PS Topologie und Funktionalanalysis

Giacomo Sodini und Gerald Teschl

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Note: References refer to the lecture notes.

1. Suppose $\sum_{n=1}^{\infty} |c_n| < \infty$. Show that
$$u(t,x) := \sum_{n=1}^{\infty} c_n \mathrm{e}^{-(\pi n)^2 t} \sin(n\pi x),$$
is continuous for $(t,x) \in [0,\infty) \times [0,1]$ and solves the heat equation for $(t,x) \in (0,\infty) \times [0,1]$. (Hint: Weierstrass M-test. When can you interchange the order of summation and differentiation?)

2. Show that $|\|f\| - \|g\|| \le \|f - g\|$.

3. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \to f$, $g_n \to g$, and $\alpha_n \to \alpha$, then $\|f_n\| \to \|f\|$, $f_n + g_n \to f + g$, and $\alpha_n g_n \to \alpha g$.

4. While $\ell^1(\mathbb{N})$ is separable, it still has room for an uncountable set of linearly independent vectors. Show this by considering vectors of the form
$$a^\alpha = (1, \alpha, \alpha^2, \dots), \qquad \alpha \in (0,1).$$
(Hint: Recall the Vandermonde determinant.)

5. Prove Young's inequality
$$\alpha^{1/p}\beta^{1/q} \le \frac{1}{p}\alpha + \frac{1}{q}\beta, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \ge 0.$$
Show that equality occurs precisely if $\alpha = \beta$. (Hint: Take logarithms on both sides.)

6. Show that $\ell^p(\mathbb{N})$, $1 \le p < \infty$, is complete.

7. Show that $\ell^\infty(\mathbb{N})$ is a Banach space.

8. Is $\ell^1(\mathbb{N})$ a closed subspace of $\ell^\infty(\mathbb{N})$ (with respect to the $\|.\|_\infty$ norm)? If not, what is its closure?

9. Show that $\ell^\infty(\mathbb{N})$ is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)

10. Show that there is equality in the Hölder inequality for $1 < p < \infty$ if and only if either $a = 0$ or $|b_j|^q = \alpha |a_j|^p$ for all $j \in \mathbb{N}$. Show that we have equality in the triangle inequality for $\ell^1(\mathbb{N})$ if and only if $a_j b_j^* \ge 0$ for all $j \in \mathbb{N}$ (here the '$*$' denotes complex conjugation). Show that we have equality in the triangle inequality for $\ell^p(\mathbb{N})$ with $1 < p < \infty$ if and only if $a = 0$ or $b = \alpha a$ with $\alpha \ge 0$.

11. Let X be a normed space. Show that the following conditions are equivalent.

    (i) If $\|x + y\| = \|x\| + \|y\|$ then $y = \alpha x$ for some $\alpha \geq 0$ or $x = 0$.

    (ii) If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\|\lambda x + (1-\lambda)y\| < 1$ for all $0 < \lambda < 1$.

    (iii) If $\|x\| = \|y\| = 1$ and $x \neq y$ then $\frac{1}{2}\|x + y\| < 1$.

    (iv) The function $x \mapsto \|x\|^2$ is strictly convex.

    A norm satisfying one of them is called strictly convex.

    Show that $\ell^p(\mathbb{N})$ is strictly convex for $1 < p < \infty$ but not for $p = 1, \infty$.

12. Show that $p_0 \leq p$ implies $\ell^{p_0}(\mathbb{N}) \subset \ell^p(\mathbb{N})$ and $\|a\|_p \leq \|a\|_{p_0}$. Moreover, show
$$\lim_{p\to\infty} \|a\|_p = \|a\|_\infty.$$

13. Consider $X = C([-1, 1])$. Which of the following subsets are subspaces of X? If yes, are they closed?

    (i) monotone functions

    (ii) even functions

    (iii) continuous piecewise linear functions

    (iv) $\{f \in C([-1, 1]) | f(c) = f_0\}$ for some fixed $c \in [-1, 1]$ and $f_0 \in \mathbb{R}$

14. Let I be a compact interval. Show that the set $Y := \{f \in C(I, \mathbb{R}) | f(x) > 0\}$ is open in $X := C(I, \mathbb{R})$. Compute its closure.

