universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

On the well-posedness of the KdV in $H^{-1}(\mathbb{R})$

verfasst von / submitted by

Florian Lang, BSc BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Dipl.-Ing. Dr. Gerald Teschl

Abstract

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation that arises in various physical and mathematical contexts. The question of global well-posedness in appropriate function spaces, ensuring the existence and uniqueness of solutions, has been the subject of extensive research. In this thesis, we delve into the work of Killip and Vişan [2018], who provided a proof of global well-posedness of the KdV equation for initial conditions in the Sobolev space $H^s(\mathbb{R})$ for $s \geq -1$. In the class of $H^s(\mathbb{R})$ spaces, this result is sharp, in the sense that the KdV equation is not globally well-posed for $s < -1$.

Zusammenfassung

Die Korteweg-de Vries (KdV) Gleichung ist eine nichtlineare partielle Differentialgleichung, die in verschiedenen physikalischen und mathematischen Kontexten auftritt. Die Frage der globalen Wohlgestelltheit in geeigneten Funktionenräumen, die die Existenz und Eindeutigkeit von Lösungen sicherstellt, war Gegenstand umfangreicher Forschung. In dieser Arbeit behandeln wir die Arbeit von Killip and Vişan [2018], die einen Beweis der globalen Wohlgestelltheit der KdV-Gleichung für Anfangsbedingungen im Sobolev-Raum $H^s(\mathbb{R})$ für $s \geq -1$ lieferten. In der Klasse der $H^s(\mathbb{R})$-Räume ist dieses Ergebnis scharf, in dem Sinne, dass die KdV-Gleichung für $s < -1$ nicht global wohlgestellt ist.

Acknowledgements

Firstly and most importantly I would like to thank my supervisor, Gerald Teschl. He suggested the topic of this thesis and provided me with a lot of helpful advice. As the lecturer of some of the more interesting courses I attended during my studies, he was also a great inspiration for me to pursue a career in mathematics. I am very grateful for his patience and support throughout my studies.
I would also like to thank my friends, family, and colleagues for the social and mathematical support they have provided me with throughout the years.
Special thanks go to Florian Fürnsinn, Markus Reibnegger, and Stephan Schneider for listening to my complaints and helping me with some of the more technical problems I have encountered.

Contents

1 Introduction

The *Korteweg-de Vries* equation

$$\partial_t q + q''' - 6qq' = 0 \qquad \text{KdV (1)}$$

is a prototypical example of the propagation of solitons. It was originally derived to describe shallow water waves but has been used to model a wide variety of phenomena, such as ion-acoustic waves in plasma or acoustic waves in a harmonic crystal [Wazwaz, 2009]. Even though it is a non-linear equation, it resembles linear dynamics in certain aspects, and under certain conditions [Erdoğan et al., 2011]. As for every partial differential equation, the question of local as well as global well-posedness in varying function spaces is of great interest. Global well-posedness for the KdV with Schwartz initial conditions has been known for a while now. For $H^s(\mathbb{R})$, $s \geq -3/4$ global well-posedness was established by Colliander et al. [2001] and Guo [2009]. For $s < -1$ Molinet [2010] showed that global well-posedness can not hold. In the cases $s < -3/4$ the solution map fails to be uniformly continuous as proven by Christ et al. [2004]. This, however, does not prevent global well-posedness for $-1 \leq s < -3/4$, as the solution map might still be continuous. This is what Killip and Vişan [2018] showed quite recently. We will review their proof, offering supplementary information that was omitted.

Notation
We will use the notation f' solely for the derivative with respect to the spatial variable.

1.1 Proof Idea

Lax Pair

For the two operators

$$L(t) = -\partial_x^2 + q(t,x) \quad \text{and} \quad A(t) = -4\partial_x^3 + 3\big(\partial_x q(t,x) + q(t,x)\partial_x\big)$$

we have that

$$[A, L] = -q''' + 6qq',$$

and thus they form a *Lax pair*

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [A, L] \iff q \text{ solves the KdV (1)}.$$

Now observe that if we define U as the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} U = AU, \quad U(0) = \mathbb{I}$$

then, as A is anti-self-adjoint ($A^* = -A$), we have that U is unitary by

$$\frac{\mathrm{d}}{\mathrm{d}t} U^* U = -U^* A U + U^* A U = 0.$$

So observing

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} U L(0) U^* &= A U L(0) U^* - U L(0) U^* A \\ &= [A, U L(0) U^*] \end{aligned}$$

we see that

$$L = U L(0) U^*.$$

For more details on this cf. Abraham and Marsden [2008].

Renormalized Fredholm Determinant

Unitarity of U suggests that the spectral properties of the Schrödinger operator L are conserved under the flow of the KdV. One might want to consider something like $\det\big(L + \kappa^2\big)$ to investigate and exploit these spectral properties, however, there is no chance of existence for such an object. So we would like to normalize it in some way, say using the resolvent of the free Schrödinger operator $R_0 = (-\partial_x^2 + \kappa^2)^{-1}$

$$\det\left(\frac{L + \kappa^2}{-\partial_x^2 + \kappa^2}\right) \rightsquigarrow \det\left(1 + \frac{q}{-\partial_x + \kappa^2}\right) \rightsquigarrow \det\Big(1 + \sqrt{R_0} q \sqrt{R_0}\Big).$$

Again, we run into the problem that this Fredholm determinant fails to exist, at least for $\sqrt{R_0} q \sqrt{R_0}$ not trace class. So we use the *renormalized Fredholm determinant* (cf. Simon [2005])

$$\det_2(1 + A) = \det(1 + A) \exp(-\operatorname{Tr}(A))$$

which is well-defined on trace class operators and extends to Hilbert-Schmidt operators via

$$-\log \det_2(1 + A) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \operatorname{Tr}\Big(A^k\Big).$$

As we will see $\sqrt{R_0} q \sqrt{R_0}$ is Hilbert-Schmidt for $q \in H^{-1}(\mathbb{R})$ and the quantity

$$\alpha(\kappa) = -\log \det_2 \Big(1 + \sqrt{R_0} q \sqrt{R_0}\Big) = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \operatorname{Tr}\Big(\big(\sqrt{R_0} q \sqrt{R_0}\big)^k\Big)$$

is conserved under the flow of the KdV.

Hamiltonian Flow

The idea is to then approximate the KdV flow by a family of commuting flows that are easier to handle. Zakharov and Faddeev [1971] showed

$$\begin{aligned}\alpha(\kappa) &= \frac{1}{4\kappa^3}\int \frac{1}{2}q(x)^2\,\mathrm{d}x + \frac{1}{16\kappa^5}\int \Big(\frac{1}{2}\big(q'(x)\big)^2 + q(x)^3\Big)\,\mathrm{d}x + O\big(\kappa^{-7}\big)\\ &= \frac{1}{4\kappa^3}P + \frac{1}{16\kappa^5}H_{\mathrm{KdV}} + O\big(\kappa^{-7}\big),\end{aligned}$$

where P is the momentum and H_{KdV} is the Hamiltonian of the KdV. This suggests that if we were to take the Hamiltonian

$$H_\kappa = -16\kappa^5\alpha(\kappa) + 4\kappa^2 P,$$

we may get a reasonable approximation for the KdV Hamiltonian for large κ.

Equicontinuity

As it turns out, $\alpha(\kappa)$ is not only conserved under the flows of both H_{KdV} and H_κ, but it also serves to bound the $H^{-1}_\kappa(\mathbb{R})$ norm of q, and it gives a fitting criterion on equicontinuity. This will then allow us to upgrade local well-posedness to global well-posedness, and the equicontinuity will help us transfer the well-posedness result from the H_κ flow to the H_{KdV} flow.

2 Preliminaries

Notation
We write

$$\begin{aligned} f \lesssim g &\iff \exists\, c>0:\ f \le cg,\\ f \gtrsim g &\iff \exists\, c>0:\ f \ge cg,\\ f \approx g &\iff f \lesssim g \quad\text{and}\quad f \gtrsim g.\end{aligned}$$

If the implicit constant depends on further parameters, and this dependency is important, we indicate this by a subscript.

2.1 Sobolev spaces and Fourier transform

We denote the *Fourier transform* of a function by $\mathcal{F}f$ and its inverse by $\mathcal{F}^{-1}f$. For the Fourier transform and its inverse we use the normalization

$$\begin{aligned}\mathcal{F}f(w) &= \frac{1}{\sqrt{2\pi}}\int f(x)\exp(-iwx)\,\mathrm{d}x\,,\\ \mathcal{F}^{-1}f(x) &= \frac{1}{\sqrt{2\pi}}\int f(w)\exp(iwx)\,\mathrm{d}w\,.\end{aligned}$$

We use

$$\langle x \rangle = \left(4 + \|x\|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \langle x \rangle_\kappa = \left(4\kappa^2 + \|x\|^2\right)^{\frac{1}{2}}$$

to regard the *Sobolev spaces* $H^s(\mathbb{R})$, $H^s_\kappa(\mathbb{R})$ as completions of $\mathcal{S}(\mathbb{R})$ with respect to the norms

$$\|f\|_{H^s(\mathbb{R})} = \|\langle \cdot \rangle^s \mathcal{F} f\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|f\|_{H^s_\kappa(\mathbb{R})} = \|\langle \cdot \rangle^s_\kappa \mathcal{F} f\|_{L^2(\mathbb{R})}.$$

Remark 2.1
We have $\langle \cdot \rangle \approx_\kappa \langle \cdot \rangle_\kappa$ and therefore $H^s(\mathbb{R}) = H^s_1(\mathbb{R}) \cong H^s_\kappa(\mathbb{R})$. Furthermore, by taking the $L^2(\mathbb{R})$ pairing on $\mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned} \left|\langle f|g\rangle_{L^2(\mathbb{R})}\right| &= \left|\langle \mathcal{F}f|\mathcal{F}g\rangle_{L^2(\mathbb{R})}\right| \\ &= \left|\left\langle \langle \cdot \rangle^s_\kappa \mathcal{F}f \middle| \langle \cdot \rangle^{-s}_\kappa \mathcal{F}g\right\rangle_{L^2(\mathbb{R})}\right| \leq \|f\|_{H^s_\kappa(\mathbb{R})} \|g\|_{H^{-s}_\kappa(\mathbb{R})}. \end{aligned}$$

So by extending this pairing, we can identify

$$H^{-s}_\kappa(\mathbb{R}) = \left(H^s_\kappa(\mathbb{R})\right)^*.$$

2.2 Hilbert-Schmidt and Trace Class operators

For a compact operator A the *Schatten p-norm* is defined as

$$\|A\|_{\mathfrak{I}_p} = \left(\sum_{n=1}^{\infty} |s_n(A)|^p\right)^{\frac{1}{p}}, \qquad p \geq 1,$$

where $s_n(A)$ are the singular values of A. The space of operators with $\|A\|_{\mathfrak{I}_p} < \infty$ is called *Schatten p-class* $\mathfrak{I}_p$. The cases $p = 1, 2$ are called *Trace Class* and *Hilbert-Schmidt* respectively. For trace class operators, we can define the *Trace* as

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} \langle e_n | A e_n \rangle \leq \|A\|_{\mathfrak{I}_1},$$

where $\{e_n\}_n$ is an orthonormal basis.

The Schatten p-classes work analogously to the ℓ^p spaces.

Lemma 2.2
We have that

$$\|\cdot\|_{\mathfrak{J}_p} \leq \|\cdot\|_{\mathfrak{J}_q}, \quad p \geq q$$

hence $\mathfrak{J}_p \subseteq \mathfrak{J}_q$ *for* $p \geq q$. *For* $A \in \mathfrak{J}_p$ *and* B *bounded we have that*

1. $\|A\| \leq \|A\|_{\mathfrak{J}_p} = \|A^*\|_{\mathfrak{J}_p}$.
2. $AB, BA \in \mathfrak{J}_p$ *with* $\|AB\|_{\mathfrak{J}_p} \leq \|A\|_{\mathfrak{J}_p}\|B\|$ *and* $\|BA\|_{\mathfrak{J}_p} \leq \|B\|\|A\|_{\mathfrak{J}_p}$.

