\beta p_\gamma))_{|\beta|\leq m,|\gamma|\leq m} = (A_o((b(Q))^\beta p_\gamma))_{|\beta|\leq m,|\gamma|\leq m}$$

$$= (A_o(\sum_\delta (\sum_{\substack{\lambda=(\lambda_{i\alpha})\in N^{n\times N^n}\\ \sum_\alpha \lambda_{i\alpha}=\beta_i\\ \sum_{i,\alpha}\lambda_{i\alpha}\cdot\alpha=\delta}} \frac{\beta!}{\lambda!} B^\lambda) Q^\delta p_\gamma))_{|\beta|\leq m,|\gamma|\leq m}$$

by 2.19, where $B = (b_{i\alpha})$,

$$= (\sum_{\substack{\lambda=(\lambda_{i\alpha})\\ \sum_\alpha \lambda_{i\alpha}=\beta_i\\ \sum_{i,\alpha}\lambda_{i\alpha}\cdot\alpha=\delta}} \frac{\beta!}{\lambda!} B^\lambda)_{|\beta|\leq m,|\delta|\leq m} \cdot (A_o(Q^\delta p_\gamma))_{|\delta|\leq m,|\gamma|\leq m}.$$

Since the product is invertible, each of the two factor matrices is invertible and 3 follows.

The last assertion of the theorem is a trivial consequence of the fact that $Q = a(R)$ and $R = b(Q)$. qed.

2.21 Proposition: Let $R = (R_1, \ldots, R_n)$ be a delta operation, let $Q = (Q_1, \ldots, Q_n)$ be another delta operation with $Q_i \in K[[R]]$, $Q_i = \sum_\alpha a_{i\alpha} R^\alpha$. Then the following conditions are satisfied:

1. $a_{io} = 0$.
2. $(a_{i,\varepsilon(j)}) \in GL(n,K)$.
3. If we set $P_{ij} = \frac{1}{n} \sum_\beta a_{i,\beta+\varepsilon(j)} R^\beta \in K[[R]]$, then $(P_{ij}) \in GL(n,K[[R]])$.
4. $Q_i = \sum_j R_j P_{ij}$, or $Q = P.R$ for short.

Proof: 1 and 2 follow from 2.20.

3. $K[[R]]$ is a commutative K - algebra and a $n \times n$ - matrix P over it is invertible if and only if $\det P$ is multiplicatively invertible in $K[[R]]$ and that is the case iff $A_o(\det P) \neq 0$ in K. $A_o: K[[R]] \longrightarrow K$ is an algebra homomorphism, so $A_o(\det P) = \det(A_o P) = \det(A_o(P_{ij}))_{i,j} = \det(\frac{1}{n} a_{i,\varepsilon(j)})_{i,j}$ and that is not zero.

4. A straightforward computation. qed.

2.22 Theorem: Let $R = (R_1, \ldots, R_n)$ be a delta operation and $Q = (Q_1, \ldots, Q_n)$ with $Q_i \in K[[R]]$. Q is a delta operation if and only if there is an invertible matrix $P = (P_{ij}) \in GL(n,K[[R]])$ with $Q = P.R$. P is in general not unique.

Proof: Necessity follows from 2.21. Sufficiency is seen after a simple computation by 2.20.

§3 The Pincherle derivative

3.1 Let $R = (R_1,\dots,R_n)$ be a delta operation with basic sequence $r = (r_\alpha)$ and let $T(r)$ be the admissible operation for r (cf. 2.2). We call $T = T(r)$ the <u>basic operation</u> for R. We define then for $1 \leq i \leq n$ linear mappings

$$\frac{\partial}{\partial T_i} : L(P_n) \longrightarrow L(P_n),$$

$$\frac{\partial}{\partial R_i} : L(P_n) \longrightarrow L(P_n),$$

by $\frac{\partial}{\partial T_i}(S) = R_i \circ S - S \circ R_i$, $S \in L(P_n)$ and

$$\frac{\partial}{\partial R_i}(S) = S \circ T_i - T_i \circ S , \quad S \in L(P_n).$$

These are called the partial Pincherle derivatives induced by r. Note the asymmetry in the definition.

3.2 <u>Lemma</u>: 1. $\frac{\partial}{\partial T_i}(T_j) = R_i T_j - T_j R_i = \delta_{ij}\,\mathrm{Id}.$

2. $\frac{\partial}{\partial R_i}(R_j) = R_j T_i - T_i R_j = \delta_{ij}\,\mathrm{Id}.$

3. $\frac{\partial}{\partial T_i} \circ \frac{\partial}{\partial T_j} = \frac{\partial}{\partial T_j} \circ \frac{\partial}{\partial T_i}$

4. $\frac{\partial}{\partial R_i} \circ \frac{\partial}{\partial R_j} = \frac{\partial}{\partial R_j} \circ \frac{\partial}{\partial R_i}$

5. $(T_i)_* \circ \frac{\partial}{\partial R_j} = \frac{\partial}{\partial R_j} \circ (T_i)_*$

6. $(R_i)_* \circ \frac{\partial}{\partial T_j} = \frac{\partial}{\partial T_j} \circ (R_i)_*$

7. $\frac{\partial}{\partial R_j}(T_i) = 0, \quad \frac{\partial}{\partial T_j}(R_i) = 0.$

<u>Proof</u>: 1,2 are clear from 2.4. 3,4 are straightforward computations. In 5 $(T_i)_*(S) = T_i \circ S$, likewise $(R_i)_*(S) = R_i \circ S$ in 6. 5 and 6 are again to be proved by straightforward computation. 7 follow from 5 and 6:

$$\frac{\partial}{\partial R_j}(T_i) = \frac{\partial}{\partial R_j} \circ (T_i)_*(\mathrm{Id}) = (T_i)_* \circ \frac{\partial}{\partial R_j}(\mathrm{Id}) = 0. \qquad \text{qed.}$$

3.3 Definition: Generalizing our notation we call $\frac{\partial}{\partial R} = (\frac{\partial}{\partial R_1}, \dots, \frac{\partial}{\partial R_n})$ and $\frac{\partial}{\partial T} = (\frac{\partial}{\partial T_1}, \dots, \frac{\partial}{\partial T_n})$ again operations (by 3.2 the constituents commute), and we write $(\frac{\partial}{\partial R})^{\alpha} = (\frac{\partial}{\partial R_1})^{\alpha_1} (\frac{\partial}{\partial R_2})^{\alpha_2} \dots (\frac{\partial}{\partial R_n})^{\alpha_n}$, $\alpha \in N^n$ and likewise for $\frac{\partial}{\partial T}$.