15. Compute the closure of the following subsets of $\ell^1(\mathbb{N})$: (i) $B_1 := \{a \in \ell^1(\mathbb{N}) | \sum_{j\in\mathbb{N}} |a_j|^2 \leq 1\}$. (ii) $B_\infty := \{a \in \ell^1(\mathbb{N}) | \sum_{j\in\mathbb{N}} |a_j|^2 < \infty\}$.

16. Which of the following bilinear forms are scalar products on $\mathbb{R}^n$?

    (i) $s(x, y) := \sum_{j=1}^n (x_j + y_j)$.

    (ii) $s(x, y) := \sum_{j=1}^n \alpha_j x_j y_j$, $\alpha \in \mathbb{R}^n$.

17. Show that the norm in a Hilbert space satisfies $\|f + g\| = \|f\| + \|g\|$ if and only if $f = \alpha g$, $\alpha \geq 0$, or $g = 0$. Hence Hilbert spaces are strictly convex.

18. Show that the maximum norm on $C[0, 1]$ does not satisfy the parallelogram law.

19. Show that $\ell^p(\mathbb{N})$, $1 \leq p \leq \infty$, is a Hilbert space if and only if $p = 2$.

20. Suppose $\mathfrak{Q}$ is a complex vector space. Let $s(f, g)$ be a sesquilinear form on $\mathfrak{Q}$ and $q(f) := s(f, f)$ the associated quadratic form. Prove the parallelogram law
$$q(f + g) + q(f - g) = 2q(f) + 2q(g)$$
and the polarization identity
$$s(f, g) = \frac{1}{4} \left(q(f + g) - q(f - g) + \mathrm{i}\, q(f - \mathrm{i}g) - \mathrm{i}\, q(f + \mathrm{i}g)\right).$$

Show that $s(f,g)$ is symmetric if and only if $q(f)$ is real-valued.

Note, that if $\mathfrak{Q}$ is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form $s(f,g) = \frac{1}{4}(q(f+g) - q(f-g))$ will only hold if $s(f,g)$ is symmetric.

21. Prove the claims made about f_n in Example 1.11.
22. Show that the integral defined in Example 1.13 satisfies
$$\int_c^e f(x)dx = \int_c^d f(x)dx + \int_d^e f(x)dx, \qquad \left|\int_c^d f(x)dx\right| \le \int_c^d |f(x)|dx.$$
How should $|f|$ be defined here?
23. Show Hölder's inequality for continuous functions and conclude that $\|.\|_p$ fulfills the requirements of a norm on $C(I)$.
24. Show that in a Banach space X a totally bounded set U is bounded.
25. Find a compact subset of $\ell^\infty(\mathbb{N})$ which does not satisfy (ii) from Fréchet's theorem (Theorem 1.13).
26. Which of the following families are relatively compact in $C[0,1]$?
    (i) $F := \{f \in C^1[0,1] | \, \|f\|_\infty \le 1\}$
    (ii) $F := \{f \in C^1[0,1] | \, \|f'\|_\infty \le 1\}$
    (iii) $F := \{f \in C^1[0,1] | \, \|f\|_\infty \le 1, \|f'\|_2 \le 1\}$
27. Show that two norms on X are equivalent if and only if they give rise to the same convergent sequences.
28. Show that a finite dimensional subspace $M \subseteq X$ of a normed space is closed.
29. Let $X := C[0,1]$. Investigate if the following operators $A : X \to X$ are linear and, if yes, compute the norm.
    (i) $f(x) \mapsto (1-x)x\, f(x^2)$.
    (ii) $f(x) \mapsto (1-x)x\, f(x)^2$.
    (iii) $f(x) \mapsto \int_0^1 (1-x)y\, f(y)dy$.
30. Let $X := C[0,1]$. Show that $\ell(f) := \int_0^1 f(x)dx$ is a linear functional. Compute its norm. Is the norm attained? What if we replace X by $X_0 := \{f \in C[0,1] | f(0) = 0\}$ (in particular, check that this is a closed subspace)?
31. Let $X := C[0,1]$. Investigate the operator $A : X \to X$, $f(x) \mapsto x\, f(x)$. Show that this is a bounded linear operator and compute its norm. What is the closure of $\mathrm{Ran}(A)$?