Lemma 2.3
For $r^{-1} = p^{-1} + q^{-1}$ *we have that* $A \in \mathfrak{J}_r$ *iff there exists* $B \in \mathfrak{J}_p$ *and* $C \in \mathfrak{J}_q$ *such that* $A = BC$. *In this case, we have* $\|A\|_{\mathfrak{J}_r} \leq \|B\|_{\mathfrak{J}_p}\|C\|_{\mathfrak{J}_q}$. *Furthermore, for A Hilbert-Schmidt we have that*

$$\|A\|_{\mathfrak{J}_2}^2 = \mathrm{Tr}(A^*A).$$

Remark 2.4
If $p \in \mathbb{N}$ and $A \in \mathfrak{J}_p$ with bounded inverse, then A is trace class as A^p is trace class by Lemma 2.3 and therefore $A = A^p A^{1-p}$ is trace class by Lemma 2.2.

Remark 2.5
For $B, C \in \mathfrak{J}_2$ we have that

$$|\mathrm{Tr}(BC)| \leq \|B\|_{\mathfrak{J}_2}\|C\|_{\mathfrak{J}_2}.$$

Lemma 2.6
For A *trace class,* B *bounded we have that*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

In the case of operators on $L^2(\mathbb{R})$ we have further characterizations of Hilbert-Schmidt operators.

Theorem 2.7 cf. Simon [2005]
An operator A *is Hilbert-Schmidt iff*

$$Af(x) = \int_{\mathbb{R}} K_A(x,y)f(y)\,\mathrm{d}y$$

where $K_A \in L^2(\mathbb{R}^2)$. *In this case, we have*

$$\|A\|_{\mathfrak{J}_2}^2 = \|K_A\|_{L^2(\mathbb{R}^2)}^2 = \iint |K_A(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y\,.$$

Remark 2.8
For $A, B \in \mathfrak{J}_2$ and $f \in L^\infty(\mathbb{R})$ we have

$$K_{fA}(x,y) = f(x)K_A(x,y),$$
$$K_{Af}(x,y) = K_A(x,y)f(y) \quad \text{and}$$
$$K_{AB}(x,y) = \int K_A(x,z)K_B(z,y)\,\mathrm{d}z\,.$$

Theorem 2.9 cf. Bernat [1972]
If A is trace class with K_A continuous then we have

$$\mathrm{Tr}(A) = \int K_A(x,x)\,\mathrm{d}x\,.$$

2.3 Differentiability

A function $A : H_1 \to H_2$ is called *Fréchet differentiable* at f if there is a linear, bounded operator $\mathrm{d}A(f) : H_1 \to H_2$ such that

$$\frac{\|A(f+g) - A(f) - \mathrm{d}A(f)g\|}{\|g\|} \xrightarrow{\|g\|\to 0} 0.$$

The operator $\mathrm{d}A(f)$ is called the *Fréchet derivative* of A at f.

Remark 2.10
If we have a Fréchet differentiable $A : H^s(\mathbb{R}) \to \mathbb{R}$ then $\mathrm{d}A(f) \in (H^s(\mathbb{R}))^*$ and hence there is a $\delta A/\delta f \in H^{-s}(\mathbb{R})$ such that

$$\mathrm{d}A(f)(g) = \int \frac{\delta A}{\delta f}(x)g(x)\,\mathrm{d}x\,.$$

We call $\delta A/\delta f$ the *functional derivative* of A at f.

Lemma 2.11
If A is Fréchet differentiable at f, then A is directional differentiable at f and

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} A(f+sg) = \mathrm{d}A(f)(g).$$

Theorem 2.12 cf. Lang [2012]
Let $A : H_1 \to H_2$ be continuously Fréchet differentiable in some neighborhood of f such that $\mathrm{d}A(f)$ is invertible as a bounded operator. Then A is a local diffeomorphism at f. Furthermore, if

$$\left\|\mathrm{d}A(f)^{-1}\right\| \|\mathrm{d}A(f) - \mathrm{d}A(g)\| < \frac{1}{2}$$

for all g in an r-ball around f, then the size of the neighborhood on which A is a diffeomorphism depends only on r.

2.4 Hamiltonian Flow

For a symplectic Manifold (M, ω) a *Poisson bracket* is a map

$$\{\,\cdot\,,\,\cdot\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M).$$

Along with other very important properties, it obeys

$$X_{\{F,G\}} = [X_F, X_G],$$

where

$$X_A = \{\,\cdot\,, A\}$$

is the *Hamiltonian vector field* of A. Meaning that if two functions Poisson commute, then their Hamiltonian vector fields, and in extension their flows, commute as well. Another important property is that G is constant along the flow of F iff $\{F, G\} = 0$. In our setting, we will work with $M = \mathcal{S}(\mathbb{R})$ and the Poisson bracket is

$$\{F, G\}(q) = \int \frac{\delta F}{\delta q} \left(\frac{\delta G}{\delta q} \right)',$$

where we have

$$\begin{aligned} \int \frac{\delta F}{\delta q} \left(\frac{\delta H}{\delta q} \right)' &= \{F, H\}(q) \\ &= X_H(F)(q) \\ &= \mathrm{d}F(q)\big(X_H(q)\big) \\ &= \int \frac{\delta F}{\delta q} X_H(q) \end{aligned}$$

i.e.,

$$X_H(q) = \left(\frac{\delta H}{\delta q} \right)'.$$

Hence, for the flow of X_H, denoted by Fl_t^H, we have

$$\partial_t q(t) = \partial_t \mathrm{Fl}_t^H(q) = X_H\big(\mathrm{Fl}_t^H(q)\big) = \left(\frac{\delta H}{\delta q(t)} \right)'$$

if we write $q(t) = \mathrm{Fl}_t^H(q)$. If a PDE is of this particular form, we call H the associated *Hamiltonian*. The exact details on this topic are not of importance to us, it mainly provides us with convenient notation for flows. For more details on this topic cf. Abraham et al. [2012], Lee [2012] or Arnold [2013].

3 Diagonal Green's function

As we have seen, the KdV is closely related to the *Schrödinger operator*, so let us first investigate the Schrödinger operator

$$L = -\partial_x^2 + q$$

with potentials

$$q \in B_\delta = \{q \in H^{-1}(\mathbb{R}) \mid \|q\|_{H^{-1}(\mathbb{R})} \leq \delta\}$$

for some $\delta > 0$ sufficiently small. We want to describe the corresponding diagonal Green's function as well as possible. So let us first look at the case $q = 0$ and recall that the resolvent

$$R_0(\kappa) = \big(-\partial_x^2 + \kappa^2 \big)^{-1}$$

with $\kappa > 0$ has the integral kernel

$$G_0(x, y; \kappa) = \frac{1}{2\kappa} \exp\big(-\kappa|x - y| \big),$$

as

$$R_0(\kappa)f = \mathcal{F}\bigg(\frac{1}{(\,\cdot\,)^2 + \kappa^2} \mathcal{F}^{-1} f \bigg) = \frac{1}{\sqrt{2\pi}} \mathcal{F}\bigg(\frac{1}{(\,\cdot\,)^2 + \kappa^2} \bigg) * f.$$

So the case $q = 0$ we already understand quite well, and we can use this to express the resolvent of L in the general case. Assuming that all the objects exist we would have

$$\begin{aligned}
\big(L + \kappa^2\big)^{-1} &= \Big(R_0(\kappa)^{-1} + q \Big)^{-1} \\
&= \sqrt{R_0(\kappa)} \Big(1 + \sqrt{R_0(\kappa)}\, q \sqrt{R_0(\kappa)} \Big)^{-1} \sqrt{R_0(\kappa)}
\end{aligned}$$

and the inverse we could express in terms of a series.

Lemma 3.1 cf. Reed and Simon [1975]
Let A be positive self-adjoint, β a symmetric quadratic form on $Q(A)$, $a < 1$ and $b \in \mathbb{R}$ such that

$$|\beta(f, f)| \leq a\langle f|Af\rangle + b\langle f|f\rangle,$$

then there exists a unique self-adjoint C with $Q(C) = Q(A)$ and

$$\langle f|Cg\rangle = \langle f|Ag\rangle + \beta(f, g).$$

Proposition 3.2
For $q \in H^{-1}(\mathbb{R})$ and given the quadratic form

$$\begin{aligned} H^1(\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \int_{\mathbb{R}} \left|f'(x)\right|^2 + q(x)|f(x)|^2 \,\mathrm{d}x \end{aligned}$$

there exists an unique associated self-adjoint operator L. If in addition $q \in B_\delta$ for $\delta > 0$ sufficiently small, then the resolvent of L is given by the series

$$R(q,\kappa) = \left(L + \kappa^2\right)^{-1} = \sum_{i=0}^{\infty} (-1)^i \sqrt{R_0(\kappa)} \left(\sqrt{R_0(\kappa)}\, q \sqrt{R_0(\kappa)}\right)^i \sqrt{R_0(\kappa)} \tag{2}$$

for $\kappa \geq 1$.

Notation
For better readability, we omit the dependence on extra parameters like q and κ whenever the context permits.

Proof. We have for $q \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} \left\|\sqrt{R_0} q \sqrt{R_0}\right\|^2 &\leq \left\|\sqrt{R_0} q \sqrt{R_0}\right\|_{\mathfrak{I}_2}^2 \\ &= \operatorname{Tr}\left(\sqrt{R_0} q R_0 q \sqrt{R_0}\right) \\ &= \operatorname{Tr}(q R_0 q R_0) \\ &= \iint q(x) G_0(x,y)^2 q(y) \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{1}{\kappa} \int q(x) \int \frac{1}{4\kappa} \exp(-2\kappa|x-y|) q(y) \,\mathrm{d}y \,\mathrm{d}x \\ &= \frac{1}{\kappa} \int q(x) R_0(2\kappa) q(x) \,\mathrm{d}x \\ &= \frac{1}{\kappa} \int q(x) \mathcal{F}^{-1}\left(\langle \cdot \rangle_\kappa^{-2} \mathcal{F} q\right)(x) \,\mathrm{d}x \\ &= \frac{1}{\kappa} \int \langle w \rangle_\kappa^{-2} |\mathcal{F} q(w)|^2 \,\mathrm{d}w = \frac{1}{\kappa} \|q\|_{H_\kappa^{-1}(\mathbb{R})}^2, \end{aligned} \tag{3}$$

and by density, it holds for all $q \in H^{-1}(\mathbb{R})$.

Now note that with $h = \sqrt{R_0}^{-1} f$ we have

$$\begin{aligned}
\langle f | q f \rangle_{L^2(\mathbb{R})} &= \left\langle \sqrt{R_0} h \middle| q \sqrt{R_0} h \right\rangle_{L^2(\mathbb{R})} \\
&= \left\langle h \middle| \sqrt{R_0} q \sqrt{R_0} h \right\rangle_{L^2(\mathbb{R})} \\
&\leq \kappa^{-\frac{1}{2}} \|q\|_{H^{-1}(\mathbb{R})} \|h\|^2_{L^2(\mathbb{R})} \\
&= \kappa^{-\frac{1}{2}} \|q\|_{H^{-1}(\mathbb{R})} \langle f | R_0^{-1} f \rangle_{L^2(\mathbb{R})} \\
&= \kappa^{-\frac{1}{2}} \|q\|_{H^{-1}(\mathbb{R})} \int |f'(x)|^2 + \kappa^2 |f(x)|^2 \, \mathrm{d}x \,.
\end{aligned}$$

So by Lemma 3.1 there exists a unique L such that

$$\langle f | L f \rangle_{L^2(\mathbb{R})} = \int |f'(x)|^2 + q(x) |f(x)|^2 \, \mathrm{d}x \,.$$

Moreover, Equation 3 shows that Equation 2 indeed converges for $q \in B_\delta$, $\delta < 1$ and $\kappa \geq 1$. □

3.1 Regularity

We would like to know if the resolvent has an integral kernel, and what its properties are. Specifically, we are interested in the diagonal of the integral kernel and its regularity.

Lemma 3.3
For $q \in B_\delta$, $\delta > 0$ sufficiently small and $\kappa \geq 1$, the resolvent R admits a continuous integral kernel $G(x, y; \kappa, q)$.

