Infinite dimensional differential geometry: The space of all Riemannian metrics on a compact manifold and several natural weak Riemannian metrics on it.

Peter W. Michor
Universität Wien

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Based on collaborations with: M. Bauer, M. Bruveris, P. Harms, D. Mumford

- 1. A short introduction to convenient calculus in infinite dimensions.
- 2. Manifolds of mappings (with compact source) and diffeomorphism groups as convenient manifolds
- 3. A diagram of actions of diffeomorphism groups
- 4. Riemannian geometries on spaces of Riemannian metrics

Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.
Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.
For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
    - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
    - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.

   We call this *scalarwise* C^∞.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called smooth or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C^∞_c.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection
$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R},U),\ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$
where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces.
   Note that this is the main assumption of variational calculus. Here it is a theorem.
7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$.
   (*Smooth uniform boundedness theorem*, see [KM97], theorem 5.26).

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

$$
\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G)\\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to\\
&\qquad\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}
$$

This ends our review of the standard results of convenient calculus. Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]

Manifolds of mappings

Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar{g}$ on N. Then

$$\begin{tikzcd}
& 0_N \arrow[dl, bend right, "\text{zero section}"'] \arrow[d, hook] & N \arrow[d, hook] \arrow[dr, bend left, "\text{diagonal}"] & \\
TN & V^N \arrow[l, hook', "\text{open}"] \arrow[r, "{(\pi_N, \exp^{\bar g})}", "\cong"'] & V^{N\times N} \arrow[r, hook, "\text{open}"'] & N \times N
\end{tikzcd}$$

$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^* TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar g})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar g}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar g}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar g}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^* TN)) = \Gamma(\mathbb{R} \times M; {\mathrm{pr}_2}^* f^* TN)$
By Cartesian Closedness after considering charts.
Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.
Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By the first lemma.
Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves
Corollary (of the chart structure):

$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N).$$

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.

A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.

- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\mathrm{Evol}^X_t := g(t) = \mathrm{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*

Proof: $\mathrm{Diff}(M) \xrightarrow{open} C^\infty(M, M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\mathrm{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t, x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.

Let $X(t, x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\mathrm{Fl}_s^{\partial_t \times X}(t, x) = (t + s, \mathrm{Evol}^X(t, x))$ satisfies the ODE $\quad \partial_t \, \mathrm{Evol}(t, x) = X(t, \mathrm{Evol}(t, x))$. If $X(s, t, x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism.

QED.

The principal bundle of embeddings

For finite dimensional manifolds M, N with M compact, $\mathrm{Emb}(M, N)$, the space of embeddings of M into N, is open in $C^\infty(M, N)$, so it is a smooth manifold. $\mathrm{Diff}(M)$ acts freely and smoothly from the right on $\mathrm{Emb}(M, N)$.
Theorem: $\mathrm{Emb}(M, N) \to \mathrm{Emb}(M, N)/\mathrm{Diff}(M)$ *is a principal fiber bundle with structure group* $\mathrm{Diff}(M)$.
Proof: Auxiliary Riem. metric $\bar{g}$ on N. Given $f \in \mathrm{Emb}(M, N)$, view $f(M)$ as submanifold of N. $TN|_{f(M)} = \mathrm{Nor}(f(M)) \oplus Tf(M)$.

$$\mathrm{Nor}(f(M)) : \xrightarrow[\cong]{\exp^{\bar{g}}} W_{f(M)} \xrightarrow{p_{f(M)}} f(M) \text{ tubular nbhd of } f(M).$$

If $g : M \to N$ is C^1-near to f, then
$\varphi(g) := f^{-1} \circ p_{f(M)} \circ g \in \mathrm{Diff}(M)$ and
$g \circ \varphi(g)^{-1} \in \Gamma(f^* W_{f(M)}) \subset \Gamma(f^* \mathrm{Nor}(f(M)))$.
This is the required local splitting. QED