32. Show that the integral operator
$$(Kf)(x) := \int_0^1 K(x,y)f(y)dy,$$
where $K(x,y) \in C([0,1]\times[0,1])$, defined on $\mathfrak{D}(K) := C[0,1]$, is a bounded operator in $X := \mathcal{L}^2_{cont}(0,1)$.

33. Let I be a compact interval. Show that the set of differentiable functions $C^1(I)$ becomes a Banach space if we set $\|f\|_{\infty,1} := \max_{x\in I}|f(x)| + \max_{x\in I}|f'(x)|$.

34. Show that $\|AB\| \le \|A\|\|B\|$ for every $A, B \in \mathfrak{L}(X)$. Conclude that the multiplication is continuous: $A_n \to A$ and $B_n \to B$ imply $A_n B_n \to AB$.

35. Suppose $B \in \mathfrak{L}(X)$ with $\|B\| < 1$. Then $\mathbb{I} + B$ is invertible with
$$(\mathbb{I}+B)^{-1} = \sum_{n=0}^{\infty}(-1)^n B^n.$$
Consequently for $A, B \in \mathfrak{L}(X,Y)$, $A+B$ is invertible if A is invertible and $\|B\| < \|A^{-1}\|^{-1}$.

36. Let
$$f(z) := \sum_{j=0}^{\infty} f_j z^j, \qquad |z| < R,$$
be a convergent power series with radius of convergence $R > 0$. Suppose X is a Banach space and $A \in \mathfrak{L}(X)$ is a bounded operator with $\limsup_n \|A^n\|^{1/n} < R$ (note that by $\|A^n\| \le \|A\|^n$ the limsup is finite). Show that
$$f(A) := \sum_{j=0}^{\infty} f_j A^j$$
exists and defines a bounded linear operator. Moreover, if f and g are two such functions and $\alpha \in \mathbb{C}$, then
$$(f+g)(A) = f(A) + g(A), \quad (\alpha f)(A) = \alpha f(a), \quad (f\,g)(A) = f(A)g(A).$$
(Hint: Problem 1.8 from the lecture notes.)

37. Show that a linear map $\ell : X \to \mathbb{C}$ is continuous if and only if its kernel is closed. (Hint: If ℓ is not continuous, we can find a sequence of normalized vectors x_n with $|\ell(x_n)| \to \infty$ and a vector y with $\ell(y) = 1$.)

38. Let X_j, $j = 1, \dots, n$, be Banach spaces. Then $(\bigoplus_{p,j=1}^n X_j)^* \cong \bigoplus_{q,j=1}^n X_j^*$, where $\frac{1}{p} + \frac{1}{q} = 1$.

39. Compute $\|[e]\|$ in $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$, where $e := (1,1,1,\dots)$.

40. Let $X := \ell^p(\mathbb{N})$ and $M := \{a \in X | a_{2n} = 0\}$, $N := \{a \in X | n\, a_{2n} = a_{2n-1}\}$. Is $M \dotplus N$ closed?

41. Let ℓ be a nontrivial linear functional. Then its kernel has codimension one.

42. Suppose $A \in \mathfrak{L}(X,Y)$. Show that $\mathrm{Ker}(A)$ is closed. Suppose $M \subseteq \mathrm{Ker}(A)$ is a closed subspace. Show that the induced map $\tilde{A}: X/M \to Y$, $[x] \mapsto Ax$ is a well-defined operator satisfying $\|\tilde{A}\| = \|A\|$ and $\mathrm{Ker}(\tilde{A}) = \mathrm{Ker}(A)/M$. In particular, $\tilde{A}$ is injective for $M = \mathrm{Ker}(A)$.

43. Show that if a closed subspace M of a Banach space X has finite codimension, then it can be complemented. (Hint: Start with a basis $\{[x_j]\}$ for X/M and choose a corresponding dual basis $\{\ell_k\}$ with $\ell_k([x_j]) = \delta_{j,k}$.)