Proof. First note that by Equation 3

$$\begin{aligned}
\left\| \sqrt{R_0}^{-1} (R - R_0) \sqrt{R_0}^{-1} \right\|_{\mathfrak{J}_2} &= \left\| \sqrt{R_0}^{-1} R \sqrt{R_0}^{-1} - 1 \right\|_{\mathfrak{J}_2} \\
&\leq \sum_{i=1}^{\infty} \left\| \sqrt{R_0} q \sqrt{R_0} \right\|^i_{\mathfrak{J}_2} < \infty
\end{aligned}$$

for $q \in B_\delta$ and δ sufficiently small. So on the one hand, we have

$$\|R - R_0\|_{\mathfrak{J}_2} \leq \left\| \sqrt{R_0} \right\| \left\| \sqrt{R_0}^{-1} (R - R_0) \sqrt{R_0}^{-1} \right\|_{\mathfrak{J}_2} \left\| \sqrt{R_0} \right\| < \infty$$

which shows that R does have an integral kernel G, while on the other hand, we have

$$\|G - G_0\|_{H^1(\mathbb{R}) \otimes H^1(\mathbb{R})} \approx \left\| \sqrt{R_0}^{-1} (R - R_0) \sqrt{R_0}^{-1} \right\|_{\mathfrak{J}_2} < \infty,$$

from which we conclude that G is continuous. □

Proposition 3.4
The diagonal of the Green's function G we denote by

$$g(x;\kappa,q) = G(x,x;\kappa,q).$$

If $\delta > 0$ is sufficiently small and $\kappa \geq 1$ then the maps

$$q \mapsto g - \frac{1}{2\kappa} \quad \text{and} \quad q \mapsto \kappa - \frac{1}{2g}$$

are real analytic diffeomorphisms from B_δ into $H^1(\mathbb{R})$.

Proof. Since $\sqrt{R_0}$ is essentially $\langle\,\cdot\,\rangle^{-1}$ on the Fourier side, it is continuous as a map on the Schwartz space. So we may extend it to $\mathcal{S}(\mathbb{R})'$ in the usual way, and we have

$$\begin{aligned}
\left|\sqrt{R_0}\delta_x(f)\right| &= \left|\delta_x\left(\sqrt{R_0}f\right)\right| \\
&\approx \left|\delta_x\left(\mathcal{F}^{-1}\langle\,\cdot\,\rangle^{-1}\mathcal{F}f\right)\right| \\
&\lesssim \int \left|\langle w\rangle^{-1}\mathcal{F}f(w)\right| \mathrm{d}w \\
&\lesssim \|f\|_{L^2(\mathbb{R})}
\end{aligned}$$

i.e., $\sqrt{R_0}\delta_x \in L^2(\mathbb{R})$. By continuity of G and Equation 2 we may write

$$\begin{aligned}
g(x) &= \langle \delta_x | R\delta_x\rangle \\
&= \frac{1}{2\kappa} + \sum_{i=1}^{\infty}(-1)^i\left\langle \sqrt{R_0}\delta_x \middle| \left(\sqrt{R_0}q\sqrt{R_0}\right)^i \sqrt{R_0}\delta_x\right\rangle_{L^2(\mathbb{R})}.
\end{aligned} \tag{4}$$

Now let $q \in B_\delta$, $\delta < 1/2$ and $\kappa \geq 1$ and observe that for all $f \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}
\left|\int f(x)\left(g(x) - \frac{1}{2\kappa}\right)\mathrm{d}x\right| &= |\mathrm{Tr}(f(R - R_0))| \\
&\leq \sum_{i=1}^{\infty}\left|\mathrm{Tr}\left(f\sqrt{R_0}\left(\sqrt{R_0}q\sqrt{R_0}\right)^i\sqrt{R_0}\right)\right| \\
&= \sum_{i=1}^{\infty}\left|\mathrm{Tr}\left(\sqrt{R_0}f\sqrt{R_0}\left(\sqrt{R_0}q\sqrt{R_0}\right)^i\right)\right| \\
&\leq \sum_{i=1}^{\infty}\left\|\sqrt{R_0}f\sqrt{R_0}\right\|_{\mathfrak{I}_2}\left\|\sqrt{R_0}q\sqrt{R_0}\right\|_{\mathfrak{I}_2}^i \\
&\leq \frac{1}{\sqrt{\kappa}}\|f\|_{H^{-1}(\mathbb{R})}\frac{\delta/\sqrt{\kappa}}{1-\delta/\sqrt{\kappa}} \\
&\leq 2\delta\kappa^{-1}\|f\|_{H^{-1}(\mathbb{R})}
\end{aligned}$$

by Equation 2 and Equation 3.

Hence, $g - 1/2\kappa$ is in $H^1(\mathbb{R})$ with

$$\left\| g - \frac{1}{2\kappa} \right\|_{H^1(\mathbb{R})} \leq 2\delta\kappa^{-1}. \tag{5}$$

This also shows the convergence of Equation 4 and thus $q \mapsto g - 1/2\kappa$ is real analytic. For $f \in H^{-1}(\mathbb{R})$ we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\big(R(q+sf)^{-1}R(q+sf)\big) \\ &= fR(q) + R(q)^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}R(q+sf) \end{aligned}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}R(q+sf) = -R(q)fR(q)$$

Taking the diagonal of the corresponding integral kernels yields

$$\mathrm{d}g(q)(f)(x) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\, g(x, q+sf) = -\int G(x,y;q)f(y)G(y,x;q)\,\mathrm{d}y\,. \tag{6}$$

Specializing to $q = 0$ we obtain

$$\mathrm{d}g(0) = -\kappa^{-1}R_0(2\kappa).$$

Using Equation 2 and Equation 3 we see that

$$\begin{aligned} &\|\mathrm{d}g(q)(f) - \mathrm{d}g(0)(f)\|_{H^1_\kappa(\mathbb{R})} \\ &\quad\approx \left\| \sqrt{R_0}^{-1} RfR\sqrt{R_0}^{-1} - \sqrt{R_0}^{-1} R_0fR_0\sqrt{R_0}^{-1} \right\|_{\mathfrak{I}_2} \\ &\quad= \left\| \sum_{i\neq 0 \vee j\neq 0} (-1)^{i+j}\big(\sqrt{R_0}q\sqrt{R_0}\big)^i\sqrt{R_0}f\sqrt{R_0}\big(\sqrt{R_0}q\sqrt{R_0}\big)^j \right\|_{\mathfrak{I}_2} \\ &\quad\leq \sum_{i\neq 0 \vee j\neq 0} \left\| \sqrt{R_0}q\sqrt{R_0} \right\|^{i+j} \left\| \sqrt{R_0}f\sqrt{R_0} \right\|_{\mathfrak{I}_2} \\ &\quad\lesssim \frac{1}{\kappa}\|q\|_{H^{-1}_\kappa(\mathbb{R})}\|f\|_{H^{-1}_\kappa(\mathbb{R})} \end{aligned}$$

i.e., we have

$$\|\mathrm{d}g(0) - \mathrm{d}g(q)\|_{H^{-1}_\kappa(\mathbb{R})\to H^1_\kappa(\mathbb{R})} \lesssim \kappa^{-1}\|q\|_{H^{-1}_\kappa(\mathbb{R})}. \tag{7}$$

Further, as $\left\|R_0(2\kappa)^{-1}\right\|_{H^1_\kappa(\mathbb{R})\to H^{-1}_\kappa(\mathbb{R})} = 1$, we have

$$\kappa^{-1}\|q\|_{H^{-1}_\kappa(\mathbb{R})} \leq \delta\left\|\big(\mathrm{d}g(0)\big)^{-1}\right\|^{-1}_{H^1_\kappa(\mathbb{R})\to H^{-1}_\kappa(\mathbb{R})}. \tag{8}$$

By Theorem 2.12 the map

$$q \mapsto g - \frac{1}{2\kappa}$$

is a local diffeomorphism from B_δ into $H^1_\kappa(\mathbb{R})$ for some δ sufficiently small. Note that Equation 7 and Equation 8 guarantee that we may choose δ independent of κ. By the Sobolev embedding and Equation 5 we may choose δ even smaller to get

$$\frac{1}{4\kappa} \leq g \leq \frac{3}{4\kappa} \tag{9}$$

for all $q \in B_\delta$, which implies that $q \mapsto \kappa - 1/2g$ is also real analytic. Using that $f \mapsto f/(1+f)$ is a diffeomorphism from a neighborhood of 0 in $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$ and observing that

$$\kappa - \frac{1}{2g} = \kappa \frac{2\kappa\big(g - 1/2\kappa\big)}{1 + 2\kappa\big(g - 1/2\kappa\big)},$$

we have that $q \mapsto \kappa - 1/2g$ is also a diffeomorphism. □

Remark 3.5
We have the inherent symmetry that translating the potential q is equivalent to translating the Green's function

$$g(x+h, q) = g(x, q(\,\cdot\, + h)). \tag{10}$$

Lemma 3.6
For $q \in \mathcal{S}(\mathbb{R})$ and a multi-index $\sigma = (\sigma_1, \ldots, \sigma_l)$ we have

$$\prod_{k=1}^{i} \left\| q^{(\sigma_k)} \right\|_{H^s(\mathbb{R})} \leq \left\| q^{(|\sigma|)} \right\|_{H^s(\mathbb{R})} \|q\|_{H^s(\mathbb{R})}^{i-1}.$$

Proof. We want to apply Hölder's inequality in every factor of the product. So take $p_k = \frac{|\sigma|}{\sigma_k}$, $\frac{1}{p_k} + \frac{1}{q_k} = 1$ then

$$\begin{aligned}
\left\| q^{(\sigma_k)} \right\|_{H^s(\mathbb{R})}^2 &= \left\| \langle \cdot \rangle^{2s} (\,\cdot\,)^{2\sigma_k} (\mathcal{F}q)^2 \right\|_{L^1(\mathbb{R})} \\
&\leq \left\| (\,\cdot\,)^{2\sigma_k} \Big(\langle \cdot \rangle^{2s} (\mathcal{F}q)^2 \Big)^{\frac{1}{p_k}} \right\|_{L^{p_k}(\mathbb{R})} \left\| \Big(\langle \cdot \rangle^{2s} (\mathcal{F}q)^2 \Big)^{\frac{1}{q_k}} \right\|_{L^{q_k}(\mathbb{R})} \\
&= \left\| (\,\cdot\,)^{2\sigma_k p_k} \langle \cdot \rangle^{2s} (\mathcal{F}q)^2 \right\|_{L^1(\mathbb{R})}^{\frac{1}{p_k}} \left\| \langle \cdot \rangle^{2s} (\mathcal{F}q)^2 \right\|_{L^1(\mathbb{R})}^{\frac{1}{q_k}} \\
&= \left\| q^{(|\sigma|)} \right\|_{H^s(\mathbb{R})}^{\frac{2}{p_k}} \|q\|_{H^s(\mathbb{R})}^{\frac{2}{q_k}}.
\end{aligned}$$

The claim follows by taking the product over k. □

Proposition 3.7
If in addition to $q \in B_\delta$ with $\delta > 0$ sufficiently small and $\kappa \geq 1$ we have $q \in \mathcal{S}(\mathbb{R})$ then

$$g - \frac{1}{2\kappa} \quad \textit{and} \quad \kappa - \frac{1}{2g}$$

are also in $\mathcal{S}(\mathbb{R})$, and we have the bounds

$$\begin{aligned} \left\|g'\right\|_{H^s(\mathbb{R})} &\lesssim_s \|q\|_{H^{s-1}(\mathbb{R})}, \\ \left\|\langle \cdot \rangle^s g'\right\|_{L^2(\mathbb{R})} &\lesssim_s \|\langle \cdot \rangle^s q\|_{H^{-1}(\mathbb{R})}. \end{aligned}$$

Proof. Using Equation 4 and Equation 10 we have for every $s \in \mathbb{N}$

$$\begin{aligned} \partial_x^s g(x,q) &= \frac{\mathrm{d}^s}{\mathrm{d}h^s}\Big|_{h=0} g(x, q(\,\cdot\, + h)) \\ &= \frac{\mathrm{d}^s}{\mathrm{d}h^s}\Big|_{h=0} \left(\frac{1}{2\kappa} + \sum_{i=1}^{\infty} (-1)^i \left\langle \sqrt{R_0}\delta_x \middle| \left(\sqrt{R_0} q(\,\cdot\, + h)\sqrt{R_0}\right)^i \sqrt{R_0}\delta_x \right\rangle \right) \\ &= \sum_{i=1}^{\infty} (-1)^i \left\langle \sqrt{R_0}\delta_x \middle| \sum_{|\sigma|=s} \binom{s}{\sigma} \prod_{k=1}^{i} \left(\sqrt{R_0} q^{(\sigma_k)} \sqrt{R_0}\right) \sqrt{R_0}\delta_x \right\rangle \end{aligned}$$

and thus for $f \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} \left| \int \partial_x^s g(x) f(x)\,\mathrm{d}x \right| &= \sum_{i=1}^{\infty} \left| \mathrm{Tr}\left(f\sqrt{R_0} \sum_{|\sigma|=s} \binom{s}{\sigma} \prod_{k=1}^{i} \left(\sqrt{R_0} q^{(\sigma_k)} \sqrt{R_0}\right) \sqrt{R_0} \right) \right| \\ &\leq \sum_{i=1}^{\infty} \sum_{|\sigma|=s} \binom{s}{\sigma} \left\| \sqrt{R_0} f \sqrt{R_0} \right\|_{\mathfrak{I}_2} \prod_{k=1}^{i} \left\| \sqrt{R_0} q^{(\sigma_k)} \sqrt{R_0} \right\|_{\mathfrak{I}_2} \\ &\leq \|f\|_{H_\kappa^{-1}(\mathbb{R})} \sum_{i=1}^{\infty} \sum_{|\sigma|=s} \binom{s}{\sigma} \prod_{k=1}^{i} \left\| q^{(\sigma_k)} \right\|_{H_\kappa^{-1}(\mathbb{R})}. \end{aligned}$$