The orbifold bundle of immersions

$\mathrm{Imm}(M,N)$, the space of immersions $M \to N$, is open in $C^\infty(M,N)$, and is thus a smooth manifold. The regular Lie group $\mathrm{Diff}(M)$ acts smoothly from the right, but no longer freely.
Theorem: [Cervera,Mascaro,M,1991] *For an immersion $f : M \to N$, the isotropy group $\mathrm{Diff}(M)_f = \{\varphi \in \mathrm{Diff}(M) : f \circ \phi = f\}$ is always a finite group, acting freely on M; so $M \xrightarrow{p} M/\mathrm{Diff}(M)_f$ is a convering of manifold and f factors to $f = \bar{f} \circ p$.*
Thus $\mathrm{Imm}(M,N) \to \mathrm{Imm}(M,N)/Diff(M)$ is a projection onto an honest infinite dimensional orbifold.

A Zoo of diffeomorphism groups on $\mathbb{R}^n$

Theorem. *The following groups of diffeomorphisms on $\mathbb{R}^n$ are regular Lie groups:*

- $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, *the group of all diffeomorphisms which differ from the identity by a function which is bounded together with all derivatives separately.*
- $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *the group of all diffeomorphisms which differ from the identity by a function in the intersection H^∞ of all Sobolev spaces H^k for $k \in \mathbb{N}_{\geq 0}$.*
- $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, *the group of all diffeomorphisms which fall rapidly to the identity.*
- $\mathrm{Diff}_c(\mathbb{R}^n)$ *of all diffeomorphisms which differ from the identity only on a compact subset. (well known since 1980)*

[M,Mumford,2013], partly [B.Walter,2012]; for Denjoy-Carleman ultradifferentiable diffeomorphisms [Kriegl, M, Rainer 2014].
In particular, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is essential if one wants to prove that the geodesic equation of a right Riemannian invariant metric is well-posed with the use of Sobolov space techniques.

The diagram

$$
\begin{tikzcd}
\mathrm{Diff}(M) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] \arrow[dd, two heads, "\mathrm{Diff}(M,\mu)"] & & \mathrm{Imm}(M,N) \arrow[dl, "\text{needs } \bar g"] \arrow[dr, "\mathrm{Diff}(M)"'] & & \mathrm{Diff}_{\mathcal{A}}(N) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dd, "\text{r-acts}"] \\
& \mathrm{Met}(M) \arrow[dr, "\mathrm{Diff}(M)"'] & & B_i(M,N) \arrow[dl, "\text{needs } \bar g"] & \\
\mathrm{Vol}^1_+(M) & & \dfrac{\mathrm{Met}(M)}{\mathrm{Diff}(M)} & & \mathrm{Met}_{\mathcal{A}}(N)
\end{tikzcd}
$$

M compact , N possibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar g$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities

About Met(M)

Let $\mathrm{Met}(M) = \Gamma(S^2_+ T^*M)$ be the space of all smooth Riemannian metrics on a compact manifold M, and let $\mathrm{Met}_{H^k(\hat{g})} = \Gamma_{H^k(\hat{g})}(S^2_+ T^*M)$ the space of all Sobolev $H^k(\hat{g})$ sections of the metric bundle, where $\hat{g}$ is a smooth background Riemann metric on M.

$$
\begin{tikzcd}
\mathrm{Diff}(M) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] \arrow[dd, two heads, "\mathrm{Diff}(M,\mu)"] & & \mathrm{Imm}(M,N) \arrow[dl, "\text{needs } \bar{g}"'] \arrow[dr, "\mathrm{Diff}(M)"'] & & \mathrm{Diff}_{\mathcal{A}}(N) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts (LDDMM)}"'] \arrow[dd, "\text{r-acts}"] \\
& {\color{red}\mathrm{Met}(M)} \arrow[dr, two heads, "\mathrm{Diff}(M)"'] & & B_i(M,N) \arrow[dl, "\text{needs } \bar{g}"] & \\
\mathrm{Vol}^1_+(M) & & \dfrac{\mathrm{Met}(M)}{\mathrm{Diff}(M)} & & \mathrm{Met}(N)
\end{tikzcd}
$$

Weak Riemann metrics on $\mathrm{Met}(M)$

All of them are $\mathrm{Diff}(M)$-invariant; natural, tautological.