3.4 Lemma: The partial Pincherle derivatives are derivations of the algebra $L(P_n)$, i.e.

$$\frac{\partial}{\partial T_i}(S_1 \circ S_2) = \frac{\partial}{\partial T_i}(S_1) \circ S_2 + S_1 \circ \frac{\partial}{\partial T_i}(S_2) \quad \text{and}$$
$$\frac{\partial}{\partial R_i}(S_1 \circ S_2) = \frac{\partial}{\partial R_i}(S_1) \circ S_2 + S_1 \circ \frac{\partial}{\partial R_i}(S_2).$$

Proof: A straightforward computation.

3.5 Proposition: 1. $(\frac{\partial}{\partial R})^{\beta} (\sum_{\alpha} a_{\alpha} R^{\alpha}) = \sum_{\alpha} a_{\alpha} (\alpha)_{\beta} R^{\alpha-\beta}$.

2. $\frac{\partial}{\partial T_i}(K[[R]]) = 0$

3. $(\frac{\partial}{\partial T})^{\beta} (\sum_{\alpha} b_{\alpha} T^{\alpha}) = \sum_{\alpha} b_{\alpha} (\alpha)_{\beta} T^{\alpha-\beta}$ for $\sum_{\alpha} b_{\alpha} T^{\alpha} \in K[T]$.

4. $\frac{\partial}{\partial R_i}(K[T]) = 0$.

Proof: $\frac{\partial}{\partial R_i}(\sum_{\alpha} a_{\alpha} R^{\alpha}) = (\sum_{\alpha} a_{\alpha} R^{\alpha}) \circ T_i - T_i \circ (\sum_{\alpha} a_{\alpha} R^{\alpha})$

$$= \sum_{\alpha} a_{\alpha} (R^{\alpha} T_i - T_i R^{\alpha})$$
$$= \sum_{\alpha} a_{\alpha} (R_1^{\alpha_1} \dots R_{i-1}^{\alpha_{i-1}} (R_i^{\alpha_i} T_i) R_{i+1}^{\alpha_{i+1}} \dots R_n^{\alpha_n} - T_i R^{\alpha})$$
$$= \sum_{\alpha} a_{\alpha} (R_1^{\alpha_1} \dots R_{i-1}^{\alpha_{i-1}} (T_i R_i^{\alpha_i} + \alpha_i R_i^{\alpha_i - 1}) R_{i+1}^{\alpha_{i+1}} \dots R_n^{\alpha_n} - T_i R^{\alpha})$$
$$= \sum_{\alpha} a_{\alpha} (T_i R^{\alpha} + \alpha_i R^{\alpha - \varepsilon(i)} - T_i R^{\alpha})$$
$$= \sum_{\alpha} a_{\alpha} \alpha_i R^{\alpha - \varepsilon(i)}.$$

We have used $R_i T_j = T_j R_i$, $i \neq j$, and $R_i^m T_i = T_i R_i^m + m R_i^{m-1}$ from the proof of 2.4. This proves1. 2 follows from 3.2.7. 3 and 4 can be proved with the same method. qed.

3.6 Remark: So $\frac{\partial}{\partial R_i}$ is just the formal partial differentiation on $K[[R]]$ in the direction R_i; hence the name derivative.

It is clear that all the formal rules of differential calculus hold for $\frac{\partial}{\partial R}$ on $K[[R]]$ like the chain rule or the Leibnitz rule:

$(\frac{\partial}{\partial R})^{\alpha} (a(R)b(R)) = \sum_{\beta} \binom{\alpha}{\beta} (\frac{\partial}{\partial R})^{\beta} (a(R)) (\frac{\partial}{\partial R})^{\alpha-\beta} (b(R))$.
Likewise for $\frac{\partial}{\partial T}$ on K[T] .

See books on modern algebraic geometry for a verification of that statement or Tutte 9 . If K = R then the validity of the formulas of elementary calculus for formal power series can be seen by the following simple argument: If one associates the infinite Taylor expansion to each germ at = of smooth functions on R^n, then this gives algebra homomorphism onto and to the composition of germs corresponds exactly the formal composition of power series. So one may just project down all the formulas of differential calculus.

3.7 Proposition: For all $a(R) \in K[[R]]$ and $b(T) \in K[T]$ we have
$$a(R)b(T) = \sum_{\alpha} \frac{1}{\alpha!} (\frac{\partial}{\partial T})^{\alpha} (b(T)) (\frac{\partial}{\partial R})^{\alpha} (a(R)).$$

This is the commutation rule for K R and K T in $L(P_n)$.

Proof: First we note that for any $S \in L(P_n)$ and the constant $1 \in P_n$
$(\frac{\partial}{\partial T_i}(S))(1) = (R_i \circ S - S \circ R_i)(1) = R_i \circ S\ (1)$, so for $\alpha \in N^n$:
$((\frac{\partial}{\partial T})^{\alpha} (S))(1) = R^{\alpha} \circ S\ (1)$.
Now let $a(R) = \sum_{\alpha} a_{\alpha} R^{\alpha} \in K[[R]]$; $b(T)$, $c(T) \in K[T]$. Then:
$$a(R)b(T)c(T)(1) = \sum_{\alpha} a_{\alpha} R^{\alpha}\, b(T)\ c(T)\ (1)$$
$$= \sum_{\alpha} a_{\alpha} ((\frac{\partial}{\partial T})^{\alpha} (b(T)c(T)))\ (1)$$
$$= \sum_{\alpha} a_{\alpha} (\sum_{\beta} \binom{\alpha}{\beta} (\frac{\partial}{\partial T})^{\beta} (b(T)) (\frac{\partial}{\partial T})^{\alpha-\beta} (c(T)))\ (1)$$
$$= \sum_{\beta} \frac{1}{\beta!} (\frac{\partial}{\partial T})^{\beta} (b(T)) (\sum_{\alpha} (\alpha)_{\beta}\, a_{\alpha} R^{\alpha-\beta} c(T))\ (1)$$
$$= \sum_{\beta} \frac{1}{\beta!} (\frac{\partial}{\partial T})^{\beta} (b(T)) (\frac{\partial}{\partial R})^{\beta} (a(R))\ c(T)\ (1).$$
c(T) (1) runs through all of P_n if c(T) runs through K[T] , so the result follows. qed.