Applying Lemma 3.6 we get

$$\begin{aligned} \left| \int \partial_x^s g(x) f(x)\,\mathrm{d}x \right| &\leq \|f\|_{H_\kappa^{-1}(\mathbb{R})} \sum_{i=1}^{\infty} i^s \left\| q^{(s)} \right\|_{H_\kappa^{-1}(\mathbb{R})} \delta^{i-1} \\ &\lesssim_s \|f\|_{H_\kappa^{-1}(\mathbb{R})} \left\| q^{(s)} \right\|_{H_\kappa^{-1}(\mathbb{R})} \end{aligned}$$

i.e., we verified the first bound

$$\|\partial_x^s g\|_{H^1} \lesssim_s \left\| q^{(s)} \right\|_{H^{-1}(\mathbb{R})} \lesssim \|q\|_{H^{s-1}(\mathbb{R})}.$$

For the second bound, we first need to verify that

$$\langle x\rangle^s R_0 = \sum_{r=0}^{s} \sqrt{R_0} A_{r,s} \sqrt{R_0} \langle x\rangle^r$$

where $\|A_{r,s}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \lesssim_s 1$ are bounded. For $s = 0$ this is trivially true and for the induction step, we note the commutator relations

$$\begin{aligned} [\langle x\rangle, R_0] &= R_0 I R_0, \\ [\langle x\rangle, I] &= S \\ \text{with} \quad I &= -\Big(\frac{x}{\langle x\rangle}\partial_x + \partial_x \frac{x}{\langle x\rangle}\Big), \\ S &= 2\frac{x^2}{\langle x\rangle^2}. \end{aligned}$$

Then we have

$$\begin{aligned} \langle x\rangle^{s+1} R_0 &= \langle x\rangle^s (R_0 I R_0 + R_0 \langle x\rangle) \\ &= \sum_{r=0}^{s} \sqrt{R_0} A_{r,s} \sqrt{R_0} \langle x\rangle^r I R_0 + \sum_{r=0}^{s} \sqrt{R_0} A_{r,s} \sqrt{R_0} \langle x\rangle^{r+1}. \end{aligned}$$

The trailing sum is already of the wanted form, so let us focus on the leading sum. Observe that

$$\langle x\rangle^r I = rS\langle x\rangle^{r-1} + I\langle x\rangle^r$$

and hence

$$\begin{aligned} \sum_{r=0}^{s} \sqrt{R_0} A_{r,s} \sqrt{R_0} \langle x\rangle^r I R_0 &= r\sum_{r=0}^{s} \sqrt{R_0} A_{r,s} \sqrt{R_0} S \langle x\rangle^{r-1} R_0 \\ &\quad + \sum_{r=0}^{s} \sqrt{R_0} A_{r,s} \sqrt{R_0} I \langle x\rangle^r R_0 \\ &= r\sum_{r=0}^{s}\sum_{k=0}^{r-1} \sqrt{R_0} A_{r,s} \sqrt{R_0} S \sqrt{R_0} A_{k,r-1} \sqrt{R_0} \langle x\rangle^k \\ &\quad + \sum_{r=0}^{s}\sum_{k=0}^{r} \sqrt{R_0} A_{r,s} \sqrt{R_0} I \sqrt{R_0} A_{k,r} \sqrt{R_0} \langle x\rangle^k. \end{aligned}$$

Thus, we only need to verify that both $\sqrt{R_0} S \sqrt{R_0}$ and $\sqrt{R_0} I \sqrt{R_0}$ are bounded. The first one is bounded as S is bounded. The second one is bounded as both $\sqrt{R_0}\partial_x$ and $\partial_x\sqrt{R_0}$ are bounded, which can be seen by conjugating with the Fourier transform and observing that they are essentially $x/\langle x\rangle$ on the Fourier side.

Using this representation, the second bound is now a straightforward calculation, we have for $f \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}
&\left| \int f(x)\langle x\rangle^s \left(g(x) - \frac{1}{2\kappa}\right) \mathrm{d}x \right| \\
&\quad = |\mathrm{Tr}(f\langle\,\cdot\,\rangle^s (R - R_0))| \\
&\quad \leq \sum_{i=1}^{\infty} \left| \mathrm{Tr}\left(f\langle\,\cdot\,\rangle^s \sqrt{R_0}\left(\sqrt{R_0}q\sqrt{R_0}\right)^i \sqrt{R_0}\right)\right| \\
&\quad = \sum_{i=1}^{\infty} \left| \mathrm{Tr}\left(\sqrt{R_0} f\langle\,\cdot\,\rangle^s R_0 q \sqrt{R_0}\left(\sqrt{R_0}q\sqrt{R_0}\right)^{i-1}\right)\right| \\
&\quad \leq \sum_{i=1}^{\infty}\sum_{r=0}^{s} \left| \mathrm{Tr}\left(\sqrt{R_0} f \sqrt{R_0} A_{r,s}\sqrt{R_0}\langle\,\cdot\,\rangle^r q \sqrt{R_0}\left(\sqrt{R_0}q\sqrt{R_0}\right)^{i-1}\right)\right| \\
&\quad \lesssim_s \sum_{i=1}^{\infty}\sum_{r=0}^{s} \|f\|_{H_\kappa^{-1}(\mathbb{R})} \|\langle\,\cdot\,\rangle^r q\|_{H_\kappa^{-1}(\mathbb{R})}\delta^{i-1} \\
&\quad \lesssim_s \|f\|_{H_\kappa^{-1}(\mathbb{R})} \|\langle\,\cdot\,\rangle^s q\|_{H_\kappa^{-1}(\mathbb{R})},
\end{aligned}$$

so we verified the second bound

$$\|\langle\,\cdot\,\rangle^s g'\|_{L^2(\mathbb{R})} \lesssim_s \left\| \langle\,\cdot\,\rangle^s \left(g - \frac{1}{2\kappa}\right)\right\|_{H^1(\mathbb{R})} \lesssim_s \|\langle\,\cdot\,\rangle^s q\|_{H^{-1}(\mathbb{R})}.$$

□

Now that we have some information about the regularity of g, we want to turn our attention to other characteristics and properties of g.

Lemma 3.8
For $\delta > 0$ sufficiently small and $\kappa \geq 1$ the diagonal Green's function satisfies

$$g''' = 2(qg)' + 2qg' + 4\kappa^2 g'.$$

Proof. For the kernel of the resolvent we have

$$\big(- \partial_x^2 + q(x)\big) G(x,y) = -\kappa^2 G(x,y) + \delta(x-y) = \big(- \partial_y^2 + q(y)\big) G(x,y).$$

Hence, differentiating with respect to x and y yields

$$\begin{aligned}
\partial_x\big(- \partial_x^2 + q(x)\big) G(x,y) &= -\partial_x \kappa^2 G(x,y) + \delta'(x-y) \\
&= \partial_x\big(- \partial_y^2 + q(y)\big) G(x,y), \\
\partial_y\big(- \partial_y^2 + q(y)\big) G(x,y) &= -\partial_y \kappa^2 G(x,y) - \delta'(x-y) \\
&= \partial_y\big(- \partial_x^2 + q(x)\big) G(x,y),
\end{aligned}$$

and adding them appropriately gives

$$\partial_x\big(\partial_x^2+3\partial_y^2\big)G(x,y)+\partial_y\big(\partial_y^2+3\partial_x^2\big)G(x,y)$$
$$=\partial_x\big(q(x)+3q(y)+4\kappa^2\big)G(x,y)+\partial_y\big(q(y)+3q(x)+4\kappa^2\big)G(x,y).$$

Therefore,

$$\begin{aligned}
&(\partial_x+\partial_y)^3G(x,y)\\
&\quad=\partial_x\big(q(x)+3q(y)+4\kappa^2\big)G(x,y)+\partial_y\big(q(y)+3q(x)+4\kappa^2\big)G(x,y)\\
&\quad=q'(x)G(x,y)+q(x)\partial_xG(x,y)+3q(x)\partial_yG(x,y)\\
&\qquad+q'(y)G(x,y)+q(y)\partial_yG(x,y)+3q(y)\partial_xG(x,y)\\
&\qquad+4\kappa^2(\partial_x+\partial_y)G(x,y)\\
&\quad=q'(x)G(x,y)+2q(x)\partial_xG(x,y)-q(x)\partial_xG(x,y)+3q(x)\partial_yG(x,y)\\
&\qquad+q'(y)G(x,y)+2q(y)\partial_yG(x,y)-q(y)\partial_yG(x,y)+3q(y)\partial_xG(x,y)\\
&\qquad+4\kappa^2(\partial_x+\partial_y)G(x,y)\\
&\quad=(q'(x)+q'(y))G(x,y)+2(q(x)+q(y))(\partial_x+\partial_y)G(x,y)\\
&\qquad-(q(x)-q(y))(\partial_x-\partial_y)G(x,y)+4\kappa^2(\partial_x+\partial_y)G(x,y).
\end{aligned}$$

For $x=y$ this yields the desired result

$$g'''=2q'g+4qg'+4\kappa^2g'.$$

□

Remark 3.9
Lemma 3.8 needed the assumptions on q and κ merely to ensure that the Green's function is well-defined.

Lemma 3.10
The Green's function satisfies

$$\int\frac{G(x,y)G(y,x)}{2g(y)^2}\,\mathrm{d}y=g(x) \tag{11}$$

for $q\in B_\delta$, $\delta>0$ sufficiently small and $\kappa\geq 1$.

Proof. Choosing δ small enough we see that both sides of Equation 11 are real analytic by Proposition 3.4. Hence, due to density, we may assume that $q\in\mathcal{S}(\mathbb{R})$. There are solutions to the equation

$$-f''+qf=-\kappa^2f$$

with

$$f_\pm = O\big(\exp(\mp\kappa x)\big)$$

for both $x \to \infty$ and $x \to -\infty$ (cf. Teschl [2014]). By differentiating the Wronskian of the two solutions we see that it is constant, further, the Sturm oscillation theorem guarantees that these solutions cannot change sign. Hence, we may normalize such that

$$f_+ f_-' - f_+' f_- = 1, \tag{12}$$

and $f_\pm > 0$. Using these solutions we can represent the Green's function as

$$G(x,y) = f_+(x \vee y) f_-(x \wedge y)$$

since we have

$$\begin{aligned}
&\partial_x^2\big(f_+(x \vee y) f_-(x \wedge y)\big) \\
&\quad = \partial_x\big(f_+'(x \vee y) f_-(x \wedge y)\theta(x-y) + f_+(x \vee y) f_-'(x \wedge y)\theta(y-x)\big) \\
&\quad = f_+''(x \vee y) f_-(x \wedge y)\theta(x-y) + f_+'(x \vee y) f_-(x \wedge y)\delta(x-y) \\
&\qquad + f_+(x \vee y) f_-''(x \wedge y)\theta(y-x) - f_+(x \vee y) f_-'(x \wedge y)\delta(y-x) \\
&\quad = (q(x \vee y) + \kappa^2) f_+(x \vee y) f_-(x \wedge y)\theta(x-y) \\
&\qquad + (q(x \wedge y) + \kappa^2) f_+(x \vee y) f_-(x \wedge y)\theta(y-x) + \delta(x-y) \\
&\quad = (q(x) + \kappa^2) f_+(x \vee y) f_-(x \wedge y) + \delta(x-y)
\end{aligned}$$

i.e., $R^{-1}\big(f_+(\,\cdot \vee y) f_-(\,\cdot \wedge y)\big) = \delta_y$. By reformulating the identity we want to prove in terms of $f_\pm$ the problem reduces to showing

$$\int \frac{f_+(x \vee y)^2 f_-(x \wedge y)^2}{2 f_+(y)^2 f_-(y)^2}\,\mathrm{d}y = f_+(x) f_-(x). \tag{13}$$

