PS Combinatorics

(Modul: “Combinatorics” (MALK))

Markus Fulmek

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Exercise 1: *Show that the number of all unlabelled ordered nonempty rooted trees with n vertices, where every* inner *vertex has* 2 *or* 3 *branches, equals*

$$\frac{1}{n}\sum_{j}\binom{n}{j}\binom{j}{3j-n+1}.$$

Hint: *Find an equation for the generating function and use Langrange's inversion formula.*

Exercise 2: *How many ways are there to (properly) parenthesize n pairwise non–commuting elements of a monoid? And how does this number change if the n elements are pairwise commuting?*
For example, consider 6 non–commuting *elements* $x_1, x_2, \ldots, x_6$. Two different *ways to parenthesize them properly would be*

$$((x_2x_5)((x_1(x_4x_6))x_3)) \text{ and } ((x_3(x_1(x_4x_6)))(x_5x_2)).$$

However, these would be equivalent *for* commuting *elements.*

Hint: *Translate parentheses to* labelled binary trees*: The outermost pair of parentheses corresponds to the root, and the elements of the monoid correspond to the leaves.*

Exercise 3: *Develop a theory for* weighted *generating functions (for labelled and unlabelled species). I.e., let $\mathcal{A}$ be some species with weight function ω which assigns to every object $A \in \mathcal{A}$ some element in a ring R (for instance, $R = \mathbb{Z}[y]$, the ring of polynomials in y with coefficients in $\mathbb{Z}$). So the generating function to be considered is*

$$\sum_{A\in\mathcal{A}} z^{\|A\|}\cdot\omega(A).$$

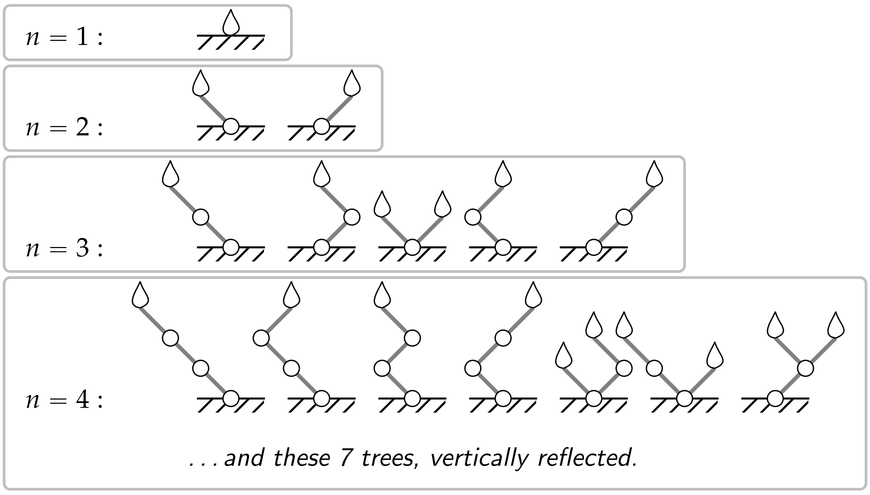
How should we define the weight function for sums, products and composition of species, so that the corresponding assertions for generating functions remain valid?

Exercise 4: *Let $f(m,n)$ be the number of all paths from $(0,0)$ to (m,n) in $\mathbb{N}\times\mathbb{N}$, where each single steps is either $(1,0)$ (step to the right) or $(0,1)$ (step to the left) or $(1,1)$ (diagonal step upwards). Use the language of species to show that*

$$\sum_{m,n\geqslant 0} f(m,n)\,x^m y^n = \frac{1}{1-x-y-x\cdot y}.$$

Exercise 5: *Determine the number of all* unlabelled ordered binary *rooted trees with n vertices and k leaves.*

Hint: *Consider the generating function in* 2 *variables z and y, where every rooted tree W with n vertices and k leaves is assigned $\omega(W) := z^n y^k$. The following picture shows these trees for $n = 1, 2, 3, 4$:*



I.e., the first terms of the generating function are:

$$T(z, y) := \sum_{W} \omega(W) = z \cdot y + z^2 \cdot 2y + z^3 \left(y^2 + 4y\right) + z^4 \left(6y^2 + 8y\right) + \cdots$$

Find an equation for this generating function T, from which the series expansion can be derived.

Exercise 6: *Determine the number of all* labelled unordered *rooted trees with n vertices and k leaves.*

Hint: Consider the exponential generating function in 2 variables z and y (as in Exercise 5) and use Lagrange's inversion formula.