§4 The formulas of Rodrigues and Lagrange

4.1 Theorem (formula of Rodrigues, Cigler [4]):

Let $R = (R_1,\dots,R_n)$ be a delta operation with basic sequence $r = (r_\alpha)$ and basic operation $T(r)$. Let $Q = (Q_1,\dots,Q_n)$ be another delta operation with basic sequence $q = (q_\alpha)$ and basic operation $T(q)$. Assume that all $Q_i \in K[[R]]$. Then $T(q)$ can be expressed as

$$T(q)_i = T(r)_1 \frac{\partial R_1}{\partial Q_i} + \dots + T(r)_n \frac{\partial R_n}{\partial Q_i} = \left\langle T(r) , \frac{\partial R}{\partial Q_i} \right\rangle .$$

Proof: By 3.7 we have $\frac{\partial R_k}{\partial Q_i} T(r)_l = T(r)_l \frac{\partial R_k}{\partial Q_i} + \frac{\partial}{\partial R_l}\left(\frac{\partial R_k}{\partial Q_i}\right)$, so

$$\begin{aligned}
&T(q)_i\, T(q)_j - T(q)_j\, T(q)_i \\
&= \sum_{k,l} T(r)_k \frac{\partial R_k}{\partial Q_i} T(r)_l \frac{\partial R_l}{\partial Q_j} - \sum_{k,l} T(r)_k \frac{\partial R_k}{\partial Q_j} T(r)_l \frac{\partial R_l}{\partial Q_i} \\
&= \sum_{k,l} T(r)_k\, T(r)_l \frac{\partial R_k}{\partial Q_i} \frac{\partial R_l}{\partial Q_j} + \sum_k T(r)_k \sum_l \frac{\partial}{\partial R_l}\left(\frac{\partial R_k}{\partial Q_i}\right) \frac{\partial R_l}{\partial Q_j} \\
&- \sum_{k,l} T(r)_k\, T(r)_l \frac{\partial R_k}{\partial Q_j} \frac{\partial R_l}{\partial Q_i} - \sum_k T(r)_k \sum_l \frac{\partial}{\partial R_l}\left(\frac{\partial R_k}{\partial Q_j}\right) \frac{\partial R_l}{\partial Q_i} \\
&= \sum_k T(r)_k \left(\frac{\partial}{\partial Q_j}\left(\frac{\partial R_k}{\partial Q_i}\right) - \frac{\partial}{\partial Q_i}\left(\frac{\partial R_k}{\partial Q_j}\right)\right) = 0,
\end{aligned}$$

where we used the chain rule for formal differentiation of power series and the fact that $\frac{\partial}{\partial Q_i} \frac{\partial}{\partial Q_j} = \frac{\partial}{\partial Q_j} \frac{\partial}{\partial Q_i}$.
So $T(q) = (T(q)_1,\dots,T(q)_n)$ is an operation, if defined by the formula of the theorem. Furthermore we have

$Q_i\, T(q)_j - T(q)_j\, Q_i = \delta_{ij}\, \mathrm{Id}$, since

$$\begin{aligned}
&Q_i \left(T(r)_1 \frac{\partial R_1}{\partial Q_j} + \dots + T(r)_n \frac{\partial R_n}{\partial Q_j}\right) \\
&\qquad - \left(T(r)_1 \frac{\partial R_1}{\partial Q_j} + \dots + T(r)_n \frac{\partial R_n}{\partial Q_j}\right) Q_i \\
&= (Q_i\, T(r)_1 - T(r)_1\, Q_i) \frac{\partial R_1}{\partial Q_j} + \dots + (Q_i\, T(r)_n - T(r)_n\, Q_i) \frac{\partial R_n}{\partial Q_j} \\
&= \frac{\partial Q_i}{\partial R_1} \frac{\partial R_1}{\partial Q_j} + \dots + \frac{\partial Q_i}{\partial R_n} \frac{\partial R_n}{\partial Q_j} = \delta_{ij}
\end{aligned}$$

by the chain rule again.

If we define now polynomials q_α for $\alpha \in N^n$ by
$q_\alpha = T(q)^\alpha\, 1 = \langle T(r), \frac{\partial R}{\partial Q_1} \rangle^{\alpha_1} \dots \langle T(r), \frac{\partial R}{\partial Q_n} \rangle^{\alpha_n}$ (1) ,
hen the formula above implies that $Q^\alpha q_\beta = (\beta)_\alpha q_{\beta-\alpha}$ as we saw in the proof of 2.4.

By definition $q_o = 1$. If $\alpha > 0$ then some $T(r)_j$ is leading the formula for q_α , so $q_\alpha(0) = 0$. Clearly q_α is of degree $\leq |\alpha|$ for each α . If we can show that $\{q_\alpha : |\alpha| \leq m\}$ is K - linearly independent in P_n for each m then $q = (q_\alpha)$ is the basic sequence for Q and T(q) is the basic operation for Q.

We show this by induction on m. For m = 0 this is obviously true. Suppose it is true for m. If there is a linear combination $\sum_{|\alpha| \leq m+1} a_\alpha q_\alpha = 0$ with some $a_\beta \neq 0$, $|\beta| \leq m+1$, choose i with $\beta_i \neq 0$. Then $\sum_{|\alpha| \leq m+1} a_\alpha \alpha_i\, q_{\alpha-\varepsilon(i)} = Q_i(\sum_\alpha a_\alpha q_\alpha) = 0$ would be a nontrivial relation in $\{q_\alpha : |\alpha| \leq m\}$ - to see that each q_α appears only once it suffices to note that $q_\alpha \longmapsto q_{\alpha-\varepsilon(i)}$ is injective on the set where it is defined; the rest is taken care of by the factor $\alpha_i = 0$. qed.

