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A note on the inverse mapping theorem of F. Berquier

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We show that the notion of strict differentiability of [1], § IV is rather restrictive. In fact, we give a complete characterization of strictly differentiable mappings and use it to give a short proof of the main theorem of [1]. Notation is from [1], we only remark, that X is a finite dimensional C^0 manifold and C(R,X) is the space of continuous realvalued functions on X with the Whitney C^0 topology.

Theorem 1: Let $\Phi: C(R,X) \to C(R,X)$ be strictly differentiable at $\varphi_0 \in C(R,X)$. Then there exists an open neighbourhood V_0 of φ_0 in C(R,X) and a continuous function $f: \Omega \to R$, where Ω is a suitable open neighbourhood of the graph of φ_0 in X×R such that $\Phi(\varphi)(x) = f(x,\varphi(x))$, $x \in X$ for all $\varphi \in V_0$ and furthermore the map $f(x,.)$ is differentiable at $\varphi(x)$ for all $x \in X$ and $(D\Phi(\varphi_0)h)(x) = df(x,.)(\varphi_0(x)).h(x)$, $x \in X$ for all $h \in C(R,X)$. If Φ is furthermore differentiable in V_0 (cf. [1], § III) then $f(x,.)$ is differentiable in $\Omega \cap \{x\}\times R$ and $df(x,.)$ is continuous on each point of $\varphi_0(X)$.

Remark: The theorem says, that each strictly differentiable mapping $\Phi: C(R,X) \to C(R,X)$ looks locally like pushing forward sections of the trivial vector bundle X×R by a suitably differentiable fibre bundle homomorphism. Of course each such map is strictly differentiable, so we have obtained a complete characterization.

Proof: First we remark that the topology on C(R,X) can be described in the followig way: C(R,X) is a topological ring and sets of the form $V_\varepsilon = \{ g \in C(R,X) : |g(x)| < \varepsilon(x) , x \in X\}$ are a base of open neighbourhoods of 0 , where $\epsilon: X \to R$ is strictly positive and continuous.

Now by definition IV-1 of [1] we may write in a neighbourhood of φ_o $\Phi(g+h) - \Phi(g) = D\Phi(\varphi_o)h + R(g,h)$ where R satisfies the following condition: For each V_ϵ there are V_δ , V_λ such that $R(g,hk) \in h.V_\epsilon$ for all $g \in \varphi_o + V_\delta$, $h \in V_\lambda$ and $k \in C(R,X)$ with $|k(x)| \leq 1$, $x \in X$.
Let $V_\epsilon = V_1$, $k = 1$, then there are V_δ , V_λ such that $R(g,h) \in hV_1$ for all $g \in \varphi_o + V_\delta$, $h \in V_\lambda$. Let $V_o = \varphi_o + (V_\delta \cap V_{\lambda/2})$.
We claim that if $\varphi_1, \varphi_2 \in V_o$ and $x \in X$ such that $\varphi_1(x) = \varphi_2(x)$ then $\Phi(\varphi_1)(x) = \Phi(\varphi_2)(x)$. This follows from the equation $\Phi(\varphi_1) - \Phi(\varphi_2) = D\Phi(\varphi_o)(\varphi_1 - \varphi_2) + R(\varphi_2, \varphi_1 - \varphi_2)$, since $[D\Phi(\varphi_o)(\varphi_1 - \varphi_2)](x) = (\varphi_1 - \varphi_2)(x).[D\Phi(\varphi_o)(1)](x) = 0$ and $R(\varphi_2,\varphi_1 - \varphi_2) \in (\varphi_1 - \varphi_2).V_1$, so $R(\varphi_2,\varphi_1 - \varphi_2)(x) = 0$.
If $\varphi \in C(R,X)$ denote the graph of φ by $X_\varphi = \{ (x,\varphi(x)): x \in X \}$.
Let $\Omega = \bigcup \{ X_\varphi : \varphi \in V_o \}$. By the form of V_o it is clear that Ω is an open neighbourhood of X_{φ_o}. For $\varphi \in V_o$ define $f_\varphi: X_\varphi \to R$ by $f_\varphi(x,\varphi(x)) = \Phi(\varphi)(x)$. By the claim above we see that we have $f_\varphi | X_\varphi \cap X_\Psi = f_\Psi | X_\varphi \cap X_\Psi$ if φ and Ψ are in V_o, so we have got a mapping $f: \Omega \to R$, and $\Phi(\varphi)(x) = f(x,\varphi(x))$ for all $\varphi \in V_o$ and $x \in X$.
We show that f is continuous. If $(x_n,t_n) \to (x,t)$ in $\Omega \subseteq X \times R$ we may choose a sequence $\varphi_n \to \varphi$ in $C(R,X)$ such that $(x_n,\varphi_n(x_n)) = (x_n,t_n)$, $\varphi(x) = t$ (remembering that a sequence converges in the Whitney C^o topology iff it coincides with its limit off a compact set K of X after a while and converges uniformly on K). But then $\Phi(\varphi_n) \to \Phi(\varphi)$ uniformly, and $x_n \to x$, so $\Phi(\varphi_n)(x_n) = f(x_n,t_n) \to \Phi(\varphi)(x) = f(x,t)$.
Now we show that f is differentiable at each point of X_φ if Φ is differentiable at φ (strict differentiability implies differentiability, see [1]). We have $\Phi(\varphi+ h) - \Phi(\varphi) = D\Phi(\varphi)h + r_\varphi(h)$, where r_φ is a "small" mapping ([1], § III), i.e. for each V_ϵ there is V_δ such