Exercise 7: *Prove Cayley's formula (the number of labelled trees on n vertices equals n^{n-2}) as follows: Take a labelled tree on n vertices and tag two vertices S and E. View S and E as the starting point and ending point of the unique path p connecting S and E in the tree. Now orient all edges belonging to p "from S to E", and all edges not belonging to p "towards p". Now travel along p from S to E and write down the labels of the vertices: Whenever a new maximal label is encountered, close a cycle (by inserting an oriented edge from the vertex before this new maximum to the start of the "current cycle") and start a new cycle. Interpret the resulting directed graph as a function $[n] \to [n]$ (i.e., a directed edge from a to b indicates that the function maps a to b).*

Exercise 8: *Show that the number of all graphs on n vertices, m edges and k components equals the coefficient of $u^n \alpha^m \beta^k / n!$ in*

$$\left(\sum_{n \geqslant 0} (1 + \alpha)^{\binom{n}{2}} \frac{u^n}{n!} \right)^{\beta}.$$

Hint: Find a connection between the generating function of all labelled graphs (weight $\omega(G) := u^{|V(G)|} \alpha^{|E(G)|}$) and the generating function of all connected *labelled graphs.*

Exercise 9: *Show that the number of labelled* unicyclic *graphs (i.e., connected graphs with exactly* one *cycle) on n vertices equals*

$$\frac{1}{2}\sum_{j=3}^{n}\binom{n}{j} j!\, n^{n-1-j}.$$

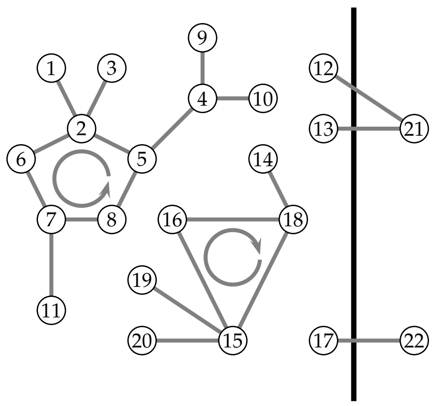
Hint: Find a representation as a composition of species.

Exercise 10: *Let $T(u)=\sum_{n\geqslant 0}(n+1)^{n-1}u^n/n!$. Prove the identity*

$$\frac{T^j(u)}{1-uT(u)}=\sum_{l\geqslant 0}(l+j)^l\,\frac{u^l}{l!}.$$

Hint: For a bijective *proof consider the species $\mathcal{W}$ of labelled trees, where the vertex with the largest label (i.e.: n, if the tree has n vertices) is tagged as the* root, *but this label is* erased, *and the root does not contribute to the* size *of the tree (i.e: if t has n vertices (including the root), then we have $\|t\|_{\mathcal{W}}=n-1$): Obviously, $T=\mathcal{GF}_W$.*
Now consider functions $f:[l]\to[l+j]$. Visualize such function f as a directed graph with vertex set $[l+j]$ and directed edges $(x,f(x))$. The following graphic illlustrates this for the case $l=20$, $j=2$ and

$$(f(n))_{n=1}^{l}=(2,6,2,5,2,7,8,5,4,4,7,21,21,18,16,18,22,15,15,15):$$



The components of this graph are rooted trees or unicyclic graphs; all *vertices in $[l+j]\setminus[l]$ appear as roots of corresponding trees.*

Exercise 11: *The* derivative *$\mathcal{A}'$ of a labelled species $\mathcal{A}$ is defined as follows: Objects of species $\mathcal{A}'$ with size $n-1$ are objects of $\mathcal{A}$ with size n, whose atoms are numbered from 1 to $n-1$* (not *from 1 to n), such that there is one atom without a label. A typical element of* `Sequences`$'$ *is*

$$(3,1,2,5,\circ,4),$$

where $\circ$ indicates the unlabelled atom.
Show: The generating function of $\mathcal{A}'$ is precisely the derivative of the generating function of $\mathcal{A}$. Moreover, show:

1. $(\mathcal{A}+\mathcal{B})'=\mathcal{A}'+\mathcal{B}'$.

2. $(\mathcal{A} \star \mathcal{B})' = \mathcal{A}' \star \mathcal{B} + \mathcal{B}' \star \mathcal{A}$.
3. $(\mathcal{A} \circ \mathcal{B})' = \left(\mathcal{A}' \circ \mathcal{B}\right) \star \mathcal{B}'$.

These equations are to be understood as size–preserving bijections.