4.2 The rest of this section is devoted to deriving the Lagrange formula, it is based on Cigler [3]. Very simple examples (just permute a basic sequence within $\{\alpha : |\alpha| = m\}$) show that the following is the most general situation where something like the Lagrange formula can hold, i.e. 4.4, where q_α depends only on r_α .

<u>Definition</u>: Let R be a delta operation, let $Q_i \in K[[R]], 1 \leq i \leq n$. $Q = (Q_1, \dots, Q_n)$ is called a <u>delta operation of diagonal type</u> in K[[R]] if the following holds: for $Q_i = \sum_\alpha a_{i\alpha} R^\alpha$:

1. $a_{io} = 0$.
2. $(a_{i,\varepsilon(j)})_{i,j}$ is an invertible diagonal matrix over K, i.e. $a_{i,\varepsilon(j)} = 0$ if $i \neq j$ and $a_{i,\varepsilon(i)} \neq 0$.

4.3 <u>Proposition</u>: <u>Let</u> $R = (R_1,\dots,R_n)$ <u>be a delta operation</u>. $Q = (Q_1,\dots,Q_n)$ <u>is a delta operation of diagonal type in</u> $K[[R]]$ <u>if and only if there are (multiplicatively) invertible operators</u> P_i , $i = 1,\dots,n$, <u>in</u> $K[[R]]$ <u>such that</u> $Q_i = R_i\, P_i$. <u>Then the</u> P_i <u>are uniquely determined</u>.

<u>Proof</u>: Let Q be a delta operation of diagonal type in $K[[R]]$, $Q_i = \sum_\alpha a_{i\alpha} R^\alpha$. Choose $P_i = \sum_\alpha a_{i,\alpha+\varepsilon(i)} R^{\alpha+\varepsilon(i)}$, then $a_{i,\varepsilon(i)} \neq 0$, so P_i is invertible in $K[[R]]$ and clearly $P_i\, R_i = Q_i$.

Suppose conversely that $Q_i = R_i\, P_i$, $P_i = \sum_\beta b_{i\beta} R^\beta$ with $b_{io} \neq 0$ for all i. Then $Q_i = R_i\, P_i = \sum_\beta b_{i\beta} R^{\beta+\varepsilon(i)}$. Then the constant term of each Q_i is zero, the (in R) linear term has the form of an invertible diagonal matrix with b_{io} on the i'th place in the main diagonal. So Q is a delta operation by 2.20 and is of diagonal type. It is clear that the P_i are uniquely determined. qed.

4.4 <u>Theorem</u> (formula of Lagrange - Good):
<u>Let</u> R <u>be a delta operation with basic sequence</u> $r = (r_\alpha)$, <u>let</u> Q <u>be a delta operation of diagonal type in</u> $K[[R]]$ <u>with basic sequence</u> $q = (q_\alpha)$. <u>Write</u> $\eta = (1,\dots,1) \in N^n$ <u>and</u> $P^{-\alpha} = (P_1^{-1},\dots,P_n^{-1})^\alpha$. <u>Then the following formula holds</u>:

$$q_\alpha = \det\left(\frac{\partial Q_j}{\partial R_i}\right)_{i,j} \cdot P^{-\alpha-\eta} \cdot r_\alpha$$

$$= \det\left(\delta_{ij} P_j^{-\alpha_j} - \frac{1}{\alpha_j}\frac{\partial}{\partial R_i}(P_j^{-\alpha_j})\, R_j\right) \cdot r_\alpha \ .$$

(note that for $\alpha_j = 0$ we have $\frac{\partial}{\partial R_i}(P_j^{-\alpha_j}) = \frac{\partial}{\partial R_j}(Id) = 0$).

<u>Proof</u>: First we show that the two expressions are equal:

$$\det\left(\frac{\partial Q_i}{\partial R_j}\right)_{i,j} \cdot P^{-\alpha-\eta} = \det\left(\frac{\partial}{\partial R_j}(R_i P_i)\right)_{i,j} \cdot P^{-\alpha-\eta}$$

$$= \det \left(\frac{\partial R_i}{\partial R_j} P_i + R_i \frac{\partial P_i}{\partial R_j} \right)_{i,j} . P^{-\alpha-\eta}$$

$$= \sum_{\pi} \operatorname{sign} \pi \prod_i \left(\delta_{i,\pi(i)} P_i + R_i \frac{\partial P_i}{\partial R_{\pi(i)}} \right) . P^{-\alpha-\eta}$$

$$= \sum_{\pi} \operatorname{sign} \pi \prod_i \left(\delta_{i,\pi(i)} P_i^{-\alpha_i} + P_i^{-\alpha_i - 1} \frac{\partial P_i}{\partial R_{\pi(i)}} R_i \right)$$

$$= \det \left(\delta_{ij} P_i^{-\alpha_i} + P_i^{-\alpha_i - 1} \frac{\partial P_i}{\partial R_j} R_i \right)$$

$$= \det \left(\delta_{ij} P_i^{-\alpha_i} - \frac{1}{\alpha_i} \frac{\partial}{\partial R_j} (P_i^{-\alpha_i}) \; R_i \right) .$$

Now we show that the polynomial sequence given ba the first formula satisfies the functional equation $Q_i \, q_\alpha = \alpha_i \, q_{\alpha - \varepsilon(i)}$:

$$Q_i \det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} P^{-\alpha-\eta} \; r_\alpha = \det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} P^{-\alpha-\eta} P_i \, R_i \, r_\alpha$$

$$= \det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} P^{-(\alpha-\varepsilon(i))-\eta} \, \alpha_i \, r_{\alpha-\varepsilon(i)}$$

$$= \alpha_i \, . \det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} P^{-(\alpha-\varepsilon(i))-\eta} \; r_{\alpha-\varepsilon(i)} .$$

That the initial condition $q_\alpha(0) = \delta_{0\alpha}$ is satisfied will follow from lemma 4.5 below. Here it remains to show that q_α is of degree $|\alpha|$ and that $\{q_\alpha\}$ is a basis of P_n. This is a trivial consequence of the fact that $\det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j}$ is invertible in $K[[R]]$ (its constant term is the determinant of the coefficient matrix of the liner part of Q in its R - expansion, cf. 2.20 and 3.5). So $\det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} P^{-\alpha-\eta}$ is invertible and degree-non-increasing and its inverse has the same property (being in $K[[R]]$), so q_α has degree $|\alpha|$. That $\{ q_\alpha : |\alpha| \leq m \}$ is linearly independent can be seen as in the end of the proof of 4.1. qed.