Splitting the integral into the parts $x < y$ and $x > y$ we have

$$\begin{aligned}
&\int_{-\infty}^{x} \frac{f_+(x)^2}{2 f_+(y)^2}\,\mathrm{d}y + \int_{x}^{\infty} \frac{f_-(x)^2}{2 f_-(y)^2}\,\mathrm{d}y \\
&\quad = \frac{f_+(x)^2}{2} \int_{-\infty}^{x} \frac{1}{f_+(y)^2}\,\mathrm{d}y + \frac{f_-(x)^2}{2} \int_{x}^{\infty} \frac{1}{f_-(y)^2}\,\mathrm{d}y\,.
\end{aligned}$$

Using the asymptotic behaviour of $f_\pm$ and that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f_-}{f_+} = \frac{1}{f_+^2} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\frac{f_+}{f_-} = -\frac{1}{f_-^2},$$

which follows from Equation 12, we indeed obtain Equation 13. □

Lemma 3.11
If in addition to $q \in B_\delta$, $\delta > 0$ sufficiently small and $\kappa \geq 1$, we have $f, q \in \mathcal{S}(\mathbb{R})$ it holds that

$$\int G(x,y)\Big(- f'''(y) + 2q(y)f' + 2\big(q(y)f(y)\big)' + 4\kappa^2 f'(y)\Big) G(y,x)\,\mathrm{d}y$$
$$= 2f'(x)g(x) - 2f(x)g'(x),$$

Proof. Applying the commutator relations $[\partial_x, f] = f'$ and $\left[\partial_x^2, f\right] = -f'' + 2\partial_x f'$ we have

$$\begin{aligned}
&R^{-1}f' - 2qf' + f'R^{-1} - 2R^{-1}f\partial_x - 2(qf)' + 2\partial_x f R^{-1} - 4\kappa^2 f' \\
&\quad = R^{-1}f' - qf' + f'R^{-1} - f'q - 2R^{-1}f\partial_x + 2qf\partial_x + 2\partial_x f R^{-1} \\
&\qquad - 2\partial_x f q - 4\kappa^2 f' \\
&\quad = \big(- \partial_x^2 + \kappa^2\big) f' + f'\big(- \partial_x^2 + \kappa^2\big) - 2\big(- \partial_x^2 + \kappa^2\big) f\partial_x \\
&\qquad + 2\partial_x f\big(- \partial_x^2 + \kappa^2\big) - 4\kappa^2 f' \\
&\quad = \big(- \partial_x^2 + \kappa^2\big) f' + \left[\partial_x^2, f'\right] - \partial_x^2 f' + \kappa^2 f' \\
&\qquad + 2\big(- \partial_x^2 + \kappa^2\big)\big([\partial_x, f] - \partial_x f\big) + 2\partial_x\Big(\left[\partial_x^2, f\right] - \partial_x^2 f + \kappa^2 f\Big) - 4\kappa^2 f' \\
&\quad = \big(- \partial_x^2 + \kappa^2\big) f' - f''' + 2\partial_x f'' - \partial_x^2 f' + \kappa^2 f' \\
&\qquad + 2\big(- \partial_x^2 + \kappa^2\big)\big(f' - \partial_x f\big) + 2\partial_x\big(- f'' + 2\partial_x f' - \partial_x^2 f + \kappa^2 f\big) - 4\kappa^2 f' \\
&\quad = -f'''.
\end{aligned}$$

Multiplying by R from left and right gives

$$\begin{aligned}
&- Rf'''R + R\,2qf'R + R\,2(qf)'R + R\,4\kappa^2 f'R \\
&\quad = f'R + Rf' - 2f\partial_x R + R\,2\partial_x f \\
&\quad = f'R + Rf' - 2f\partial_x R + 2\partial_x R f - 2[\partial_x, R]f.
\end{aligned}$$

Taking the diagonals for the corresponding integral kernels yields the result. □

Remark 3.12
Lemma 3.11 also holds for $f + c \in \mathcal{S}(\mathbb{R})$ for some constant c, as the equation is linear in f and the case of f being a constant is easy to check using Equation 10.

3.2 Introducing α and ρ

Proposition 3.13
For $\kappa \geq 1$ and $q \in B_\delta$, $\delta > 0$ sufficiently small it holds that

$$\rho(x;\kappa,q) \coloneqq \kappa - \frac{1}{2g(x;\kappa,q)} + 2\kappa R_0(2\kappa)q(x) \quad \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}).$$

Moreover, if we define

$$\alpha(\kappa,q) = \int \rho(x;\kappa,q)\,\mathrm{d}x\,,$$

then α is real analytic with

$$\alpha\big(\kappa,q\big) \approx \frac{1}{\kappa}\|q\|^2_{H^{-1}_\kappa(\mathbb{R})}$$

uniformly for $q \in B_\delta$.

Proof. First note that ρ is in $H^1(\mathbb{R})$ by Proposition 3.4. To see that $\rho \in L^1(\mathbb{R})$ we rewrite

$$\begin{aligned}\rho &= \kappa - \frac{1}{2g} + 2\kappa R_0(2\kappa)q \\ &= 2\kappa^2\left(g - \frac{1}{2\kappa} + \frac{1}{\kappa}R_0(2\kappa)q\right) - \frac{2\kappa^2}{g}\left(g - \frac{1}{2\kappa}\right)^2\end{aligned}$$

from which it is clear that the latter term is in $L^1(\mathbb{R})$ as $g - 1/2\kappa \in L^2(\mathbb{R})$ by Proposition 3.4. For the first term take $f \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}&\left|\int\left(g(x) - \frac{1}{2\kappa} + \frac{1}{\kappa}R_0(2\kappa)q(x)\right)f(x)\,\mathrm{d}x\right| \qquad (14)\\ &\quad = \big|\mathrm{Tr}\,\big(f(R - R_0 + R_0 q R_0)\big)\big| \\ &\quad \leq \sum_{i\geq 2}\left|\mathrm{Tr}\left(f\sqrt{R_0}\left(\sqrt{R_0}q\sqrt{R_0}\right)^i\sqrt{R_0}\right)\right| \\ &\quad \leq \|f\|_{L^\infty(\mathbb{R})}\left\|\sqrt{R_0}\right\|^2\left\|\sqrt{R_0}q\sqrt{R_0}\right\|^2_{\mathfrak{I}_2}\sum_{i\geq 2}\|q\|^{i-2}_{H^{-1}_\kappa(\mathbb{R})} \\ &\quad \lesssim \|f\|_{L^\infty(\mathbb{R})}\left\|\sqrt{R_0}\right\|^2\end{aligned}$$

and by duality, we have $\rho \in L^1(\mathbb{R})$.

By Equation 6 and Lemma 3.10 we have

$$\begin{aligned}
\mathrm{d}\alpha(q)(f) &= \int \mathrm{d}\rho(q)(f)(x)\,\mathrm{d}x \qquad (15)\\
&= \int \Big(- \frac{1}{2g(x)^2} \int G(x,y)f(y)G(y,x)\,\mathrm{d}y + 2\kappa R_0(2\kappa)f(x)\Big)\,\mathrm{d}x\\
&= \iint \Big(- \frac{G(x,y)G(y,x)}{2g(x)^2} + 2\kappa G_0(x,y;2\kappa)\Big)\,\mathrm{d}x\, f(y)\,\mathrm{d}y\\
&= \int \Big(\frac{1}{2\kappa} - g(y)\Big) f(y)\,\mathrm{d}y\,.
\end{aligned}$$

Note that both α and $\mathrm{d}\alpha$ vanish at $q=0$. Moreover, α is real analytic by Proposition 3.4. To find an approximate representation of α in some small neighborhood B_δ around 0 we thus only need to consider the Hessian of α at 0. For the Hessian, we have by Equation 6

$$\begin{aligned}
\mathrm{d}^2\alpha(q)(f,f) &= \int \big(- \mathrm{d}g(q)(f)(x)\big) f(x)\,\mathrm{d}x\\
&= \iint G(x,y)f(y)G(y,x)f(x)\,\mathrm{d}y\,\mathrm{d}x\,,
\end{aligned}$$

and in particular, for $q=0$ we have

$$\begin{aligned}
\mathrm{d}^2\alpha(0)(f,f) &= \int \kappa^{-1} R_0(2\kappa) f(x) f(x)\,\mathrm{d}x\\
&= \kappa^{-1}\|f\|^2_{H^{-1}_\kappa(\mathbb{R})},
\end{aligned}$$

by a calculation similar to the one in Equation 3. So we do indeed have that

$$\alpha(\kappa,q) \approx \frac{1}{\kappa}\|q\|^2_{H^{-1}_\kappa(\mathbb{R})}$$

in some neighborhood of 0. Moreover, Equation 7 allows us to control the modulus of continuity of the Hessian

$$\begin{aligned}
\big|\mathrm{d}^2\alpha(q)(f,f) - \mathrm{d}^2\alpha(0)(f,f)\big| &= \left| \int \big(\mathrm{d}g(q)(f)(x) - \mathrm{d}g(0)(f)(x)\big) f(x)\,\mathrm{d}x \right|\\
&\le \|\mathrm{d}g(q)(f) - \mathrm{d}g(0)(f)\|_{H^1(\mathbb{R})}\|f\|_{H^{-1}(\mathbb{R})}\\
&\lesssim \kappa^{-1}\|q\|_{H^{-1}(\mathbb{R})}\|f\|^2_{H^{-1}_\kappa(\mathbb{R})},
\end{aligned}$$

which verifies that the size of the neighborhood can be chosen independently of κ. □

4 Dynamics

By virtue of Equation 15 we have that

$$\frac{\delta\alpha}{\delta q} = \frac{1}{2\kappa} - g(x). \tag{16}$$

Moreover, by Equation 4 we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(-\log\det_2\Big(1+\sqrt{R_0}(q+sf)\sqrt{R_0}\Big)\Big)\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\operatorname{Tr}\Big(\Big(\sqrt{R_0}(q+sf)\sqrt{R_0}\Big)^k\Big)\\
&\quad= \sum_{k=2}^{\infty}(-1)^k\operatorname{Tr}\Big(\Big(\sqrt{R_0}q\sqrt{R_0}\Big)^{k-1}\sqrt{R_0}f\sqrt{R_0}\Big)\\
&\quad= \int\Big(\frac{1}{2\kappa}-g(x)\Big)f(x)\,\mathrm{d}x\,.
\end{aligned}$$

Combining these two equations we see that

$$\frac{\delta\alpha}{\delta q} = \frac{1}{2\kappa} - g(x) = \frac{\delta}{\delta q}\Big(-\log\det_2\Big(1+\sqrt{R_0}q\sqrt{R_0}\Big)\Big)$$

and hence

$$\alpha = -\log\det_2\Big(1+\sqrt{R_0}q\sqrt{R_0}\Big)$$

is indeed the quantity we wanted to look out for. So, we expect α to be conserved under the flow of the KdV.