Exercise 12: *Show the following identities for labelled species:*

1. $\texttt{oPar}' = \texttt{oPar}^2 \star \texttt{Sets}$, *where* `oPar` *denotes the species of* ordered *set partitions (i.e., the order of the blocks of the partitions matters).*
2. $\texttt{Polyp}' = \texttt{Sequences}\,(\texttt{Atom}) \star \texttt{Sequences}\,(\texttt{2Atom})$, *where* `Polyp` *denotes the species*

$$\texttt{Cycles}\left(\texttt{Sequences}_{\geqslant 1}\right);$$

i.e., an object of `Polyp` *is a cycle, where there is a nonempty sequence attached to each atom of the cycle.*

Exercise 13: *Let $\mathcal{A}$ be the (labelled) species of (unordered) rooted trees, $\mathcal{U}$ the (labelled) species of trees (without root) and $\mathcal{F}$ the (labelled) species of rooted forests. Show the following equations:*

1. $\mathcal{A}' = \mathcal{F} \star \texttt{Sequences}\,(\mathcal{A})$,
2. $\mathcal{U}'' = \mathcal{F} \star \mathcal{A}'$,
3. $\mathcal{A}'' = \left(\mathcal{A}'\right)^2 + \left(\mathcal{A}'\right)^2 \star \texttt{Sequences}\,(\mathcal{A})$.

Exercise 14: *Compute all derivatives of* $\texttt{Sets}^2$ *and of* `Sequences`.

Exercise 15: *How many* different *necklaces of n pearls in k colours are there? (This should be understood "as in real life", where rotations* and *reflections of necklaces are considered equal; in contrast to the presentation in the lecture course.)*

Exercise 16: *Determine the cycle index series of the species* `Fixfree` *of permutations* without *fixed points.*

Hint: *Show the relation* $\texttt{Sets} \cdot \texttt{Fixfree} = \texttt{Permutations}$.

Exercise 17: *Determine the cycle index series for the species "set partitions".*

Exercise 18: *Given some arbitrary species $\mathcal{A}$, show the formula*

$$\mathrm{Z}_{\mathcal{A}'}\,(x_1, x_2, \dots) = \left(\frac{\partial}{\partial x_1} \mathrm{Z}_{\mathcal{A}}\right)(x_1, x_2, \dots).$$

Exercise 19: *Show:*

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \exp\left(\sum_{n \geqslant 1} \frac{1}{n} \frac{x^n}{1-x^n}\right).$$

<u>*Hint:*</u> `Permutations` = `Sets(Cycles)`.

Exercise 20: *Let P be some finite poset and $f : P \to P$ an order–preserving bijection. Show that f^{-1} is* also *order–preserving.*
Show that this is not true in general *for infinite posets.*

Exercise 21: *(a) Find a (finite) poset P where*

- *the length of a* longest *chain is l,*
- every *element of P belongs to a chain of length l,*

which nevertheless has a maximal *chain of length $< l$.*
(b) Let P be a (finite) poset with connected Hasse–diagram, where the longest *chain has length l (there might be several longest chains). Moreover, assume that for all $x, y \in P$ such that $y \gtrdot x$ (y covers x), x and y belong to a chain of length l: Show that under this assumption* all *maximal chains have length l.*

Exercise 22: *Consider the "zigzag–poset" Z_n with elements $x_1, x_2, \ldots, x_n$ and cover relations*

$$x_{2i-1} \lessdot x_{2i} \text{ for } i \geqslant 1, 2i \leqslant n \text{ and } x_{2i} \gtrdot x_{2i+1} \text{ for } i \geqslant 1, 2i+1 \leqslant n$$

a) How many order ideals are there in Z_n?
b) Let $W_n(q)$ be the rank generating function of the lattice of order ideals $J(Z_n)$ of Z_n. For instance, $W_0(q) = 1$, $W_1(q) = 1 + q$, $W_2(q) = 1 + q + q^2$, $W_3(q) = 1 + 2q + q^2 + q^3$. Show:

$$W(q,z) := \sum_{n=0}^{\infty} W_n(q)\, z^n = \frac{1 + (1+q)z - q^2 z^3}{1 - (1 + q + q^2)z^2 + q^2 z^4}.$$

c) Let e_n be the number of all linear extensions of Z_n. Show:

$$\sum_{n=0}^{\infty} e_n \frac{z^n}{n!} = \tan z + \frac{1}{\cos z}.$$

Exercise 23: *Let P, Q be graded posets, let r and s be the maximal ranks of P and Q, respectively, and let $F(P,q)$ and $F(Q,q)$ be the corresponding rank generating functions. Show:*
a) If $r = s$ (otherwise maximal chains would be of different lengths), then $F(P+Q,q) = F(P,q) + F(Q,q)$.
b) $F(P \oplus Q, q) = F(P,q) + q^{r+1} F(Q,q)$.
c) $F(P \times Q, q) = F(P,q) \cdot F(Q,q)$.
d) $F(P \otimes Q, q) = F\left(P, q^{s+1}\right) \cdot F(Q,q)$.