4.5 We need a convenient terminology (Tutte,[9]): by a cyclic map we mean a pair $L = (W, \rho)$ where $W \subseteq \{1, \ldots, n\}$ and ρ is a permutation of W. Let $c(L)$ be the number of cycles of ρ. Let $\varepsilon(L)_i = 1$ or 0 accordig as i is or is not in W and let $\varepsilon(L) = (\varepsilon(L)_1, \ldots, \varepsilon(L)_n) \in N^n$.

Suppose now the data of theorem 4.4 be given and let

$L(P_i^{-\alpha_i}) = \frac{\partial}{\partial R_{\rho(i)}} (P_i^{-\alpha_i})$ if $i \in W$ and

$L(P_i^{-\alpha_i}) = P_i^{-\alpha_i}$ if $i \notin W$, for $L = (W, \rho)$.

Lemma: With the assumptions of theorem 4.4 we have (1) = (2) for all $\alpha \in N^n$ and (2) = (3) for $\alpha > 0$, where:

(1) $\det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} \; P^{-\alpha-\eta} \; r_\alpha$

(2) $\sum\limits_{\substack{L = (W,\rho) \\ W \subseteq U_\alpha}} (-1)^{c(L)} \; L(P_1^{-\alpha_1}) \; \dots \; L(P_n^{-\alpha_n}) \; r_{\alpha-\varepsilon(L)}$,

where $U_\alpha = \{ i : \alpha_i > 0 \} \subseteq \{1,\dots,n\}$.

(3) $\sum\limits_{\substack{L = (W,\rho) \\ i \in W \subseteq U_\alpha}} (-1)^{c(L)} \; T(r)_{\rho(i)} \; P_i^{-\alpha_i} \prod\limits_{j \neq i} L(P_j^{-\alpha_j}) \; r_{\alpha-\varepsilon(L)}$,

where i is a fixed element of U_α .

We first indicate how theorem 4.4 follows from this lemma:
If $\alpha = 0$ then from (1) = (2) we see that $q_o = P_1^{o} \dots P_n^{o} \; r_o = 1$.
If $\alpha \neq 0$ then for some $i \in U_\alpha$ we have (1) = (3) and in the sum (3) for q_α each term begins with some $T(r)_j$, so $q_\alpha(0) = 0$.
Note that (2) and (3) are additional expressions for q_α which are perhaps useful for some purpose.

Proof: $\det \left(\frac{\partial Q_i}{\partial R_j} \right)_{i,j} \cdot P^{-\alpha-\eta} \; r_\alpha$

$$= \sum_\pi \operatorname{sign}\pi \prod_i \left(\frac{\partial Q_i}{\partial R_{\pi(i)}} \cdot P_i^{-1} \right)_{i,j} \cdot P^{-\alpha} \; r_\alpha$$

$$= \det \left(\frac{\partial Q_i}{\partial R_j} P_i^{-1} \right)_{i,j} \cdot P^{-\alpha} r_\alpha$$

$$= \det \left(\frac{\partial}{\partial R_j} (Q_i \; P_i^{-1}) - Q_i \frac{\partial}{\partial R_j} (P_i^{-1}) \right)_{i,j} \cdot P^{-\alpha} r_\alpha$$

$$= \det \left(\frac{\partial R_i}{\partial R_j} - Q_i \frac{\partial}{\partial R_j} (P_i^{-1}) \right)_{i,j} \cdot P^{-\alpha} r_\alpha$$

$$= \det \left(\delta_{ij} - Q_i \frac{\partial}{\partial R_j} (P_i^{-1}) \right)_{i,j} \cdot P^{-\alpha} r_\alpha$$

$$= \sum_\pi \operatorname{sign}\pi \prod_i \left(\delta_{i,\pi(i)} - Q_i \frac{\partial}{\partial R_{\pi(i)}} (P_i^{-1}) \right) \cdot P^{-\alpha} \; r_\alpha$$

$$= \sum_{L = (W,\rho)} (-1)^{c(L)} \prod_{i \in W} \left(Q_i \frac{\partial}{\partial R_{\rho(i)}} (P_i^{-1}) \right) \cdot P^{-\alpha} \; r_\alpha$$

since $c(L) = k + |W| \pmod 2$ where $|W|$ is the number of elements of W, ρ the restriction of π to $W \supseteq \{ i : \pi(i) \neq i \}$ and k is the number of cycles of negative sign of π . The last expression equals:

$$\sum_{L=(W,\rho)} (-1)^{c(L)} \prod_{i\in W} \left((P_i^{-1})^{\alpha_i-1} \frac{\partial}{\partial R_{\rho(i)}}(P_i^{-1})\right) \cdot$$

$$\cdot \prod_{j\notin W} (P_j^{-\alpha_j}) \cdot R^{\varepsilon(L)} \; r_\alpha$$

$$= \sum_{\substack{L=(W,\rho)\\ W\subseteq U_\alpha}} (-1)^{c(L)} \prod_i L(P_i^{-\alpha_i})\, r_{\alpha-\varepsilon(L)} \cdot$$

This is (2).