44. Let $I := [a,b]$ be a compact interval and consider $C^1(I)$. Which of the following is a norm? In case of a norm, is it equivalent to $\|.\|_{1,\infty}$?
    (i) $\|f'\|_\infty$
    (ii) $|f(a)| + \|f'\|_\infty$
    (iii) $\int_a^b |f(x)|dx + \|f'\|_\infty$

45. Given some vectors $f_1, \dots, f_n$ we define their Gram determinant as
$$\Gamma(f_1, \dots, f_n) := \det\left(\langle f_j, f_k\rangle\right)_{1\le j,k\le n}.$$
Show that the Gram determinant is nonzero if and only if the vectors are linearly independent. Moreover, show that in this case
$$\mathrm{dist}(g, \mathrm{span}\{f_1, \dots, f_n\})^2 = \frac{\Gamma(f_1, \dots, f_n, g)}{\Gamma(f_1, \dots, f_n)}$$
and
$$\Gamma(f_1, \dots, f_n) \le \prod_{j=1}^n \|f_j\|^2.$$
with equality if the vectors are orthogonal. (Hint: First establish $\Gamma(f_1, \dots, f_j + \alpha f_k, \dots, f_n) = \Gamma(f_1, \dots, f_n)$ for $j \ne k$ and use it to investigate how Γ changes when you apply the Gram–Schmidt procedure?)

46. Give an example of a nonempty closed bounded subset of a Hilbert space which does not contain an element with minimal norm. Can this happen in finite dimensions? (Hint: Look for a discrete set.)

47. Show that the set of vectors $\{c^n := (1, n^{-1}, n^{-2}, \dots)\}_{n=2}^\infty$ is total in $\ell^2(\mathbb{N})$. (Hint: Use that for any $a \in \ell^2(\mathbb{N})$ the functions $f(z) := \sum_{j\in\mathbb{N}} a_j z^{j-1}$ is holomorphic in the unit disc.)

48. Let $I = (a,b)$ be some interval and consider the scalar product
$$\langle f, g\rangle := \int_a^b f(x)^* g(x) w(x) dx$$
associated with some positive weight function $w(x)$. Let $P_j(x) = x^j + \dots$ be the corresponding monic orthogonal polynomials obtained by applying the Gram–Schmidt procedure (without normalization) to the monomials:
$$\int_a^b P_i(x)P_j(x)w(x)dx = \begin{cases} \alpha_j^2, & i = j,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\bar{P}_j(x) := \alpha_j^{-1} P_j(x)$ be the corresponding orthonormal polynomials and show that they satisfy the three term recurrence relation

$$a_j \bar{P}_{j+1}(x) + b_j \bar{P}_j(x) + a_{j-1}\bar{P}_{j-1}(x) = x\bar{P}_j(x),$$

or equivalently

$$P_{j+1}(x) = (x - b_j)P_j(x) - a_{j-1}^2 P_{j-1}(x),$$

where

$$a_j := \int_a^b x\bar{P}_{j+1}(x)\bar{P}_j(x)w(x)dx, \qquad b_j := \int_a^b x\bar{P}_j(x)^2 w(x)dx.$$

Here we set $P_{-1}(x) = \bar{P}_{-1}(x) \equiv 0$ for notational convenience.

49. Consider $\mathfrak{H} := \ell^2(\mathbb{Z})$. A sequence $a \in \mathfrak{H}$ is called even if $a_j = a_{-j}$ for all $j \in \mathbb{Z}$. Show that the set of even sequences M forms a closed subspace. Compute P_M and $M^\perp$.

50. Let M_1, M_2 be two subspaces of a Hilbert space $\mathfrak{H}$. Show that $(M_1 + M_2)^\perp = M_1^\perp \cap M_2^\perp$. If in addition M_1 and M_2 are closed, show that $(M_1 \cap M_2)^\perp = \overline{M_1^\perp + M_2^\perp}$.

51. Show that $\ell(a) = \sum_{j=1}^\infty \frac{a_j + a_{j+2}}{2^j}$ defines a bounded linear functional on $X := \ell^2(\mathbb{N})$. Compute its norm.

52. Suppose $P \in \mathfrak{L}(\mathfrak{H})$ satisfies

$$P^2 = P \qquad \text{and} \qquad \langle Pf, g\rangle = \langle f, Pg\rangle$$

and set $M := \operatorname{Ran}(P)$. Show

- $Pf = f$ for $f \in M$ and M is closed,
- $\operatorname{Ker}(P) = M^\perp$

and conclude $P = P_M$.