Notation
We will abbreviate

$$G(t,x,y) = G(x,y;q(t)), \quad g(t,x) = g(x;q(t)) \quad \text{and} \quad \rho(t,x) = \rho(x;q(t)).$$

Proposition 4.1
For a Schwartz solution $q(t)$ of the KdV (1) with initial condition $q(0) \in B_\delta$, $\delta > 0$ sufficiently small and $\kappa \geq 1$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t,x) = -2q'(t,x)g(t,x) + 2q(t,x)g'(t,x) - 4\kappa^2 g'(t,x) \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2g(t,x)} = \left(\frac{q(t,x)}{g(t,x)} - \frac{2\kappa^2}{g(t,x)} + 4\kappa^3\right)' \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,x) = \left(6\kappa R_0(2\kappa)q^2(t,x) + 2q(t,x)\left(\kappa - \frac{1}{2g(t,x)}\right) - 4\kappa^2\rho(t,x)\right)' \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha\big(\kappa, q(t)\big) = 0. \tag{20}$$

Proof. Equation 17 follows from Equation 6, Lemma 3.11 and by virtue of $q(t)$ being a solution of the KdV (1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}g(t,x) &= \mathrm{d}g(q(t))\big(\partial_t q(t)\big)(x)\\ &= -\int G(t,x,y)\partial_t q(t,y)G(t,y,x)\,\mathrm{d}y\\ &= -\int G(t,x,y)\big(-q'''(t,y)+6q'(t,y)q(t,y)\big)G(t,y,x)\,\mathrm{d}y\\ &= -2q'(t,x)g(t,x)+2q(t,x)g'(t,x)+4\kappa^2\int G(t,x,y)q'(t,y)G(t,y,x)\,\mathrm{d}y\\ &= -2q'(t,x)g(t,x)+2q(t,x)g'(t,x)-4\kappa^2 g'(t,x).\end{aligned}$$

For Equation 18, just note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2g(t)} &= -\frac{1}{2g(t)^2}\frac{\mathrm{d}}{\mathrm{d}t}g(t)\\ &= \frac{2q'(t)g(t)-2q(t)g'(t)+4\kappa^2 g'(t)}{2g(t)^2}\\ &= \left(\frac{q(t)}{g(t)}-\frac{2\kappa^2}{g(t)}\right)'.\end{aligned}$$

Using this and that $q(t)$ satisfies the KdV (1) we get for Equation 19

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho &= -\left(\frac{q}{g}-\frac{2\kappa^2}{g}+4\kappa^3\right)'+2\kappa R_0(2\kappa)(\partial_t q)\\ &= -\left(\frac{q}{g}-\frac{2\kappa^2}{g}+4\kappa^3\right)'+2\kappa R_0(2\kappa)\big(-q'''\big)+\kappa R_0(2\kappa)\big(3q^2\big)'\\ &= \left(-\left(\frac{q}{g}-\frac{2\kappa^2}{g}+4\kappa^3\right)+2\kappa R_0(2\kappa)\big(-q''+4\kappa^2 q-4\kappa^2 q\big)+\kappa R_0(2\kappa)\big(3q^2\big)\right)'\\ &= \left(-\left(\frac{q}{g}-\frac{2\kappa^2}{g}+4\kappa^3\right)+2\kappa q-8\kappa^3 R_0(2\kappa)q+2\kappa R_0(2\kappa)\big(3q^2\big)\right)'\\ &= \left(6\kappa R_0(2\kappa)\big(q^2\big)+2q\left(\kappa-\frac{1}{g}\right)-4\kappa^2\left(\kappa-\frac{1}{2g}+2\kappa R_0(2\kappa)q\right)\right)'.\end{aligned}$$

Equation 20 is just a simple consequence of Equation 19 after integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha\big(q(t)\big)=\int\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,x)\,\mathrm{d}x=0$$

as $\mathrm{d}\rho/\mathrm{d}t$ is a spatial derivative of a Schwartz function. □

Corollary 4.2
For $\delta > 0$ there is a $\delta_0 > 0$ such that for every Schwartz solution $q(t)$ of the KdV (1) with initial condition $q(0) \in B_{\delta_0}$ one has

$$\sup_{t\in\mathbb{R}} \|q(t)\|_{H^{-1}(\mathbb{R})} \leq \delta.$$

Proof. Choosing δ_0 such that Proposition 3.13 and Proposition 4.1 are applicable, we have

$$\|q(t)\|_{H^{-1}(\mathbb{R})} \approx \alpha(1, q(t)) = \alpha(1, q(0)) \approx \|q(0)\|_{H^{-1}(\mathbb{R})} \leq \delta_0$$

and the claim follows after updating δ_0 if necessary. □

The Hamiltonian associated with the KdV is

$$H_{\text{KdV}} = \int \left(\frac{1}{2} q'(x)^2 + q(x)^3\right) \mathrm{d}x$$

as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} &\int \frac{1}{2}\Big(\big(q(x) + sf(x)\big)'\Big)^2 + \big(q(x) + sf(x)\big)^3 \,\mathrm{d}x \\
&= \int q'(x) f'(x) + 3q(x)^2 f(x) \,\mathrm{d}x \\
&= \int -q''(x) f(x) + 3q(x)^2 f(x) \,\mathrm{d}x \,.
\end{aligned}$$

As noted previously, we want to look at the Hamiltonian

$$H_\kappa = -16\kappa^5 \alpha(\kappa) + 4\kappa^2 P, \tag{21}$$

to approximate the KdV Hamiltonian. Note that since α is preserved under the flow of the KdV we know that $\{H_{\text{KdV}}, \alpha\} = 0$. Furthermore, the momentum

$$P = \int \frac{1}{2} q(x)^2 \,\mathrm{d}x$$

also Poisson commutes with H_{KdV} as

$$\frac{\delta P}{\delta q} = q, \tag{22}$$

and

$$\{H_{\text{KdV}}, P\} = \int \big(-q''(x) + 3q^2(x)\big) q'(x) \,\mathrm{d}x = \int \left(\frac{1}{2} q'(x)^2 + q^3(x)\right)' \mathrm{d}x = 0.$$

So by the linearity of the Poisson bracket, we also have $\{H_{\text{KdV}}, H_\kappa\} = 0$.

Proposition 4.3
The flow associated with H_κ is

$$\frac{\mathrm{d}}{\mathrm{d}t}q(t,x) = 16\kappa^5 g'(t,x;\kappa) + 4\kappa^2 q'(t,x), \tag{23}$$

and it is globally well-posed for $q(0) \in B_\delta$ for δ sufficiently small and independent of κ. The flow commutes with that of the KdV (1), and $\alpha(\varkappa)$ is conserved under this flow for all $\varkappa \geq 1$. For Schwartz initial conditions, the diagonal Green's function obeys

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2g(t,x;\varkappa)} = -\frac{4\kappa^5}{\kappa^2-\varkappa^2}\left(\frac{g(t,x;\kappa)}{g(t,x;\varkappa)} - \frac{\varkappa}{\kappa}\right)' + 4\kappa^2\left(\frac{1}{2g(t,x;\varkappa)} - \varkappa\right)'$$

as long as $\varkappa \neq \kappa$.

Proof. By Equation 16, Equation 21 and Equation 22 we have

$$\begin{aligned}\frac{\delta H_\kappa}{\delta q} &= -16\kappa^5\frac{\delta\alpha}{\delta q} + 4\kappa^2\frac{\delta P}{\delta q}\\ &= -16\kappa^5\left(\frac{1}{2\kappa} - g\right) + 4\kappa^2 q\end{aligned}$$

from which we get Equation 23. Using that

$$\begin{aligned}\partial_s q(s, x+4\kappa^2(t-s)) &= q_t(s, x+4\kappa^2(t-s)) - 4\kappa^2 q'(s, x+4\kappa^2(t-s))\\ &= 16\kappa^5 g'(x+4\kappa^2(t-s), q(s))\end{aligned}$$

we can integrate over s to obtain the equivalent integral equation

$$q(t,x) = q(0, x+4\kappa^2 t) + \int_0^t 16\kappa^5 g'(x+4\kappa^2(t-s), q(s))\,\mathrm{d}s\,.$$

From Proposition 3.4 we get

$$\left\|g'(q) - g'(\tilde{q})\right\|_{H^{-1}(\mathbb{R})} \lesssim \|g(q) - g(\tilde{q})\|_{H^1(\mathbb{R})} \lesssim \|q - \tilde{q}\|_{H^{-1}(\mathbb{R})}$$

and hence local well-posedness follows from Picard iteration. Once we proved the conservation of $\alpha(\varkappa)$, global well-posedness follows from Proposition 3.13, as conservation of $\alpha(\varkappa)$ gives a bound on $\|q(t)\|_{H^{-1}(\mathbb{R})}$. From Equation 6 and Equation 23 we get for the diagonal Green's function

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2g(x;\varkappa)} &= -\frac{1}{2g(x;\varkappa)^2}\mathrm{d}g(q;\varkappa)\big(\partial_t q\big)(x)\\ &= -\frac{8\kappa^5}{g(x;\varkappa)^2}\mathrm{d}g(q;\varkappa)\big(g'(\kappa)\big)(x) - \frac{4\kappa^2}{2g(x;\varkappa)^2}\mathrm{d}g(q(t);\varkappa)\big(q'\big)(x)\\ &= \frac{8\kappa^5}{g(x;\varkappa)^2}\int G(x,y;\varkappa)g'(y;\kappa)G(y,x;\varkappa)\,\mathrm{d}y - \frac{4\kappa^2}{2g(x;\varkappa)^2}g'(x;\varkappa).\end{aligned}$$

By Lemma 3.8 we have

$$\big(4\kappa^2 - 4\varkappa^2\big)g'(\kappa) = -\Big(- g'''(\kappa) + 2\big(qg(\kappa)\big)' + 2qg'(\kappa) + 4\varkappa^2 g'(\kappa)\Big).$$

Substituting this into the above equation and calling for Lemma 3.11 gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2g(\varkappa)} &= -\frac{8\kappa^5}{g(\varkappa)^2}\frac{1}{(4\kappa^2 - 4\varkappa^2)}\Big(2g'(\kappa)g(\varkappa) - 2g(\kappa)g'(\varkappa)\Big) + 4\kappa^2\left(\frac{1}{2g(\varkappa)}\right)' \\ &= -\frac{4\kappa^5}{\kappa^2 - \varkappa^2}\left(\frac{g(\kappa)}{g(\varkappa)}\right)' + 4\kappa^2\left(\frac{1}{2g(\varkappa)}\right)'.\end{aligned}$$

Using this as well as Equation 23 we can see immediately that $\alpha(\varkappa)$ is conserved

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(\varkappa) &= \int \frac{\mathrm{d}}{\mathrm{d}t}\rho(x;\varkappa)\,\mathrm{d}x \\ &= \int \frac{\mathrm{d}}{\mathrm{d}t}\left(\varkappa - \frac{1}{2g(x;\varkappa)} + 2\varkappa R_0(2\varkappa)q(x)\right)\mathrm{d}x = 0\end{aligned}$$

as $\mathrm{d}\rho/\mathrm{d}t$ is a spatial derivative of a Schwartz function.
As mentioned previously, the two Hamiltonians H_κ and H_{KdV} Poisson commute, so we conclude further that their flows commute. □

5 Equicontinuity

To transfer the well-posedness result from the H_κ flow to that of H_{KdV}, we need equicontinuity.

Definition 5.1
We call $Q \subseteq H^s(\mathbb{R})$ *equicontinuous* in $H^s(\mathbb{R})$ if

$$q(\,\cdot\, + h) \xrightarrow[H^s(\mathbb{R})]{h\to 0} q$$

uniformly for $q \in Q$.

Lemma 5.2
For $\sigma < s$ and Q bounded in $H^s(\mathbb{R})$

1. *the following are equivalent.*
   (a) *Q is equicontinuous in $H^s(\mathbb{R})$.*
   (b) *It holds*

$$\int_{|w|\geq\kappa} \langle w\rangle^{2s} |\mathcal{F}q(w)|^2 \, \mathrm{d}w \xrightarrow{\kappa\to\infty} 0 \tag{24}$$

   uniformly for $q \in Q$.