In order to prove (3) we look at (2) and observe that $L(P_i^{-\alpha_i}) = \mathrm{Id}$ if $\alpha_i = 0$. By deleting those i with $\alpha_i = 0$ we may suppose that $\alpha_i > 0$ for all i. Furthermore all the $L(P_i^{-\alpha_i})$ commute, so we may suppose that the fixed i in (3) is just 1 (the formulas are easier to write down then). Write $T(r) = T = (T_1,\dots,T_n)$ for short. Then (2) equals

$$(4)\quad \sum_{L=(W,\ )} (-1)^{c(L)}\, L(P_1^{-\alpha_1}) \dots L(P_n^{-\alpha_n})\, T^{\eta-\varepsilon(L)}\, r_{\alpha-\eta} \ .$$

Now for $i\in W$ we have $L(P_i^{-\alpha_i}) = \frac{\partial}{\partial R_{\rho(i)}}(P_i^{-\alpha_i}) =$ $P_i^{-\alpha_i}\, T_{\rho(i)} - T_{\rho(i)}\, P_i^{-\alpha_i}$. If we insert this in (4) and multiply out we get a sum of expressions of the form

$$(5)\quad (-1)^{c(L)} (-1)^{|\mu|}\, T_{\rho(1)}^{\mu_1} P_1^{-\alpha_1} T_{\rho(1)}^{\delta_1}\, T_{\rho(2)}^{\mu_2} P_2^{-\alpha_2} T_{(2)}^{\delta_2} \dots$$
$$\dots T_{\rho(n)}^{\mu_n} P_n^{-\alpha_n} T_{\rho(n)}^{\delta_n}\, T^{\eta-\varepsilon(L)}\, r_{\alpha-\eta} \ ,$$

where $\mu,\delta\in\{0,1\}^n$ with $\mu+\delta = \varepsilon(L)$ and where we extend ρ from W to the whole of $\{1,\dots,n\} = U_\alpha$ by $\rho(i) = i$ for $i\notin W$.

Our purpose is to show that all terms of the form (5) that appear and have $\mu_1 = 0$ cancel out, so all terms that begin with $P_1^{-\alpha_1}$ can be neglected and it is clear that (3) remains.

For this end let a term (5) be given and let $L = (W,\rho)$ be a cyclic map for which this term appears. We shall construct a uniquely determined cyclic map $L^* = (W^*,\rho^*)$ for which the same term appears with the opposite sign. Our construction will be such that $L^{**} = L$.

Suppose first that there is some $i \in W$ with $\mu_i + \delta_{i+1} = 2$. In this case choose the smallest i with this property and let $W = W^*$ and $\varrho^* = \varrho \circ (i,i+1)$. Since sign $\varrho \neq$ sign ϱ^*, $|W| = |W^*|$ and $\mu^* = \mu$ the corresponding terms have opposite signs.

Now suppose that there is no $i \in W$ with $\mu_i = \delta_{i+1} = 1$. Consider first the case $\delta_n = 0$ and $W \neq [1,n]$. let k be the largest i that is not in W. Let $W^* = W \cup \{k\}$ and $\varrho^* = \varrho \circ (k,k+1,\dots,n)$. Then $\delta_n^* = 1$, $\varrho^*(n) = k$, $\delta^*_{\varrho^*(n)} = 1$ and $[k,n] \subseteq W^*$. Furthermore we have sign $\varrho^* = (-1)^{n-k}$ sign ϱ, $|W^*| = |W| + 1$ and $|\mu^*| = |\mu| - (n-k)$, so the corresponding terms have opposite signs.

If on the othe hand we have $\delta_n = 1$, $\varrho(n) = k$, $\delta_{\varrho(n)} = 1$ and $[k,n] \subseteq W$, we define $W^* = W \setminus \{k\}$ and $\varrho^* = \varrho \circ (k,k+1,\dots,n)^{-1}$. This is just the opposite construction to the last one above, so $L^{**} = L$ in this case, the signs of the corresponding terms are opposite as we saw above.

Now consider the case $\delta_n = 1$, $\delta_{\varrho(n)} = 1$, and there is some $j \notin W$ such that all inner points of $[j, \varrho(n)]$ $(j < i < \varrho(n))$ belong to W. Let $k = \varrho(n)$ and define $W^* = (W \cup \{j\}) \setminus \{k\}$ and $\varrho^* = \varrho \circ (j,j+1,\dots,k-1,k,n)$. Then we have $\varrho^*(n) = \varrho(j) = j$, $\delta_1^* = 1$, $\delta^*_{\varrho^*(n)} = \delta_j^* = 0$, further sign $\varrho^* = (-1)^{k-j+1}$ sign ϱ, $|W^*| = |W|$ and $|\mu^*| = |\mu| + k - j$ such that the corresponding terms have again opposite signs.

Now **consider** the case $\delta_1 = 1$, $\delta_{\varrho(1)} = 0$ and there is $k \notin W$ such that all inner points of $[\varrho(n),k]$ belong to W. Let $j = \varrho(n)$ and define $W^* = (W \cup \{k\}) \setminus \{j\}$ and $\varrho^* = \varrho \circ (j,j+1,\dots,k-1,k,n)^{-1}$. It is again clear that this is the opposite construction to the one above, so $L^{**} = L$ and the signs are opposite.

The only case that remains is $_{(n)}$ = 0 and all i (n) belong to W. This imlies $_1$ = 1 which we have excluded. qed.

§5 Shift invariant operators

The following results are a very straightforward generalisation of the theory in one variable (cf. [4], [6]). For completeness' sake we include proofs too.

5.1 In 1.8 we had for a K^n the shift operator $E_a: P_n \longrightarrow P_n$, given by $(E_a f)(x) = f(x+a) = \sum_\alpha \frac{a_\alpha}{\alpha!} (D^\alpha f)(x) = (e^{\langle a,D\rangle} f)(x)$.

Definition: $F \in L(P_n)$ is called shift invariant if $F \circ E_a = E_a \circ F$ for all $a \in K^n$. We denote by $L(P_n)^{K^n}$ the subalgebra of $L(P_n)$ consisting of all shift invariant operators.

By 1.8 we have $E_a \in K[[D]]$ for all a, $K[[D]]$ is a commutative subalgebra of $L(P_n)$, so $K[[D]] \subseteq L(P_n)^{K^n}$.

5.2 Definition: An admissible sequence $p = (p_\alpha)$ is called of binomial type or a binomial sequence if

$p_\alpha(x+y) = \sum_\beta \binom{\alpha}{\beta} p_\beta(x)\, p_{\alpha-\beta}(y)$ holds for all $\alpha \in N^n$.

Lemma: If $p = (p_\alpha)$ is a binomial sequence then it is a basic sequence.

Proof: $p_\alpha(x) = p_\alpha(x+0) = \sum_\beta \binom{\alpha}{\beta} p_\beta(x)\, p_{\alpha-\beta}(0)$.
Since $\{p_\alpha\}$ is a basis of P_n and all $\binom{\alpha}{\beta} \neq 0$ for $\beta \leqslant \alpha$ we conclude that $p_{\alpha-\beta}(0) = 0$ for $0 \leqslant \beta < \alpha$ and $p_0(0) = 1$. qed.