$$G_g(h,k) = \int_M g_2^0(h,k)\,\mathrm{vol}(g) = \int \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g), \quad L^2\text{-metr.}$$

$$\text{or } = \Phi(\mathrm{Vol(g)}) \int_M g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{conformal}$$

$$\text{or } = \int_M \Phi(\mathrm{Scal}^g).g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{curvature modified}$$

$$\text{or } = \int_M g_2^0((1+\Delta^g)^p h,k)\,\mathrm{vol}(g) \qquad \text{Sobolev order } p$$

$$\text{or } = \int_M \Big(g_2^0(h,k) + g_3^0(\nabla^g h, \nabla^g k) + \dots + g_p^0((\nabla^g)^p h, (\nabla^g)^p k)\Big)\,\mathrm{vol}(g)$$

where $\Phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is total volume of (M,g), Scal is scalar curvature, and g_2^0 is the induced metric on $\binom{0}{2}$-tensors.

$\Delta^g h := (\nabla^g)^{*,g} \nabla^g h = -\operatorname{Tr}^{g^{-1}}((\nabla^g)^2 h)$ is the Bochner-Laplacian. It can act on all tensor fields h, and it respects the degree of the tensor field it is acting on.
We consider Δ^g as an unbounded self-adjoint positive semidefinite operator on the Hilbert space H^0 with compact resolvent. The domain of definition of Δ^g is the space

$$H^2 = H^{2,g} := \{h \in H^0 : (1 + \Delta^g) h \in H^0\} = \{h \in H^0 : \Delta^g h \in H^0\}$$

which is again a Hilbert space with inner product

$$\int_M g_2^0((1 + \Delta^g) h, k) \operatorname{vol}(g).$$

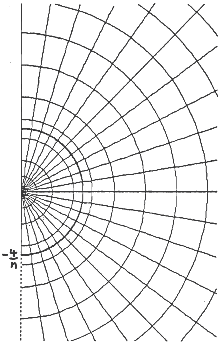
Again H^2 does not depend on the choice of g, but the inner products for different g induce different but equivalent norms on H^2. Similarly we have

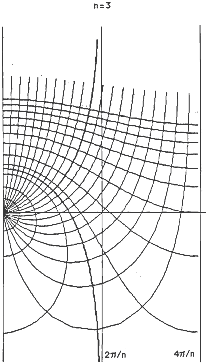
$$\begin{aligned} H^{2k} = H^{2k,g} : &= \{h \in H^0 : (1 + \Delta^g)^k h \in H^0\} \\ &= \{h \in H^0 : \Delta^g h, (\Delta^g)^2, \dots (\Delta^g)^k \in H^0\} \end{aligned}$$

The L^2-metric on the space of all Riemann metrics

[Ebin 1970]. Geodesics and curvature [Freed Groisser 1989]. [Gil-Medrano Michor 1991] for non-compact M. [Clarke 2009] showed that geodesic distance for the L^2-metric is positive, and he determined the metric completion of $\mathrm{Met}(M)$.
The geodesic equation is completely decoupled from space, it is an ODE:

$$g_{tt} = g_t g^{-1} g_t + \tfrac{1}{4} \operatorname{Tr}(g^{-1} g_t g^{-1} g_t)\, g - \tfrac{1}{2} \operatorname{Tr}(g^{-1} g_t)\, g_t$$





$$A = g^{-1}a \qquad \text{for } a \in T_g\mathrm{Met}(M)$$

$$\exp_0(A) = \tfrac{2}{n}\log\left(\left(1+\tfrac{1}{4}\,\mathrm{Tr}(A)\right)^2 + \tfrac{n}{16}\,\mathrm{Tr}(A_0^2)\right) Id$$
$$+\frac{4}{\sqrt{n\,\mathrm{Tr}(A_0^2)}}\arctan\left(\frac{\sqrt{n\,\mathrm{Tr}(A_0^2)}}{4+\mathrm{Tr}(A)}\right)A_0.$$

Back to the the general metric on $\mathrm{Met}(M)$.