2. *a sequence $\{q_n\}_n \subseteq H^s(\mathbb{R})$ is convergent iff it converges in $H^\sigma(\mathbb{R})$, and it is equicontinuous in $H^s(\mathbb{R})$.*

Proof. Property 1b implies Property 1a follows from

$$\|q(\,\cdot\,+h) - q\|_{H^s(\mathbb{R})} = \int |\exp(iwh) - 1|^2 \langle w\rangle^{2s} |\mathcal{F}q(w)|^2 \, \mathrm{d}w$$

noting that

$$|\exp(iwh) - 1|^2 = 2 - 2\cos(wh) \lesssim \kappa^2 h^2, \quad |w| < \kappa$$
and
$$|\exp(iwh) - 1| \lesssim 1, \quad |w| \geq \kappa.$$

Hence, by splitting the integral into two parts we get

$$\begin{aligned}\|q(\,\cdot\,+h) - q\|_{H^s(\mathbb{R})} &\lesssim \kappa^2 h^2 \int_{|w|<\kappa} \langle w\rangle^{2s} |\mathcal{F}q(w)|^2 \, \mathrm{d}w \\ &\quad + \int_{|w|\geq\kappa} \langle w\rangle^{2s} |\mathcal{F}q(w)|^2 \, \mathrm{d}w \\ &\leq \kappa^2 h^2 \|q\|^2_{H^s(\mathbb{R})} + \int_{|w|\geq\kappa} \langle w\rangle^{2s} |\mathcal{F}q(w)|^2 \, \mathrm{d}w\,.\end{aligned}$$

For $\kappa = 1/\sqrt{h}$ and $h \to 0$ we see that this vanishes uniformly as Q is bounded, so we have equicontinuity in $H^s(\mathbb{R})$. The direction Property 1a implies Property 1b follows from

$$\begin{aligned}\int |\exp(iwh) - 1|^2 \kappa \exp(-2\kappa|h|) \, \mathrm{d}h &= \int \big(2 - 2\cos(wh)\big) \kappa \exp(-2\kappa|h|) \, \mathrm{d}h \\ &= 2\int_{\mathbb{R}_+} \big(2 - 2\cos(wh)\big) \kappa \exp(-2\kappa h) \, \mathrm{d}h \\ &= \frac{2w^2}{4\kappa^2 + w^2} \gtrsim 1 - \chi_{[-\kappa,\kappa]}(w)\end{aligned}$$

and

$$\begin{aligned}\int \|q(\,\cdot\,+h)-q\|_{H^s(\mathbb{R})}^2 \kappa \exp(-2\kappa|h|)\,\mathrm{d}h \\ = \iint |\exp(iwh)-1|^2 \langle w\rangle^{2s} |\mathcal{F}q(w)|^2 \kappa \exp(-2\kappa|h|)\,\mathrm{d}w\,\mathrm{d}h \\ \gtrsim \int_{|w|\geq\kappa} \langle w\rangle^{2s} |\mathcal{F}q(w)|^2\,\mathrm{d}w\,.\end{aligned}$$

Since $\kappa\exp(-2\kappa|h|)$ forms a good kernel, we know that the left-hand side converges to zero as $\kappa\to\infty$ uniformly for $q\in Q$.
For Property 2 let q_n be a sequence in $H^s(\mathbb{R})$ convergent. Then it clearly converges in $H^\sigma(\mathbb{R})$ and equicontinuity in $H^s(\mathbb{R})$ holds by Equation 24 and the fact that Q is bounded in $H^s(\mathbb{R})$. Conversely, let q_n converge in $H^\sigma(\mathbb{R})$ and equicontinuous in $H^s(\mathbb{R})$. Using $\langle w\rangle^{2s} \lesssim \langle\kappa\rangle^{2s-2\sigma}\langle w\rangle^{2\sigma}$ for $w<\kappa$ we get

$$\begin{aligned}\|q_n-q_m\|_{H^s(\mathbb{R})}^2 &= \int \langle w\rangle^{2s} |\mathcal{F}q_n-\mathcal{F}q_m|^2\,\mathrm{d}w \\ &\leq \langle\kappa\rangle^{2s-2\sigma} \int \langle w\rangle^{2\sigma} |\mathcal{F}q_n-\mathcal{F}q_m|^2\,\mathrm{d}w \\ &\quad + \int_{|w|\geq\kappa} \langle w\rangle^{2s} |\mathcal{F}q_n-\mathcal{F}q_m|^2\,\mathrm{d}w\,.\end{aligned}$$

Since the q_n are equicontinuous in $H^s(\mathbb{R})$ we may take κ large enough so that the last term is smaller than some $\epsilon>0$, and then let $n,m\to\infty$. □

As mentioned previously, equicontinuity is strongly tied to $\alpha(\kappa)$.

Lemma 5.3
A bounded set $Q\subseteq B_\delta$ is equicontinuous in $H^{-1}(\mathbb{R})$ iff

$$\kappa\alpha(\kappa) \xrightarrow{\kappa\to\infty} 0$$

uniformly for $q\in Q$.

Proof. As a result of Proposition 3.13, we want to show that equicontinuity of Q in $H^{-1}(\mathbb{R})$ is equivalent to

$$\lim_{\kappa\to\infty}\sup_{q\in Q}\|q\|_{H_\kappa^{-1}(\mathbb{R})}^2 = 0.$$

That this implies equicontinuity in $H^{-1}(\mathbb{R})$ follows immediately from Lemma 5.2 and the fact that

$$\int_{|w|\geq\kappa} \langle w\rangle^{-2} |\mathcal{F}q(w)|^2\,\mathrm{d}w \lesssim \|q\|_{H_\kappa^{-1}(\mathbb{R})}^2.$$

Conversely, if Q is equicontinuous in $H^{-1}(\mathbb{R})$ we have by Lemma 5.2 that

$$\int_{|w|\geq\kappa} \langle w\rangle^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w \xrightarrow{\kappa\to\infty} 0$$

uniformly for $q \in Q$. So using the estimate

$$\begin{aligned}\int \langle w\rangle_\kappa^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w \lesssim \frac{\varkappa^2}{\kappa^2}\int_{|w|<\varkappa} \langle w\rangle^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w \\ + \int_{|w|\geq\varkappa} \langle w\rangle^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w\,,\end{aligned}$$

which tends to zero uniformly on Q as $\varkappa^2 = \kappa \to \infty$, we get the desired result. □

Proposition 5.4
For a bounded set of Schwartz functions $Q \subseteq B_\delta \cap \mathcal{S}(\mathbb{R})$ that are equicontinuous in $H^{-1}(\mathbb{R})$ we have that

$$Q^* = \left\{\mathrm{Fl}_t^{H_{\mathrm{KdV}}}\mathrm{Fl}_s^{H_\kappa} q \mid q \in Q,\ t,s \in \mathbb{R},\ \kappa \geq 1\right\} \tag{25}$$

is equicontinuous in $H^{-1}(\mathbb{R})$ and

$$4\kappa^3\left(\frac{1}{2\kappa} - g\right) \xrightarrow[H^{-1}(\mathbb{R})]{\kappa\to\infty} q$$

uniformly for $q \in Q^*$.

Proof. Lemma 5.3 shows that equicontinuity is a property that is contained in the asymptotic behavior of α. As $\alpha(\kappa)$ is conserved under the flows of H_κ and H_{KdV}, equicontinuity is therefore also conserved under these flows and hence Q^* is equicontinuous. By Equation 14 we have

$$\begin{aligned}\left\|\frac{1}{2\kappa} - g - \frac{1}{\kappa}R_0(2\kappa)q\right\|_{L^1(\mathbb{R})} &\lesssim \left\|\sqrt{R_0}\right\|^2\alpha(\kappa) \\ &\leq \frac{1}{\kappa^2}\alpha(\kappa)\end{aligned}$$

and using Lemma 5.3 we get

$$\kappa^3\left\|\frac{1}{2\kappa} - g - \frac{1}{\kappa}R_0(2\kappa)q\right\|_{L^1(\mathbb{R})} \xrightarrow{\kappa\to\infty} 0$$

uniformly for $q \in Q^*$.

Since

$$\left\|4\kappa^3\left(\frac{1}{2\kappa}-g\right)-q\right\|_{H^{-1}(\mathbb{R})} \le 4\kappa^3\left\|\frac{1}{2\kappa}-g-\frac{1}{\kappa}R_0(2\kappa)q\right\|_{H^{-1}(\mathbb{R})} + \left\|4\kappa^2R_0(2\kappa)q-q\right\|_{H^{-1}(\mathbb{R})}$$

we only need to show that

$$\left\|4\kappa^2R_0(2\kappa)q-q\right\|_{H^{-1}(\mathbb{R})} \xrightarrow{\kappa\to\infty} 0$$

uniformly for $q \in Q^*$, which follows from

$$\begin{aligned}\left\|4\kappa^2R_0(2\kappa)q-q\right\|^2_{H^{-1}(\mathbb{R})} &= \int \left(\frac{4\kappa^2}{w^2+4\kappa^2}-1\right)^2 \langle w\rangle^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w \\ &= \int \left(\frac{w^2}{w^2+4\kappa^2}\right)^2 \langle w\rangle^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w \\ &\le \int \langle w\rangle_\kappa^{-2}|\mathcal{F}q(w)|^2\,\mathrm{d}w \\ &= \|q\|^2_{H^{-1}_\kappa(\mathbb{R})} = \kappa\alpha(\kappa) \xrightarrow{\kappa\to\infty} 0\end{aligned}$$

by Lemma 5.3. □

6 Well-posedness

Theorem 6.1
Given Schwartz solutions $q_n(t)$ of the KdV (1) with initial conditions $q_n(0)$ converging in $H^{-1}(\mathbb{R})$ we also have that $q_n(t)$ converges in $H^{-1}(\mathbb{R})$ uniformly for compact time intervals.

Proof. First note that we may assume that $q_n(0) \in B_\delta$ for $\delta > 0$ small enough that all the previous results are applicable, since the scaling

$$q \mapsto q_\lambda(t,x) = \lambda^2 q(\lambda^3 t, \lambda x)$$

maps solutions of the KdV to solutions with

$$\begin{aligned}\|q_\lambda(0)\|^2_{H^{-1}(\mathbb{R})} &= \int \frac{|\mathcal{F}q_\lambda(0,w)|^2}{w^2+4}\,\mathrm{d}w \\ &= \lambda^2\int \frac{|\mathcal{F}q(0,w/\lambda)|^2}{w^2+4}\,\mathrm{d}w \\ &= \lambda\int \frac{|\mathcal{F}q(0,w)|^2}{w^2+4\lambda^{-2}}\,\mathrm{d}w\,,\end{aligned}$$

which can be made arbitrarily small.

Setting $Q = \{q_n(0)\}_n$ we have that Q is equicontinuous in $H^{-1}(\mathbb{R})$ as it converges there, By Proposition 5.4 we have that Q^*, as defined in Equation 25, is equicontinuous in $H^{-1}(\mathbb{R})$ as well. What we want is that

$$\|q_n(t) - q_m(t)\|_{H^{-1}(\mathbb{R})} \xrightarrow{n,m\to\infty} 0$$

uniformly for compact time intervals. So we bound

$$\begin{aligned}
&\|q_n(t) - q_m(t)\|_{H^{-1}(\mathbb{R})} \\
&\quad = \left\| \mathrm{Fl}_t^{H_{\mathrm{KdV}}} q_n(0) - \mathrm{Fl}_t^{H_{\mathrm{KdV}}} q_n(0) \right\|_{H^{-1}(\mathbb{R})} \\
&\quad \le \left\| \mathrm{Fl}_t^{H_\kappa} q_n(0) - \mathrm{Fl}_t^{H_\kappa} q_m(0) \right\|_{H^{-1}(\mathbb{R})} \\
&\qquad + \left\| \mathrm{Fl}_t^{H_{\mathrm{KdV}}} q_n(0) - \mathrm{Fl}_t^{H_\kappa} q_n(0) \right\|_{H^{-1}(\mathbb{R})} \\
&\qquad + \left\| \mathrm{Fl}_t^{H_{\mathrm{KdV}}} q_n(0) - \mathrm{Fl}_t^{H_\kappa} q_m(0) \right\|_{H^{-1}(\mathbb{R})} \\
&\quad \le \left\| \mathrm{Fl}_t^{H_\kappa} q_n(0) - \mathrm{Fl}_t^{H_\kappa} q_m(0) \right\|_{H^{-1}(\mathbb{R})} \\
&\qquad + 2 \sup_{q\in Q^*} \left\| \mathrm{Fl}_t^{H_{\mathrm{KdV}}} \mathrm{Fl}_{-t}^{H_\kappa} q - q \right\|_{H^{-1}(\mathbb{R})}
\end{aligned}$$

and taking the supremum over t we have

$$\begin{aligned}
\sup_{|t|\le T} \|q_n(t) - q_m(t)\|_{H^{-1}(\mathbb{R})} \le \sup_{|t|\le T} & \left\| \mathrm{Fl}_t^{H_\kappa} q_n(0) - \mathrm{Fl}_t^{H_\kappa} q_m(0) \right\|_{H^{-1}(\mathbb{R})} \\
& + 2 \sup_{q\in Q^*} \sup_{|t|\le T} \left\| \mathrm{Fl}_t^{H_{\mathrm{KdV}}} \mathrm{Fl}_{-t}^{H_\kappa} q - q \right\|_{H^{-1}(\mathbb{R})}.
\end{aligned}$$

Now by the well-posedness of the H_κ flow, we see that the first term will converge to zero as $n, m \to \infty$. The second term is a bit more involved. Fix $\kappa - 1 > \varkappa > 1$ and write $q(t) = \mathrm{Fl}_t^{H_{\mathrm{KdV}}} \mathrm{Fl}_{-t}^{H_\kappa} q$ for $q \in Q^*$.