that $r_\varphi(h) \in h.V_\epsilon$ for all $h \in V_\delta$. Evaluating this equation at x we get $f(x,\varphi(x) + h(x)) - f(x,\varphi(x)) = [D\Phi(\varphi)(1)](x).h(x) + r_\varphi(h)(x)$. It is clear that the map $h(x) \to r_\varphi(h)(x)$ is $o(h(x))$ by the "smallness" of r_φ , so $f(x,.)$ is differentiable at $\varphi(x)$ and $[D\Phi(\varphi)h](x) =$ $= df(x,.)(\varphi(x)).h(x)$.

It remains to show that $df(x,.)$ is continuous at each point of X_{φ_o} . This follows easily from Proposition IV-2 of [1] with the method we just applied to show that f is continuous. qed.

<u>Theorem 2</u>: Let $\Phi: C(R,X) \to C(R,X)$ be differentiable in a neighbourhood of $\varphi_o \in C(R,X)$ and strictly differentiable at φ_o and suppose that $D\Phi(\varphi_o)$ is surjective. Then there exists a neighbourhood V_o of φ_o and a neighbourhood W_o of $\Phi(\varphi_o)$ in $C(R,X)$ such that $\Phi: V_o \to W_o$ is a homeomorphism onto. Furthermore the map $\Phi^{-1}: W_o \to V_o$ is differentiable on W_o, strictly differentiable at $\Phi(\varphi_o)$ and for each $\varphi \in V_o$ we have $D(\Phi^{-1})(\Phi(\varphi)) = (D\Phi(\varphi))^{-1}$.

<u>Proof</u>: By theorem 1 we have that $\Phi(\varphi)(x) = f(x,\varphi(x))$ and $D\Phi(\varphi)(1)(x) =$ $df(x,.)(\varphi(x))$. Since $D\Phi(\varphi_o)$ is surjective we conclude that $df(x,.)(\varphi_o(x)) \neq 0$ for all $x \in X$, and since $df(x,.)$ is continuous at $\varphi_o(x)$ it is $\neq 0$ on a neighbourhood of $\varphi_o(x)$ in R. Writing $f_x = f(x,.)$ we see that f_x^{-1} exists and is differentiable on some neighbourhood of $\Phi(\varphi_o)(x)$ in R by the ordinary inverse function theorem. So the map $(x,t) \to (x,f(x,t))$ is locally invertible at each point of the graph X_{φ_o} of φ_o ; one may construct a neighbourhood Ω of X_{φ_o} in $X\times R$ such that this map is invertible there (considering neighbourhoods $U_x \times V_{\varphi_o(x)}$ of $(x,\varphi_o(x))$ where $Id \times f$ is invertible and taking $\Omega = \bigcup_x U_x \times V_{\varphi_o(x)}$). Then $\Phi^{-1}(\Psi)(x) = f_x^{-1}(\Psi(x))$; all other claims of the theorem are easily checked up. qed.

Remark: Theorem 2 is a little more general than the result in [1]. The method of proof is adapted from [2], 4.1 and 4.2 where we treated an anlogous smooth result.

References

[1] F. BERQUIER: Un theoreme d'inversion locale, to appear in the Proceedings of the Conference on Categorical Topology, Mannheim 1975.

[2] P.MICHOR: Manifolds of smooth maps, to appear.

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