5.2 Proposition: Let $p = (p_\alpha)$ be the basic sequence of a delta operation $R = (R_1,\dots,R_n)$. p is of binomial type if and only if R (i.e. each R_i) is shift invariant.

Proof: Let R be shift invariant. By the Taylor formula 2.18 we have:

$$p_\alpha(x+y) = \sum_\beta \frac{p_\beta(x)}{\beta!} A_0(R^\beta p_\alpha(.+y)) = \sum_\beta \frac{p_\beta(x)}{\beta!} A_0(R^\beta E_y p_\alpha)$$
$$= \sum_\beta \frac{p_\beta(x)}{\beta!} A_0(E_y R^\beta p_\alpha) = \sum_\beta \binom{\alpha}{\beta} p_\beta(x)\, p_{\alpha-\beta}(y).$$

Let conversely $p = (p_\alpha)$ be of binomial type. Then we have:

$$p_\alpha(x+y) = \sum_\beta \binom{\alpha}{\beta} p_\beta(x)\, p_{\alpha-\beta}(y) = \sum_\beta \frac{p_\beta(x)}{\beta!} (\alpha)_\beta\, p_{\alpha-\beta}(y)$$
$$= \sum_\beta \frac{p_\beta(x)}{\beta!} (R^\beta p_\alpha)(y).$$

This equation (for fixed y) is linear in p_α and $\{p_\alpha\}$ is a basis, so we get for any $f \in P_n$:

$$f(x+y) = \sum_\beta \frac{p_\beta(x)}{\beta!} (R^\beta f)(y) = \sum_\beta \frac{p_\beta(y)}{\beta!} (R^\beta f)(x)$$

by symmetry. Insert $R_i f$ into this equation:

$$(E_y R_i f)(x) = R_i f(x+y) = \sum_\beta \frac{p_\beta(y)}{\beta!} (R^\beta R_i f)(x)$$
$$= R_i \Big(\sum_\beta \frac{p_\beta(y)}{\beta!} R^\beta f\Big)(x) = R_i (E_y f)(x).$$

So R_i is shift invariant. qed.

5.4 Theorem: (expansion for shift invariant operators):

Let R be a shift invariant delta operation with basic sequence p. Then for any shift invariant operator $F \in L(P_n)^{K^n}$ we have

$$F = \sum_\beta \frac{A_0(F p_\beta)}{\beta!} R^\beta .$$

Proof: p is of binomial type, so we have as in the proof of 5.3:

$$E_y f = \sum_\beta \frac{p_\beta}{\beta!} (R^\beta f)(y) \text{ for any } f \in P_n \text{, so}$$
$$A_0(F E_y f) = \sum_\beta \frac{A_0(F p_\beta)}{\beta!} (R^\beta f)(y), \text{ i.e.}$$
$$(F f)(y) = A_0(E_y F f) = A_0(F E_y f) = \sum_\beta \frac{A_0(F p_\beta)}{\beta!} (R^\beta f)(y). \text{ qed.}$$

5.5 Corollary: $L(P_n)^{K^n} = K[[R]]$ for any shift invariant delta operation R.

5.6 Corollary: Let R be a shift invariant delta operation with basic sequence $p = (p\ )$. Let $R = a(D)$ be the power series expansion (i.e. $R_i = a_i(D) \in K[[D]]$), let a^{-1} be the inverse power series. Then for $t = (t_1, \dots, t_n)$ we have for $y\ \ K^n$:

$$e^{\langle y\,,\, a^{-1}(t)\rangle} = \sum_\alpha \frac{p_\alpha(y)}{\alpha!} t^\alpha$$

<u>Proof</u>: We extend E_y in a power series in $K[[D]]$ by 5.4:

$$E_y = \sum_\alpha \frac{A_0(E_y\, p_\alpha)}{\alpha!} R^\alpha = \sum_\alpha \frac{p_\alpha(y)}{\alpha!} R^\alpha .$$

Now $R^\alpha = (a(D))^\alpha$, thus

$$e^{\langle y,D\rangle} = E_y = \sum_\alpha \frac{p_\alpha(y)}{\alpha!} (a(D))^\alpha .$$

Insert t for D to get

$$e^{\langle y,t\rangle} = \sum_\alpha \frac{p_\alpha(y)}{\alpha!} (a(t))^\alpha .$$

Now a^{-1} exists by 2.20; insert $a^{-1}(t)$ for t to get

$$e^{\langle y,\, a^{-1}(t)\rangle} = \sum_\alpha \frac{p_\alpha(y)}{\alpha!} t^\alpha . \qquad \text{qed.}$$

§6 G - invariant operators

6.1 Let G be a group and $G \longrightarrow GL(n,K)$ be a representation of G on K^n. This gives an action of G on $P_n = K[x]$ by $(g.f)(x) = f(g^{-1}.x)$, $f \in P_n$, $x \in K^n$, $g \in G$.

Definition: We denote by P_n^G the K -algebra of all G - invariant polynomials f, i.e. $g.f = f$.

Let $f \in P_n$. We decompose f into its homogeneous parts $f = f_o + f_1 + \dots + f_m$, where $m = \deg f$. This composition makes P_n into a graded algebra. Clearly we have: $f \in P_n^G$ if and only if each $f_j \in P_n^G$. So P_n^G is a graded subalgebra of P_n. This is at the basis of the following theorem, for the proof see Springer [7] or Poènaru [5] .

6.2 Theorem (Hilbert - Nagata): Let $G \longrightarrow GL(n,K)$ be a completely reducible representation. Then P_n^G is a finitely generated K - algebra.

That means the following: there are finitely many polynomials $v_1,\dots,v_k \in P_n^G$ such that each $f \in P_n^G$ may be written as a polynomial in $v_1,\dots,v_k$: $f(x) = h(v_1(x),\dots,v_k(x))$, $h \in P_k$. One may assume that all v_i are homogeneous and of degree>0.
Another way to express this theorem is the following:
let $v = (v_1,\dots,v_k): K^n \longrightarrow K^k$ be the polynomial map.
Then $0 \longleftarrow P_n^G \xleftarrow{v^*} P_k$ surjective, where $v^*(f) = f \circ v$.