53. Let $\mathfrak{H}$ be a Hilbert space and K a nonempty closed convex subset. Prove that K has a unique element of minimal norm.

54. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and let $u \in \mathfrak{H}_1$, $v \in \mathfrak{H}_2$. Show that the operator

$$Af := \langle u, f\rangle v$$

is bounded and compute its norm. Compute the adjoint of A.

55. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and $A \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$. Prove

$$\|A\| = \sup_{\|g\|_{\mathfrak{H}_2} = \|f\|_{\mathfrak{H}_1} = 1} |\langle g, Af\rangle_{\mathfrak{H}_2}| \le C.$$

(Hint: Use $\|f\| = \sup_{\|g\|=1} |\langle g, f\rangle|$ — compare Theorem 1.5.)

56. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and suppose $A \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ has a bounded inverse $A^{-1} \in \mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. Show $(A^{-1})^* = (A^*)^{-1}$.

57. Show
$$\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^\perp.$$

58. Show Theorem 3.1.

59. Is the left shift $(a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots)$ compact in $\ell^2(\mathbb{N})$?

60. Is the operator $\frac{d}{dx} : C^k[0,1] \to C[0,1]$ compact for $k = 1, 2$? (Hint: Problem 44 and Example 3.3 from the lecture notes.)

61. Find the eigenvalues and eigenfunctions of the integral operator $K \in \mathfrak{L}(\mathcal{L}^2_{cont}(0,1))$ given by
$$(Kf)(x) := \int_0^1 u(x)v(y)f(y)dy,$$
where $u, v \in C([0,1])$ are some given continuous functions.

62. Find the eigenvalues and eigenfunctions of the integral operator $K \in \mathfrak{L}(\mathcal{L}^2_{cont}(0,1))$ given by
$$(Kf)(x) := 2\int_0^1 (2xy - x - y + 1)f(y)dy.$$

63. Let $\mathfrak{H} := \mathcal{L}^2_{cont}(0,1)$. Show that the Volterra integral operator $K : \mathfrak{H} \to \mathfrak{H}$ defined by
$$(Kf)(x) := \int_a^x K(x,y)f(y)dy,$$
where $K(x,y) \in C([a,b] \times [a,b])$, has no eigenvalues except for 0. Show that 0 is no eigenvalue if $K(x,y)$ is C^1 and satisfies $K(x,x) > 0$. Why does this not contradict Theorem 3.6? (Hint: Gronwall's inequality.)

64. Show that the inverse $(A-z)^{-1}$ (provided it exists and is densely defined) of a symmetric operator A is again symmetric for $z \in \mathbb{R}$. (Hint: $g \in \mathfrak{D}(R_A(z))$ if and only if $g = (A-z)f$ for some $f \in \mathfrak{D}(A)$.)

65. Prove that every subset of a meager set is again meager and every superset of a fat set is fat.

66. Let X be a complete metric space. Prove that the complement of a meager set is dense.

67. Let X be a Banach space and Y, Z normed spaces. Show that a bilinear map $B : X \times Y \to Z$ is bounded, $\|B(x,y)\| \le C\|x\|\|y\|$, if and only if it is separately continuous with respect to both arguments. (Hint: Uniform boundedness principle.)

68. Show that a compact symmetric operator in an infinite-dimensional Hilbert space cannot be surjective.

69. Let $X := \mathbb{C}^3$ equipped with the norm $|(x,y,z)|_1 := |x| + |y| + |z|$ and $Y := \{(x,y,z) \in X | x + y = 0,\ z = 0\}$. Find at least two extensions of $\ell(x,y,z) := x$ from Y to X which preserve the norm. What if we take $Y := \{(x,y,z) \in X | x + y = 0\}$?

70. Consider $X := C[0,1]$ and let $f_0(x) := 1 - 2x$. Find at least two linear functionals with norm one such that $\ell(f_0) = 1$.

71. Show that the extension from Corollary 4.13 is unique if X^* is strictly convex. (Hint: Problem 11.)

72. Let X be some normed space. Show that
$$\|x\| = \sup_{\ell \in V,\, \|\ell\|=1} |\ell(x)|,$$
where $V \subset X^*$ is some dense subspace. Show that equality is attained if $V = X^*$.