Exercise 24: *Let P, Q, R be posets. Find order isomorphisms for the following relations:*
a) $P \times (Q + R) \simeq (P \times Q) + (P \times R)$.
b) $R^{P+Q} \simeq R^P \times R^Q$.
c) $\left(R^Q\right)^P \simeq R^{Q \times P}$.

Exercise 25: *Let P be a finite poset and define $G_P(q,t) := \sum_I q^{|I|} t^{m(I)}$, where the summation range is the set of all order ideals I of P, and where $m(I)$ denotes the number of maximal elements of I. (For instance: $G_P(q,1)$ is the rank generating function of $\mathcal{J}(P)$.)*
a) Let Q be a poset with n elements. Show:

$$G_{P \otimes Q}(q,t) = G_P\left(q^n, q^{-n} \cdot \left(G_Q(q,t) - 1\right)\right),$$

where $P \otimes Q$ denotes the ordinal *product.*
b) Let P be a poset with p elements. Show:

$$G_P\left(q, \frac{q-1}{q}\right) = q^p.$$

Exercise 26: *Let L be a finite lattice. Show that the following three conditions are equivalent for all $x, y \in L$:*
(a) L is graded (i.e.: all maximal chains have in L the same length), and for the rank function $\mathbf{rk}$ of L there holds

$$\mathbf{rk}(x) + \mathbf{rk}(y) \geqslant \mathbf{rk}(x \wedge y) + \mathbf{rk}(x \vee y).$$

(b) If y covers the element $x \wedge y$, then $x \vee y$ covers the element x.
(c) If x and y both cover element $x \wedge y$, then $x \vee y$ covers both elements x and y.
(A lattice L obeying one of these conditions is called semimodular.*)*

Hint: Employ an indirect proof for (c) $\implies$ (a): For the first assertion in (a), if there are intervals which are not graded, *then we may choose an interval $[u,v]$ among them which is* minimal *with respect to set–inclusion (i.e., every sub–interval* is *graded). Then there are two elements $x_1, x_2 \in [u,v]$, which both cover u, and the length of all maximal chains in $[x_i, v]$ is ℓ_i, such that $\ell_1 \neq \ell_2$. Now apply (b) or (c) to x_1, x_2. For the second assertion in (a), take a pair $x, y \in L$ with*

$$\mathbf{rk}(x) + \mathbf{rk}(y) < \mathbf{rk}(x \wedge y) + \mathbf{rk}(x \vee y), \tag{0.1}$$

such that the length of the interval $[x \wedge y, x \vee y]$ is minimal, and under all such pairs, also $\mathbf{rk}(x) + \mathbf{rk}(y)$ is minimal. Since it is impossible that both *x and y cover $x \wedge y$ (why?), w.l.o.g. there is an element x' with $x \wedge y < x' < x$. Show that $X = x, Y = x' \vee y$ is a pair such that $\mathbf{rk}(X) + \mathbf{rk}(Y) < \mathbf{rk}(X \wedge Y) + \mathbf{rk}(X \vee Y)$, but where the length of the interval $[X \wedge Y, X \vee Y]$ is less than the length of $[x \wedge y, x \vee y]$.*

Exercise 27: *Let L be a finite semimodular lattice. Show that the following two conditions are equivalent:*
a) For all elements $x, y, z \in L$ with $z \in [x,y]$ (i.e., $x \leqslant y$) there is an element $u \in [x,y]$, such that $z \wedge u = x$ and $z \vee u = y$ (u is a "complement" of z in the interval $[x,y]$).
b) L is atomic, *i.e.:* Every *element can be represented as the supremum of atoms.*
(A finite semimodular lattice obeying one of these conditions is called geometric.*)*

Exercise 28: *Let G be a (labelled) graph on n vertices. A partition of the vertex–set $V(G)$ is called connected if every block of the partition corresponds to a* connected *induced subgraph of G. The set of all connected partitions is a subposet of the poset of partitions of $V(G)$, and thus a poset itself. (If G is the complete graph, the poset of connected partitions of $V(G)$ is the same as the poset of* all *partitions of $V(G)$.)*
Show that the poset of connected partitions of G is a geometric lattice.