We describe all these metrics uniformly as

$$\begin{aligned} G_g^P(h,k) &= \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) \\ &= \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g), \end{aligned}$$

where

$$P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$$

is a positive, symmetric, bijective pseudo-differential operator of order $2p$, $p \geq 0$, depending smoothly on the metric g, and also $\mathrm{Diff}(M)$-equivariantly:

$$\varphi^* \circ P_g = P_{\varphi^* g} \circ \varphi^*$$

The geodesic equation in this notation:

$$
\begin{aligned}
g_{tt} = P^{-1}\Big[&(D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\
&+ \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - (D_{(g,g_t)}P)g_t \\
&- \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\Big]
\end{aligned}
$$

We can rewrite this equation to get it in a slightly more compact form:

$$
\begin{aligned}
(Pg_t)_t &= (D_{(g,g_t)}P)g_t + Pg_{tt} \\
&= (D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\
&\quad + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t
\end{aligned}
$$

Well posedness of geodesic equation.

Assumptions *Let $P_g(h)$, $P_g^{-1}(k)$ and $(D_{(g,.)}Ph)^*(m)$ be linear pseudo-differential operators of order $2p$ in m, h and of order $-2p$ in k for some $p \geq 0$.*
As mappings in the foot point $g \in \mathrm{Met}(M)$, we assume that they are non-linear, smooth, even smooth in $g \in \mathrm{Met}^k(M)$ for $k > \dim(M)2$, and that they are a composition of operators of the following type:
(a) Non-linear differential operators of order $l \leq 2p$, i.e.,

$$A(g)(x) = A\big(x, g(x), (\hat\nabla g)(x), \ldots, (\hat\nabla^l g)(x)\big),$$

(b) Linear pseudo-differential operators of order $\leq 2p$,
such that the total (top) order of the composition is $\leq 2p$.
Since $h \mapsto P_g h$ induces a weak inner product, it is a symmetric and injective pseudodifferential operator. We *assume* that it is elliptic and selfadjoint. Then it is Fredholm and has vanishing index. Thus it is invertible and $g \mapsto P_g^{-1}$ is smooth
$H^k(S^2_+ T^*M) \to L(H^k(S^2 T^*M), H^{k+2p}(S^2 T^*M))$ by the implicit function theorem on Banach spaces.

Theorem. [Bauer, Harms, M. 2011] *Let the assumptions above hold. Then for $k > \frac{\dim(M)}{2}$, the initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold $\mathrm{Met}^{k+2p}(M)$ of H^{k+2p}-metrics. The solutions depend C^∞ on t and on the initial conditions $g(0,\ .\ ) \in \mathrm{Met}^{k+2p}(M)$ and $g_t(0,\ .\ ) \in H^{k+2p}(S^2T^*M)$. The domain of existence (in t) is uniform in k and thus this also holds in $\mathrm{Met}(M)$.*
Moreover, in each Sobolev completion $\mathrm{Met}^{k+2p}(M)$, the Riemannian exponential mapping $\exp^P$ exists and is smooth on a neighborhood of the zero section in the tangent bundle, and $(\pi, \exp^P)$ is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in $\mathrm{Met}^{k+2p}(M) \times \mathrm{Met}^{k+2p}(M)$. All these neighborhoods are uniform in $k > \frac{\dim(M)}{2}$ and can be chosen H^{k_0+2p}-open, where $k_0 > \frac{\dim(M)}{2}$. Thus all properties of the exponential mapping continue to hold in $\mathrm{Met}(M)$.

Conserved Quantities on $\mathrm{Met}(M)$.

Right action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ given by

$$(g, \phi) \mapsto \phi^* g.$$

Fundamental vector field (infinitesimal action):

$$\zeta_X(g) = \mathcal{L}_X g = -2\,\mathrm{Sym}\,\nabla(g(X)).$$

If metric G^P is invariant, we have the following conserved quantities

$$\begin{aligned}
\mathrm{const} &= G^P(g_t, \zeta_X(g)) \\
&= -2 \int_M g_1^0\big(\nabla^* \,\mathrm{Sym}\, P g_t, g(X)\big)\,\mathrm{vol}(g) \\
&= -2 \int_M g\big(g^{-1} \nabla^* P g_t, X\big)\,\mathrm{vol}(g)
\end{aligned}$$