73. Let $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ be the subspace of sequences which converge to 0, and $c(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ the subspace of convergent sequences.
    (i) Show that $c_0(\mathbb{N})$, $c(\mathbb{N})$ are both Banach spaces and that $c(\mathbb{N}) = \operatorname{span}\{c_0(\mathbb{N}), e\}$, where $e := (1,1,1,\dots) \in c(\mathbb{N})$.
    (ii) Show that every $l \in c_0(\mathbb{N})^*$ can be written as
    $$l(a) = \sum_{j\in\mathbb{N}} b_j a_j$$
    with some unique $b \in \ell^1(\mathbb{N})$ which satisfies $\|b\|_1 = \|\ell\|$.
    (iii) Show that every $l \in c(\mathbb{N})^*$ can be written as
    $$l(a) = \sum_{j\in\mathbb{N}} b_j a_j + b_0 \lim_{j\to\infty} a_j$$
    with some $b \in \ell^1(\mathbb{N})$ which satisfies $|b_0| + \|b\|_1 = \|\ell\|$.

74. Show that if X, Y are Banach spaces and $A \in \mathfrak{L}(X,Y)$, then $\operatorname{Ran}(A)^\perp = \operatorname{Ker}(A')$ and $\operatorname{Ran}(A')_\perp = \operatorname{Ker}(A)$.

75. Let X be a Banach space and let $\ell_n, \ell \in X^*$. Let us write $\ell_n \overset{*}{\rightharpoonup} \ell$ provided the sequence converges pointwise, that is, $\ell_n(x) \to \ell(x)$ for all $x \in X$. Let $N \subseteq X^*$ and suppose $\ell_n \overset{*}{\rightharpoonup} \ell$ with $\ell_n \in N$. Show that $\ell \in (N_\perp)^\perp$.

76. Consider the multiplication operator $A : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ with $(Aa)_j := \frac{1}{j} a_j$. Show that $\operatorname{Ran}(A)$ is not closed but dense while $\operatorname{Ran}(A')$ is neither closed nor dense. In particular, show $\operatorname{Ker}(A)^\perp = \{0\}^\perp = \ell^\infty(\mathbb{N}) \supset \overline{\operatorname{Ran}(A')} = c_0(\mathbb{N})$.

77. Let X be a normed space. Suppose $\ell_n \to \ell$ in X^* and $x_n \rightharpoonup x$ in X. Prove that $\ell_n(x_n) \to \ell(x)$. Similarly, suppose s-lim $\ell_n = \ell$ and $x_n \to x$. Prove that $\ell_n(x_n) \to \ell(x)$. Does this still hold if s-lim $\ell_n = \ell$ and $x_n \rightharpoonup x$?

78. Let X, Y be normed spaces. Show that $x_n \rightharpoonup x$ in X implies $Ax_n \rightharpoonup Ax$ for $A \in \mathfrak{L}(X,Y)$. Conversely, show that if $x_n \to 0$ in X implies $Ax_n \rightharpoonup 0$ for a linear operator $A : X \to Y$, then $A \in \mathfrak{L}(X,Y)$.

79. Establish Lemma 4.34 in the case of weak convergence. (Hint: The formula
$$\|A\| = \sup_{x\in X,\, \|x\|=1;\, \ell\in V,\, \|\ell\|=1} |\ell(Ax)|,$$
might be useful.)

80. Show that if $\{\ell_j\} \subseteq X^*$ is total in X^* for a Banach space X, then $x_n \rightharpoonup x$ in X if and only if x_n is bounded and $\ell_j(x_n) \to \ell_j(x)$ for all j. Show that this is wrong without the boundedness assumption. (Hint: Take e.g. $X = \ell^2(\mathbb{N})$.)

81. Let X be a Banach algebra. Show $\sigma(x^{-1}) = \sigma(x)^{-1}$ if $x \in X$ is invertible.

82. An element $x \in X$ from a Banach algebra satisfying $x^2 = x$ is called a projection. Compute the resolvent and the spectrum of a projection.