Exercise 29: *A lattice L is called* modular *if it is graded and for all $x, y \in L$ there holds:*

$$\mathbf{rk}(x) + \mathbf{rk}(y) = \mathbf{rk}(x \wedge y) + \mathbf{rk}(x \vee y). \tag{0.2}$$

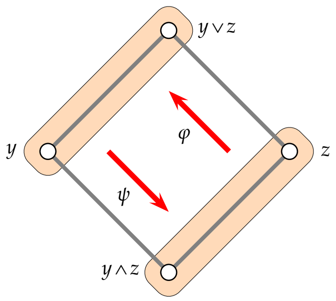
(In particular, the lattice $L(V)$ of subspaces of a finite vector space is modular.)
Show: A finite lattice L is modular if and only if for all $x, y, z \in L$ with $x \leqslant z$ there holds:

$$x \vee (y \wedge z) = (x \vee y) \wedge z. \tag{0.3}$$

Hint: *Show that* (0.3) *implies the* Diamond Property*: The mappings*

$$\psi : [y, y \vee z] \to [y \wedge z, z], \psi(x) = x \wedge z$$
$$\varphi : [y \wedge z, z] \to [y, y \vee z], \varphi(x) = x \vee y$$

are order preserving bijections *with $\phi \circ \psi = \mathrm{id}$, see the following picture:*



Exercise 30: *Show: The lattice Π_n of all partitions of an n–element set is* not *modular.*

Exercise 31: *Prove the "NBC–Theorem" ("Non–broken circuit theorem") of G.–C. Rota: Let L be geometric lattice. We assume that the atoms of L are* labelled *(with natural numbers $1, 2, \ldots$). A set B of atoms is called* independent*, if $\mathbf{rk}(\bigvee B) = |B|$, otherwise it is called* dependent*. A set C of atoms is called a* circuit*, if C is a* minimal *dependent set. A* broken circuit *is a set corresponding to a circuit from which its largest atom (with respect to the labeling of atoms) was removed. A* non–broken circuit *is a set B of atoms which does not contain a broken circuit. Then Rota's Theorem states:*

$$\mu(\hat{0}, x) = (-1)^{\mathbf{rk}(x)} \cdot \# \left(\text{non–broken circuits } B \text{ with } \bigvee B = x\right).$$

Exercise 32: *Show: The Möbius function* $\mu(x,y)$ *of a semimodular lattice is alternating, i.e.*

$$(-1)^{\text{length of } [x,y]}\, \mu(x,y) \geqslant 0.$$

Moreover, show that the Möbius function of a geometric lattice is strictly *alternating, i.e.*

$$(-1)^{\text{length of } [x,y]}\, \mu(x,y) > 0.$$

Hint: *Use the following formula for the Möbius function of a lattice:*

$$\sum_{x: x\vee a=\hat{1}} \mu(\hat{0},x) = 0 \text{ for all } a \in L. \tag{0.4}$$

Show that if a *is an atom, then there follows:*

$$\mu(\hat{0},\hat{1}) = -\sum_{\substack{x \text{ coatom} \\ x \ngeqslant a}} \mu(\hat{0},x). \tag{0.5}$$

Exercise 33: *Let* $f,g : \mathbb{N} \to \mathbb{R}^+$ *be two functions (from the natural numbers to the nonnegative reals). Which of the following rules is valid for* $n \to \infty$ *(under which preconditions)?*

$$O(f(n)) + O(g(n)) = O(f(n)+g(n)) \qquad O(f(n)) - O(g(n)) = O(f(n)-g(n))$$

$$O(f(n)) \cdot O(g(n)) = O(f(n)\cdot g(n)) \qquad \frac{O(f(n))}{O(g(n))} = O(f(n)/g(n))$$

$$O(f(n))^{O(g(n))} = O\left(f(n)^{g(n)}\right) \qquad \exp(O(f(n))) = O(\exp(f(n)))$$

$$\sqrt{O(f(n))} = O\left(\sqrt{f(n)}\right) \qquad g(O(f(n))) = O(gf(n))$$

$$e^{f(n)+O(g(n))} = e^{f(n)}(1+O(g(n))) \qquad \log(f(n)+g(n)) = \log(f(n)) + O(g(n)/f(n)).$$

(The "equations" should be interpreted as follows: $O(f(n)) + O(g(n))$ *is the class of all functions of the form* $f^\star(n) + g^\star(n)$*, where* $f^\star(n) = O(f(n))$ *and* $g^\star(n) = O(g(n))$*; the first "equation" means, that this class is* contained *in the class* $O(f(n)+g(n))$*.)*

Exercise 34: *Same question as in the preceding exercise, where* $O(.)$ *is replaced by* $o(.)$*.*