Since this holds for all vector fields X,

$$\boxed{(\nabla^* P g_t)\,\mathrm{vol}(g) \in \Gamma(T^* M \otimes_M \mathrm{vol}(M)) \text{ is const. in } t.}$$

On $\mathbb{R}^n$: The pullback of the Ebin metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider here the right action
$r : \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n) \times \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n) \to \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ which is given by
$r(g, \varphi) = \varphi^* g$, together with its partial mappings
$r(g, \varphi) = r^{\varphi}(g) = r_g(\varphi) = \mathrm{Pull}^g(\varphi)$.

Theorem. *If $n \geq 2$, the image of $\mathrm{Pull}^{\bar{g}}$, i.e., the $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$-orbit through $\bar{g}$, is the set $\mathrm{Met}^{\mathrm{flat}}_{\mathcal{A}}(\mathbb{R}^n)$ of all flat metrics in $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.*

The pullback of the Ebin metric to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X, Y) = 4 \int_{\mathbb{R}^n} \mathrm{Tr}\left((\mathrm{Sym}\, dX).(\mathrm{Sym}\, dY)\right) dx = \int_{\mathbb{R}^n} \langle X, PY \rangle dx$$

Using the inertia operator P we can write the metric as $\int_{\mathbb{R}^n} \langle X, PY \rangle dx$, with

$$P = -2(\mathrm{grad}\,\mathrm{div} + \Delta)\,.$$

The pullback of the general metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider now a weak Riemannian metric on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ in its general form

$$G^P_g(h,k) = \int_M g^0_2(P_g h, k)\,\mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is as described above. *If the operator P is equivariant for the action of $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$, then the induced pullback metric $(\mathrm{Pull}^{\bar{g}})^* G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is right invariant:*

$$G_{\mathrm{Id}}(X,Y) = -4\int_{\mathbb{R}^n} \partial_j (P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j . Y^i\,dx \qquad (1)$$

Thus we we get the following formula for the corresponding inertia operator $(\tilde{P}X)^i = \sum_j \partial_j (P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j$. Note that the pullback metric $(\mathrm{Pull}^{\bar{g}})^* G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is always of one order higher then the metric G^P on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.

The Sobolev metric of order p.

The Sobolev metric G^P

$$G^P_g(h,k) = \int_{\mathbb{R}^n} \mathrm{Tr}(g^{-1}.((1+\Delta)^p h).g^{-1}.k)\,\mathrm{vol}(g)\,.$$

The pullback of the Sobolev metric G^P to the diffeomorphism group is a right invariant metric G given by

$$G(X,Y) = -2\int_{\mathbb{R}^n} \Big\langle (\operatorname{grad}\operatorname{div} + \Delta)(1-\Delta)^p X, Y \Big\rangle dx\,.$$

Thus the inertia operator is given by

$$\tilde{P} = -2(1-\Delta)^p(\Delta + \operatorname{grad}\operatorname{div}) = -2(1-\Delta)^p(\Delta + \operatorname{grad}\operatorname{div})\,.$$

It is a linear isomorphism $H^s(\mathbb{R}^n)^n \to H^{s-2p-2}(\mathbb{R}^n)^n$ for every s.

Theorem

Module properties of Sobolev spaces.

Let $(M^m, \hat{g})$ be a connected smooth Riemannian manifold of bounded geometry, and let E_1, E_2 be natural vector bundles of order 1 over M. Then the tensor product of smooth sections extends to a bounded bilinear mapping

$$\Gamma_{H^{k_1}(\hat{g})}(E_1) \times \Gamma_{H^{k_2}(\hat{g})}(E_2) \to \Gamma_{H^k(\hat{g})}(E_1 \otimes E_2)$$

in the following cases:

- *If $k = 0$ then $k_1, k_2 > 0$ and $k_1 + k_2 \geq \frac{m}{2}$, or $k_1 \geq 0$ and $k_2 > \frac{m}{2}$, or $k_1 > \frac{m}{2}$ and $k_2 \geq 0$.*
- *If $k > 0$ then $k_1, k_2 > k$ and $k_1 + k_2 \geq k + \frac{m}{2}$, or $k_1, k_2 \geq k$ and $k_1 + k_2 > k + \frac{m}{2}$.*