A representation is completely reducible if each invariant subspace has an invariant complement. It is well known that each continuous representation of a compact group is completely reducible. Furthermore information is available for so called reductive algebraic groups, see [7].

There is a group G and a representation of G such that P_n^G is not finitely generated (Nagata).
For finite groups E. Noether gave an explicit construction of a generating system, see [5] .

6.3 G acts on $K[[x]]$ by acting on the homogeneous parts of a formal power series, which are polynomials in x.

<u>Theorem</u>: <u>Let</u> $G \longrightarrow GL(n,K)$ <u>be a completely reducible representation. Let</u> $v = (v_1,\dots,v_k)$ <u>be the polynomial map consisting of generators of</u> P_n^G. <u>Then</u> $v^*: K[[y]] \longrightarrow K[[x]]^G$ <u>is surjective, where</u> $y = (y_1,\dots,y_k)$.

For the proof see again 5 .

6.4 <u>Definition</u>: Let $G \longrightarrow GL(n,K)$ be a representation and let $F \in L(P_n)$. F is called G - invariant if F is a G - modul homomorphism $P_n \longrightarrow P_n$, i.e. $F(g.f) = g.F(f)$ for all $f \in P_n$, $g \in G$. We denote the subalgebra of all G - invariant operators by $L(P_n)^G$.

This notation is compatible with P_n^G , in the latter case G acts trivially on K.

6.5 Now we look at $K[[D]]$ and let G act on it (where $D = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$): if $g \in G$ then the action of g on K^n is given by a matrix: $g(x) = (g_1(x),\dots,g_n(x))$, where $g_i(x) = \sum_j g_{ij}\, x_j$, (g_{ij}) being an invertible matrix.
Now we let g act formally on D:
$^tg\,(D) = (^tg_1(D),\dots,{}^tg_n(D))$, $^tg_i(D) = \sum_j g_{ji}\, D_j$.
This induces an action of G on $K[[D]]$: for $a(D) \in K[[D]]$ we have $(^tg.a)(D) = a(^tg^{-1}(D))$. This action is the

transposed action of the original one, taking care of the fact that the $D_j = \frac{\partial}{\partial x_j}$ are "contravariant vectorfields". With this action a remarkable formula holds:

Theorem: Let $G \longrightarrow GL(n,K)$ be a representation, let $f \in P_n$ and $a(D) \in K[[D]]$. Then we have for $g \in G$:

$$a(D)(g.f) = g.((^t g.a)(D)f)$$

Proof: Let $f = \sum_\alpha f_\alpha x^\alpha \in P_n$, $g \in G$, then:

$$D_i(g^{-1}.f)(x) = \frac{\partial}{\partial x_i}(f(gx)) = \sum_\alpha f_\alpha \frac{\partial}{\partial x_i} (g_1(x),\dots,g_n(x))^\alpha$$

$$= \sum_\alpha f_\alpha \sum_j \frac{\partial}{\partial y_j}(y^\alpha)\Big|_{y=g(x)} \cdot \frac{\partial}{\partial x_i}(g_j(x)) \quad \text{by the chain rule,}$$

$$= \sum_\alpha f_\alpha \sum_j \frac{\partial}{\partial y_j}(y^\alpha)\Big|_{y=g(x)} \cdot g_{ji}$$

$$= \Big(\sum_j g_{ji} \frac{\partial}{\partial y_j} \Big)\Big(\sum_\alpha f_\alpha y^\alpha \Big)\Big|_{y=g(x)}$$

$$= {}^t g_i(D)(f)(gx) = (g^{-1}.({}^t g_i(D)f))(x).$$

Furthermore we get:

$$D_j D_i (g^{-1}.f)(x) = D_j(g^{-1}.({}^t g_i(D)f)(x)$$

$$= (g^{-1}.({}^t g_j(D)\ {}^t g_i(D)\ f))(x).$$

Thus for any $\alpha \in N^n$ we have $D^\alpha(g^{-1}.f) = g^{-1}.(({}^t g(D))^\alpha f)$.

Replace now g^{-1} by g and apply it to a formal power series to get the result. qed.

6.6 Corollary: $L(P_n)^G \cap L(P_n)^{K^n} = K[[D]]^G$.

Proof: Let $a(D) \in K[[D]] = L(P_n)^{K^n}$ (cf.5.5) and $g \in G$. Then $a(D)(g.f) = g.(({}^t g.a)(D)\ f)$ by 6.5, so $a(D)(g.f) = g.(a(D)f)$ iff ${}^t g.a = a$ in $K[[D]]$, i.e. $a \in K[[D]]^G$. qed.

6.7 Corollary: *Let $G \longrightarrow GL(n,K)$ be a completely reducible representation, let $v_1,\dots,v_k$ be generating polynomials for $P_n^{\,{}^tG}$, where ${}^tG$ symbolizes G with the transposed action. Then any shift and G - invariant operator F can be written as a formal power series in $v_1(D),\dots,v_k(D)$.*

Proof: The transposed action $g \longmapsto {}^tg \in GL(n,K)$ is also completely reducible since it is the induced action on the dual K^n . So the result follows from 6.6 and 6.3. qed.

6.8 Remark: *Let $G \longrightarrow GL(n,K)$ be a non trivial representation. Then there is no G - invariant delta operation on P_n .*

Proof: Assume that $R = (R_1,\dots,R_n)$ is G - invariant and a delta operation with basic sequence $r = (r_\alpha)$. By the Taylor formula (2.18) we have for each $f \in P_n$ and $g \in G$:

$$g.f = \sum_\alpha A_o(R^\alpha (g.f)) \frac{r_\alpha}{\alpha!} = \sum_\alpha A_o(g.(R^\alpha f)) \frac{r_\alpha}{\alpha!}$$
$$= \sum_\alpha A_o(R^\alpha f) \frac{r_\alpha}{\alpha!} = f.$$

We have used that g leaves invariant the constant terms of polynomials since it acts linearly. So $P_n^G = P_n$ and G has to act trivially on K^n. qed.

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