Exercise 35: *Let* f_1, f_2, g_1, g_2 *be functions* $\mathbb{N} \to \mathbb{C}$*, such that* $f_1(n) \sim f_2(n)$ *and* $g_1(n) \sim g_2(n)$ *for* $n \to \infty$*. Which of the following rules are valid for* $n \to \infty$ *(under which preconditions)?*

$$f_1(n) + g_1(n) \sim f_2(n) + g_2(n) \qquad f_1(n) - g_1(n) \sim f_2(n) - g_2(n)$$

$$f_1(n) \cdot g_1(n) \sim f_2(n) \cdot g_2(n) \qquad \frac{f_1(n)}{g_1(n)} \sim \frac{f_2(n)}{g_2(n)}$$

$$f_1(n)^{g}{}_1(n) \sim f_2(n)^{g_2(n)} \qquad \exp(f_1(n)) \sim \exp(f_2(n))$$

$$\sqrt{f_1(n)} \sim \sqrt{f_2(n)} \qquad g_1(f_1(n)) \sim g_2(f_2(n))$$

$$\log(f_1(n)) \sim \log(f_2(n)).$$

Exercise 36: *Let f, g be complex functions which are analytic on some given domain. Which of the following rules are valid (under which preconditions)?*

$$\mathrm{Sing}\,(f \pm g) \subseteq \mathrm{Sing}\,(f) \cup \mathrm{Sing}\,(g) \qquad \mathrm{Sing}\,(f \cdot g) \subseteq \mathrm{Sing}\,(f) \cup \mathrm{Sing}\,(g)$$

$$\mathrm{Sing}\,(f/g) \subseteq \mathrm{Sing}\,(f) \cup \mathrm{Sing}\,(g) \cup \mathrm{Null}\,(g) \qquad \mathrm{Sing}\,(f \circ g) \subseteq \mathrm{Sing}\,(g) \cup g^{(-1)}\,(\mathrm{Sing}\,(f))$$

$$\mathrm{Sing}\left(\sqrt{f}\right) \subseteq \mathrm{Sing}\,(f) \cup \mathrm{Null}\,(f) \qquad \mathrm{Sing}\,(\log f) \subseteq \mathrm{Sing}\,(f) \cup \mathrm{Null}\,(f)$$

$$\mathrm{Sing}\left(f^{(-1)}\right) \subseteq f\,(\mathrm{Sing}\,(f)) \cup f\,(\mathrm{Null}\,(f'))$$

Here, $\mathrm{Sing}\,(f)$ *denotes the set of singular points of* f*, and* $\mathrm{Null}\,(f)$ *denotes the set of zeroes of* f*.*

Exercise 37: *Let $p\,(n)$ be the number of (integer) partitions of n. We know that*

$$\sum_{n=0}^{\infty} p\,(n)\, z^n = \prod_{i=1}^{\infty} \frac{1}{1-z^i}.$$

What are the (dominant) singular points of this generating function? What does this imply for the asymptotic behaviour of $p\,(n)$ for $n \to \infty$?

Exercise 38: *Let w_n be the number of possibilities of paying an amount of n Euro using 1–Euro–coins, 2–Euro–coins and 5–Euro–notes (the* order *of the coins and notes is irrelevant).*

1. *Determine the generating function $\sum_{n \geqslant 0} w_n z^n$.*
2. *Determine the asymptotic behaviour of w_n for $n \to \infty$.*

Exercise 39: *Let $D_{n,k}$ be the number of permutations of $[n]$, whose disjoint cycle decomposition does not contain any cycle of length $\leqslant k$. (So $D_{n,1}$ is the number of fixed–point–free permutations of $[n]$.)*

1. *Show:*

$$\sum_{n \geqslant 0} \frac{D_{n,k}}{n!} z^n = \frac{e^{-z-\frac{z^2}{2}-\cdots-\frac{z^k}{k}}}{1-z}.$$

2. *For k fixed, what is the asymptotic behaviour of $D_{n,k}$ for $n \to \infty$?*

Exercise 40: *Let $w_{n,k}$ be the number of possibilities of paying an amount of n Euro using 1–Euro–coins, 2–Euro–coins and 5–Euro–notes, where exactly k coins or notes are used (again, the* order *of the coins and notes is irrelevant).*

1. *Determine the generating function $\sum_{n,k \geqslant 0} w_{n,k} z^n t^k$.*
2. *If we assume that all possibilities which are enumerated by $w_n = \sum_k w_{n,k}$ have the same probability: What is the asymptotic behaviour of the* expected value *for the number of coins and notes which are used to pay an amount of n Euro, for $n \to \infty$?*