83. If $X := \mathfrak{L}(L^p(I))$, then every $x \in C(I)$ gives rise to a multiplication operator $M_x \in X$ defined as $M_x f := x\, f$. Show $r(M_x) = \|M_x\| = \|x\|_\infty$ and $\sigma(M_x) = \mathrm{Ran}(x)$.

84. If $X := \mathfrak{L}(\ell^p(\mathbb{N}))$, $1 \le p \le \infty$, then every $m \in \ell^\infty(\mathbb{N})$ gives rise to a multiplication operator $M \in X$ defined as $(Ma)_n := m_n a_n$. Show $r(M) = \|M\| = \|m\|_\infty$ and $\sigma(M) = \overline{\mathrm{Ran}(m)}$.

85. Let X be a C^* algebra and $x \in X$ be self-adjoint. Show that the following are equivalent:
    (i) $\sigma(x) \subseteq [0, \infty)$.
    (ii) x is positive.
    (iii) $\|\lambda - x\| \le \lambda$ for all $\lambda \ge \|x\|$.
    (iv) $\|\lambda - x\| \le \lambda$ for one $\lambda \ge \|x\|$.

86. Let X be a C^* algebra. Show that if $x \in X$ is unitary then $\sigma(x) \subseteq \{\alpha \in \mathbb{C} | |\alpha| = 1\}$.

87. Let X be a C^* algebra and suppose that $x \in X$ is self-adjoint. Show that
$$\|(x-\alpha)^{-1}\| = \frac{1}{\mathrm{dist}(\alpha, \sigma(x))},$$
with the convention that $1/0 = \infty$ and that $\|(x-\alpha)^{-1}\| = \infty$ if $x - \alpha$ is not invertible.

88. Let $X := \mathbb{N}$. Which of the following families of subsets form the open sets of a topology when augmented with the empty set and X?
    (i) $U_n := \{j \in \mathbb{N} | j \le n\}$, $n \in \mathbb{N}$.
    (ii) $U_n := \{j \in \mathbb{N} | j \ge n\}$, $n \in \mathbb{N}$.
    (iii) all finite subsets of $\mathbb{N}$.
    (iv) all infinite subsets of $\mathbb{N}$.

89. Show that the closure satisfies the Kuratowski closure axioms.

90. Let X be a topological space and let $A, A_\alpha, B \subset X$, where α is any index. Show:

    (i) $A \subseteq B \quad \Rightarrow \quad A^\circ \subseteq B^\circ$

    (ii) $A \subseteq B \quad \Rightarrow \quad \overline{A} \subseteq \overline{B}$

    (iii) $\left(\bigcap_\alpha A_\alpha\right)^\circ \subseteq \bigcap_\alpha (A_\alpha)^\circ$

    (iv) $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$

91. Let $X := \{1,2,3\}$ and define $c : \mathfrak{P}(X) \to \mathfrak{P}(X)$ via $c(\{1\}) = \{1\}$, $c(\{2\}) = \{1,2\}$, $c(\{3\}) = \{2,3\}$ and

    $$c(U) = \bigcup_{x\in U} c(\{x\}), \quad U \subset X,$$

    with the convention $c(\emptyset) = \emptyset$.

    Which of the Kuratowski closure axioms does c satisfy? What are the closed sets according to c (i.e. the sets $U \subset X$ such that $c(U) = U$)? Do they give raise to a topology? If yes, do we have $\overline{U} = c(U)$ with respect to this topology?

92. Let X be a topological space and let $U \subset X$. Show that the closure and interior operators are dual in the sense that

    $$X \setminus \overline{U} = (X \setminus U)^\circ \qquad \text{and} \qquad X \setminus U^\circ = \overline{(X \setminus U)}.$$

    In particular, the closure is the set of all points which are not interior points of the complement. (Hint: De Morgan's laws.)

93. Let X be some nonempty set and define $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$. Show that (X,d) is a metric space. When are sequences convergent? When is X separable?

94. Show that in a (nonempty) Hausdorff space X singleton sets $\{x\}$ (with $x \in X$) are closed.

95. Let X be a metric space and denote by $B(X)$ the set of all bounded functions $X \to \mathbb{C}$. Introduce the metric

    $$d(f,g) = \sup_{x\in X} |f(x) - g(x)|, \quad f, g \in B(X).$$

    Show that $B(X)$ is complete.