Then we have by Proposition 4.1 and Proposition 4.3

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2g(\varkappa)} \\
&= \left(\frac{q}{g(\varkappa)} - \frac{2\varkappa^2}{g(\varkappa)} + 4\varkappa^3\right)' + \frac{4\kappa^5}{\kappa^2-\varkappa^2}\left(\frac{g(\kappa)}{g(\varkappa)} - \frac{\varkappa}{\kappa}\right)' - 4\kappa^2\left(\frac{1}{2g(\varkappa)} - \varkappa\right)' \\
&= \left(\frac{q}{g(\varkappa)} - \frac{2\varkappa^2}{g(\varkappa)} + 4\varkappa^3 + \frac{4\kappa^5}{\kappa^2-\varkappa^2}\frac{g(\kappa)}{g(\varkappa)} - \frac{4\kappa^5}{\kappa^2-\varkappa^2}\frac{\varkappa}{\kappa} - 4\kappa^2\frac{1}{2g(\varkappa)} + 4\kappa^2\varkappa\right)' \\
&= \left(\frac{q}{g(\varkappa)} - \frac{2\varkappa^2}{g(\varkappa)} + \frac{4\kappa^5}{\kappa^2-\varkappa^2}\frac{g(\kappa)}{g(\varkappa)} - 2\kappa^2\frac{1}{g(\varkappa)} - \frac{4\varkappa^5}{\kappa^2-\varkappa^2}\right)' \\
&= \left(\frac{1}{g(\varkappa)}\left(q - 2\varkappa^2 + \frac{4\kappa^5}{\kappa^2-\varkappa^2}g(\kappa) - 2\kappa^2 - \frac{4\varkappa^5}{\kappa^2-\varkappa^2}g(\varkappa)\right)\right)' \\
&= \left(\frac{1}{g(\varkappa)}\left(q - 2\frac{\kappa^4}{\kappa^2-\varkappa^2} + \frac{4\kappa^5}{\kappa^2-\varkappa^2}g(\kappa) + 2\frac{\varkappa^4}{\kappa^2-\varkappa^2} - \frac{4\varkappa^5}{\kappa^2-\varkappa^2}g(\varkappa)\right)\right)' \\
&= \left(\frac{1}{g(\varkappa)}\left(q + \frac{4\kappa^5}{\kappa^2-\varkappa^2}\left(g(\kappa) - \frac{1}{2\kappa}\right) - \frac{4\varkappa^5}{\kappa^2-\varkappa^2}\left(g(\varkappa) - \frac{1}{2\varkappa}\right)\right)\right)'.
\end{aligned}
$$

Therefore, using Equation 9, we can conclude that

$$
\begin{aligned}
\left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\varkappa - \frac{1}{2g(\varkappa)}\right)\right\|_{H^{-2}(\mathbb{R})} \lesssim_\varkappa &\left\| q(t) + 4\kappa^3\left(g(\kappa) - \frac{1}{2\kappa}\right)\right\|_{H^{-1}(\mathbb{R})} \\
&+ \kappa\left\| g(\kappa) - \frac{1}{2\kappa}\right\|_{H^{-1}(\mathbb{R})} \\
&+ \kappa^{-2}\left\| g(\varkappa) - \frac{1}{2\varkappa}\right\|_{H^{-1}(\mathbb{R})}
\end{aligned}
$$

uniformly for $q \in Q^*$. Using the fundamental theorem of calculus and taking suprema and limits we get using Proposition 5.4

$$
\begin{aligned}
&\lim_{\kappa\to\infty}\sup_{q\in Q^*}\sup_{|t|<T}\left\| \frac{1}{2g(0,\varkappa)} - \frac{1}{2g(t,\varkappa)}\right\|_{H^{-2}(\mathbb{R})} \\
&\lesssim T\lim_{\kappa\to\infty}\sup_{q\in Q^*}\sup_{|t|<T}\left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\varkappa - \frac{1}{2g(\varkappa)}\right)\right\|_{H^{-2}(\mathbb{R})} \\
&= 0.
\end{aligned} \tag{26}
$$

Note that by Proposition 3.4 and by Equation 10

$$
\left\{ \varkappa - \frac{1}{2g(\varkappa, q(t))} \mid q \in Q^*,\ t \in \mathbb{R}\right\}
$$

is equicontinuous in $H^1(\mathbb{R})$, so we may upgrade the convergence in Equation 26 to $H^1(\mathbb{R})$ by virtue of Lemma 5.2.

And again by Proposition 3.4 we may conclude

$$\begin{aligned}
&\lim_{\kappa\to\infty} \sup_{q\in Q^*} \sup_{|t|<T} \left\| \mathrm{Fl}_t^{H_{\mathrm{KdV}}} \mathrm{Fl}_{-t}^{H_\kappa} q - q \right\|_{H^{-1}(\mathbb{R})} \\
&\quad = \lim_{\kappa\to\infty} \sup_{q\in Q^*} \sup_{|t|<T} \left\| \frac{1}{2g(0,\varkappa)} - \frac{1}{2g(t,\varkappa)} \right\|_{H^1(\mathbb{R})} \\
&\quad = 0.
\end{aligned}$$

□

Theorem 6.2 Killip and Vişan [2018]
The KdV (1) is globally well-posed in $H^{-1}(\mathbb{R})$ in the sense that the solution map

$$\Phi : \mathbb{R} \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

extends uniquely to a continuous map

$$\Phi : \mathbb{R} \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R}).$$

The orbits $\{\Phi(t,q) \mid t \in \mathbb{R}\}$ are bounded and equicontinuous in $H^{-1}(\mathbb{R})$ with

$$\sup_t \|q(t)\|_{H^{-1}(\mathbb{R})} \lesssim \|q(0)\|_{H^{-1}(\mathbb{R})} + \|q(0)\|^3_{H^{-1}(\mathbb{R})}$$

and Φ fulfills

$$\Phi(t+s) = \Phi(t) \circ \Phi(s).$$

Proof. For $q \in H^{-1}(\mathbb{R})$ choose some $q_n(t) \in \mathcal{S}(\mathbb{R})$ which solve the KdV (1) and whose initial conditions converge to q

$$q_n(0) \xrightarrow[H^{-1}(\mathbb{R})]{n\to\infty} q.$$

Then we define the extension of Φ as

$$\Phi(t,q) = \lim_{n\to\infty} q_n(t)$$

which exists in $H^{-1}(\mathbb{R})$, is independent of the choice of q_n and convergence is uniform for compact time intervals by Theorem 6.1. To show continuity, take $q_n \to q \in H^{-1}(\mathbb{R})$ and $T > 0$. By Theorem 6.1 we know that there are Schwartz solutions $\tilde{q}_n$ with

$$\sup_{|t|\le T} \|\tilde{q}_n(t) - \Phi(t,q_n)\|_{H^{-1}(\mathbb{R})} \xrightarrow{n\to\infty} 0.$$

Especially, $\tilde{q}_n(0) \xrightarrow[H^{-1}(\mathbb{R})]{n\to\infty} q$, and again by Theorem 6.1 we get convergence

$$\sup_{|t|\le T} \|\tilde{q}_n(t) - \Phi(t,q)\|_{H^{-1}(\mathbb{R})} \xrightarrow{n\to\infty} 0.$$

Since the $\tilde{q}_n(t)$ are $H^{-1}(\mathbb{R})$ continuous in time, we see that Φ is continuous. The group property follows from continuity and the group property of Φ on $\mathcal{S}(\mathbb{R})$. For an orbit $\{\Phi(t,q) \mid t \in \mathbb{R}\}$ we know that $\alpha(\kappa) \approx \frac{1}{\kappa}\|\Phi(t,q)\|^2_{H^{-1}_\kappa(\mathbb{R})}$ is conserved. Hence, we see that the orbits stay bounded and are equicontinuous by Lemma 5.3. We especially have

$$\|q(t)\|_{H^{-1}(\mathbb{R})} \approx \alpha(1) \approx \|q(0)\|_{H^{-1}(\mathbb{R})}$$

for sufficiently small initial conditions and by scaling we have for initial conditions with $\|q\|_{H^{-1}(\mathbb{R})} \geq \delta$

$$\|q(t)\|_{H^{-1}(\mathbb{R})} \lesssim \|q(0)\|^3_{H^{-1}(\mathbb{R})}.$$

In general, we thence have

$$\|q(t)\|_{H^{-1}(\mathbb{R})} \lesssim \|q(0)\|_{H^{-1}(\mathbb{R})} + \|q(0)\|^3_{H^{-1}(\mathbb{R})}.$$

□

Corollary 6.3
In the same sense as in Theorem 6.2 the KdV (1) is globally well-posed in $H^s(\mathbb{R})$ for all $s \geq -1$.

Proof. We will restrict ourselves to $s < 0$, for a more direct proof. Let $q_n(t) \in \mathcal{S}(\mathbb{R})$ be solutions with $q_n(0)$ convergent in $H^s(\mathbb{R})$. Using Theorem 6.1 we get that $q_n(t)$ converges in $H^{-1}(\mathbb{R})$ uniformly for compact time intervals. Integrate

$$\begin{aligned}\int_{\kappa_0}^{\infty} \alpha(\kappa)\kappa^{2+2s}\,\mathrm{d}\kappa &\approx \int_{\kappa_0}^{\infty} \|q\|^2_{H^{-1}(\mathbb{R})}\kappa^{1+2s}\,\mathrm{d}\kappa \\ &= \int |\mathcal{F}q|^2 \int_{\kappa_0}^{\infty} \frac{1}{w^2+4\kappa^2}\kappa^{1+2s}\,\mathrm{d}\kappa\,\mathrm{d}w \\ &\approx_s \int |\mathcal{F}q|^2 (w^2+4\kappa_0^2)^s\,\mathrm{d}w\end{aligned}$$

and observe that the left-hand side is conserved under the KdV flow. So we obtain

$$\int |\mathcal{F}q_n(0)|^2 (w^2+4\kappa_0^2)^s\,\mathrm{d}w \approx_s \int |\mathcal{F}q_n(t)|^2 (w^2+4\kappa_0^2)^s\,\mathrm{d}w\,.$$

Now let $\kappa_0 \to \infty$, then the left-hand side converges to 0 uniformly in n as the initial conditions are $H^s(\mathbb{R})$ convergent and thus equicontinuous. Therefore, the same holds for the right-hand side uniformly in t and n so

$$\{q_n(t) \mid n \in \mathbb{N}, t \in \mathbb{R}\}$$

is equicontinuous in $H^s(\mathbb{R})$ and combined with the convergence in $H^{-1}(\mathbb{R})$ we obtain uniform convergence of $q_n(t)$ in $H^s(\mathbb{R})$ by Lemma 5.2. □

References

R. Abraham and J. E. Marsden. *Foundations of mechanics.* American Mathematical Soc., 2008.

R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media, 2012.

V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.

P. Bernat. *Représentations des groupes de Lie résolubles.* Dunod, 1972.

M. Christ, J. Colliander, and T. Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, 2004.

J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Sharp Global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, 2001.

M. Erdoğan, N. Tzirakis, and V. Zharnitsky. Nearly linear dynamics of nonlinear dispersive waves. *Physica D: Nonlinear Phenomena*, 240(17):1325–1333, 2011.

Z. Guo. Global well-posedness of korteweg-de vries equation in $H^{-3/4}(\mathbb{R})$, 2009.

R. Killip and M. Vişan. KdV is wellposed in H^{-1}, 2018. URL https://arxiv.org/abs/1802.04851.

S. Lang. *Real and functional analysis*, volume 142. Springer Science & Business Media, 2012.

J. M. Lee. *Smooth Manifolds*, pages 1–31. Springer New York, New York, NY, 2012. ISBN 978-1-4419-9982-5. doi: 10.1007/978-1-4419-9982-5_1. URL https://doi.org/10.1007/978-1-4419-9982-5_1.

L. Molinet. A note on ill-posedness for the KdV equation, 2010.

M. Reed and B. Simon. *II: Fourier Analysis, Self-Adjointness: Volume 2*. Methods of Modern Mathematical Physics. Academic Press, San Diego, CA, Nov. 1975.

B. Simon. *Trace ideals and their applications.* American Mathematical Soc., 2005.

G. Teschl. *Mathematical methods in quantum mechanics*, volume 157. American Mathematical Soc., 2014.

A.-M. Wazwaz. *The Family of the KdV Equations*, pages 503–556. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009. ISBN 978-3-642-00251-9. doi: 10.1007/978-3-642-00251-9_13. URL https://doi.org/10.1007/978-3-642-00251-9_13.

V. E. Zakharov and L. D. Faddeev. Korteweg-de Vries equation: A completely integrable Hamiltonian system. *Functional Analysis and Its Applications*, 5(4):280–287, Oct 1971. ISSN 1573-8485. doi: 10.1007/BF01086739. URL https://doi.org/10.1007/BF01086739.