Exercise 41: *Let R_n be the number of possibilities of (completely) tiling a $2 \times n$ rectangle by 1×1 squares and 1×2 rectangles (dominoes).*

1. *Determine the generating function $\sum_{n\geqslant 0} R_n z^n$.*
2. *Determine the asymptotic behaviour of R_n for $n \to \infty$?*

Hint: *The fact that the dominant singularity can not be calculated explicitly is not an obstacle. One has to continue to calculate with the dominant singularity symbolically.*

Exercise 42: *Let $p(n,k)$ be the number of all (integer) partitions of n with at most k summands. Show:*

$$\sum_{n\geqslant 0} p(n,k)\, z^n = \frac{1}{(1-z)(1-z^2)\cdots(1-z^k)}.$$

Determine the asymptotic behaviour of $p(n,k)$ for fixed k and $n \to \infty$.

Exercise 43: *The exponential generating function of the Bernoulli numbers b_n is*

$$\sum_{n\geqslant 0} b_n z^n = \frac{z}{\mathrm{e}^z - 1}.$$

Determine the asymptotic behaviour of b_n for $\to \infty$.

Exercise 44: *The fraction $\frac{1}{\Gamma(z)}$ is an entire function with zeroes $0, -1, -2, \dots$. Show Weierstraß' product representation:*

$$\frac{1}{\Gamma(s)} = s\mathrm{e}^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) \mathrm{e}^{-s/n},$$

where γ denotes Eulers constant

$$\gamma = \lim_{n\to\infty} (H_n - \log n)$$

and

$$H_n = \sum_{j=1}^{n} \frac{1}{j}$$

denotes the n–th harmonic number.

Exercise 45: *Show the* reflection formula *for the gamma function:*

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{0.6}$$

Hint: *Use the product representation of the sine:*

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right). \tag{0.7}$$

Exercise 46: *Show the* duplication formula *for the gamma function:*

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z)$$

and its generalisation

$$\prod_{j=0}^{m}\Gamma\left(z+\frac{j}{m}\right) = m^{\frac{1}{2}-mz}\,(2\pi)^{\frac{m-1}{2}}\,\Gamma(mz)\,.$$

Exercise 47: *Show using Stirling's Formula for the* Γ*–Function*

$$\binom{n+\alpha-1}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right)$$

Hint: *Here is Stirling's Formula:*

$$\Gamma(s+1) \sim s^s e^{-s}\sqrt{2\pi s}\left(1+O\left(\frac{1}{s}\right)\right).$$

Exercise 48: *A* Motzkin path *is a lattice path, where every step is of the form* $(1,0)$, $(1,1)$, $(1,1)$ *(horizontal, up– and down–steps), which starts at the origin, returns to the* x*–axis and never goes below the* x*–axis. Let* M_n *be the number of all Motzkin paths of length (i.e., number of steps)* n*. Show that the generating function for Motzkin paths is given by*

$$\sum_{n\geqslant 0} M_n z^n = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}.$$

Derive an explizit formula for M_n*. Use the generating function to determine the asymptotic behaviour of* M_n *for* $n\to\infty$*.*

Exercise 49: *A* Schröder path *is a lattice path consisting of steps* $(2,0)$, $(1,1)$ *and* $(1,-1)$ *(i.e., double horizontal, upward and downward steps) which starts at the origin, returns to the* x*–axis but never falls below the* x*–axis. If we assume that all Schröder paths of length* n *have the same probability: What is the asymptotics of the expected value of the number of steps for a Schröder path of length* n *for* $n\to\infty$*?*

Exercise 50: *Consider the number of cycles in the disjoint cycle decomposition of permutations of* $[n]$ on average*: What is the asymptotics for this average for* $n\to\infty$*?*

Exercise 51: *Let* $H_n = \sum_{j=1}^{n} j^{-1}$ *be the* n*–th* harmonic number*. Show that*

$$\sum_{n\geqslant 0} H_n z^n = \frac{1}{1-z}\log\frac{1}{1-z},$$

and use this result together with singularity analysis to obtain an asymptotic expansion of H_n *for* $n\to\infty$*.*

Exercise 52: *Let u_n be the number of permutations of $[n]$, which only have cycles of* odd *length in their decomposition into disjoint cycles. Determine the asymptotic behaviour of u_n for $n \to \infty$.*

Hint: Observe that the generating function is analytic in a "Double–Delta–Domain" (i.e., in a disk with two "dents" at the two singularities), so we have two contributions according to the Transfer Theorem.