Lemma

Let $(M^m, \hat{g})$ *be a compact smooth Riemannian manifold, and let* E_1, E_2 *be natural vector bundles of order 1 over* M*. Let* $U \subset E_1$ *be an open neighborhood of the image of a smooth section, and let* $F : U \to E_2$ *be a fiber smooth mapping whose restriction to each fiber is real analytic. Let* $k > \frac{\dim(M)}{2}$*.*
Then $\Gamma_{H^k(\hat{g})}(U) := \{s \in \Gamma_{H^k(\hat{g})}(E_1) : s(M) \subset U\}$ *is open in* $\Gamma_{H^k(\hat{g})}(E_1)$ *and the mapping*

$$F_* : \Gamma_{H^k(\hat{g})}(U) \to \Gamma_{H^k(\hat{g})}(E_2), \quad s \mapsto F \circ s,$$

is real analytic.

Riemannian metric of Sobolev order on M

For $k > \frac{\dim(M)}{2}$, the subset $\mathrm{Met}_{H^k(\hat{g})}(M) = \Gamma_{H^k(\hat{g})}(S^2_+ T^*M) \subset \Gamma_{H^k(\hat{g})}(S^2 T^*M)$ of Riemannian metrics of Sobolev order k is open in the space of all H^k-sections, since these are continuous.
In each chart, the first derivative of $g \in \mathrm{Met}_{H^k(\hat{g})}(M)$ is of class H^{k-1} only, and thus no longer continuous. Nevertheless, *the Levi-Civita covariant derivative ∇^g for the metric g exists and is H^{k-1}.* This can be seen in several ways.

(1) Using the Levi-Civita covariant derivative $\nabla^{\hat{g}}$ for a smooth background Riemannian metrig $\hat{g}$, we express the Levi-Civita connection of $g \in \mathrm{Met}_{H^k(\hat{g})}(M)$ as

$$\nabla^g_X = \nabla^{\hat{g}}_X + A^g(X, \quad)$$

for a suitable
$A^g \in \Gamma_{H^{k-1}(\hat{g})}(T^*M \otimes T^*M \otimes TM) = \Gamma_{H^{k-1}(\hat{g})}(T^*M \otimes L(TM, TM)$.
This tensor field A has to satisfy the following conditions (for smooth vector fields X, Y, Z):

$$\begin{aligned} (\nabla^{\hat{g}}_X g)(Y, Z) = g(A(X, Y), Z) + g(Y, A(X, Z)) \quad &\iff \quad \nabla^g_X g = 0, \\ A(X, Y) = A(Y, X) \quad &\iff \quad \nabla^g \quad \text{is torsionfree.} \end{aligned}$$

We take the cyclic permutations of the first equation sum them with signs $+, +, -$, and use symmetry of A to obtain

$$2g(A(X, Y), Z) = (\nabla^{\hat{g}}_X g)(Y, Z) + (\nabla^{\hat{g}}_Y g)(Z, X) - (\nabla^{\hat{g}}_Z g)(X, Y);$$

this equation determines A uniquely as a H^{k-1}-tensor field. It is easy checked that it satisfies the two requirements above.

(2) For each local chart $u : U \to \mathbb{R}^m$ which extends to a compact neighborhood of $U \subset M$, the Christoffel forms are given by the usual formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \Big(\frac{\partial g_{ij}}{\partial u^l} - \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{il}}{\partial u^j} \Big) \in H^{k-1}(U, \mathbb{R}) .$$

They transform as the last part in the second tangent bundle. The associated spray S^g is an H^{k-1}-section of both $\pi_{TM} : T^2M \to TM$ and $T(\pi_M) : T^2M \to TM$. If $k > \frac{\dim(M)}{2} + 1$, then the S^g is continuous and we have local existence (but not uniqueness) of geodesics in each chart separately, by Peano's theorem. If $k > \frac{\dim(M)}{2} + 2$, we have the usual existence and uniqueness of geodesics, by Picard-Lindelöf, since then S^g is C^1 and thus Lipschitz.