96. Let X be a metric space and $B(X)$ as in the previous problem. Consider the embedding $J : X \hookrightarrow B(X)$ defind via

    $$y \mapsto J(x)(y) = d(x,y) - d(x_0,y)$$

    for some fixed $x_0 \in X$. Show that this embedding is isometric. Hence $\overline{J(X)}$ is another (equivalent) completion of X.

97. Let X, Y be topological spaces and $U \subseteq X$, $V \subseteq Y$. Show that if $f : X \to Y$ is continuous and $\mathrm{Ran}(f) \subseteq V$, then the restriction $f : U \to V$ is continuous when U, V are equipped with the relative topology.

98. Let X, Y be topological spaces. Show that if $f : X \to Y$ is continuous at $x \in X$ then it is also sequential continuous at $x \in X$. Show that the converse holds if X is first countable.

99. Let X, Y be topological spaces and let $f : X \to Y$ be continuous. Show that $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subset X$.

100. Let X be a topological space and $f : X \to \overline{\mathbb{R}}$. Let $x_0 \in X$ and let $\mathcal{B}(x_0)$ be a neighborhood base for x_0. Define

$$\liminf_{x\to x_0} f(x) := \sup_{U\in\mathcal{B}(x_0)} \inf_U f, \qquad \limsup_{x\to x_0} f(x) := \inf_{U\in\mathcal{B}(x_0)} \sup_U f.$$

Show that both are independent of the neighborhood base and satisfy

(i) $\liminf_{x\to x_0}(-f(x)) = -\limsup_{x\to x_0} f(x)$.
(ii) $\liminf_{x\to x_0}(\alpha f(x)) = \alpha \liminf_{x\to x_0} f(x)$, $\alpha \ge 0$.
(iii) $\liminf_{x\to x_0}(f(x) + g(x)) \ge \liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x)$.

Moreover, show that

$$\liminf_{n\to\infty} f(x_n) \ge \liminf_{x\to x_0} f(x), \qquad \limsup_{n\to\infty} f(x_n) \le \limsup_{x\to x_0} f(x)$$

for every sequence $x_n \to x_0$ and there exists a sequence attaining equality if X is a metric space.

101. Let X, Y be topological spaces and $A \subseteq X$, $B \subseteq Y$. Show that $(A\times B)^\circ = A^\circ \times B^\circ$ and $\overline{A \times B} = \overline{A} \times \overline{B}$.

102. Let X, Y be topological spaces. Show that $X \times Y$ has the following properties if both X and Y have.

(i) Hausdorff
(ii) separable
(iii) first countable
(iv) second countable

103. Show that a subset K of a topological space X is compact if and only if it is compact with respect to the relative topology.

104. Consider $X := \mathbb{R}$ with the lower semicontinuity topology $\mathcal{O} := \{(a,\infty)|a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Show that this is indeed a topology. Show that a set $C \subset X$ is compact if and only if it has a minimum.

105. Show that a nonempty subset of $\mathbb{R}$ is connected if and only if it is an interval.

106. Let X be a metric space and $Y \subseteq X$. Show $\operatorname{dist}(x, Y) = \operatorname{dist}(x, \overline{Y})$. Moreover, show $x \in \overline{Y}$ if and only if $\operatorname{dist}(x, Y) = 0$.

107. Let X_α be topological spaces and let $X := \bigtimes_{\alpha\in A} X_\alpha$ with the product topology. Show that the product $\bigtimes_{\alpha\in A} C_\alpha$ of closed sets $C_\alpha \subseteq X_\alpha$ is closed.

108. Let $\{(X_j, d_j)\}_{j\in\mathbb{N}}$ be a sequence of metric spaces. Show that

$$d(x,y) := \sum_{j\in\mathbb{N}} \frac{1}{2^j} \frac{d_j(x_j,y_j)}{1+d_j(x_j,y_j)} \quad \text{or} \quad d(x,y) := \sup_{j\in\mathbb{N}} \frac{1}{2^j} \frac{d_j(x_j,y_j)}{1+d_j(x_j,y_j)}$$

is a metric on $X = \bigtimes_{n\in\mathbb{N}} X_n$ which generates the product topology. Show that X is complete if all X_n are.