Exercise 53: *Determine the asymptotic behaviour of expected value and variance of the number of connected components of 2–regular labelled graphs with n vertices for $n \to \infty$.*

Hint: Use the following additional information:

Definition 0.0.1. *A function $G(z)$ which is analytic at 0, has only non–negative coefficients and finite radius of convergence ρ, is said to be of logarithmic type with parameters (κ, λ), where $\kappa, \lambda \in \mathbb{R}$, $\kappa \neq 0$, if the following conditions hold:*

1. *the number ρ is the unique singularity of $G(z)$ on $|z| = \rho$,*
2. *$G(z)$ is continuable to a Δ–domain at ρ,*
3. *$G(z)$ satisfies*

$$G(z) = \kappa \cdot \log \frac{1}{1-z} + \lambda + O\left(\frac{1}{(\log(1 - z/\rho))^2}\right) \text{ as } z \to \rho \text{ in } \Delta. \tag{0.8}$$

Definition 0.0.2. *The labelled construction*

$$\mathcal{F} = \mathtt{Sets}(\mathcal{G})$$

is called a (labelled) exp–log–scheme if the exponential generating function $G(z)$ of $\mathcal{G}$ is of logarithmic type.
The unlabelled construction

$$\mathcal{F} = \mathtt{Multisets}(\mathcal{G})$$

is called an (unlabelled) exp–log–scheme if the ordinary generating function $G(z)$ of $\mathcal{G}$ is of logarithmic type, with $\rho < 1$.
In both cases (labelled and unlabelled), the quantities (κ, λ) from (0.8) *are called the* parameters *of the scheme.*

Theorem 0.0.3 (Exp–log scheme)**.** *Consider an exp–log scheme with parameters (κ, λ).*
Then we have

$$[\![z^n]\!]\, G(z) = \frac{\kappa}{n \cdot \rho^n} \cdot \left(1 + O\left((\log n)^{-2}\right)\right),$$
$$[\![z^n]\!]\, F(z) = \frac{e^{\lambda + r_0}}{\Gamma(\kappa)} \cdot n^{\kappa - 1} \cdot \rho^{-n} \cdot \left(1 + O\left((\log n)^{-2}\right)\right),$$

where $r_0 = 0$ in the labelled case and $r_0 = \sum_{j \geqslant 2} \frac{G(\rho^j)}{j}$ in the unlabelled case.
If we consider the number X of $\mathcal{G}$–components in a (randomly chosen) $\mathcal{F}$–object of size n, then the expected value *of X is*

$$\kappa \cdot (\log n - \Psi(\kappa)) + \lambda + r_1 + O\left((\log n)^{-1}\right) \text{ (where } \Psi(s) = \frac{d}{ds}\Gamma(s)\text{)},$$

where $r_1 = 0$ in the labelled case and $r_0 = \sum_{j \geqslant 2} G(\rho^j)$ in the unlabelled case. The variance of X is $O(\log n)$.

Exercise 54: *Determine the asymptotic behaviour of the sum*

$$f_n = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k}$$

for $n \to \infty$.

<u>Hint:</u> Compute the generating function of these sums, i.e., multiply the above expression by z^n and sum over all $n \geqslant 0$; apply the binomial theorem for simplifying the double sum thus obtained.

Exercise 55: *Denote by I_n the number of all involutions (an* involution *is a self–inverse bijection) on $[n]$.*

1. *Show*
$$\sum_{n \geqslant 0} I_n \frac{z^n}{n!} = e^{z + \frac{z^2}{2}}.$$

2. *Use the saddle point method to determine the asymptotic behaviour of I_n for $n \to \infty$.*

Exercise 56: *The exponential generating function of the Bell–numbers B_n (B_n is the number of all partitions of $[n]$) is*

$$\sum_{n \geqslant 0} B_n \frac{z^n}{n!} = e^{e^z - 1}.$$

Use the saddle point method to determine the asymptotic behaviour of B_n for $n \to \infty$.

Exercise 57: *Determine the asymptotic behaviour of the sum*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}$$

for $n \to \infty$.

<u>Hint:</u> Determine the generating function for this sum!

Exercise 58: *The saddle point method can also be used for the asymptotics of the Motzkin numbers M_n (see exercise 48) for $n \to \infty$:*
Show

$$M_n = \left[\!\left[z^0\right]\!\right] \left(z + 1 + z^{-1}\right)^n - \left[\!\left[z^2\right]\!\right] \left(z + 1 + z^{-1}\right)^n$$

and obtain a complex contour integral for M_n, which can be dealt with using the saddle point method.