(3) $\nabla^g : (X, Y) \mapsto \nabla^g_X Y$ is a bilinear bounded mapping

$$\Gamma_{H^k(\hat{g})}(TM) \times \Gamma_{H^l(\hat{g})}(TM) \to \Gamma_{H^{l-1}(\hat{g})}(TM) \quad \text{for } 1 \leq l \leq k;$$

we write $\nabla^g \in L^2(\Gamma_{H^k(\hat{g})}(TM), \Gamma_{H^l(\hat{g})}(TM); \Gamma_{H^{l-1}(\hat{g})}(TM))$ to express this fact. Moreover, ∇^g has the expected properties

$$\begin{aligned} \nabla^g_{fX} Y &= f \nabla^g_X Y && \text{for } f \in H^k(M, \mathbb{R}), \\ \nabla^g_X (fY) &= df(X) Y + f \nabla^g_X Y && \text{for } f \in H^k(M, \mathbb{R}). \end{aligned}$$

Its expression in a local chart is

$$\nabla_{X^i \partial_{u^i}} Y^j \partial_{u^j} = X^i (\partial_{u^i} Y^j) \partial_{u^j} - X^i Y^j \Gamma^k_{ij} \partial_{u^k}.$$

The global implicit equation holds for $X, Z \in \Gamma_{H^k(\hat{g})}(TM)$ and $Y \in \Gamma_{H^l(\hat{g})}(TM)$ for $\frac{\dim(M)}{4} + \frac{1}{2} < l \leq k$:

$$\begin{aligned} 2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Note that $[X, Y], [Z, Y] \in \Gamma_{H^{l-1}(\hat{g})}(TM)$, by differentiation and the module properties of Sobolev spaces.

(4) Moreover, for each $g \in \mathrm{Met}_{H^k(\hat{g})}(M)$, where $k > \frac{\dim(M)}{2}$, the *Bochner-Laplacian* is defined as

$$\Delta^g h := (\nabla^g)^{*,g} \nabla^g h = -\operatorname{Tr}^{g^{-1}}((\nabla^g)^2 h) = -\operatorname{Tr}(g^{-1}.\nabla^g.\nabla^g h).$$

Here we view $\nabla^g h \in \Gamma_{H^{l-1}(\hat{g})}(T^*M \otimes E)$; on a local chart (U, u) we have $\nabla^g_X Y = \sum_i X^i \nabla_{\partial_{u^i}} Y \in H^k(M)|_U \otimes \Gamma_{H^{l-1}(\hat{g})}(E)|_U$.
The operator Δ^g can act on all tensor fields h of Sobolev order l for $2 \leq l \leq k$, and it respects the degree and the symmetry properties of the tensor field it is acting on. If $h \in \Gamma_{H^l(\hat{g})}(S^2 T^*M)$ then the expression in a local chart (U, u^i) is as follows, using summation convention:

$$\begin{aligned}
&\Delta^g \left(h_{kl}\, du^k \otimes du^l \right) = \\
&\quad = \Bigg[\frac{\partial}{\partial u^i} \left(-\frac{1}{2} g^{ij} \frac{\partial h_{kl}}{\partial u^j} \right) - g^{ij} \Bigg(-\frac{1}{2} \frac{\partial g_{jm}}{\partial u^i} g^{mn} \frac{\partial h_{kl}}{\partial u^n} + \frac{1}{2} \frac{\partial h_{kl}}{\partial u^m} \Gamma^m_{ij} + 2 \frac{\partial h_{kn}}{\partial u^j} \Gamma^n_{il} + \\
&\qquad + h_{nl} \frac{\partial \Gamma^n_{jk}}{\partial u^i} + h_{ml} \Gamma^m_{nk} \Gamma^n_{ij} + h_{nl} \Gamma^n_{jm} \Gamma^m_{ik} + h_{mn} \Gamma^m_{jk} \Gamma^n_{il} \Bigg) \Bigg] (du^k \otimes du^l + du^l \otimes du^k)
\end{aligned}$$

Theorem

Let $k > \frac{\dim(M)}{2}$ *and let* $E \to M$ *be a natural bundle of first order. Then* $g \mapsto \nabla^g$ *is a real analytic mapping:*

$$\nabla : \mathrm{Met}_{H^k(\hat{g})}(M) \to L^2(\Gamma_{H^k(\hat{g})}(TM), \Gamma_{H^l(\hat{g})}(E); \Gamma_{H^{l-1}(\hat{g})}(E)),$$
$$\nabla : \mathrm{Met}_{H^k(\hat{g})}(M) \to L(\Gamma_{H^l(\hat{g})}(E); \Gamma_{H^{l-1}(\hat{g})}(T^*M \otimes E)),$$

for $1 \leq l \leq k$*.*
Consequently, $g \mapsto \Delta^g$ *is a real analytic mapping*

$$\mathrm{Met}_{H^k(\hat{g})}(M) \to L(\Gamma_{H^l(\hat{g})}(E), \Gamma_{H^{l-2}(\hat{g})}(E)),$$

for $2 \leq l \leq k$*, where* E *is a first oder natural bundle over* M*; i.e.,* E *is a bundle associated to the linear frame bundle of* M *for any finite dimensional representation of* $GL(\dim(M))$*.*

By the real analytic uniform boundedness theorem of convenient calculus this means one of the two equivalent assertions:

- For each smooth curve $g(t)$ of Sobolev Riemannian metrics in $\mathrm{Met}_{H^k(\hat{g})}(M)$ and for all fixed $X \in \Gamma_{H^{l-1}(\hat{g})}(TM)$ and $s \in \Gamma_{H^l(\hat{g})}(E)$ the mapping $t \mapsto \nabla_X^{g(t)} s \in \Gamma_{H^{l-1}(\hat{g})}(E)$ is smooth. And for each real analytic curve $g(t)$ of Sobolev Riemannian metrics in $\mathrm{Met}_{H^k(\hat{g})}(M)$ and for all fixed $X \in \Gamma_{H^{l-1}(\hat{g})}(TM)$ and $s \in \Gamma_{H^l(\hat{g})}(E)$ the mapping $t \mapsto \nabla_X^{g(t)} s \in \Gamma_{H^{l-1}(\hat{g})}(E)$ is real analytic.
- The mapping is real analytic from the open subset $\mathrm{Met}_{H^k(\hat{g})}(M)$ in the Hilbert space $\Gamma(S^2 T^* M)$ into the Banach space of all bounded bilinear operators $L^2(\Gamma_{H^{l-1}(\hat{g})}(TM), \Gamma_{H^l(\hat{g})}(E); \Gamma_{H^{l-1}(\hat{g})}(E))$ in the sense that is is locally given by convergent power series.

Similarly for the map $g \mapsto \Delta^g$.

Some Sobolev spaces can be described by $(1+\Delta^g)$ for $g \in \mathrm{Met}_{H^k(\hat{g})}(M)$

The following results were made possible by [Olaf Müller, 2015]. Let $k > \frac{m}{2}$. Obviously, $\Gamma_{H^0(g)}(E)$ and $\Gamma_{H^0(\hat{g})}(E)$ are isomorphic, but not isometric. The unbounded operator $(1+\Delta^g) : \Gamma_{H^0(g)}(E) \to \Gamma_{H^0(g)}(E)$ is positive, self-adjoint, elliptic, invertible. This is a k-save differential operator in the sense of [O.Müller, 2015] with Sobolev coefficients.

Theorem

Let $g \in \mathrm{Met}_{H^k(\hat{g})}(M)$*. If* $k > \frac{m}{2}$ *then for* $2 \leq l \leq k$ *the Laplacian is a bounded linear operator:*

$$\Delta^g = \mathrm{Tr}^{g^{-1}}.\nabla^g.\nabla^g : \Gamma_{H^l(\hat{g})}(E) \to \Gamma_{H^{l-2}(\hat{g})}(E)$$

and $1 + \Delta^g : \Gamma_{H^l(\hat{g})}(E) \to \Gamma_{H^{l-2}(\hat{g})}(E)$ *has a bounded inverse.*

Thank you for your attention